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Compression Fracture–Mechanics of Damage Localization and Size Effect

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ABSTRACT

The present paper summarizes several results on compression fracture and its scaling recently obtained at Northwestern University. They deal with the development of approximate asymptotic formulas for the nominal strength of quasibrittle structures, and focuses on concrete and fiber composites. One approach exploits the approximate method of energy release zones, and another approach is based on asymptotic expansions of quasi-LEFM solutions. The compression fracture is assumed to be caused either by axial splitting or by lateral propagation of a band of axial splitting cracks. The residual stress transmitted across the band is governed by internal buckling. The size effect curve of the logarithm of nominal strength versus the logarithm of size is found to terminate with a horizontal asymptote.

1 INTRODUCTION

The fracture of quasibrittle materials due to compressive stress is one of the most difficult aspects of fracture mechanics. In compression fracture, one must distinguish two distinct phenomena:

- micromechanics of initiation of compression fracture, and
- mechanics of global compression fracture causing failure.

The first problem has been investigated much more than the second, and various micromechanical mechanisms that initiate fracture under compressive stresses have been identified; e.g., the growth of axial splitting cracks from voids (Cotterell 1972, Sammis and Ashby 1986, Kemeny and Cook 1987, 1991, Steif 1984, Wittmann and Zaitsev, 1981, Zaitsev 1985, Fairhurst and Cornet 1981, Ingraffea Heuzé 1980, Nesetova and Lajtai 1973, Carter 1992, Yuan et al. 1993) or near inclusions, the creation of axial splitting cracks by groups of hard inclusions, and the formation of wing-tip cracks from sliding inclined surfaces (Hawkes and Mellor 1970, Ingraffea 1977, Ashby and Hallam 1986, Horii and Nemat-Nasser 1982, 1986, Sanderson 1988, Schulson 1990,

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Costin 1991, Batto and Schulson 1993. Schulson and Nickolayev 1995. Lehner and Kachanov 1996, and a critique by Nixon 1996).

It must be realized, however, that these mechanisms do not explain the global failure of the structure. They can cause only a finite extension of the axial splitting cracks whose length is of the same order of magnitude as the size of the void, the inclusion, or the inclined microcrack. Each of these mechanism can produce a zone of many splitting cracks approximately parallel to the uniaxial compressive stress or, under triaxial stress states, to the compressive principal stress of the largest magnitude.

Biot (1965) proposed that the cause of compression failure may consist of three-dimensional internal buckling which can occur either in the bulk of specimen or within an inclined band. However, he considered only elastic behavior and did not conduct any energy analysis of fracture. Finite strain analysis of compression failure caused by internal buckling of an orthotropically damaged material or orthotropic laminate was analyzed by Bažant (1967). Kendall (1978) showed that, with the consideration of buckling phenomena under eccentric compressive loads, the energy balance condition of fracture mechanics yields realistic predictions of compression fracture of test cylinders loaded only on a part of the end surface.

The purpose of the present paper is to outline some results on compression fracture and its scaling that have been recently obtained at Northwestern University. The symposium lecture will also provide a broad overview of the problem.

2 MECHANISM OF SIZE EFFECT

The global compression fracture has been analyzed (Bažant 1993. Bažant and Xiang 1997, Bažant and Planas 1997, Bažant and Chen 1997) under the hypothesis that the aforementioned micromechanisms create a band of axial splitting cracks as shown in Fig. 1, which propagates laterally, in a direction either inclined or normal to the direction of the compressive stress of the largest magnitude (Bažant, 1993. Bažant and Xiang, 1997). In the post-peak regime, the axial splitting cracks interconnect to produce what looks as a shear failure although there is no shear slip before the post-peak softening (in fact, shear failure per se is probably impossible in concrete). The energy analysis of the propagating band of axial splitting cracks shows that, inevitably, there ought to be a size effect. Let us discuss it for the prismatic specimen shown in Fig. 1.

