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Material versus local incompressibility and its influence on glacial-isostatic adjustment

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SUMMARY

We present an analytical form of the layer propagator matrix for the response of a locally incompressible, layered, linear-viscoelastic sphere to an external load assuming that the initial density stratification $\varrho_0(r)$ within each layer is parametrized by Darwin's law. From this, we show that the relaxation of a sphere consisting of locally incompressible layers is governed by a discrete set of viscous modes. The explicit dependence of the layer propagator matrix on the Laplace transform variable allows us to determine the amplitudes of the viscous modes analytically. Employing Darwin's parametrization, we construct three simplified earth models with different initial density gradients that are used to compare the effects of the local incompressibility constraint, div $(\varrho_0\mathbf{u})=0$, and the material incompressibility constraint, div $\mathbf{u}=0$, on viscoelastic relaxation. We show that a locally incompressible earth model relaxes faster than a materially incompressible model. This is a consequence of the fact that the perturbations of the initial density are zero during viscoelastic relaxation of a locally incompressible medium, so that there are no internal buoyancy forces associated with the continuous radial density gradients, only the buoyancy forces generated by internal density discontinuities. On the other hand, slowly decaying internal buoyancy forces in a materially incompressible earth model cause it to reach the hydrostatic equilibrium after a considerably longer time than a locally incompressible model. It is important to note that the approximation of local incompressibility provides a solution for a *compressible* earth model that is superior to the conventional solutions for a compressible earth with homogeneous layers because it is based on an initial state that is consistent with the assumption of compressibility.

Key words: Darwin's law, glacial-isostatic adjustment, hydrostatic equilibrium, incompressibility, layer propagator, viscoelasticity.

1 INTRODUCTION

The theory of the glacial-isostatic adjustment process belongs to the classical problems of solid earth geophysics and has been developed over several decades. To make the forward and inverse problems tractable, the Earth is usually modelled as linear-viscoelastic, self-gravitating and spherically symmetric. Particular attention has been paid to a materially incompressible Maxwell viscoelastic fluid, since this is the simplest rheology that describes the short-time (elastic) and long-time (fluid) limits of the Earth's response correctly.

The literature dealing with the theory of gravitational viscoelastic relaxation of a materially incompressible earth model is extensive. A short overview of the studies on incompressible earth models completed during the past two decades is as follows. Wu & Peltier (1982) derived the analytical formulae for the response of a homogeneous Maxwell sphere. Sabadini *et al.* (1982) extended this study and introduced semi-analytical solutions for the relaxation of two- and three-layer models of the Earth. Spada *et al.* (1992) inverted the fundamental matrix associated with the field equations using symbolic manipulation software and gave an analytical expression for the inverse of the fundamental matrix. Wolf (1984) and Amelung & Wolf (1994) presented closed-form solutions for the viscoelastic relaxation of an earth model composed of a viscoelastic mantle and an inviscid core or an elastic lithosphere. Wu (1990) analysed gravitational viscoelastic perturbations of two-layer spheres with arbitrary contrasts of density,

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shear modulus and viscosity across the interfaces. Wu & Ni (1996) provided analytical solutions for the viscoelastic relaxation of two-layer, non-gravitating spherical earth models. Vermeersen *et al.* (1997) studied the rotational response of the Earth to Pleistocene deglaciation by means of a multilayered, materially incompressible earth model derived from the Preliminary Reference Earth Model (PREM).

In contrast to this, the viscoelastic relaxation of a locally incompressible earth model has been studied less extensively. There may be two explanations for this. First, the local incompressibility constraint reduces to the material incompressibility constraint for a material with a homogeneous initial density distribution. Second, it is more laborious to solve the viscoelastic relaxation problem for an initial density gradient in a locally incompressible body by an analytical method than to find an analytical solution to the problem for a constant density in a materially incompressible body. The concept of local incompressibility was introduced into the theory of glacial-isostatic adjustment by Wolf (1991), who derived the associated incremental field equations describing gravitational viscoelastic perturbations. Recently, Wolf & Kaufmann (2000) have solved these equations for load-induced Maxwell-viscoelastic perturbations of a non-gravitating half-space with a compressional or compositional initial density gradient. They showed that, for local incompressibility, the elastic limit of the radial surface displacement may overshoot the hydrostatic limit for very long-wavelength perturbations. Li & Wolf (unpublished manuscript, 2000) found the fundamental solution to the incremental field equations for local incompressibility using a self-gravitating spherical earth model and Darwin's distribution of initial density. Nakada (1999) evaluated the effects of a non-adiabatic density gradient in the upper mantle and a non-adiabatic density jump at the 670 km discontinuity on the viscoelastic response of the Earth to a surface load.

The present study is also concerned with viscoelastic earth models with a continuous density stratification. We extend the work of Li & Wolf (unpublished manuscript, 2000) and estimate the effect of local and material incompressibility on viscoelastic relaxation. To solve the problem analytically by the propagator matrix approach, simple earth models are chosen. We thus use spherical earth models with a distribution of the initial density according to Darwin's law that are perturbed by an axisymmetric surface load.

