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MATCHINGS IN INFINITE GRAPHS[1])

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1. Introduction. Let G be a graph with vertex-set V and edge-set E. All terms undefined in this paper have their meaning in [1]. Thus, the graph G has no loops or multiple edges. It may be finite or infinite. If G is infinite and each vertex of G has finite degree, then G is called *locally finite.*

A *matching* in G is a set M of edges of G such that each vertex of G is incident with at most one edge in M. A matching M in G is called *perfect* if each vertex of G is incident with exactly one edge in M. A perfect matching in G is also called a *1-factor* of G. A matching M in G *meets* a vertex v of G if v is incident with an edge in M. If d is a positive integer and if M is a matching in G which fails to meet exactly d vertices of G, then M is said to have *defect d.*

In this paper we prove several theorems concerning the existence of matchings in infinite graphs. In section 2 we obtain a general result and apply it to squares and line graphs of infinite graphs. In section 3 we give new proofs of two known results, and in section 4 we extend a result of Plesník on finite regular graphs to infinite graphs.

2. Squares and line graphs. We use the following version of Rado's selection theorem: it is proved by modifying only slightly the proof of Rado's theorem in [5, p. 52].

Let $(A_e : e \in E)$ *be a family of nonempty finite subsets of a set S. Let* $\{E_i : i \in I\}$ *be a collection of finite subsets of E such that if* $i, j \in I$ *then there exists* $k \in I$ *such that* $E_i \cup E_j \subseteqq E_k$, *and* $E = \bigcup\{E_i : i \in I\}$. *For each* $i \in I$, *let* $f_i : E_i \to S$ *be a mapping such that* $f_i(e) \in A_e$ *for each* $e \in E_i$. *Then there exists a mapping* $f : E \to S$ *such that*

(a) $f(e) \in A_e$ *for each* $e \in E$, *and*

(b) *for each* $i \in I$ *there exists* $j \in I$ *such that* $E_i \subseteqq E_j$ *and* $f \mid E_i = f_j \mid E_i$.

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If H is a subgraph of G, we denote its vertex-set by $V(H)$ and its edge-set by $E(H)$. If $F \subseteq E$, we denote the set $E(H) \cap F$ by $H \cap F$. Note that if M is a matching in G, then $H \cap M$ is a matching in H.

Theorem 1. *Let $\{H_i : i \in I\}$ be a collection of finite subgraphs of G, each without isolated vertices, such that*

(*) *if $i, j \in I$ then there exists $k \in I$ such that $E(H_i) \cup E(H_j) \subseteq E(H_k)$, and $E = \bigcup\{E(H_i) : i \in I\}$.*

For each $i \in I$, let M_i be a matching in H_i. Then there is a matching M in G such that for each $i \in I$ there exists $j \in I$ such that H_i is a subgraph of H_j and $H_i \cap M = H_i \cap M_j$. Furthermore, let d be a nonnegative integer and A a set of vertices of finite degree of G. If M_i fails to meet at most d vertices in $V(H_i) \cap A$ for each $i \in I$, then M fails to meet at most d vertices in A. In particular, if M_i meets all vertices in $V(H_i) \cap A$ for each $i \in I$, then M meets all vertices in A.

Proof. We apply the selection theorem with $A_e = \{0, 1\}$ for each $e \in E$, and $E_i = E(H_i)$ for each $i \in I$. Note that each $E(H_i)$ is nonempty, since the vertex-set of a graph is always nonempty and H_i has no isolated vertices. For each $i \in I$, let $f_i : E(H_i) \to \{0, 1\}$ be defined by $f_i(e) = 1$ if $e \in M_i$ and $f_i(e) = 0$ if $e \notin M_i$. By the selection theorem there is a mapping $f : E \to \{0, 1\}$ such that for each $i \in I$ there exists $j \in I$ such that $E(H_i) \subseteq E(H_j)$ and $f \mid E(H_i) = f_j \mid E(H_i)$. Since H_i has no isolated vertices, it is a subgraph of H_j. Let $M = \{e : e \in E \text{ and } f(e) = 1\}$. If $e_1, e_2 \in E$, $e_1 \neq e_2$, then there is an $i \in I$ such that $e_1, e_2 \in E(H_i)$. With $j \in I$ as above, we have $f(e_1) = f_j(e_1)$ and $f(e_2) = f_j(e_2)$. Thus, if e_1 and e_2 are adjacent we cannot have both $f(e_1) = 1$ and $f(e_2) = 1$. Therefore, M is a matching in G.