Formation of the axial splitting cracks causes a narrowing of the band and, in an approximate sense, a buckling of the slabs of the material between the splitting cracks as shown in the figure (alternatively, this can be modeled as internal buckling of damaged continuum). This causes a reduction of stress, which may be considered to occur approximately in the shaded triangular areas (where the strain energy density drops from that given by area 0120 to that given by area 0340 in Fig. 1d). For the calculation of the energy change within the crack band one needs to take into account the fact that the slabs of material between the axial splitting cracks ought to undergo significant post-buckling deflections corresponding to the horizontal line 3-5. Thus, the energy change in the splitting crack band is given by the difference of the areas 0120 and 03560 in Fig. 1e (the fact that there is a residual stress σ_{cr} in compression fracture is an important difference from a similar analysis of tensile crack band propagation). The energy released must be consumed and dissipated by

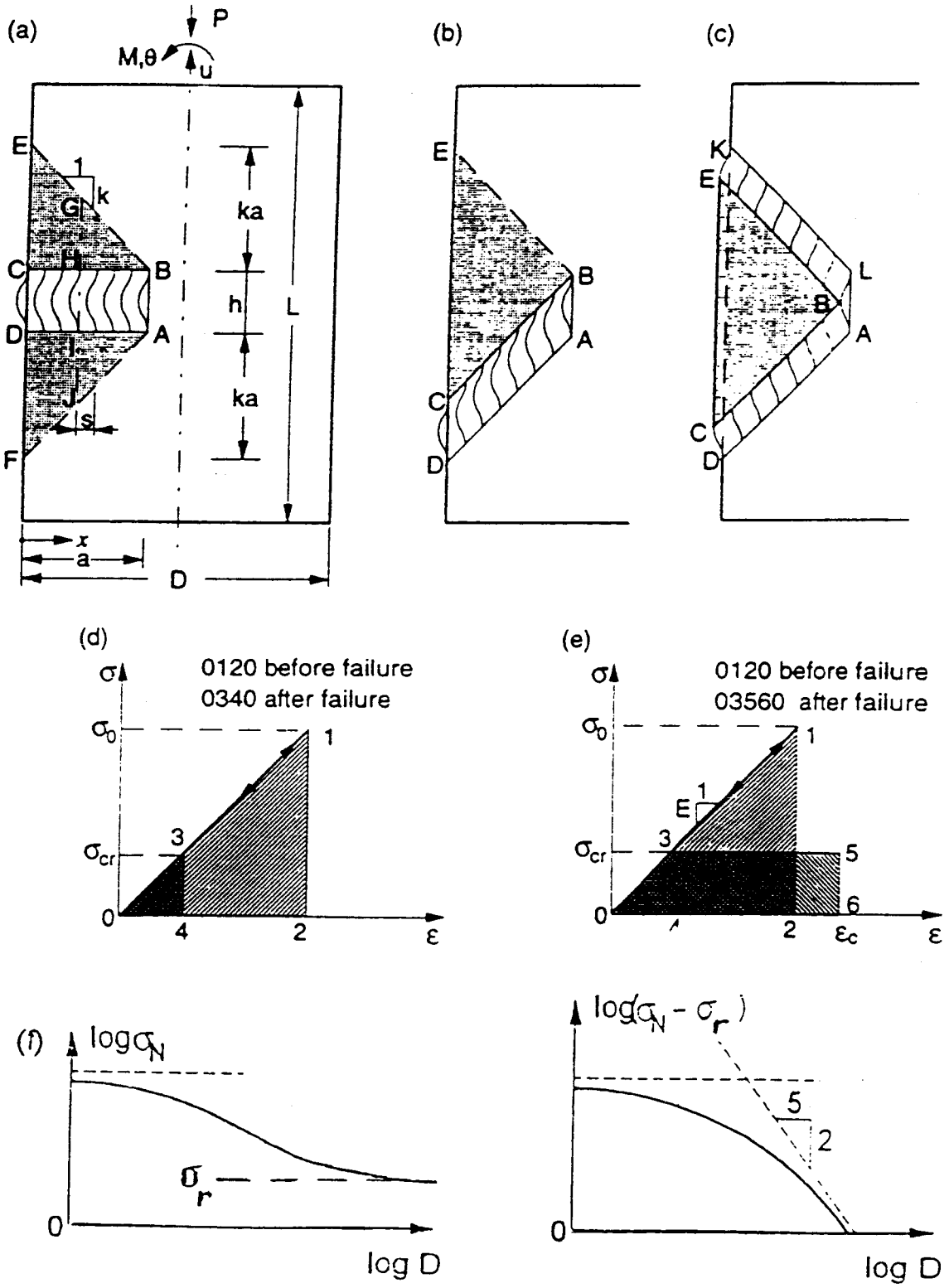

Figure 1: Simplified analysis of energy release in compression fracture.

the axial splitting cracks in the band. This is one condition for the analysis.

Another condition is that the narrowing of the band due to microslab buckling must be compatible with the expansion of the adjacent triangular areas due to the stress relief. One needs to write the condition that the shortening of segment HI in Fig. 1 on top left is compensated for by the extension of segments GH and IJ, which is a compatibility condition. The energy release from the crack band is given by the change of the areas under the stress-strain diagrams in the middle of Fig. 1 (bottom), caused by the drop of stress from the initial compressive stress σ_0 to the final compressive stress σ_{cr} carried by the band of splitting cracks.