2 LOCAL AND MATERIAL DENSITY INCREMENTS

For any physical quantity q, the local (Eulerian) and material (Lagrangian) increments, q^{E} and q^{L}, respectively, are defined as follows (Wolf 1991; Dahlen & Tromp 1998, Section 3.2.1):

$$q^{\mathrm{E}}(\mathbf{r},\, t)=q(\mathbf{r},\, t)-q(\mathbf{r},\, 0)\,, \tag{1}$$

$$q^{\mathrm{L}}(\mathbf{x},\, t)=q(\mathbf{x},\, t)-q(\mathbf{x},\, 0)\,, \tag{2}$$

where t is the current time and $\mathbf{r}(\mathbf{x},\, t)$ is the current position of a particle located at position $\mathbf{x}$ at the initial time $t=0$. Writing the current position $\mathbf{r}(\mathbf{x},\, t)$ in the form $\mathbf{r}(\mathbf{x},\, t)=\mathbf{x}+\mathbf{u}(\mathbf{x},\, t)$, where $\mathbf{u}(\mathbf{x},\, t)$ is the displacement, the first-order relationship between the material and local increments is

$$q^{\mathrm{L}}=q^{\mathrm{E}}+\mathbf{u}\cdot \operatorname{grad} q(\mathbf{x},\, 0)\,. \tag{3}$$

We have dropped the dependence of q^{L} and q^{E} on the position vectors $\mathbf{x}$ and $\mathbf{r}$, respectively, since, in first-order theory, it is immaterial whether the increments q^{L} and q^{E} are regarded as functions of $\mathbf{r}$ or $\mathbf{x}$.

Adopting this concept for the volume mass density ϱ, the material and local increments in density, ϱ^{L} and ϱ^{E}, respectively, can be expressed in the forms

$$\varrho^{\mathrm{L}}=-\varrho_0 \operatorname{div}\mathbf{u}\,, \quad \varrho^{\mathrm{E}}=-\operatorname{div}(\varrho_0\mathbf{u})\,, \tag{4}$$

where $\varrho_0=\varrho_0(\mathbf{x})$ is the initial density. The deformation is called *materially* and *locally incompressible* if the material and local density increments, ϱ^{L} and ϱ^{E}, respectively, are equal to zero in the deformed state. Note that the constraint of the locally incompressible deformation, $\operatorname{div}(\varrho_0\mathbf{u})=0$, for a homogeneous initial density distribution reduces to that of the materially incompressible deformation, $\operatorname{div}\mathbf{u}=0$. Since $\operatorname{div}\mathbf{u}$ is the relative change in volume for a particle initially at $\mathbf{x}$, this volume does not change for materially incompressible deformation. The local incompressibility condition gives a rather accurate approximation of the long-period viscous behaviour of the Earth. It has therefore been adopted for studying long-period viscous flow in the mantle (e.g. Li & Yuen 1987; Wu & Yuen 1991).

3 BOUNDARY-VALUE PROBLEM FOR VISCOELASTIC RELAXATION

Our intention is to study the response of a self-gravitating, Maxwell viscoelastic, locally incompressible sphere to a surface mass load. This classical problem of solid earth geophysics can be formulated mathematically as an initial boundary-value problem. In this section, we briefly recall the formulation of this problem and refer to the extensive literature for more details (e.g. Farrell 1972; Wu & Peltier 1982; Sabadini *et al.* 1982; Wu 1990; Wu & Ni 1996).

Let us assume that a viscoelastic sphere B approximates the Earth with the initial volume mass density ϱ_0, the shear modulus μ^e and the dynamic viscosity ν. Let a load, represented by the surface density σ, be placed on the surface ∂B of the earth model B. The viscoelastic response of model B to the surface mass load is governed by the equation of linear momentum conservation and by Poisson's equation for small perturbations of a hydrostatically pre-stressed and self-gravitating continuum in a non-rotating reference frame. Their material-local forms are as follows (e.g. Wolf 1991):

$$\mathrm{div}\,\boldsymbol{\tau} - \varrho_0\,\mathrm{grad}\,\phi_1 + \mathrm{div}\,(\varrho_0\mathbf{u})\,\mathrm{grad}\,\phi_0 - \mathrm{grad}\,(\varrho_0\mathbf{u}\cdot\mathrm{grad}\,\phi_0) = 0 \quad \text{in } B\,, \tag{5}$$

$$\nabla^2\phi_1 + 4\pi G\,\mathrm{div}\,(\varrho_0\mathbf{u}) = 0 \quad \text{in } B\,, \tag{6}$$

where $\boldsymbol{\tau}$ is the material (Lagrangian) increment of the Cauchy stress tensor, ϕ_1 is the sum of the local (Eulerian) increment of the initial gravitational potential ϕ_0 and the potential ϕ_2 of the externally applied gravitational force field and G is Newton's gravitational constant. We will assume that the initial density ϱ_0 is a radially dependent function only, $\varrho_0 = \varrho_0(r)$, which implies that $\phi_0 = \phi_0(r)$.