Now let $i \in I$, and let $j \in I$ be as above. If $e \in E(H_i)$ then $f(e) = f_j(e)$. Thus, $e \in H_i \cap M$ if and only if $e \in H_i \cap M_j$. Hence, $H_i \cap M = H_i \cap M_j$.

Finally, suppose that d and A are as in the statement of the theorem, and suppose that M fails to meet $d + 1$ vertices in A, say $v_1, \ldots, v_{d+1}$. Then, these vertices and all their incident edges belong to some H_i. If $j \in I$ is such that H_i is a subgraph of H_j and $H_i \cap M = H_i \cap M_j$, then M_j fails to meet $v_1, \ldots, v_{d+1}$, all of which are in $V(H_j) \cap A$.

We denote by G^2 the *square* of G [1, p. 46]. It was proved in [3] and in [6] that if G is a finite connected graph with an even number of vertices, then G^2 has a perfect matching. Our next results show that this holds for locally finite graphs, but not for all infinite graphs.

Theorem 2. *If G is a connected infinite graph, then G^2 has a matching which meets each vertex of finite degree.*

Proof. Let $\{V_i : i \in I\}$ be the set of all finite subsets of V with an even number of elements and which induce connected subgraphs of G. Let H_i be the subgraph of G

induced by V_i, and let K_i be the subgraph of G^2 induced by V_i. Then $\{K_i : i \in I\}$ satisfies the condition $(*)$ of Theorem 1 relative to G^2. Since H_i^2 is a spanning subgraph of K_i, and H_i^2 has a perfect matching, so also does K_i. Therefore, this theorem follows from Theorem 1.

Corollary. *The square of a connected locally finite graph has a perfect matching.*

The following is an example of an infinite graph whose square has no perfect matching.

Example 1. Let G_1 be the graph with vertices $v_0, v_1, v_2, \ldots, u_1, u_2, \ldots$, and edges $u_n v_n$ and $v_0 v_n$, $n = 1, 2, \ldots$. The graph G^2 has the additional edges $v_0 u_n$, $n = 1, 2, \ldots$, and $v_m v_n$, $m, n = 1, 2, \ldots, m \neq n$. The set $\{u_n v_n : n = 1, 2, \ldots\}$ is a matching of defect one in G_1 and in G_1^2. Let M be a matching in G_1^2 which meets v_0: then for some m, either $v_0 u_m$ or $v_0 v_m$ is in M. If M meets u_n for all $n \neq m$, then $u_n v_n \in M$ for all $n \neq m$: hence, either u_m or v_m is not met by M. Thus, M is not a perfect matching, and we conclude that G_1^2 has no perfect matching.

We denote by $L(G)$ the *line graph* of G [1, p. 182]. It was proved in [3] and in [6] that a finite connected line graph with an even number of vertices has a perfect matching. This also can be extended to locally finite graphs, but not to all infinite graphs. First we note a more general result.

Theorem 3. *If G is a connected infinite graph with no induced $K(1, 3)$, then G has a perfect matching which meets each vertex of finite degree.*

Proof. Let $\{H_i : i \in I\}$ be the same collection of subgraphs of G used in the proof of Theorem 2: it satisfies condition $(*)$ of Theorem 1. Since G has no induced $K(1, 3)$, and since each H_i is an induced subgraph of G, each H_i has no induced $K(1, 3)$. Hence, by [6, Corollary 2], each H_i has a perfect matching. Thus, this theorem follows from Theorem 1.

Corollary 1. *A connected locally finite graph with no induced $K(1, 3)$ has a perfect matching.*

Corollary 2. *A connected infinite line graph has a matching which meets each vertex of finite degree.*

Corollary 3. *If G is a connected locally finite graph, then $L(G)$ has a perfect matching.*

Corollary 2 follows from the fact that a line graph has no induced $K(1, 3)$. Corollary 3 is a consequence of the fact that the line graph of a connected locally finite graph is itself connected and locally finite. In fact, these are the only connected infinite graphs with locally finite line graphs. The next example shows that the line graph of a connected infinite graph need not have a perfect matching.

