nasa-cr-178,022

NASA Contractor Report 178022

ICASE REPORT NO. 85-53

NASA-CR-178022
19860009179

ICASE

MATCHED ASYMPTOTIC EXPANSIONS
TO SIMILARITY SOLUTIONS OF SHOCK DIFFRACTION

Eduard Harabetian

Contract No. NAS1-17070
November 1985

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

NASA
National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

MATCHED ASYMPTOTIC EXPANSIONS TO SIMILARITY SOLUTIONS

OF SHOCK DIFFRACTION

Eduard Harabetian

Courant Institute of Mathematical Sciences

and

Institute for Computer Applications in Science and Engineering

ABSTRACT

We match formal asymptotic expansions with differently scaled variables to obtain a uniform approximation to the similarity solution of the shock-wedge diffraction problem.

Research was supported in part by the National Aeronautics and Space Administration under NASA Contract No. NAS1-17070 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665-5225 and in part by NSF-MCS-82-01599 while the author was at the Courant Institute of Mathematical Sciences, New York.

1. INTRODUCTION

The time-dependent compressible Euler equations are a hyperbolic system of five conservation laws

$$(1.1) \qquad u_t + \sum_{i=1}^{3} \left(f_i(u)\right)_{x_i} = 0, \quad u \in \mathbb{R}^5.$$

In spherical coordinates $(r,\theta) \in R^+ \times S^2$, the equations assume the following form

$$(1.2) \qquad u_t + F(\theta,u)_r + \frac{1}{r} \sum_{i=1}^{2} G_i(\theta,u)_{\theta_i} + \frac{1}{r} H(\theta,u) = 0.$$

Since they are rotationally invariant, there exists $K(\theta)$ such that

$$K^{-1} F(\theta,Ku) = \tilde{F}(u)$$

$$K^{-1} G_i(\theta,Ku) = \tilde{G}_i(u)$$

$$-\sum K^{-1}_{\theta_i} G_i(\theta,Ku) + K^{-1}(\theta)H(\theta,Ku) = \tilde{H}(u)$$

for some $\tilde{F}$, $\tilde{G}_i$, $\tilde{H}$. Then, by setting $u = K(\theta)v$ we can express (1.2) as

$$(1.3) \qquad v_t + \left(\tilde{F}(v)\right)_r + \frac{1}{r} \sum \tilde{G}_i(v)_{\theta_i} + \frac{1}{r} \tilde{H}(v) = 0.$$

For convenience we suppress the tildas in (1.3). A similarity solution of (1.3) depends only on $s = \frac{r}{t}$, θ and satisfies

$$(1.4) \qquad \left(F(v) - sv\right)_s + \frac{1}{s} \left(G(v)\right)_\theta + \frac{1}{s} H(v) + v = 0$$

where

$$G_\theta = \sum (G_i)_{\theta_i},$$

or, in nonconservative form

$$\left(F_v(v) - s\right)v_s + \frac{1}{s} G_v(v)v_\theta + \frac{1}{s} H(v) = 0.$$

Let $L(v)F_v(v) = \Lambda(v)L(v)$ with

$$\Lambda(v) = \begin{bmatrix} \lambda_1(v) & & 0 \\ & \ddots & \\ 0 & & \lambda_5(v) \end{bmatrix}.$$

If we multiply by $L(v)$ on the left

$$(1.5) \qquad (\Lambda - s)Lv_s + \frac{1}{s} LG_v\, v_\theta + \frac{1}{s} LH = 0.$$

Suppose one substitutes

$$(1.6) \qquad v(\theta,s) = v_0 + \varepsilon v_1(\theta,s) + \varepsilon^2\, v_2 + \cdots$$

into (1.5), with v_0 a constant solution. Then $v_1(\theta,s)$ will satisfy the linear equation

$$\left(\Lambda(v_0) - s\right)L(v_0)(v_1)_s + \frac{1}{s} L(v_0)G_v(v_0)(v_1)_\theta + \frac{1}{s} (LH)_v(v_0)v_1 = 0.$$

Suppose we had the appropriate boundary condition, and subsequently solved the boundary value problem for v_1. Note that the system degenerates where $s = \lambda_j(v_0)$, $j = 1,\cdots,5$; therefore, one suspects the solution

$v_1(\theta,s)$ will develop a singularity at that point. Later we will see that the singularity is typically of the form $\sqrt{s-\lambda_j}$. This shows that v_{1_s} blows up at λ_j and therefore the expansion (1.6) is no longer valid when s is too close to λ_j. In our case "too close" means within ε^2.

In order to see nonlinear phenomena such as shocks and expansion waves, one needs to let s vary near λ_j. Thus we are led to an ansatz different than (1.6).

Near an eigenvalue $\lambda_j(v_0)$ we rescale

$$s = \lambda_j(v_0) + \varepsilon\nabla\lambda_j(v_0)\cdot v_1(\theta,\lambda_j(v_0)) + \varepsilon^2\,\overline{s}$$

and we let

$$v = v_0 + \varepsilon v_1(\theta,\lambda_j(v_0)) + \varepsilon^2\,\overline{v}(\theta,\overline{s}) + \cdots. \tag{1.7a}$$

Note that if

$$v_1(\theta,s) = v_1(\theta,\lambda_j(v_0)) + \sqrt{s-\lambda_j(v_0)}\;\tilde{v}_1(\theta) + \cdots$$

then in the expansion (1.6)

$$v = v_0 + \varepsilon v_1(\theta,\lambda_j(v_0)) + \varepsilon\sqrt{s-\lambda_j(v_0)}\;\tilde{v}_1(\theta) + \cdots. \tag{1.7b}$$

Our application is the shock-wedge diffraction problem in two dimensions (Figure 1.2). We will substitute (1.7a) and solve for $\overline{v}$ explicitly by matching its boundary values with the third term in (1.7b), which is obtained from Keller and Blank [1] where the solution to the linearized problem is given. The term $\overline{v}$ will capture the position of the diffracted shock, within the order of approximation. It will contain vorticity generated by the curved shock. The $O(\varepsilon)$ approximation, v_1, is too crude to see such effects.

To be specific we consider the isentropic, two-dimensional Euler equations in polar coordinates and self similar form (see (1.4)):

$$(1.8)\qquad \begin{bmatrix} \rho(R-r) \\ \rho(R-r)R+p \\ \rho(R-r)\Theta \end{bmatrix}_r + \frac{1}{r}\begin{bmatrix} \rho\Theta \\ \rho\Theta R \\ \rho\Theta^2+p \end{bmatrix}_\theta + \begin{bmatrix} \rho \\ \rho R \\ \rho\Theta \end{bmatrix} + \frac{1}{r}\begin{bmatrix} \rho R \\ \rho R^2-\rho\Theta^2 \\ 2\rho\Theta R \end{bmatrix} = 0.$$

Here $r = \frac{\sqrt{x^2+y^2}}{t}$, $\theta = \tan^{-1}\frac{y}{x}$,

$$\begin{bmatrix} R \\ \Theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix},$$

and the equation of state is $p = A\rho^\gamma$.

The assumption of constant entropy is no loss of generality because the changes in entropy are of order lower than the order of our approximation.

Differentiating (1.8) and simplifying

$$(1.9a)\qquad \begin{bmatrix} R-r & \rho & 0 \\ \frac{c^2}{\rho} & R-r & 0 \\ 0 & 0 & R-r \end{bmatrix}\begin{bmatrix} \rho \\ R \\ \Theta \end{bmatrix}_r + \frac{1}{r}\begin{bmatrix} \Theta & 0 & \rho \\ 0 & \Theta & 0 \\ \frac{c^2}{\rho} & 0 & \Theta \end{bmatrix}\begin{bmatrix} \rho \\ R \\ \Theta \end{bmatrix}_\theta$$

$$+ \frac{1}{r}\begin{bmatrix} 0 & \rho & 0 \\ 0 & 0 & -\Theta \\ 0 & \Theta & 0 \end{bmatrix}\begin{bmatrix} \rho \\ R \\ \Theta \end{bmatrix} = 0.$$

Multiplying on the left by the eigenvector matrix

$$\begin{bmatrix} c & \rho & 0 \\ c & -\rho & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} R - r + c & 0 & 0 \\ 0 & R - r - c & 0 \\ 0 & 0 & R - r \end{bmatrix} \begin{bmatrix} c & \rho & 0 \\ c & -\rho & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ R \\ \Theta \end{bmatrix}_r$$

(1.9b)

$$+ \frac{1}{r} \begin{bmatrix} c\Theta & \rho\Theta & c\rho \\ c\Theta & -\rho\Theta & c\rho \\ \frac{c^2}{\rho} & 0 & \Theta \end{bmatrix} \begin{bmatrix} \rho \\ R \\ \Theta \end{bmatrix}_\theta + \frac{1}{r} \begin{bmatrix} 0 & c\rho & -\rho\Theta \\ 0 & c\rho & \rho\Theta \\ 0 & \Theta & 0 \end{bmatrix} \begin{bmatrix} \rho \\ R \\ \Theta \end{bmatrix} = 0.$$

Consider now a weak shock impinging on a wedge so that at time $t = 0$ the shock reaches the wedge. The initial values at $t = 0$, comprising of the states ahead and behind the shock, are shown in Figure 1.1.