The resulting size effect on the nominal strength σ_N of large structures failing in compression has, according to this analysis, the form:

$$\sigma_N = C_1 D^{-2/5} + C_0 \tag{1}$$

where C_1, C_0 = constants and D = characteristic dimension of the structure.

Mathematical formulation of the foregoing arguments (Bažant, 1993; Bažant and Xiang, 1997) provided a formula for the compression failure which exhibits a size effect. This size effect is plotted in Fig. 1(f), with the logarithm of size D as a coordinate and either $\log\sigma_N$ or $\log(\sigma_N - \sigma_r)$ as the ordinate. In the latter plot (Fig. 1f), the size effect is shown to approach an asymptote of slope $-2/5$. This is another interesting feature, which results from the fact that the spacing of the axial splitting cracks is not constant but depends on the overall energy balance. The solution of the nominal strength of σ_N has been obtained under the assumption of arbitrary spacing s, and it was noted that σ_N exhibits a minimum for a certain spacing s, which depends on size D. It is this condition of minimum which causes the asymptotic slope to be $-2/5$ instead of $-1/2$.

The foregoing approximate theoretical results, given by simple formulas (Bažant, 1993), have been compared (Bažant and Xiang, 1997: Fig. 2) to the test results (Bažant and Kwon, 1994) on size effect in reduced-scale tied reinforced concrete columns of three different sizes (in the ratio 1:2:4) and three different slendernesses, $\lambda = 19.2, 35.8$ and 52.5. The columns were made of concrete with reduced aggregate size. The test results indicated a significant size effect which is ignored by the current design codes. Recently a similar size effect has been confirmed for large columns made with normal size aggregate by the tests of B.I.G. Barr and S. Şener at the University of Wales, Cardiff (private communication, 1997).