The constitutive property of model B corresponds to that of an incompressible linear viscoelastic body. In the Laplace transform domain s, the constitutive equation has the form

$$\boldsymbol{\tau} = \Pi\boldsymbol{I} + \mu\,(\mathrm{grad}\,\mathbf{u} + \mathrm{grad}^{\mathrm{T}}\mathbf{u}) \quad \text{in } B\,, \tag{7}$$

where Π is the material (Lagrangian) increment of the pressure, $\boldsymbol{I}$ is the second-order identity tensor and the superscript T denotes transposition. For the Maxwell viscoelasticity considered here, the s-dependent shear modulus $\mu = \mu(s)$ has the form

$$\mu := \frac{\mu^e s}{s + \mu^e/\nu}\,. \tag{8}$$

In the following, we assume that the elastic shear modulus and the viscosity are only radially variable functions, i.e. $\mu^e = \mu^e(r)$ and $\nu = \nu(r)$. In addition, we assume that earth model B is locally incompressible,

$$\mathrm{div}\,(\varrho_0\mathbf{u}) = 0 \quad \text{in } B\,. \tag{9}$$

This condition is a kinematic constraint that restricts possible displacements. In particular, it imposes an additional restriction on the displacement gradient, whose components are not all independent. To take into account this dependence, the linear viscoelastic constitutive equation for a compressible body must be modified by including an additional field variable, the perturbation pressure Π, in order to match the kinematic constraint (9). The pressure Π enters into the minimization of the Helmholtz free energy as the Lagrange multiplier when the constitutive equation for the locally incompressible body is formulated. For more details see e.g. Eringen (1980, Section 5.5). Adopting constraint (9) and the identity div grad $\mathbf{u}$ = grad div $\mathbf{u}$ − rot rot $\mathbf{u}$, eqs (5)–(7) can be combined to give

$$\mathrm{grad}\,\Pi + \mu\,(2\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \mathrm{rot}\,\mathrm{rot}\,\mathbf{u}) + \mathrm{grad}\,\mu\cdot(\mathrm{grad}\,\mathbf{u} + \mathrm{grad}^{\mathrm{T}}\mathbf{u}) - \varrho_0\,\mathrm{grad}\,\phi_1 - \mathrm{grad}\,(\varrho_0\mathbf{u}\cdot\mathrm{grad}\,\phi_0) = 0 \quad \text{in } B\,, \tag{10}$$

$$\nabla^2\phi_1 = 0 \quad \text{in } B\,. \tag{11}$$

On an internal discontinuity Σ, the interface conditions for the displacement, the traction, the perturbed gravitational potential and the perturbed normal component of gravitation are (e.g. Dahlen 1974)

$$[\mathbf{u}]_-^+ = 0 \quad \text{and} \quad [\mathbf{n}\cdot\boldsymbol{\tau}]_-^+ = 0 \quad \text{on } \Sigma\,, \tag{12}$$

$$[\phi_1]_-^+ = 0 \quad \text{and} \quad [(\mathrm{grad}\,\phi_1 + 4\pi G\varrho_0\mathbf{u})\cdot\mathbf{n}]_-^+ = 0 \quad \text{on } \Sigma\,, \tag{13}$$

where $\mathbf{n}$ is the outward unit normal to Σ and the symbol $[\,f\,]_-^+$ indicates the outward jump of quantity f on Σ. If an inviscid fluid core is included, the continuity of the normal component of the displacement, $\mathbf{n}\cdot\mathbf{u}$, the continuity of the normal component of the stress vector, $\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n}$, and the free-slip behaviour, $\boldsymbol{\tau}\cdot\mathbf{n} - (\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n})\mathbf{n} = 0$, are appropriate instead of condition (12).

To complete the specification of the problem, boundary conditions are prescribed on the external surface ∂B. We assume that earth model B is loaded at time $t = 0+$ by an impulsive unit point mass with the surface density σ applied on the outer surface ∂B such that

$$\sigma = \frac{1}{a^2 \sin\vartheta}\,\delta(\vartheta)\delta(\varphi) \quad \text{on } \partial B\,, \tag{14}$$

where ϑ and φ are the co-latitude and the longitude, respectively, a is the radius of the sphere ∂B and $\delta(\cdot)$ is the Dirac delta function. Since the time dependence of the surface load is a delta function, the boundary conditions in the Laplace domain are independent of the Laplace variable and are of the form (Longman 1963; Farrell 1972)

$$\mathbf{e}_r\cdot\boldsymbol{\tau}^-\cdot\mathbf{e}_r = -g_0(a)\sigma \quad \text{on } \partial B\,, \tag{15}$$