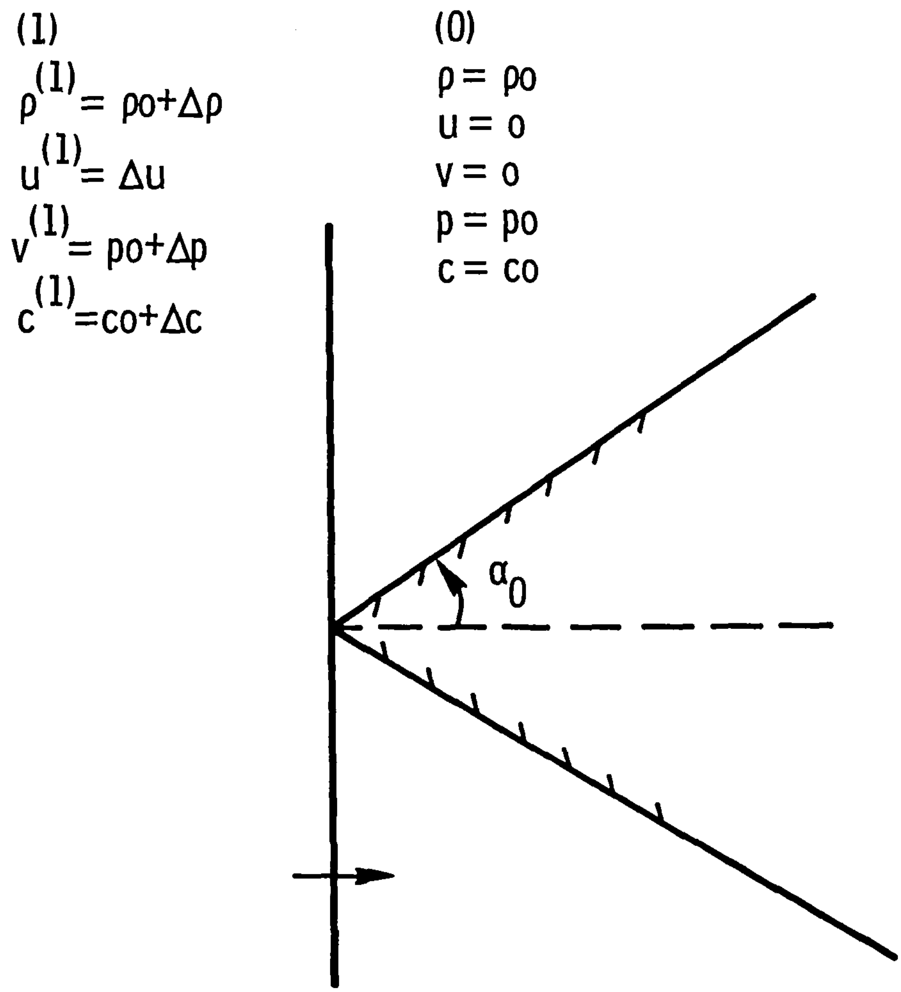

Figure 1.1

The problem above could be viewed as the simplest Riemann Problem in two dimensions that cannot be reduced to one dimension. A Riemann Problem is an initial value problem with piecewise constant states and straight discontinuity interfaces. The existence of a solution remains an open question; however, it is presumed the solutions are of similarity type. We wish to obtain the first few terms in the uniform formal asymptotic expansion approximating the similarity solution in Figure 1.1. The solution is symmetric about the real axis. Its upper half plane restriction is shown in similarity coordinates in Figure 1.2. It consists of the reflected shock (S_R) which interacts with the expansion wave E and curves into the diffracted shock S_D.

The state (2) behind the reflected shock is constant and could be obtained exactly, together with the shock angle, from shock polars [4]. We only need the first two asymptotic terms which we derive in Section 2. The curve separating state (2) from the expansion wave is characteristic, and the solution is expected to have a gradient discontinuity across it, in analogy with the one-dimensional rarefactions.

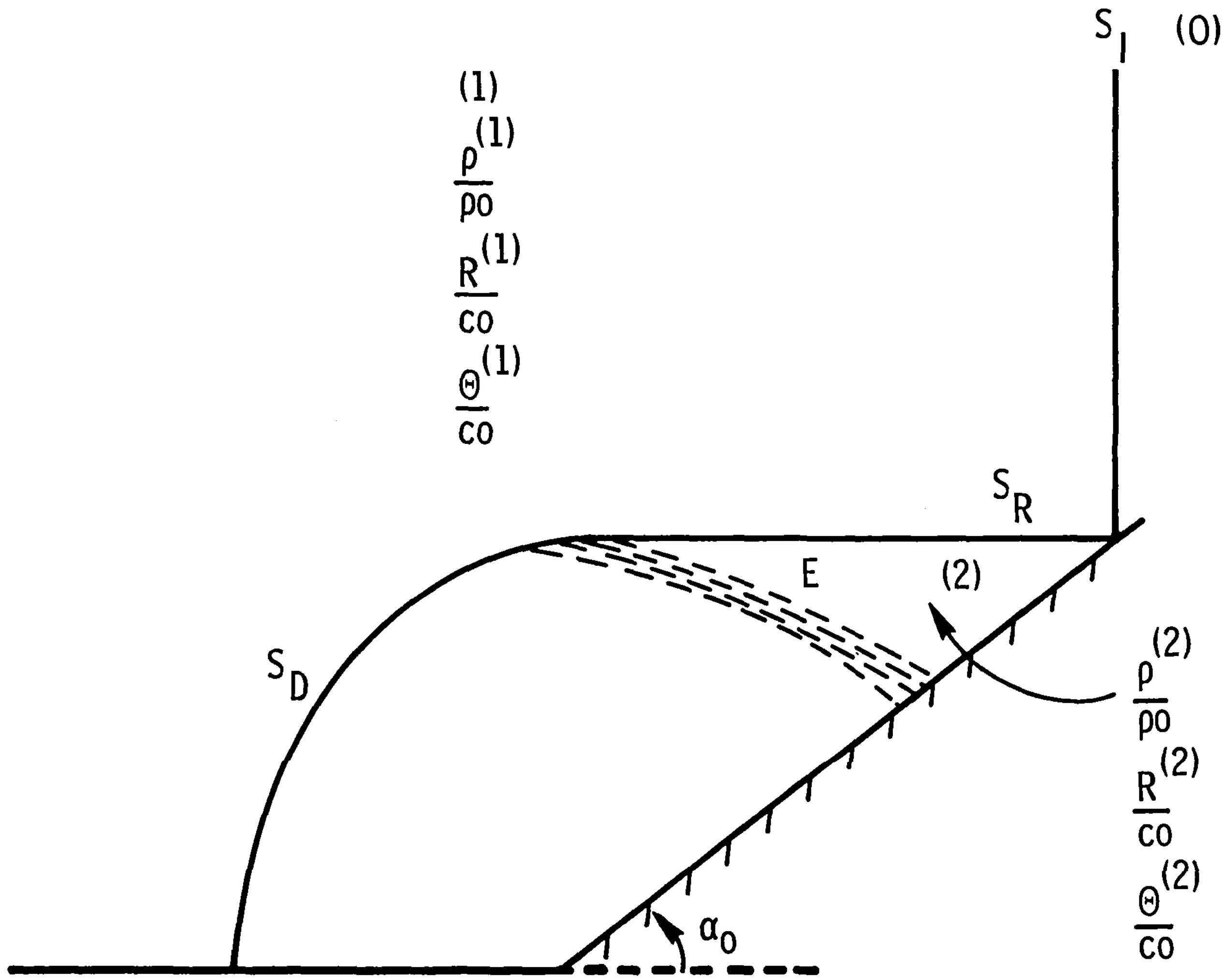

Figure 1.2

Away from the "corner" where E meets S_D, we scale the radial variable in accordance with (1.7a) and obtain the asymptotic behavior of S_D and E. The region where E meets S_D requires a special ansatz where one needs to scale θ appropriately as well. This is to be expected since changes in the tangential direction are substantial there. As a result we obtain a transonic equation, resembling the small disturbance equation in steady flows, for which we can provide boundary condition by matching with the outside. Solving this boundary value problem analytically, or even proving the existence of a solution, remains an open question. One encounters difficulties analogous to the boundary value problem for the small disturbance equation.