Example 2. Let P be the path with three vertices and for $n = 1, 2, \ldots$, let W_n be $K(1, 3)$: we assume that P and the W_n's have mutually disjoint vertex-sets. Let v_0 be the middle vertex of P and let v_n be the center of the star W_n. Let u be a vertex of none of these graphs, and let G_2 be constructed by joining u to each of $v_0, v_1, v_2, \ldots$. The line graph $L(G_2)$ consists of a complete graph K with a denumerable number of vertices, a complete graph C_0 with three vertices, and complete graphs C_n, $n = 1, 2, \ldots$, with four vertices. The graphs $C_0, C_1, C_2, \ldots$ have mutually disjoint vertex-sets, each has one vertex in common with K, and each vertex of K is a vertex of exactly one of them. The set of edges of $L(G_2)$ consisting of a maximum matching in each of $C_0, C_1, C_2, \ldots$ is a matching of defect one in $L(G_2)$. A matching in $L(G_2)$ which meets all vertices of $C_1, C_2, \ldots$ cannot meet all three vertices of C_0. Therefore, $L(G_2)$ has no perfect matching.

Note that it is not simply the existence of a vertex of infinite degree in a connected infinite graph that causes its square or line graph to fail to have a perfect matching. For, both $L(G_1)$ and G_2^2 have perfect matchings.

3. Locally finite graphs. The two theorems proved in this section are due essentially to BRUALDI [2, Theorems 4 and 5]: he stated them in the case $d = 0$, but his proofs can be adapted to the more general case. Brualdi's proof of Theorem 4 used Rado's theorem applied to sets of vertices. Our proof makes use of Theorem 1, i.e., it makes use of Rado's theorem applied to sets of edges.

Theorem 4. *Let A be a set of vertices of finite degree of a connected infinite graph G. Let d be a nonnegative integer. Suppose that for each finite subset A' of A there is a matching in G which fails to meet at most d vertices in A'. Then, there is a matching in G which fails to meet at most d vertices in A.*

Proof. For each finite subset A' of A, let $H(A')$ be the subgraph of G induced by the set of edges of G incident with vertices in A'. Then $H(A')$ is a finite subgraph of G without isolated vertices. Now consider all subgraphs of G which can be obtained by adding a finite number of edges of G not incident with vertices in A, and their incident vertices, to $H(A')$ for some A'. The collection $\{H_i : i \in I\}$ of subgraphs of G so obtained satisfies condition $(*)$ of Theorem 1. If a vertex v in A is a vertex of H_i, then every edge of G incident with v is an edge of H_i. Hence, by hypothesis, H_i has a matching which fails to meet at most d vertices in $V(H_i) \cap A$. This theorem follows now from Theorem 1.

If $X \subseteq V$, then $G - X$ denotes the graph that remains after the vertices in X and their incident edges have been removed from G. An *odd component* of a graph is one with an odd number of vertices. Let $A \subseteq V$. For a finite subset X of V we denote by $p_{G,A}(X)$ the number of odd components of $G - X$ all of whose vertices are contained in A. We write p_G for $p_{G,V}$.

Theorem 5. *Let A be a set of vertices of finite degree of a connected infinite graph G. Let d be a nonnegative integer. There is a matching in G which fails to* *meet at most d vertices in A if and only if* $p_{G,A}(X) \leqq |X| + d$ *for every finite subset X of V.*

When G is locally finite, $d = 0$, and $A = V$, Theorem 5 is a result due to TUTTE [9].

Theorem 5′. *A connected locally finite graph G has a perfect matching if and only if* $p_G(X) \leqq |X|$ *for every finite subset X fo V.*

We shall prove Theorem 5 using Theorem 4 and the finite graph version of Theorem 5. The latter was proved in [4] to be a direct consequence of the finite graph version of Theorem 5′, which is also due to Tutte [8].

Proof of Theorem 5. Suppose that there is a matching M in G which fails to meet at most d vertices in A. Let $X \subseteqq V$. Let $C_1, \ldots, C_t$ be odd components of $G - X$ all of whose vertices are contained in A. Each C_i has at least one vertex, say v_i, not met by an edge in M having both end-vertices in $V(C_i)$. At least $t - d$ of the vertices $v_1, \ldots, v_t$ are joined to distinct vertices in X by edges in M. Hence, $t - d \leqq |X|$, i.e., $t \leqq |X| + d$. This proves the necessity of the condition.