$$\boldsymbol{\tau}^-\cdot\mathbf{e}_r - (\mathbf{e}_r\cdot\boldsymbol{\tau}^-\cdot\mathbf{e}_r)\,\mathbf{e}_r = 0 \quad \text{on } \partial B\,, \tag{16}$$

$$[\phi_1]_-^+ = 0 \quad \text{on } \partial B\,, \tag{17}$$

$$[\mathrm{grad}\,\phi_1]_-^+\cdot\mathbf{e}_r - 4\pi G\varrho_0^-(\mathbf{u}^-\cdot\mathbf{e}_r) = 4\pi G\sigma \quad \text{on } \partial B\,, \tag{18}$$

where $\boldsymbol{\tau}^-$, ϱ_0^- and $\mathbf{u}^-$ denote the stress tensor, the initial density and the displacement, respectively, on the interior side of ∂B, $\mathbf{e}_r$ is the unit vector in the radial direction, and $g_0(r)$ is the initial gravitation defined by $g_0(r):=d\phi_0(r)/dr$.

4 SPHERICAL HARMONIC REPRESENTATION

The fundamental solution to eq. (11) is given by the solid spherical harmonics $r^n Y_n(\Omega)$ and $r^{-n-1} Y_n(\Omega)$,

$$\phi_1(r,\Omega)=\sum_n F_n(r)\,Y_n(\Omega)\,, \tag{19}$$

with

$$F_n(r):=\phi_{1,n}r^n+\phi_{2,n}r^{-n-1} \tag{20}$$

and $\phi_{1,n}$ and $\phi_{2,n}$ coefficients independent of r and Ω. Here, $Y_n(\Omega)$, with $\Omega:=(\vartheta,\varphi)$, ϑ the co-latitude and φ the longitude, are scalar spherical harmonics normalized according to Edmonds (1957) or Varshalovich *et al.* (1989, Chapter 5). The index n abbreviates the pair of angular degree n and azimuthal order m. Likewise, the spherical harmonic representation of the pressure Π is

$$\Pi(r,\Omega)=\sum_n \Pi_n(r)\,Y_n(\Omega)\,. \tag{21}$$

The displacement $\mathbf{u}$ can be represented in terms of spheroidal, $\mathbf{S}_n^{(\pm 1)}(\Omega)$, and toroidal, $\mathbf{S}_n^{(0)}(\Omega)$, vector spherical harmonics (see Appendix A):

$$\mathbf{u}(r,\Omega)=\sum_n\left[U_n(r)\mathbf{S}_n^{(-1)}(\Omega)+V_n(r)\mathbf{S}_n^{(1)}(\Omega)\right]+\sum_n W_n(r)\mathbf{S}_n^{(0)}(\Omega)=\mathbf{u}_S(r,\Omega)+\mathbf{u}_T(r,\Omega)\,, \tag{22}$$

where $U_n(r)$, $V_n(r)$ and $W_n(r)$ are spherical harmonic expansion coefficients of the spheroidal part $\mathbf{u}_S$ and toroidal part $\mathbf{u}_T$ of $\mathbf{u}$. For the spherically symmetric viscoelastic boundary-value problem considered here and also for an axisymmetric problem, it has been demonstrated (e.g. Martinec & Wolf 1999) that the toroidal displacement $\mathbf{u}_T$ is equal to zero in B.

The spherical harmonic parametrizations (19)–(22) allow us to transform the partial differential equations (9)–(11) into a set of six simultaneous first-order ordinary differential equations of the form (e.g. Peltier 1974; Cathles 1975)

$$\frac{d\mathbf{y}}{dr}=\mathbf{A}\mathbf{y}\,, \tag{23}$$

where the elements of the vector $\mathbf{y}(r,s)=(U_n,V_n,T_{rn},T_{\vartheta n},F_n,Q_n)^{\mathrm{T}}$ are the spherical harmonic coefficients of the radial and tangential displacements, the radial and tangential stresses, and the perturbations in gravitational potential and gravitational potential gradient, respectively. On the assumption of local incompressibility, the 6×6 matrix $\mathbf{A}$, whose elements depend on r and s, has the form

$$\mathbf{A}=\begin{pmatrix}\frac{\beta-2}{r} & \frac{n(n+1)}{r} & 0 & 0 & 0 & 0\\ -\frac{1}{r} & \frac{1}{r} & 0 & \frac{1}{\mu} & 0 & 0\\ \frac{4[(3-\beta)\mu-r\varrho_0 g_0]}{r^2} & \frac{n(n+1)(-6\mu+r\varrho_0 g_0)}{r^2} & 0 & \frac{n(n+1)}{r} & -\frac{(n+1)\varrho_0}{r} & \varrho_0\\ \frac{-2\mu(3-\beta)+r\varrho_0 g_0}{r^2} & \frac{2\mu(2n^2+2n-1)}{r^2} & -\frac{1}{r} & -\frac{3}{r} & \frac{\varrho_0}{r} & 0\\ -4\pi G\varrho_0 & 0 & 0 & 0 & -\frac{n+1}{r} & 1\\ -\frac{4\pi G\varrho_0(n+1)}{r} & \frac{4\pi G\varrho_0 n(n+1)}{r} & 0 & 0 & 0 & \frac{n-1}{r}\end{pmatrix}, \tag{24}$$

where the dimensionless function β denotes the negative radial derivative of the logarithmic density,

$$\beta(r):=-\frac{r}{\varrho_0}\frac{d\varrho_0}{dr}\,z\,. \tag{25}$$

After solving the system of equations (23), the spectral coefficients of the perturbation pressure can be determined as follows:

$$\Pi_n = \frac{2\mu}{r}(2-\beta)U_n - \frac{2\mu\, n(n+1)}{r} V_n + T_{rn} . \tag{26}$$