Hunter and Keller have applied weakly nonlinear geometrical optics, [2], to the shock-wedge diffraction problem [3]. Their ansatz makes use of additional fast varying variables in analogy with the geometric optics construction for highly oscillatory solutions. This method, however, did not apply to the interaction at the "corner" (see Figure 1.2). In contrast with [3] we consider the equivalent boundary value problem for the pseudosteady solution in similarity coordinates and systematically match asymptotic expansions in which the original variables are rescaled to reflect nonlinear behavior. This way we can obtain the uniform asymptotic behavior over the whole space.

We mention that in one space dimension DiPerna and Majda [5] have justified the weakly nonlinear geometric optics construction.

2. SHOCK REFLECTION

Here we will obtain the first two terms in the asymptotic approximation to the shock reflection at the wall (see Figure 1.2). The small parameter, throughout this paper, is $\frac{\Delta\rho}{\rho_0} = \varepsilon$. From normal shock relations, one easily computes that for state (1):

$$\frac{\rho^{(1)}}{\rho_0} = 1 + \varepsilon$$

$$\frac{u^{(1)}}{c_0} = \varepsilon + \frac{\gamma - 3}{4}\varepsilon^2 + 0(\varepsilon^3)$$

$$v^{(1)} = 0 \tag{2.1}$$

$$\frac{c^{(1)}}{c_0} = 1 + \frac{\gamma - 1}{2}\varepsilon + \frac{(\gamma - 1)(\gamma - 3)}{8}\varepsilon^2 + 0(\varepsilon^3)$$

$$\frac{S_I}{c_0} = 1 + \frac{\gamma + 1}{4}\varepsilon + 0(\varepsilon^2)$$

where S_I is the speed of the incident shock.

We will now show how to obtain the asymptotic terms for state (2). The first term is simple and easy to compute. It corresponds to the linear theory of reflection. In particular it shows that the angle of reflection equals the angle of incidence. The second term is rather tedious to compute.

We rotate the plane so that the wall becomes horizontal (see Figure 2.1).

The Rankine-Hugoniot conditions across the reflected shock in Cartesian coordinates are:

$$\sin\beta\begin{bmatrix}\rho(u-x_0)\\ \rho u(u-x_0)+p\\ \rho u(u-x_0)\end{bmatrix}_{S_R} + \cos\beta\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\end{bmatrix}_{S_R} = 0, \tag{2.2}$$

where x_0 is the velocity of the incident shock along the ramp, β is the angle of reflection and the subscript at the brackets denotes a jump at the reflected shock.

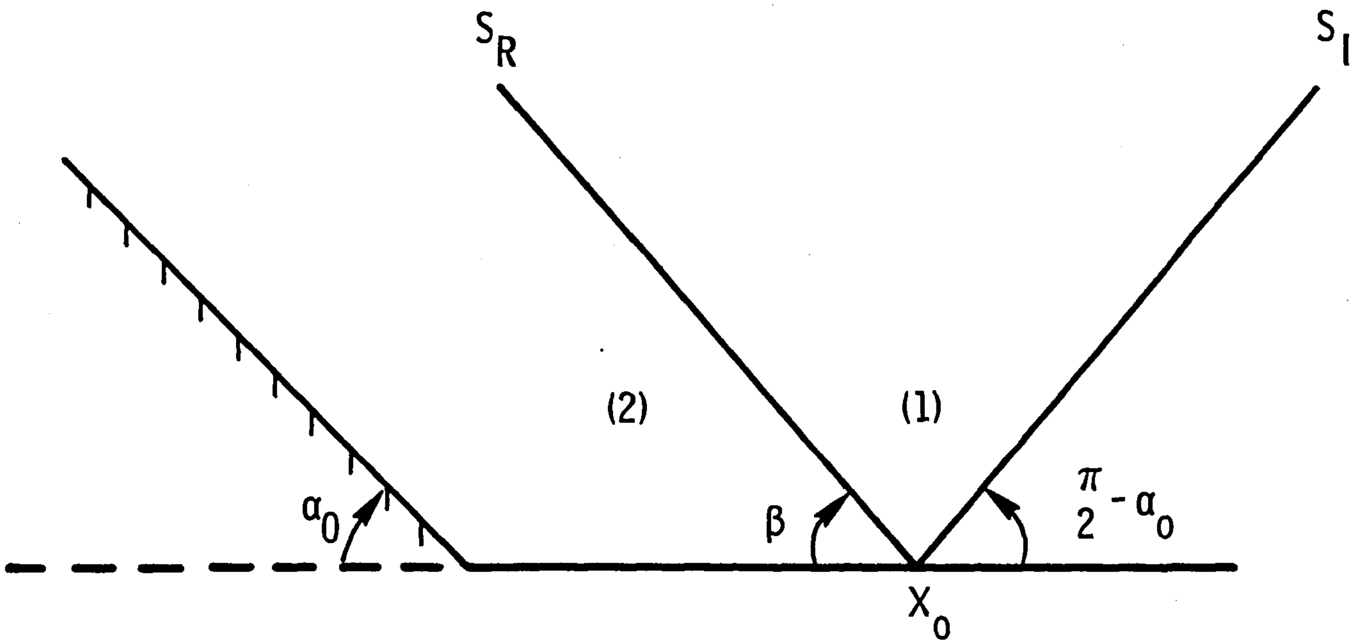

Figure 2.1

From (2.1) we obtain the rotated velocities for state (1):

$$\frac{u^{(1)}}{c_0} = (\varepsilon + \frac{\gamma - 3}{4}\varepsilon^2)\cos\theta + O(\varepsilon^3),$$

$$\frac{v^{(1)}}{c_0} = -(\varepsilon + \frac{\gamma - 3}{4}\varepsilon^2)\sin\alpha_0 + O(\varepsilon^3),$$

and

$$\frac{x_0}{c_0} = \frac{1}{\cos\alpha}\left[1 + \frac{\gamma+1}{4}\varepsilon\right] + 0(\varepsilon^2).$$