A size effect is known to occur also in the breakout of boreholes in rock, as experimentally demonstrated by Nesetova and Lajtai (1992), Carter (1992), Carter et al. (1992), Yuan et al. (1992), and Haimson and Herrick (1989). It is known from the studies of Kemeny and Cook (1987, 1981) and others that the break out of boreholes occurs due to the formation of splitting cracks parallel to the direction of the compressive stress of the largest magnitude, $\sigma_{y\infty}$. This mechanism of failure has been analyzed in similar manner as just explained, however, an analytical solution was made possible by the approximate assumption that the growing cracking zones on the sides of the borehole are elliptical (although in reality these zones are narrower and closer to triangles). The assumption of an elliptical boundary permitted the energy release from the surrounding infinite solid to be easily calculated according to Eshelby's theorem for eigenstrains in ellipsoidal inclusions (Bažant, Lin and Lippmann, 1993). According to the theorem, the energy release from the infinite rock mass can

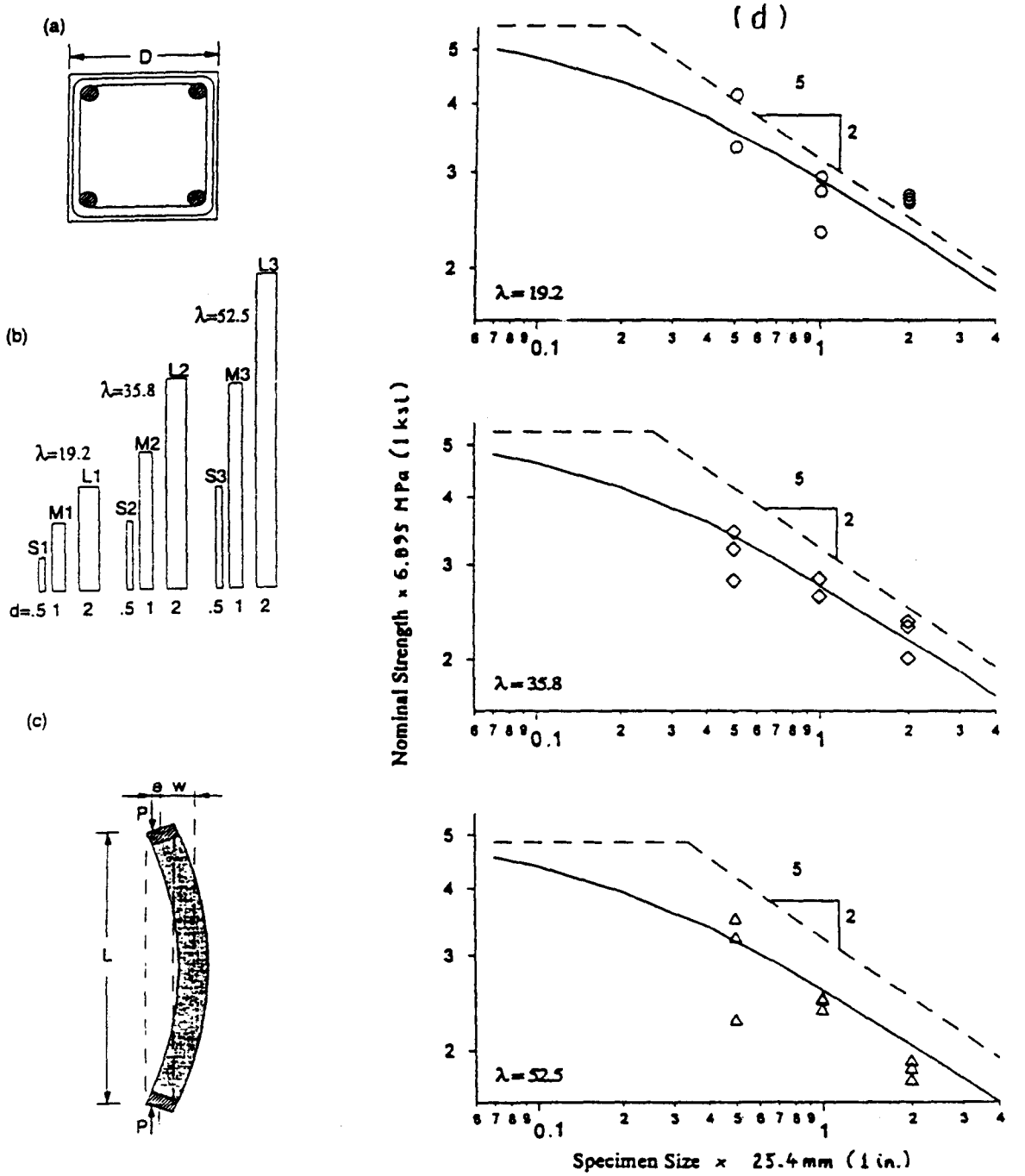

Figure 2: Comparison of theoretical size effect with the test results of Bažant and Kwon (1996): after Bažant and Xiang, 1997.