5 DARWIN'S LAW

We now assume that earth model B is composed of N spherical layers bounded by spheres Σ_k of radii $r_1 = b < r_2 < \cdots < r_N = a$. The elastic shear modulus and the viscosity are assumed to be constant in the kth layer, i.e. $\mu^e = \mu_k^e$ and $\nu = \nu_k$, respectively. In contrast to this, the initial density may change with radial distance within each layer. We will parametrize the radial dependence of the initial density by Darwin's law (Bullen 1975, Section 6.5.3),

$$\varrho_0(r) = \alpha_k r^{-\beta_k} \quad \text{for } r_{k-1}^+ \le r \le r_k^- , \tag{27}$$

where $\alpha_k > 0$ and $0 \le \beta_k < 3$. Note that for $\beta_k < 0$, $d\rho_0/dr$ is positive, $\beta_k = 0$ is the case of constant density, for $\beta_k = 3$ the mass of the layer and the initial gravitation within the layer are infinite, and for $\beta_k > 3$ these quantities are negative. The parameters α_k and β_k may be determined from the boundary values of the initial density,

$$\alpha_k = \varrho_k^- r_k^{\beta_k} , \quad \beta_k = \frac{\ln \dfrac{\varrho_{k-1}^+}{\varrho_k^-}}{\ln \dfrac{r_k}{r_{k-1}}} , \tag{28}$$

where ϱ_{k-1}^+ and ϱ_k^- denote the initial density on the external side of boundary Σ_{k-1} of radius $r = r_{k-1}$ and on the internal side of boundary Σ_k of radius $r = r_k$, respectively. The initial gravitation within the kth layer ($r_{k-1} \le r \le r_k$, $k \ge 2$) is given by

$$g_0(r) = \frac{4\pi G}{r^2}\left[\frac{\alpha_k}{3-\beta_k} r^{3-\beta_k} + \sum_{i=2}^{k}\left(\frac{\alpha_{i-1}}{3-\beta_{i-1}} r_{i-1}^{3-\beta_{i-1}} - \frac{\alpha_i}{3-\beta_i} r_{i-1}^{3-\beta_i}\right)\right] . \tag{29}$$

Moreover, we assume that the central sphere ($0 \le r \le r_1$) is homogeneous, i.e. $\beta_1 = 0$, and we have

$$g_0(r) = \frac{4\pi G}{3} \alpha_1 r . \tag{30}$$

If the initial density stratification is attributed to the compressibility of the material only, then the initial density depends on the initial hydrostatic pressure only and the density gradient is governed by the isochemical adiabatic isentropic Williamson–Adams equation (Bullen 1975, Section 6.4.1; Wolf & Kaufmann 2000),

$$\frac{d\varrho_0}{dr} = -\frac{\varrho_0^2 g_0}{\kappa} . \tag{31}$$

Having prescribed the initial density by eq. (27), the bulk modulus κ associated with the hydrostatically pre-stressed state can be determined from eq. (31):

$$\kappa(r) = \frac{r\varrho_0(r) g_0(r)}{\beta_k} , \tag{32}$$

which holds for $r_{k-1} \le r \le r_k$.

6 LAYER PROPAGATOR

The general solution to eq. (23) can be expressed as the sum of six linearly independent fundamental solution vectors,

$$\mathbf{y}(r, s) = \mathbf{M}(r, s)\mathbf{c}(s) , \tag{33}$$

where $\mathbf{M}(r, s)$ is the 6×6 matrix whose columns are the six fundamental solution vectors, and $\mathbf{c}(s)$ is a 6×1 column vector of arbitrary constants that are to be determined by satisfying the appropriate boundary conditions. For a viscoelastic spherical layer with Darwin's distribution of the initial density, the fundamental solutions are constructed in Appendix B. The fundamental matrix $\mathbf{M}$ can be partitioned with respect to the variables r and μ into the product of three 6×6 matrices:

$$\mathbf{M}(r, s) = \mathbf{U}(r, \mu)\mathbf{V}\mathbf{R}(r, \mu) , \tag{34}$$

where the analytical forms of the component matrices are listed in Appendix C. Note that the three fundamental solutions associated with the second, fourth and sixth columns of matrix $\mathbf{M}$ are singular at the origin $r = 0$. In order to construct the propagator matrix for a multilayered, locally incompressible viscoelastic sphere, the inverse of matrix $\mathbf{M}$ must be found. The inverse of the matrix product (34) is

$$\mathbf{M}^{-1}(r, s) = \mathbf{R}^{-1}(r, \mu)\mathbf{V}^{-1}\mathbf{U}^{-1}(r, \mu), \tag{35}$$

where the analytical forms of the inverses of the component matrices are given in Appendix C.