For state (2) we set

$$\frac{\rho^{(2)}}{\rho_0} = 1 + \varepsilon\rho_1^{(2)} + \varepsilon^2\rho_2^{(2)} + 0(\varepsilon^3),$$

$$\frac{u^{(2)}}{c_0} = 1 + \varepsilon u_1^{(2)} + \varepsilon^2 u_2^{(2)} + 0(\varepsilon^3),$$

$$v^{(2)} = 0,$$

and

$$\tan\beta = \tan\beta_0 + \varepsilon\tan\beta_1 + 0(\varepsilon^2).$$

Substituting in (2.2) we obtain the $0(\varepsilon)$ equations

$$\sin\beta_0\left(-\sin^2\alpha_0 - \cos\alpha_0\, u_1^{(2)} + \rho_1^{(2)}\right) - \cos\beta_0\sin\alpha_0\cos\alpha_0 = 0$$

$$u_1^{(2)} - \cos\alpha_0\,\rho_1^{(2)} = 0$$

$$\sin\beta_0\tan\alpha_0 + \cos\beta_0\left(1 - \rho_1^{(2)}\right) = 0$$

and the solution

$$\rho_1^{(2)} = 2$$

$$u_1^{(2)} = 2\cos\alpha_0$$

$$\beta_0 = \frac{\pi}{2} - \alpha_0.$$

After some algebra we obtain the following equations from $O(\varepsilon^2)$ terms:

$$\frac{-\rho_2^{(2)}}{\cos\alpha_0} - u_2^{(2)} + \tan^2\alpha_0 \sin\alpha_0 \tan\beta_1 + \cos\alpha_0 \frac{\gamma - 7}{2} = 0,$$

$$-\rho_2^{(2)} + \frac{u_2^{(2)}}{\cos\alpha_0} + 2\sin^2\alpha_0 + \frac{5 - 3\gamma}{2} = 0,$$

$$-\rho_2^{(2)} + \tan\alpha_0 \tan\beta_1 + \sin^2\alpha_0 - \cos^2\alpha_0 + 2 - \gamma = 0$$

from which we can obtain the solution $\rho_2^{(2)}$, $u_2^{(2)}$, β_1, as long as $0 < \alpha_0 < \frac{\pi}{2}$. For simplicity of notation we won't write them down explicitly. In the original, unrotated coordinates:

$$\frac{\rho^{(2)}}{\rho_0} = 1 + 2\varepsilon + \rho_2^{(2)}\varepsilon^2,$$

$$\frac{u^{(2)}}{c_0} = 2\cos^2\alpha_0\,\varepsilon + \cos\alpha_0\, u_2^{(2)}\varepsilon^2,$$

$$(2.3) \qquad \frac{v^{(2)}}{c_0} = 2\sin\alpha_0\cos\alpha_0\,\varepsilon - \sin\alpha_0\, u_2^{(2)}\varepsilon^2,$$

$$\frac{c^{(2)}}{c_0} = 1 + (\gamma - 1)\varepsilon + \left(\frac{\gamma - 1}{2}\rho_2^{(2)} + \tfrac{1}{2}(\gamma - 1)(\gamma - 3)\right)\varepsilon^2,$$

$$\tan\beta = c_0 + \alpha_0 + \varepsilon\tan\beta_1 + O(\varepsilon^2),$$

β = angle of S_R with wall, and from (2.1) and (2.3)

$$(2.4)\qquad \begin{aligned} \frac{\rho^{(i)}}{\rho_0} &= 1 + \rho_1^{(i)}\varepsilon + \rho_2^{(i)}\varepsilon^2 + \cdots \\ \frac{R^{(i)}}{c_0} &= R_1^{(i)}\varepsilon + R_2^{(i)}\varepsilon^2 + \cdots \\ \frac{\Theta^{(i)}}{c_0} &= \Theta_1^{(i)}\varepsilon + \Theta_2^{(i)}\varepsilon^2 + \cdots, \qquad i = 1,2 \end{aligned}$$

and

$$\rho_1^{(i)} = \begin{cases} 1 & i = 1 \\ 2 & i = 2 \end{cases}, \qquad \rho_2^{(i)} = \begin{cases} 0 & i = 1 \\ \rho_2^{(2)} & i = 2 \end{cases}$$

$$R_1^{(i)} = \begin{cases} \cos\theta & i = 1 \\ 2\cos\alpha_0 \cos(\theta - \alpha_0) & i = 2 \end{cases}, \qquad R_2^{(i)} = \begin{cases} \frac{\gamma - 3}{4}\cos\theta & i = 1 \\ R_2^{(2)}(\theta) & i = 2 \end{cases}$$

$$\Theta_1^{(i)} = \begin{cases} -\sin\theta & i = 1 \\ -2\cos\alpha_0 \sin(\theta - \alpha_0) & i = 2 \end{cases}, \qquad \Theta_2^{(i)} = \begin{cases} -\frac{\gamma - 3}{4}\sin\theta & i = 1 \\ \Theta_2^{(2)}(\theta) & i = 2 \end{cases}$$

where $\rho_2^{(2)}$, $R_2^{(2)}$, $\Theta_2^{(2)}$ can be explicitly computed.

3. INTERIOR ANSATZ

We start with an "interior" approximation analogous to (1.6):

$$\frac{r}{c_0} = \bar{r}$$

$$\frac{\rho}{\rho_0} = 1 + \varepsilon\bar{\rho} + \varepsilon^2\bar{\bar{\rho}} + \cdots \tag{3.1}$$

$$\frac{R}{c_0} = \varepsilon\bar{R} + \varepsilon^2\bar{\bar{R}} + \cdots$$

$$\frac{\Theta}{c_0} = \varepsilon\bar{\Theta} + \varepsilon^2\bar{\bar{\Theta}} + \cdots.$$

Substituting in (1.9a) and collecting $O(\varepsilon)$ terms, we obtain the linear equations

$$-\bar{r}\,\bar{\rho}_{\bar{r}} + \bar{R}_{\bar{r}} + \frac{1}{\bar{r}}(\bar{\Theta}_\theta + \bar{R}) = 0$$

$$\bar{\rho}_{\bar{r}} - \bar{r}\,\bar{R}_{\bar{r}} = 0 \tag{3.2}$$

$$-\bar{r}\,\bar{\Theta}_{\bar{r}} + \frac{1}{\bar{r}}\bar{\rho}_\theta = 0,$$

which imply the following equation in ρ alone

$$\bar{r}^2[(1 - \bar{r}^2)\bar{\rho}_{\bar{r}}]_{\bar{r}} + \bar{\rho}_{\theta\theta} + \bar{r}\,\bar{\rho}_{\bar{r}} = 0. \tag{3.3}$$

<u>Remark</u>: Solutions to (3.3) are just similarity solutions to the wave equation $\bar{\rho}_{tt} = \Delta\bar{\rho}$.

We go further and collect $O(\varepsilon^2)$ terms and obtain

$$(3.4)\qquad \bar{r}^2[(1-\bar{r}^2)\bar{\bar{\rho}}_{\bar{r}}]_{\bar{r}} + \bar{\bar{\rho}}_{\theta\theta} + \bar{r}\,\bar{\bar{\rho}}_{\bar{r}} = G,$$

where G has nonlinear dependence on $\bar{\rho}_{\bar{r}\bar{r}}$, $\bar{\rho}_{\bar{r}}$, ρ, ρ_θ.

In order to obtain the $O(\varepsilon)$ uniform approximation we seek solutions to (3.3) with $\frac{\partial\bar{\rho}}{\partial\vec{n}} = 0$ on the solid body and by Huygen's Principle (see [1] and the above Remark) and (2.4):

$$(3.5)\qquad \bar{\rho}(1,\theta) = \rho_1 = \begin{cases} 1 & 2\alpha_0 < \theta \leq \pi \\ 2 & \alpha_0 < \theta < 2\alpha_0 \end{cases}.$$

The solution was obtained by Keller and Blank [1] using the Busemann transformation

$$\ell = \frac{\bar{r}}{1+\sqrt{1-\bar{r}^2}},$$

which takes (3.3) into the Laplace equation

$$\Delta\rho = \ell(\ell\bar{\rho}_\ell)_\ell + \bar{\rho}_{\theta\theta} = 0.$$

The solution they obtained is

$$(3.6)\qquad \begin{aligned}\bar{\rho} = 1 &+ \frac{1}{\pi}\,\mathrm{tg}^{-1}\left\{\frac{(1-\ell^{2\lambda})\cos\lambda\pi}{-(1+\ell^{2\lambda})\sin\lambda\pi - 2\ell^\lambda\sin\lambda(\theta-\pi)}\right\} \\ &+ \frac{1}{\pi}\,\mathrm{tg}^{-1}\left\{\frac{-(1-\ell^{2\lambda})\cos\lambda\pi}{(1+\ell^{2\lambda})\sin\lambda\pi - 2\ell^\lambda\sin\lambda(\theta-\pi)}\right\}\end{aligned}$$

with $\lambda = \frac{\pi}{2}\left(\frac{1}{\pi - \alpha_0}\right)$, and $tg^{-1}\colon \mathbb{R} \to [0,\pi]$. Note that $\sin \lambda(\pi - 2\alpha_0) = \sin \lambda\pi$. We are interested in the asymptotic behavior of (3.6) for ℓ near 1.