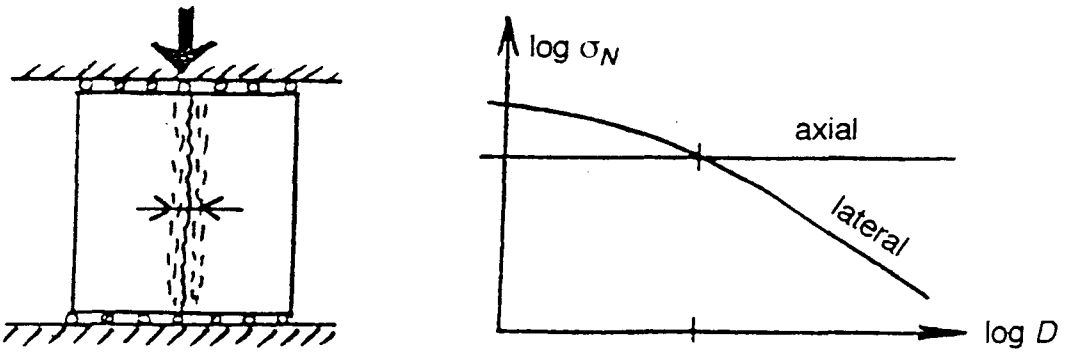

Figure 3: Propagation of an axial splitting crack band (left) and the corresponding size effect plot compared to size effect for lateral propagation (right).

be approximated as

$$\Delta\Pi = -\pi[(a+2R)R\sigma_{x\infty}^2 + (2a+R)a\sigma_{y\infty}^2 - 2aR\sigma_{x\infty}\sigma_{y\infty} - 2a^2\sigma_{cr}^2](1-\nu^2)/2E \tag{2}$$

in which R = borehole radius, a = principal axis of the ellipse, $\sigma_{x\infty}$ and $\sigma_{y\infty}$ = remote principal stresses, E = Young's modulus of the rock, and ν = Poisson ratio. A similar energy balance analysis as that for the propagating band of axial splitting cracks, already explained, has provided a formula for the breakout stress which has a plot similar to those in Fig. 1(f), and has the asymptotic behavior described by Eq. (1).

3 LACK OF SIZE EFFECT IN AXIAL SPLITTING

Why do small uniaxial compression specimens fail by an axial splitting crack and exhibit no size effect?

In a uniform uniaxial stress field, a sharp planar axial crack does not change the stress and thus releases no energy. Therefore a damage band of finite width (Fig. 3 left) must precede the formation of an axial splitting crack. The energy is released only from this band but not from the adjacent undamaged solid. Therefore, the energy release is proportional to the length of the axial splitting crack, which implies that there is no size effect (Fig. 3 right).

Consequently, the lateral propagation of a band of splitting cracks, which involves a size effect, must prevail for a sufficiently large specimen size (Fig. 1 left and Fig. 3 right, Bažant and Xiang 1996). The reason that the axial splitting prevails for a small enough size is that the overall fracture energy consumed (and dissipated) by a unit axial extension of the splitting crack band is smaller than that consumed by a unit lateral extension, for which new cracks must nucleate.