The relation between $\mathbf{y}(r, s)$ on the inner boundary at r_{k-1} and the outer boundary at r_k of the kth layer can be written in the form (Gantmacher 1959; Gilbert & Backus 1966)

$$\mathbf{y}(r_k, s) = \mathbf{P}(r_k, r_{k-1}, s)\,\mathbf{y}(r_{k-1}, s), \tag{36}$$

where the layer propagator matrix $\mathbf{P}(r_k, r_{k-1}, s)$ is given in terms of matrices $\mathbf{M}$ and $\mathbf{M}^{-1}$ as

$$\mathbf{P}(r_k, r_{k-1}, s) := \mathbf{M}(r_k, s)\mathbf{M}^{-1}(r_{k-1}, s). \tag{37}$$

Substituting for $\mathbf{M}$ and $\mathbf{M}^{-1}$ from eqs (34) and (35), we obtain

$$\mathbf{P}(r_k, r_{k-1}, s) = \mathbf{U}(r_k, \mu_k)\mathbf{V}\mathbf{G}(r_k, r_{k-1})\mathbf{V}^{-1}\mathbf{U}^{-1}(r_{k-1}, \mu_k), \tag{38}$$

where $\mathbf{G}(r_k, r_{k-1})$ is defined by $\mathbf{G}(r_k, r_{k-1}) := \mathbf{R}(r_k, \mu_k)\mathbf{R}^{-1}(r_{k-1}, \mu_k)$. Taking into account eqs (C3) and (C8), matrix $\mathbf{G}$ is diagonal with the elements

$$\mathbf{G}(r_k, r_{k-1}) = \mathrm{diag}\,(h_k^{\sigma}, h_k^{\tau}, h_k^{n+1}, h_k^{-n}, h_k^{n+1-\beta_k}, h_k^{-n-\beta_k}), \tag{39}$$

where $h_k := r_k/r_{k-1}$, β_k is the second parameter in Darwin's law, and the powers σ and τ are defined by eq. (B7). Note that the dependence of matrix $\mathbf{R}$ and its inverse $\mathbf{R}^{-1}$ on the s-dependent shear modulus μ has been cancelled by their multiplication, so that the product matrix $\mathbf{G}$ is independent of μ.

The solution in the central sphere ($0 \le r \le b$) is given by eq. (33), which takes the particular form

$$\mathbf{y}(r, s) = \mathbf{M}_c(r, s)\mathbf{c}_c(s), \tag{40}$$

where $\mathbf{M}_c(r, s)$ is the 6×3 matrix whose columns are the first, third and fifth columns of matrix $\mathbf{M}$ and $\mathbf{c}_c(s)$ is a 3×1 column vector of constants to be determined. The solution in the kth layer then follows from the upward continuation of the solution from the central sphere and the continuity of $\mathbf{y}(r, s)$ on the interfaces at $r_1, \ldots, r_N$. This corresponds to the multiplication of the product of the layer propagator matrices with the central sphere solution taken at $r = b$. In the kth layer, for which $r_{k-1} \le r \le r_k$, $k \ge 2$, applies, we obtain

$$\mathbf{y}(r, s) = \mathbf{L}(r, b, s)\mathbf{y}(b, s), \tag{41}$$

where

$$\mathbf{L}(r, b, s) := \mathbf{P}(r, r_{k-1})\mathbf{P}(r_{k-1}, r_{k-2}) \ldots \mathbf{P}(r_2, b). \tag{42}$$

Substituting for $\mathbf{y}(b, s)$ from eq. (40), we have

$$\mathbf{y}(r, s) = \mathbf{L}(r, b, s)\mathbf{M}_c(b, s)\mathbf{c}_c(s). \tag{43}$$

The constants in $\mathbf{c}_c(s)$ can be determined from the surface boundary conditions (15)–(18) for an impulsive point load (Longman 1963; Farrell 1972):

$$\mathbf{b} := \begin{pmatrix} T_{rn}(a) \\ T_{\vartheta n}(a) \\ Q_n(a) \end{pmatrix} = -\sqrt{\frac{4\pi}{2n+1}} \begin{pmatrix} \dfrac{(2n+1)g_0(a)}{4\pi a^2} \\ 0 \\ \dfrac{(2n+1)G}{a^2} \end{pmatrix}. \tag{44}$$