For $\theta \neq 2\alpha_0$

$$(3.7) \qquad \bar{\rho} = \rho_1 - \frac{1}{\pi} \frac{\sqrt{2}\,\lambda \sin 2\lambda\pi}{(\sin \lambda\pi)^2 - \left(\sin \lambda(\pi - \theta)\right)^2} \sqrt{1 - \bar{r}} + O(1 - \bar{r}).$$

The approximation (3.7) is not valid for θ near $2\alpha_0$. We observe, however, that if $\bar{r} = 1 + \varepsilon r'$ and $\theta = 2\alpha_0 + \varepsilon^{1/2}\theta'$, then

$$(3.8) \qquad \bar{\rho} = 1 + \frac{1}{\pi}\, tg^{-1} \frac{\sqrt{-2r'}}{\theta'} + O(\varepsilon^{1/2}), \quad r' < 0.$$

Since $-\bar{r}^2\, \bar{\Theta}_{\bar{r}} + \bar{\rho}_\theta = 0$ we also have

$$\bar{\Theta} = \Theta_1(2\alpha_0) + O(\varepsilon^{1/2}) \frac{\sqrt{-2r'}}{\theta'} + \cdots.$$

The two expansions (3.7), (3.8) match in the region where $\theta = 2\alpha_0 + \eta\tilde{\theta}$ and $\varepsilon^{1/2} \ll \eta \ll 1$.

Using (3.6) in (3.4) and the Busemann variables,

$$\Delta\bar{\bar{\rho}} = G$$

and $G = \dfrac{G_0(\theta)}{(1 - \ell)^3} + G_1(\theta,\ell)$ for (θ,ℓ) near $(\theta_0,1)$ with a fixed $\theta_0 \neq 2\alpha_0$ and $(1 - \ell)^2\, G_1(\theta,\ell)$ continuous at $\ell = 1$. Let $\tilde{\rho} = \bar{\bar{\rho}} - \frac{1}{2}\dfrac{G_0(\theta)}{(1 - \ell)}$. Then $\Delta\tilde{\rho} \in H^{-2}_{loc}$ where H^{-2}_{loc} is the local Sobolev space for a neighborhood of $(\theta_0,1)$.

Using local regularity results for Δ we obtain $\tilde{\rho} \in L^2_{loc}$ and therefore the asymptotic behavior of $\bar{\bar{\rho}}$ for ℓ near 1, $\theta \neq 2\alpha_0$:

$$\bar{\bar{\rho}} \sim \tfrac{1}{2} G_0(\theta) \frac{1}{1 - \ell} \sim \tfrac{1}{2} G_0(\theta) \frac{1}{\sqrt{1 - \bar{r}}} .$$

This shows that in the expansion (3.1), we have

$$\rho/\rho_0 = 1 + \varepsilon\bar{\rho} + \varepsilon^2 \bar{\bar{\rho}} + \cdots = 1 + \varepsilon\rho_1 + \varepsilon 0\left(\left(\sqrt{1 - \bar{r}}\right)\right) + \varepsilon^2\, 0\left(\frac{1}{\sqrt{1 - \bar{r}}}\right) + \cdots \quad \text{for} \quad \theta \neq 2\alpha_0. \tag{3.9}$$

In order to justify the boundary values in (3.5), that is, match (3.1) and (2.4) to $0(\varepsilon)$, it is necessary that $\varepsilon^2 \frac{1}{\sqrt{1 - \bar{r}}} \ll \varepsilon$. Therefore, we restrict the linear solution (3.6) to

$$\varepsilon^2 \ll 1 - \bar{r} \ll 1. \tag{3.10}$$

We have not yet completed the $0(\varepsilon)$ asymptotics since we haven't matched the "corner" expansion (3.8). We postpone it until the last section. With θ not near $2\alpha_0$ we wish to move in closer to the eigenvalue and match the $\sqrt{1 - \bar{r}}$ term in the expansion of $\bar{\rho}$ (3.7).

4. EXTERIOR ANSATZ

In this section we consider θ away from $2\alpha_0$. Following the procedure outlined in the Introduction we rescale the radial variable close to an eigenvalue. The eigenvalue in question is $R + c = c_0(1 + c_1\varepsilon + R_1\varepsilon) + 0(\varepsilon^2)$.

Consequently our ansatz is

$$
\begin{aligned}
&\frac{r}{c_0} = 1 + c_1\,\varepsilon + R_1(\theta)\varepsilon + \varepsilon^2\,\bar{r} \\
&\frac{\rho}{\rho_0} = 1 + \varepsilon\rho_1 + \varepsilon^2\,\bar{\rho}(\bar{r},\theta) + O(\varepsilon^3) \\
(4.1)\qquad &\frac{R}{c_0} = \varepsilon R_1(\theta) + \varepsilon^2\,\bar{R}(\bar{r},\theta) + O(\varepsilon^3) \\
&\frac{\Theta}{c_0} = \varepsilon\Theta_1(\theta) + \varepsilon^2\,\bar{\Theta}(\bar{r},\theta) + \varepsilon^3\,\bar{\bar{\Theta}}(\bar{r},\theta) + O(\varepsilon^4) \\
&\frac{c}{c_0} = 1 + c_1\,\varepsilon + \bar{c}\varepsilon^2 + \cdots
\end{aligned}
$$

with ρ_1, R_1, Θ_1 from (2.4).

Substituting (4.1) in (1.9b) we obtain the following equations in nondimensional (barred) variables

$$
\begin{bmatrix}
\varepsilon^2(\bar{R} - \bar{r} + \bar{c}) + \cdots & 0 & 0 \\
0 & -2 + \cdots & 0 \\
0 & 0 & -1 + \cdots
\end{bmatrix}
\begin{bmatrix}
\bar{\rho} + \bar{R} + \cdots \\
\bar{\rho} - \bar{R} + \cdots \\
\bar{\Theta} + \varepsilon\bar{\bar{\Theta}} + \cdots
\end{bmatrix}_{\bar{r}}
$$

$$
+ \begin{bmatrix}
\varepsilon\Theta_1 + \cdots & \varepsilon\Theta_1 + \cdots & 1 + \varepsilon(c_1 + \rho_1) + \cdots \\
\varepsilon\Theta_1 + \cdots & -\varepsilon\Theta_1 + \cdots & 1 + \cdots \\
1 + \cdots & 0 & \varepsilon\Theta_1 + \cdots
\end{bmatrix}
\left\{
\begin{bmatrix}
\varepsilon^2\,\bar{\rho} + \cdots \\
\varepsilon R_1 + \cdots \\
\varepsilon\Theta_1 + \varepsilon^2\,\bar{\Theta} + \cdots
\end{bmatrix}_{\theta}
\right.
$$

$$
\left.
- R_{1_\theta}
\begin{bmatrix}
\varepsilon\bar{\rho} + \cdots \\
\varepsilon\bar{R} + \cdots \\
\varepsilon\bar{\Theta} + \varepsilon^2\,\bar{\bar{\Theta}} + \cdots
\end{bmatrix}_{\bar{r}}
\right\}
$$

$$+\begin{bmatrix} 0 & 1+\varepsilon(c_1+\rho_1)+\cdots & -\varepsilon\Theta_1+\cdots \\ 0 & 1+\cdots & \varepsilon\Theta_1+\cdots \\ 0 & \varepsilon\Theta_1+\cdots & 0 \end{bmatrix}\begin{bmatrix} 1+\varepsilon\rho_1+\cdots \\ \varepsilon R_1+\varepsilon^2\,\bar{R}+\cdots \\ \varepsilon\Theta_1+\cdots \end{bmatrix} = 0.$$

Consider the third equation first. One obtains from $O(1)$ terms:

(4.2a) $$-\bar{\Theta}_{\bar{r}} = 0 \Rightarrow \bar{\Theta} = \Theta_2(\theta),$$

and from $O(\varepsilon)$ terms

(4.2b) $$-\bar{\bar{\Theta}}_{\bar{r}} - R_{1_\theta}\,\bar{\rho}_{\bar{r}} = 0 \Rightarrow \bar{\bar{\Theta}}_{\bar{r}} = -R_{1_\theta}\,\bar{\rho}_{\bar{r}}.$$

The second equation yields

(4.3) $$\bar{\rho}_{\bar{r}} - \bar{R}_{\bar{r}} = 0 \Rightarrow \bar{R} - R_2(\theta) = \bar{\rho} - \rho_2$$

from $O(1)$ terms.