4 SIZE EFFECT FOR THE CASE OF MANY INDEPENDENT LOADS

Before analyzing compression fracture it is useful to deduce the approximate size effect law for a structure with many independently varying loads P_i. The energy release rates of different loads are not additive, but their stress intensity factors are. By superposition, it can be shown that

$$\sum_i \sigma_{Ni}\sqrt{Dg_i(\alpha_0+\theta)} = \sqrt{EG_f} \tag{3}$$

where E = Young's elastic modulus, G_f = fracture energy of the material (for tensile fracture), $\alpha_0 = a_0/D$ = relative notch length of relative length of the initial traction-free crack (a_0 = actual notch length), $\theta = c_f/D$ = relative size of the fracture process zone (whose actual size is c_f), $\sigma_{Ni} = P_i/bD$ = nominal stresses, and $g_i(\alpha)$ are the nondimensionalized LEFM energy release rate functions corresponding to loads P_i (i=1,2,...n). It is now convenient to define the nominal design stresses as $\sigma_{Di} = \sigma_{Ni}/\mu$ where μ = safety factor, and expand functions G_i into a Taylor series as follows:

$$g_i(\alpha_0+\theta) = g_i(\alpha_0) + g'(\alpha_i)\theta + \frac{1}{2}g_i''(\alpha_0)\theta^2 + \ldots \tag{4}$$

Truncating the series after the second term, one gets

$$\mu = \sqrt{EG_f}\,(\rho_1\sigma_{D1} + \rho_2\sigma_{D2} + \ldots + \rho_n\sigma_{Dn})^{-1/2} \tag{5}$$

where

$$\rho_i^2 = g_i(\alpha_0)D + g_i'(\alpha_0)c_f \tag{6}$$

These equations for the safety factor define the failure of the structure and its dependence on structure size D.

It should be noted that the interaction diagram defining the failure as a function of the nominal stresses is linear. This is a necessary consequence of the additivity of stress intensity factors.

For the case of macroscopic crack initiation from a smooth surface, we have $g_i(0) = 0$. Therefore, the series expansions cannot be truncated after the linear term. We may truncate them after the quadratic terms. A similar procedure as before then yields for μ the same expression (5) as before, but with

$$\rho_i = \sqrt{g_i'(0)c_f + \frac{1}{2}g_i''(0)\frac{c_f^2}{D}} \tag{7}$$

Equations (5) with (6) and (7) represent the large-size asymptotic approximations of size effect for the case of many loads. Small-size asymptotic approximations for the case of many loads can be derived similarly, replacing the variable θ with $\eta = 1/\theta$.

Similar to the case of one load, it is further possible to find, for the case of many loads, a universal size effect law that has the correct asymptotic properties for large as well as small sizes and large cracks as well as crack initiation. It may again be written in the form of (6) but with

$$\rho_i = r_i\left[1+\left(\frac{D_{0i}}{D}\right)^r\right]^{1/2r}\left\{1+s\left[\left(\bar{\eta}+\frac{D}{D_{bi}}\right)\left(1+\frac{D}{D_{0i}}\right)\right]^{-1}\right\}^{-1/s} \tag{8}$$

$$r_i = [c_f g_i'(\alpha_0)]^{-1/2}, \quad D_{0i} = c_f g_i'(\alpha_0)/g_i(\alpha_0) \qquad (9)$$
$$D_{bi} = c_f(-g_i''(\alpha_0))/4g_i'(\alpha_0)$$

Here r, s and $\bar{\eta}$ are empirical constants whose magnitudes are of the order of 1.

5 ASYMPTOTIC SCALING FOR TENSILE OR COMPRESSIVE CRACKS WITH RESIDUAL COHESIVE STRESS

In the case of compression fracture due to lateral propagation of a band of axial splitting cracks, a residual stress given by the critical stress for internal buckling in the band remains. This stress can be regarded as a second load applied on the structure, which means one can apply the preceding analysis of failure under many loads.