Taking eq. (43) on the earth's surface, $r = a$, we have

$$\mathbf{b} = \mathbf{T}(s)\mathbf{c}_c(s), \tag{45}$$

where $\mathbf{T}(s)$ is a 3×3 matrix defined by

$$\mathbf{T}(s) := \mathbf{\Lambda}(a, b, s)\mathbf{M}_c(b, s), \tag{46}$$

and the 3×6 matrix $\mathbf{\Lambda}$ corresponds to matrix $\mathbf{L}$ with rows one, two and five deleted. Determining $\mathbf{c}_c$ from eq. (45) and substituting the corresponding expression into eq. (43) gives the solution in the kth layer:

$$\mathbf{y}(r, s) = \mathbf{L}(r, b, s)\mathbf{M}_c(b, s)\mathbf{T}^{-1}(s)\mathbf{b}. \tag{47}$$

Denoting the matrix of co-factors of $\mathbf{T}(s)$ by $\mathbf{T}^{\dagger}(s)$, this solution can be written in an alternative form,

$$\mathbf{y}(r,s)=\frac{\mathbf{W}(r,s)}{\det\mathbf{T}(s)}\,, \tag{48}$$

where

$$\mathbf{W}(r,s):=\mathbf{L}(r,b,s)\mathbf{M}_c(b,s)\mathbf{T}^{\dagger}(s)\mathbf{b}\,. \tag{49}$$

7 TIME-DOMAIN SOLUTION

According to Wu (1978, 1990), Peltier (1985) and Wolf (1985), the solution in the Laplace transform domain can be expressed in the time domain as follows:

$$\mathbf{y}(r,t)=\mathbf{y}^{\mathrm{E}}(r)\delta(t)+\sum_j \mathbf{y}_j^{\mathrm{V}}(r)\,\mathrm{e}^{s_j t}\,, \tag{50}$$

where $\delta(t)$ is the delta function, $\mathbf{y}^{\mathrm{E}}(r)$ is the elastic amplitude,

$$\mathbf{y}^{\mathrm{E}}(r)=\lim_{s\to-\infty}\mathbf{y}(r,s)\,, \tag{51}$$

$\mathbf{y}_j^{\mathrm{V}}(r)$ is the viscous amplitude spectrum and $-s_j$ is the inverse relaxation time spectrum. The inverse relaxation times $-s_j$ can be determined from the roots s_j of the secular determinant $\det\mathbf{T}(s)=0$. The viscous amplitudes $\mathbf{y}_j^{\mathrm{V}}(r)$ are then obtained using the residue theorem:

$$\mathbf{y}_j^{\mathrm{V}}(r)=\frac{\mathbf{W}(r,s_j)}{\dfrac{d}{ds}\left[\det\mathbf{T}(s)\right]\Big|_{s=s_j}}\,, \tag{52}$$

where simple zeros s_j are assumed. In Appendix D, we derive the analytical formula for the s-derivative of the secular determinant $\det\mathbf{T}(s)$.

8 EARTH MODELS

To illustrate the significance of the local incompressibility condition (9), we study the viscoelastic relaxation of three simple earth models. All three models consists of a viscoelastic heterogeneous mantle and an inviscid homogeneous core. The parameter β inside the core is equal to zero, which means that the conditions of local and material incompressibility coincide. The common parameter values of the models used in our computational examples are as follows: the radius of the earth's surface $a=6371$ km, the radius of the core $b=3480$ km, the elastic shear modulus of the mantle $\mu^e=1.4519\times10^{11}$ Pa, the viscosity of the mantle $\nu=10^{21}$ Pa s, the initial density of the core $\varrho_{0,c}=10.9869\times10^3$ kg m^{-3} and Newton's gravitational constant $G=6.67259\times10^{-11}$ m^3 kg^{-1} s^{-2}.

The mantle density is parametrized according to Darwin's law (27). The parameters α and β may be determined by imposing various constraints on the earth model such as prescribing the total mass of the model, defining the density jumps at the core–mantle boundary (CMB) and at the earth's surface or prescribing the moment of inertia. In this paper, we require that the total mass of the core and of the mantle are equal to those of the earth. These constraints guarantee the correct gravitational values at the CMB and the earth's surface. Since the mass-conservation principle does not determine Darwin's parameters α and β completely, we add additional constraints on the mantle density distribution. For the first two-layer model (model C), we choose the mantle density at the CMB equal to the value of the PREM model. For the second two-layer model (model S), we assume that the surface density jump is equal to that of the PREM model. Since the density jump at the earth's surface controls the radial surface displacement in the limit of hydrostatic equilibrium, model S can be employed to simulate the viscoelastic response of the PREM model. Finally, we approximate the mantle density distribution of the PREM model by the four-layer model SPR with density jumps at depths of 400, 670 and 2891 km. The parameters determining the density distributions of models C, S and SPR are listed in Table 1 and the density distributions are plotted for the mantle in Fig. 1.