Since $\Theta_{1_\theta} + R_1 = 0$ (see (2.4)) and $\bar{\Theta}_{\bar{r}} = 0$, the first equation has only terms of $O(\varepsilon^2)$ or higher. From $O(\varepsilon^2)$ terms one obtains

$$(\bar{R} - \bar{r} + \bar{c})(\bar{\rho} + \bar{R})_{\bar{r}} + \Theta_1\,R_{1_\theta} + \bar{\Theta}_\theta - \Theta_1\,R_{1_\theta}(\bar{\rho} + \bar{R})_{\bar{r}} - R_{1_\theta}\,\bar{\bar{\Theta}}_{\bar{r}} + \bar{R} - \Theta_1^2 = 0.$$

In view of (4.2b), (4.3) and the fact that

$$R_{i_\theta} - \Theta_i = R_i + \Theta_{i_\theta} = 0, \quad i = 1,2, \qquad \bar{c} = \frac{\gamma - 1}{2}\bar{\rho} + \frac{(\gamma - 1)(\gamma - 3)}{8}$$

one obtains

$$((\gamma + 1)\tilde{\rho} - 2x)\tilde{\rho}_x + \tilde{\rho} = 0 \tag{4.4}$$

where

$$\tilde{\rho} = \bar{\rho} - \rho_2, \quad x = \bar{r} - R_2 - \frac{\gamma - 1}{2}\rho_2 - \frac{(\gamma - 1)(\gamma - 3)}{8} + \tfrac{1}{2}\Theta_1 R_{1_\theta}.$$

The solution to (4.4) is given by

$$x - (\gamma + 1)\tilde{\rho} = K\tilde{\rho}^2 \tag{4.5}$$

with $K = K(\theta)$ the constant of integration, from which

$$\tilde{\rho} = \frac{-(\gamma + 1) \pm \sqrt{(\gamma + 1)^2 + 4Kx}}{2K}. \tag{4.6}$$

The solution $\tilde{\rho}$ is uniquely determined by boundary conditions at $x \to \pm\infty$ which are obtained from matching the expansion in (4.1) with (2.4) and (3.1). To match (2.4), $\tilde{\rho} = 0$ as $x \to +\infty$. This immediately shows that $K(\theta) < 0$ and that $\tilde{\rho} \equiv 0$ for $x > x_S$, for some x_S. For $x < x_S$ $\tilde{\rho}$ must pick one of the two branches in (4.6). This can be done discontinuously, by a jump, or continuously when $x_S = 0$ (see Figure 4.1).

In the former case certain jump and entropy conditions must be satisfied. We write (4.4) in conservative form

$$\left(\frac{\gamma + 1}{2} \tilde{\rho}^2 - 2x\tilde{\rho}\right)_x + 3\tilde{\rho} = 0.$$

The jump conditions are

$$\left[\frac{\gamma + 1}{2} \tilde{\rho}^2 - 2x\tilde{\rho}\right]_{x=x_S} = 0$$

and the entropy condition $\tilde{\rho}(x_S^+) < \tilde{\rho}(x_S^-)$ which says that density increases upon crossing the shock. Since $\tilde{\rho}(x_S^+) = 0$ we obtain

$$\tilde{\rho}(x_S^-) = \frac{4}{\gamma + 1} x_S > 0.$$

Therefore $x_S \geq 0$ in general with $x_S = 0$ for continuous solutions. Substituting in (4.5) we obtain

$$x_S(\theta) = -\frac{3}{K(\theta)} \left(\frac{\gamma + 1}{4}\right)^2 \tag{4.7}$$

for shocks. The solution $\tilde{\rho}$ could now be uniquely determined if we knew $K(\theta)$ and which branch to choose in case of a jump. These two will be obtained from matching with the interior expansion (3.7). We will see that for a shock the solution must pick the upper branch whereas for an expansion wave the solution is continuous; it picks the lower branch at $x = 0$ (see Figure 4.1).

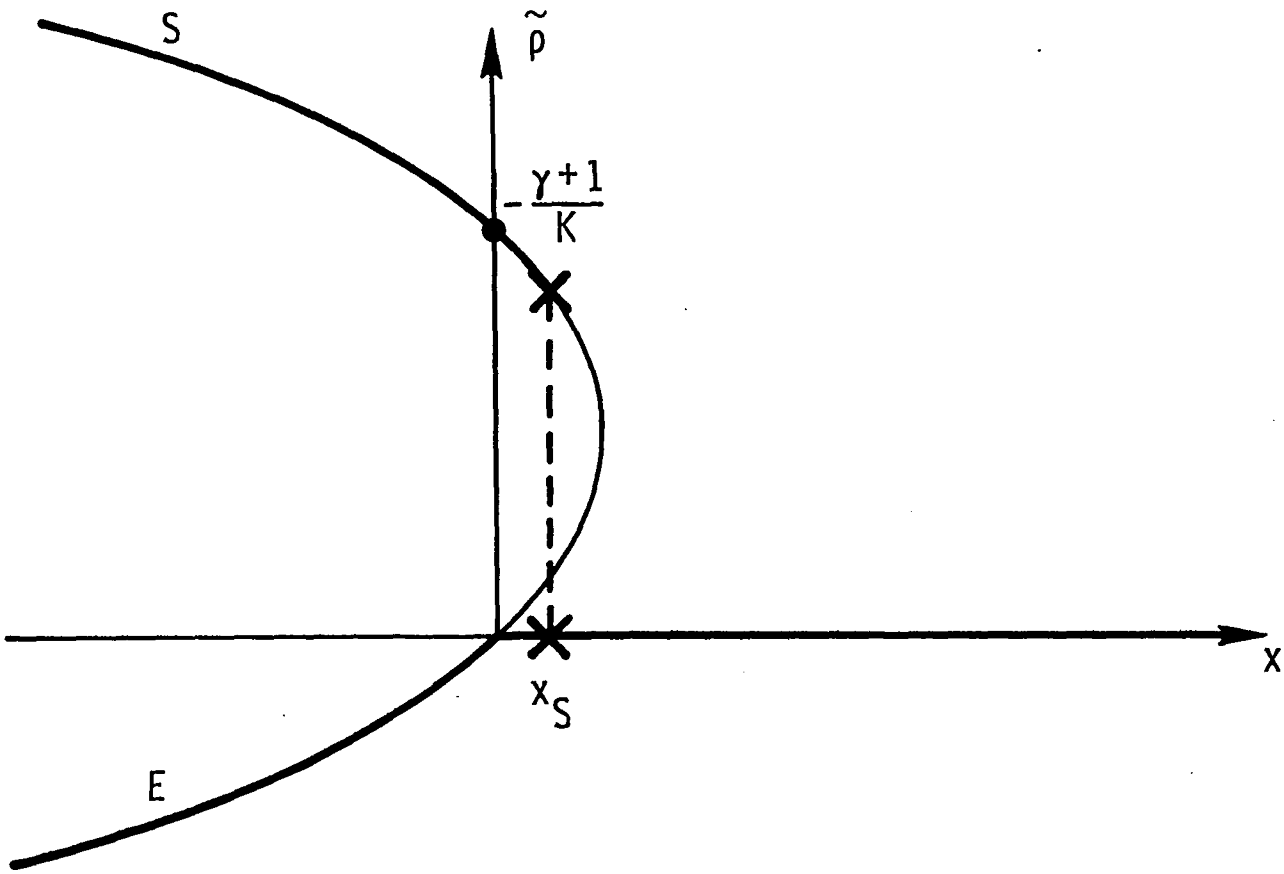

Figure 4.1

For $\bar{r} \to -\infty$, from (4.6)

$$\rho \sim \pm \sqrt{\frac{r}{K}} + \cdots;$$

therefore, from (4.1), in the exterior

$$\frac{\rho}{\rho_0} = 1 + \varepsilon\rho_1 + \varepsilon^2 \left(\pm \sqrt{\frac{1}{K} \left[\left(\frac{r}{c_0} - 1\right) \frac{1}{\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right) \right]} \right)$$

$$= 1 + \varepsilon\rho_1 \pm \varepsilon \left(\sqrt{\left(\frac{r}{c_0} - 1\right) \frac{1}{K}} \right) + O(\varepsilon^2).$$