Conversely, suppose that the condition holds. Let A' be a finite subset of A, let B be the set of all vertices of G adjacent to vertices in A', and let H be the subgraph of G induced by $A' \cup B$. Then H is a finite subgraph of G. If we show that there is a matching in H which fails to meet at most d vertices in A', then we will have verified the hypothesis of Theorem 4, and the required matching in G will exist.

Let $X \subseteqq A' \cup B$ and let C be an odd component of $H - X$ with $V(C) \subseteqq A'$. If a vertex of $G - X$ is adjacent to a vertex of C, then it is in B. Hence, it is a vertex of C, and so we conclude that C in an odd component of $G - X$ with $V(C) \subseteqq A$. Therefore, $p_{H,A'}(X) \leqq p_{G,A}(X) \leqq |X| + d$. Consequently, by the finite graph version of Theorem 5, there is a matching in H which fails to meet at most d vertices in A'.

Corollary. *Let G and A be as in Theorem 5. There is a matching in G which fails to meet only a finite number of vertices in A if and only if*

$$\sup \{p_{G,A}(X) - |X| : X \subseteqq V \text{ and } X \text{ finite}\}$$

is finite.

4. Perfect matchings in infinite regular graphs. The finite graph version of the following theorem was proved by PLESNÍK [7].

Theorem 6. *Let G be an* $(r - 1)$*-edge connected infinite regular graph of degree* $r > 0$. *If* $F \subseteqq E$ *and* $|F| = r - 1$, *then G has a perfect matching M with* $M \cap F = \emptyset$.

Proof. Let G' be the graph with vertex-set V and edge-set $E \backslash F$. Let $X \subseteqq V$, X finite, and let $C_1, \ldots, C_n$ be odd components of $G' - X$. Let k be the number of edges of G having exactly one end in X: then $k \leqq r|X|$ since G is regular. Let $Y = V \backslash (X \cup$

$\cup V_1 \cup \dots \cup V_n)$, where $V_i = V(C_i)$, $i = 1, \dots, n$. Since V is infinite, $Y \neq \emptyset$. For $i = 1, \dots, n$, let

s_i = the number of edges of G with one end in V_i and the other in X,

t_i = the number of edges of G with one end in V_i and the other in $V \setminus (X \cup V_i)$,

q_i = the number of edges of G with one end in V_i and the other in Y,

and let

q = the number of edges of G with one end in Y and the other in X.

If d_i is the sum of the degrees of the vertices of C_i, then since G is $(r - 1)$-edge connected,

$$d_i = r|V_i| - (s_i + t_i) \leqq r|V_i| - (r - 1) = r(|V_i| - 1) + 1 .$$

The number on the right is odd, while d_i is even: hence $r - 1 < s_i + t_i$, i.e., $r \leqq s_i + t_i$. (This part of the argument is due to Plesník.) Thus, $rn \leqq \sum(s_i + t_i)$ (all summations are from 1 to n). It follows also from the fact that G is $(r - 1)$-edge connected that $r - 1 \leqq q + \sum q_i$.

Now let us estimate $\sum t_i$. All edges of G with one end in some V_i and the other in $V \setminus (X \cup V_i)$ are in F. Some of these edges have one end in some V_i and the other in Y: they are $\sum q_i$ in number. Some of these edges have one end in some V_i and the other in some V_j with $j \neq i$: they are at most $r - 1 - \sum q_i$ in number, and they are counted twice in determining the sum of the t_i's. Thus,

$$\sum t_i \leqq \sum q_i + 2(r - 1 - \sum q_i) = 2(r - 1) - \sum q_i .$$

Then,

$$k = q + \sum s_i = q + \sum(s_i + t_i) - \sum t_i \geqq q + rn - 2(r - 1) + \sum q_i \geqq$$
$$\geqq rn - 2(r - 1) + r - 1 = rn - r + 1 .$$

Thus, $|X| \geqq n - 1 + 1/r > n - 1$, i.e., $|X| \geqq n$. Therefore, by Theorem 5', G' has a perfect matching, which is also a perfect matching in G and which contains no edge in F.

Corollary. *Let G be as in Theorem 6. If $e \in E$, then G has a perfect matching M with $e \in M$.*

Proof. Let v be the vertex at one end of e, and let $F = \{e' : e' \in E, e'$ is incident with v, and $e' \neq e\}$. Since G is regular of degree r, $|F| = r - 1$. By the theorem, G has a perfect matching M with $M \cap F = \emptyset$. Since M meets v, we must have $e \in M$.

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