Lumping the fracturing strains distributed over a band into a line, one may approximately treat such a fracture as a line crack in which interpenetration of the opposite faces is allowed and the softening compressive stress-displacement law terminates with a plateau of residual constant stress σ_Y. A constant residual stress σ_Y may be assumed for characterizing the tensile stress-displacement law for a crack in a fiber-reinforced composite (e.g. fiber-reinforced concrete).

The asymptotic formulae (5)–(7) for the case of many loads can be applied to this case because the uniform pressure σ_Y along the crack can be regarded as one of two loads applied on the structure. We write the stress intensity factors due to the applied load P and the uniform crack pressure σ_Y as $K_I^2 = \sigma_N^2 D g(\alpha_0 + \theta)$ (with $\theta = c_f/D$), and $K_I^2 = \sigma_Y^2 \gamma(\alpha_0 + \theta)$, respectively, where g and γ are dimensionless functions taking the role of g_1 and g_2 in the preceding formulae. In this manner, (6) and (7) yield, after rearrangements, the following formula for the size effect (and shape effect) in the case of a large crack:

$$\sigma_N = \frac{\sqrt{EG_f} + \sigma_Y \sqrt{\gamma'(\alpha_0)c_f + \gamma(\alpha_0)D}}{\sqrt{g'(\alpha_0)c_f + g(\alpha_0)D}} \qquad (10)$$

For geometrically similar structures and size-independent α_0, this formula yields a size effect curve that terminates, in the $\log D$ scale, with a horizontal asymptote on the right and begins with a higher horizontal asymptote on the left.

In the case of initiation of a crack with uniform residual stress σ_Y, equations (5) and (7) can be reduced to the following size (and shape) effect formula:

$$\sigma_N = \frac{\sqrt{EG_f} + \sigma_Y \sqrt{\gamma'(0)c_f + \frac{1}{2}\gamma''(0)\frac{c_f^2}{D}}}{\sqrt{g'(0)c_f + \frac{1}{2}g''(0)\frac{c_f^2}{D}}} \qquad (11)$$

whose logarithmic plot also terminates with a horizontal asymptote.

If the residual stress is compressive and is determined by internal buckling in a band of axial splitting cracks of arbitrary spacing, then σ_Y in the foregoing equations is not constant. As already explained, minimization of σ_N with respect to the crack spacing s shows that the crack spacing in the band should vary as $D^{5/4}$. For such variable spacing it is found that, in the foregoing equations (10) and (11),

$$\sqrt{EG_f} \quad \text{must be replaced by} \quad \sqrt{EG_f}D^{1/10} \qquad (12)$$

Furthermore, the σ_Y value also depends on the crack spacing, according to the formula for the critical buckling load. The overall trend is well approximated by (1), and in particular σ_N approaches the large-size asymptotic limit as $D^{-2/5}$.

6 THE QUESTION OF SIZE EFFECT ON KINK BAND FAILURE OF COMPOSITES

The last two formulae ought to be also applicable to the compression failure cause by fiber micro-buckling in a propagating kink band, as observed in wood or in composites reinforced by parallel fibers. This problem has so far been treated by elasto-plasticity, and solutions of failure loads which give good agreement with the existing test data have been presented (Rosen 1965, Argon 1972, Budianski 1983, Budianski et al. 1997, Budianski and Fleck 1994, Kyriakides et al. 1995, Christensen and DeTeresa, 1997). There is nevertheless good reason to suspect that a size effect exists, and that it may have been missed in previous experiments because of insufficient specimen sizes. This is indicated by observing that:

- the shear slip and fracture along the fibers in the kink band exhibits post-peak softening, i.e., a gradual reduction of the shear stress, which approaches some final asymptotic value, and
- the kink band does not form simultaneously along the entire kink band but has a front that propagates, in the manner of the band of parallel compression splitting cracks.