From (3.7), (3.9), in the interior,

$$\frac{\rho}{\rho_0} = 1 + \varepsilon\rho_1 - \varepsilon \frac{1}{\pi} \frac{\sqrt{2}\lambda \sin 2\lambda\pi}{(\sin \lambda\pi)^2 - \left(\sin \lambda(\pi - \theta)\right)^2} \sqrt{1 - \frac{r}{c_0}} + \varepsilon 0\left(1 - \frac{r}{c_0}\right)$$

$$+ \varepsilon^2 \, 0\left(\frac{1}{\sqrt{1 - \frac{r}{c_0}}}\right) + \cdots.$$

Now we can match $\pm \sqrt{\left(\frac{r}{c_0} - 1\right)\frac{1}{K}}$ to

$$- \frac{1}{\pi} \frac{\sqrt{2}\lambda \sin 2\lambda\pi}{(\sin \lambda\pi)^2 - \left(\sin \lambda(\pi - \theta)\right)^2} \sqrt{1 - \frac{r}{c_0}}$$

if we further restrict r so that

$$\frac{\varepsilon^2}{\sqrt{1 - \frac{r}{c_0}}} \ll \varepsilon\sqrt{1 - \frac{r}{c_0}}$$

or

(4.8) $$\varepsilon \ll 1 - \frac{r}{c_0} \ll 1,$$

see (3.10). Note that

$$(\sin \lambda\pi)^2 - \left(\sin \lambda(\pi - \theta)\right)^2 = \begin{cases} < 0 & \theta < 2\alpha_0 \\ > 0 & \theta > 2\alpha_0 \end{cases}$$

and therefore

$$- \frac{1}{\pi} \frac{\sqrt{2}\lambda \sin 2\lambda\pi}{(\sin \lambda\pi)^2 - \left(\sin \lambda(\pi - \theta)\right)^2} = \frac{1}{\sqrt{-K}} \qquad \theta < 2\alpha_0 \quad \text{(expansion)}$$

$$= - \frac{1}{\sqrt{-K}} \qquad \theta > 2\alpha_0 \quad \text{(shock).}$$

This shows that

$$K(\theta) = -\left(\frac{\pi(\sin\lambda\pi)^2 - \sin\lambda(\pi-\theta)^2}{\sqrt{2}\lambda\sin 2\lambda\pi}\right)^2$$

and that

$$\tilde{\rho} = \frac{-(\gamma+1) + \sqrt{(\gamma+1)^2 + 4Kx}}{2K} \qquad \theta < 2\alpha_0 \quad \text{(expansion)}$$

$$= \frac{-(\gamma+1) - \sqrt{(\gamma+1)^2 + 4Kx}}{2K} \qquad \theta > 2\alpha_0 \quad \text{(shock)}.$$

In particular the shock position is recovered from (4.7). The gradient discontinuity occurs when $x = 0$.

Given $\overline{\rho}$ one can recover the velocities $\overline{R}$, $\overline{\Theta}$, $\overline{\overline{\Theta}}$ ((4.2), (4.3)). The vorticity ω, generated by the curved shock, is of $O(\varepsilon^2)$ and is given by

$$\omega = \overline{R}_\theta - (r\overline{\Theta})_r = \left(R_{2_\theta} + \overline{\rho}_\theta\right) - \Theta_2(\theta) = \overline{\rho}_\theta.$$

5. CORNER ANSATZ

As we mentioned in the Introduction, near the corner one needs to stretch both $\overline{r}$ and θ variables. Motivated by (3.8) we have the following ansatz:

$$\frac{r}{c_0} = 1 + \varepsilon\overline{r}$$

$$\theta = 2\alpha_0 + \varepsilon^{1/2}\overline{\theta}$$

(5.1) $$\frac{\rho}{\rho_0} = 1 + \varepsilon\overline{\rho}(\overline{r},\overline{\theta}) + O(\varepsilon^2)$$

$$\frac{R}{c_0} = \varepsilon\bar{R}(\bar{r},\bar{\theta}) + O(\varepsilon^2)$$

$$\frac{\Theta}{c_0} = \varepsilon\Theta_1(2\alpha_0) + \varepsilon^{3/2}\,\bar{\Theta}(\bar{r},\bar{\theta}) + O(\varepsilon^2)$$

$$\frac{c}{c_0} = 1 + \varepsilon\bar{c} + \cdots.$$

The equations we obtain by substituting in (1.9b) are

$$(\bar{R} - \bar{r} + \bar{c})\left(\bar{\rho}_{\bar{r}} + \bar{R}_{\bar{r}}\right) + \bar{\Theta}_{\bar{\theta}} + \bar{R} = 0 \qquad O(\varepsilon),$$

$$\bar{R}_{\bar{r}} - \bar{\rho}_{\bar{r}} = 0 \qquad O(1),$$

$$-\bar{\Theta}_{\bar{r}} + \bar{\rho}_{\bar{\theta}} = 0 \qquad O(\varepsilon^{1/2}).$$

To obtain the correct boundary conditions, one must match with the outside (2.4) and the interior (3.1), (3.8) expansions. Integrating the second equation

$$R = \bar{\rho} - \rho_1^1 + R_1^1(2\alpha_0).$$

Since $\bar{c} = \frac{\gamma - 1}{2}\bar{\rho}$ we obtain

$$\text{(5.2a)} \qquad \begin{cases} ((\gamma + 1)\bar{\rho} - 2x)\bar{\rho}_x + \bar{\Theta}_{\bar{\theta}} + \bar{\rho} - \rho_1^1 + R_1^1(2\alpha_0) = 0 \\ \bar{\Theta}_x - \bar{\rho}_{\bar{\theta}} = 0 \end{cases}$$

where $x = \bar{r} - R_1^1 + \rho_1^1$.

For smooth solutions we can eliminate $\overline{\Theta}$ by differentiating and obtain

$$(((\gamma + 1)\overline{\rho} - 2x)\overline{\rho}_x)_x + \overline{\rho}_{\overline{\theta}\,\overline{\theta}} + \overline{\rho}_x = 0, \tag{5.2b}$$

which is the first approximation to the flow in the corner.

To obtain the boundary conditions we match with (2.4) and (3.1). From (2.3), (5.1) one obtains

$$\overline{r} = \tfrac{1}{2}\overline{\theta}^2 + \tilde{\beta}_1 + O(\varepsilon^{1/2})$$

as the equation for S_R in $(\overline{\theta},\overline{r})$ where $\tilde{\beta}_1$ can be explicitly computed from β_1. We see that $\overline{r} = \tfrac{1}{2}\overline{\theta}^2$ is the dominating term as $\theta \to \infty$. The boundary conditions for $\overline{\rho}$ can now be formulated

$$\lim_{\substack{\overline{\theta}\to\infty \\ \alpha>0}} \overline{\rho}\left(\alpha \frac{\overline{\theta}^2}{2}, \overline{\theta}\right) = \rho_1^i \qquad \begin{array}{l} i = 1,\ \alpha > 1 \\ i = 2,\ 0 < \alpha < 1 \end{array} \tag{5.3}$$

and

$$\lim_{\substack{\overline{\theta}\to\infty \\ \alpha<0}} \overline{\rho}\left(\alpha \frac{\overline{\theta}^2}{2}, \overline{\theta}\right) = 1 + \frac{1}{\pi}\,\mathrm{tg}^{-1}(-\alpha)$$

(see (3.8)). Provided a solution exists, the expansion (5.1) matches the other expansions to $O(\varepsilon)$ in the region

$$\varepsilon^2 \ll 1 - \frac{r}{c_0} \ll 1,$$

$$\varepsilon^{1/2} \ll \theta - 2\alpha_0 \ll 1.$$

Writing (5.2a) in conservation form we obtain the shock conditions

$$[\frac{\gamma+1}{2}\bar{\rho}^2 - 2x\bar{\rho}]_S\, n_x + [\bar{\theta}]_S\, n_\theta = 0$$

$$[\bar{\theta}]_S\, n_x - [\bar{\rho}]_S\, n_\theta = 0$$

$\vec{n}$ normal to shock curve $\bar{x} = S(\theta)$ which in turn give

$$[(S_\theta)^2\,\bar{\rho} + \frac{\gamma+1}{2}\bar{\rho}^2 - 2S\bar{\rho}]_S = 0. \tag{5.4}$$

The existence of a solution to the boundary value problem (5.2b), (5.2) satisfying (5.4) across discontinuties remains an unsolved problem. However, we remark the following: The equation (5.2b) is of mixed type, hyperbolic when $(\gamma + 1)\bar{\rho} - 2x < 0$ and elliptic when $(\gamma + 1)\bar{\rho} - 2x > 0$. When $\bar{\rho} = \rho_1^i = 1,2,\ i = 1,2$ the sonic lines are $x = \frac{\gamma+1}{2}$ and $x = \gamma + 1$ respectively. We note that

$$x = S_R(\theta) = \frac{(\bar{\theta} - \bar{\theta}_0)^2}{2} + \frac{3}{4}(\gamma + 1)$$

and

$$\begin{aligned} \rho &= 1 \qquad x > S(\theta) \\ &= 2 \qquad x < S(\theta) \end{aligned}$$

is a solution which satisfies the first boundary condition in (5.3) for $\bar{\theta} > 0$ and some θ_0 (see Figure 5.1). The equations are hyperbolic as long as

$$x > \frac{\rho_1^{(i)}(\gamma + 1)}{2}\,.$$

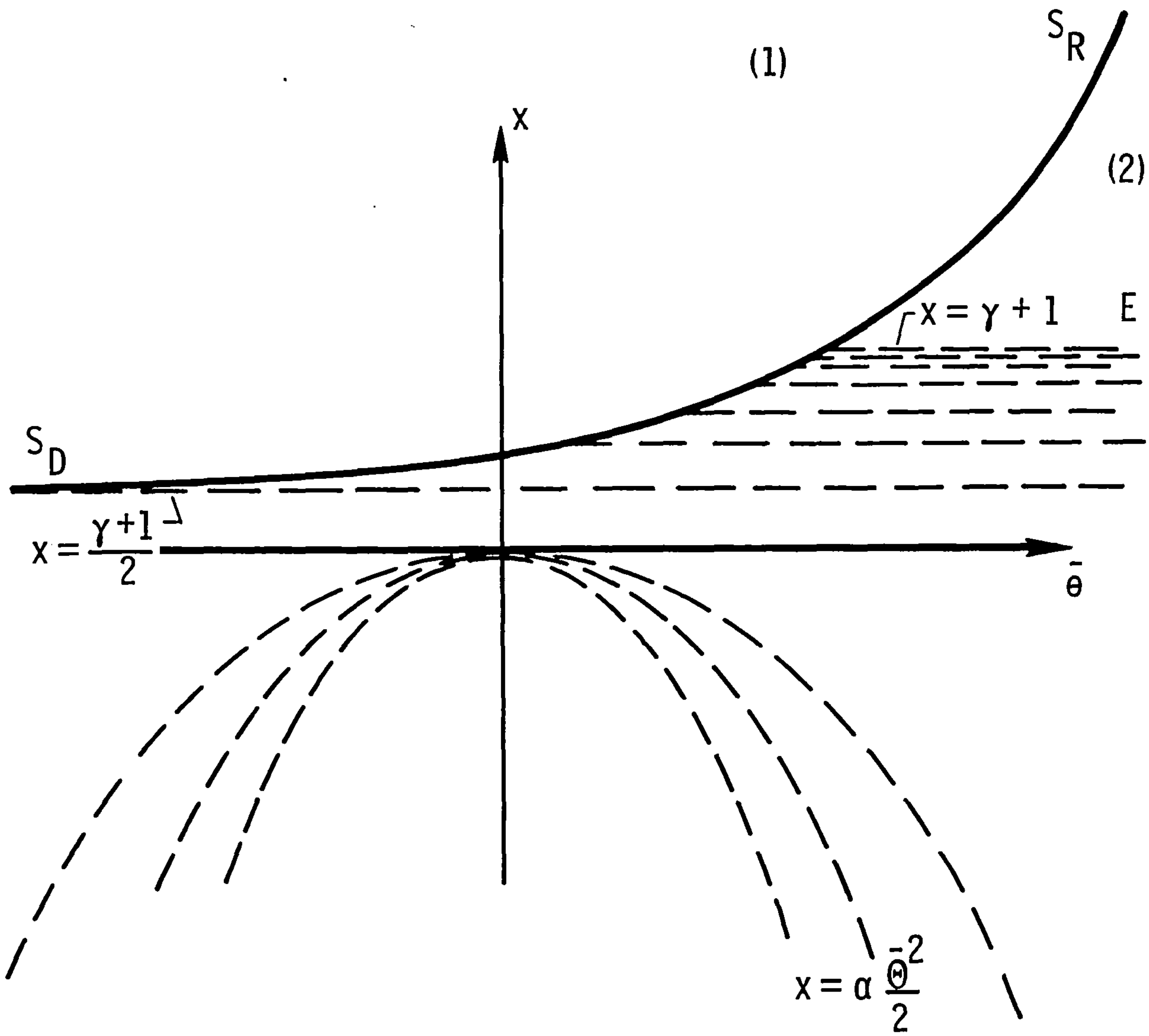

Figure 5.1

We expect the equations to change type across the sonic line $x = (\gamma + 1)$ and the expansion wave to weaken and bend S_R until it becomes asymptotic to the other sonic line $x = \frac{\gamma + 1}{2}$ (see Figure 5.1). Note that the diffracted shock is weak and appears only in the $O(\varepsilon^2)$ approximation. We finally remark that if $\bar{\rho}(x,\bar{\theta})$ is a solution then so is $\frac{1}{\alpha^2}\,\bar{\rho}(\alpha^2 x,\alpha\bar{\theta})$ and therefore (5.2b) admits similarity solutions of the form $\bar{\rho}(x,\bar{\theta}) = \bar{\theta}^2\, f(\frac{n}{\bar{\theta}^2})$ with

$$((\gamma + 1)f - 2\xi + 4\xi^2)f'' + (\gamma + 1)(f')^2 - (1 + 2\xi)f' + 2f = 0$$

$$\xi = \frac{x}{\theta^2} .$$

ACKNOWLEDGEMENTS

I benefited a great deal from several stimulating discussions with James Glimm and Cahtleen Morawetz. I thank them very much.

REFERENCES

[1] Keller, J. B. and Blank, A. A., "Diffraction and reflection of pulses by wedges and corners," Comm. Pure Appl. Math. 4, 1951, pp. 75-94.

[2] Hunter, J. and Keller, J. B., "Weakly nonlinear high frequency waves," Comm. Pure Appl. Math. 36, 1983, pp. 547-569.

[3] Hunter, J. and Keller, J. B., "Weak shock diffraction," Wave Motion 6, 1984, pp. 79-89.

[4] Courant, R. and Friedrichs, K. O., Supersonic Flow and Shock Waves, Springer-Verlag, New York, 1976.

[5] DiPerna, R. J. and Majda, A., "The validity of nonlinear geometric optics for weak solutions of conservation laws," Comm. Math. Phys. Vol. 98, No. 3, 1985, pp. 331-347.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-178022 ICASE Report No. 85-53		

4. Title and Subtitle

MATCHED ASYMPTOTIC EXPANSIONS TO SIMILARITY SOLUTIONS OF SHOCK DIFFRACTION

5. Report Date
November 1985

6. Performing Organization Code

7. Author(s)

Eduard Harabetian

8. Performing Organization Report No.
85-53

10. Work Unit No.

9. Performing Organization Name and Address

Institute for Computer Applications in Science
and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, VA 23665-5225

11. Contract or Grant No.
NAS1-17070

13. Type of Report and Period Covered
Contractor Report

12. Sponsoring Agency Name and Address

National Aeronautics and Space Administration
Washington, D.C. 20546

14. Sponsoring Agency Code
505-31-83-01

15. Supplementary Notes

Langley Technical Monitor:
J. C. South Jr.
Final Report

Submitted to Comm. Pure & Appl. Math.

16. Abstract

We match formal asymptotic expansions with differently scaled variables to obtain a uniform approximation to the similarity solution of the shock-wedge diffraction problem.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
matched asymptotics similarity solutions shock diffraction	34 - Fluid Mechanics & Heat Transfer 59 - Mathematical & Computer Sciences (General) Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	32	A03