In view of these observations, the size effect should be describable by equations (10) and (11) in which, however, G_f and σ_Y have different meaning (Bažant, Kim, Daniel and Becq-Giraudon 1997);

$$G_f = \hat{G}_f \frac{w}{s}, \qquad \sigma_Y = \frac{G_{xy}}{1 + \frac{\bar{\phi} + (\bar{o},\ s)}{\gamma_r}} \tag{13}$$

Here w = width of the kink band, s = typical spacing of axial cracks between the fibers in the kink band, $\hat{G}_f$ = shear fracture energy for the axial cracks between fibers in the kink band; G_{xy} = elastic shear modulus of the fiber composites relative to the axial and transverse axes x and y. $\bar{o}$ = initial misalignment angle of the fibers, δ_f = slip displacement at which the shear stress on (cohesive) axial is reduced to the residual stress τ_r, and $\gamma_r = \tau_r / G_{xy}$.

Tests just completed at Northwestern University (Bažant, Kim, Daniel and Becq-Giraudon 1997) confirm that notched geometrically similar uni-directionally reinforced specimens of carbon-PEEK composites exhibit a strong size effect, and that the size effect approximately agrees with (10) and (13); Fig. 4.

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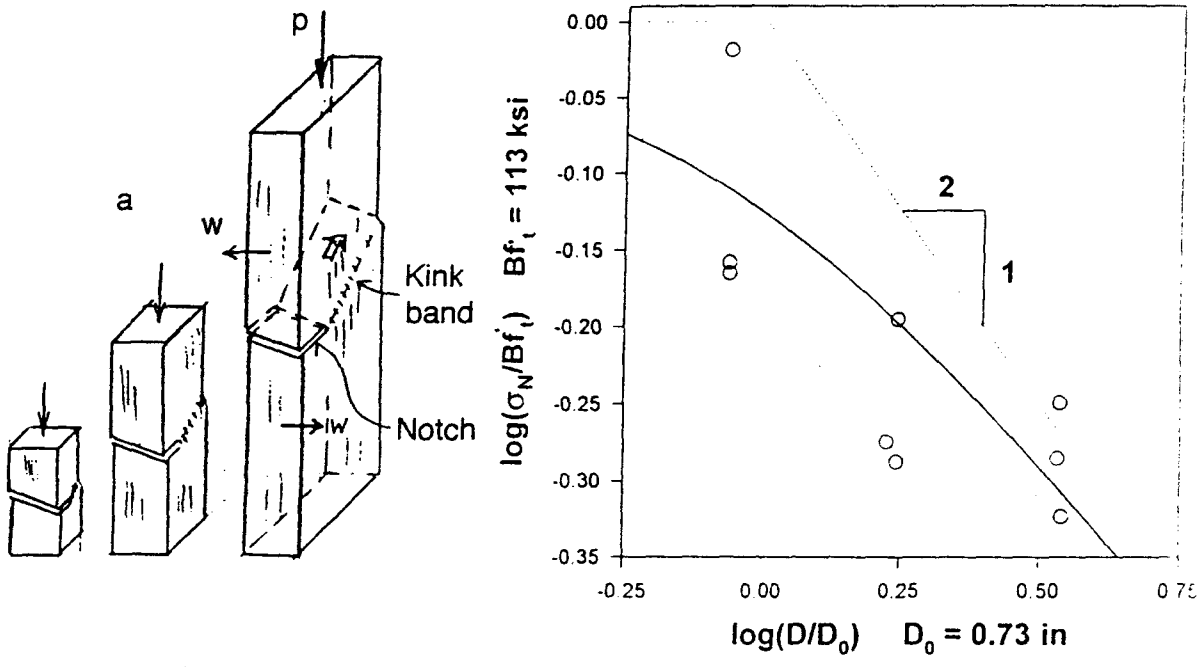

Figure 4: Notched Carbon-PEEK specimens failing by propagation of out-of-plane kink band, and size effect data obtained, compared to equations (10) with (13); after Bažant Kim, Daniel and Becq-Giraudon 1997.

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