9 NUMERICAL RESULTS

We now demonstrate numerically the difference between the viscoelastic relaxation of earth models with local incompressibility, div $(\varrho_0\mathbf{u})=0$, and those with material incompressibility, div $\mathbf{u}=0$. We emphasize that the initial density distribution is identical for both types of incompressibility, so that we determine only the effects of the particular types of incompressibility on the solution. As we have seen in the previous sections, the viscoelastic solution for a spherically symmetric earth model with the initial density distribution according to Darwin's law and the local incompressibility condition has a discrete Laplace spectrum. This implies that

Table 1. The parameters α and β and the density jumps for earth models C, S and SPR.

Region	Radius (km)	α (kg m$^{\beta-3}$)	β	ϱ_0^{bot} (kg m^{-3})	ϱ_0^{top} (kg m^{-3})
Model C					
Core	0–3480	10.9869×10^3	0	10.9869×10^3	10.9869×10^3
Mantle	3480–6371	3.87220×10^7	0.587378	5.56645×10^3	3.90226×10^3
Model S					
Core	0–3480	10.9869×10^3	0	10.9869×10^3	10.9869×10^3
Mantle	3480–6371	1.920791×10^{18}	2.185193	9.74691×10^3	2.6×10^3
Model SPR					
Core	0–3480	10.9869×10^3	0	10.9869×10^3	10.9869×10^3
Lower mantle	3480–5701	8.138945×10^6	0.482720	5.66×10^3	4.46×10^3
Upper mantle	5701–5971	1.663040×10^{16}	1.866854	4.06×10^3	3.724×10^3
	5971–6371	1.548686×10^{14}	1.570324	3.543×10^3	3.2×10^3

the inverse Laplace transformation can be implemented analytically and leads to a relaxation characterized by exponentially decaying normal modes. On the other hand, when the material incompressibility condition is assumed, the solution in the Laplace domain has a continuous spectrum and the inverse Laplace transform can be computed only numerically, which may cause numerical problems (Han & Wahr 1995; Hanyk *et al.* 1996). In this case, it is advantageous to solve the problem directly in the time domain. Recently, Martinec (2000) has proposed the spectral finite element approach with an explicit time-differencing scheme for the Maxwell rheology to model the viscoelastic relaxation of a spherical earth with a 3-D viscosity structure. Here, we will use this approach simplified for a spherically symmetric viscosity distribution in order to compute the time-domain response of an earth model with material incompressibility and a continuous stratification of the mantle density.

Fig. 2 shows the relaxation times $-1/s_j$, the elastic amplitude $-y_1^{E}(a)$ and the viscous amplitudes $-y_{1,j}^{V}(a)/s_j$ of the radial surface displacement as functions of the angular degree n for earth models C, S and SPR. The viscoelastic relaxation of models C and S is carried by two buoyancy modes, M0 and C0, associated with the density discontinuities on the earth's surface and at the CMB (e.g. Wu & Peltier 1982; Wu 1990; Wu & Ni 1996). In addition to the M0 and C0 modes, the buoyancy modes M1 and M2 associated with the density jumps at depths of 400 and 670 km appear for model SPR (e.g. Peltier 1982; Wolf 1985; Wu 1990; Han & Wahr 1995; Johnston *et al.* 1997). For small values of the angular degree n, the amplitudes of the C0 and M1 modes are close to that of the M0 mode. However, for $n > 10$, the deformation of the CMB and of the 400 and 670 km discontinuities is insignificant and the total relaxation is mainly carried by the M0 mode.

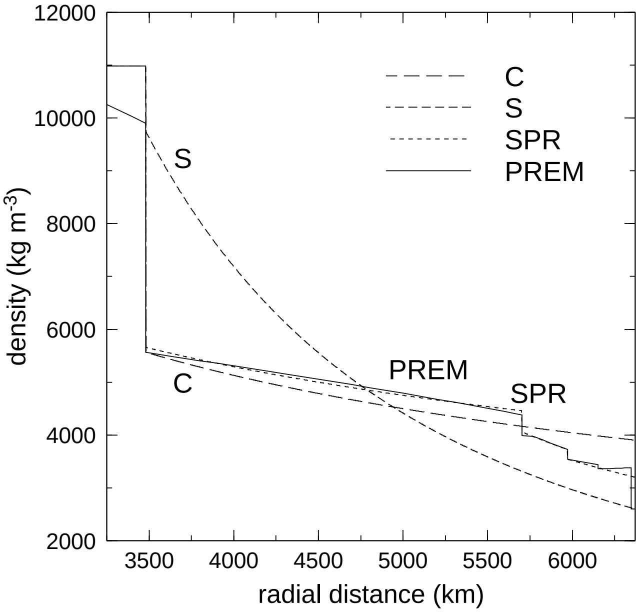

Figure 1. The mantle density distribution of earth models C, S and SPR parametrized by Darwin's law and the density of the PREM model. The density in model C at the CMB on the mantle side is equal to the PREM value, model S has the same density jump at the earth's surface as the PREM model, and model SPR approximates the mantle density of PREM by three layers. The core is considered homogeneous for models C, S and SPR. Parameter values are given in Table 1.

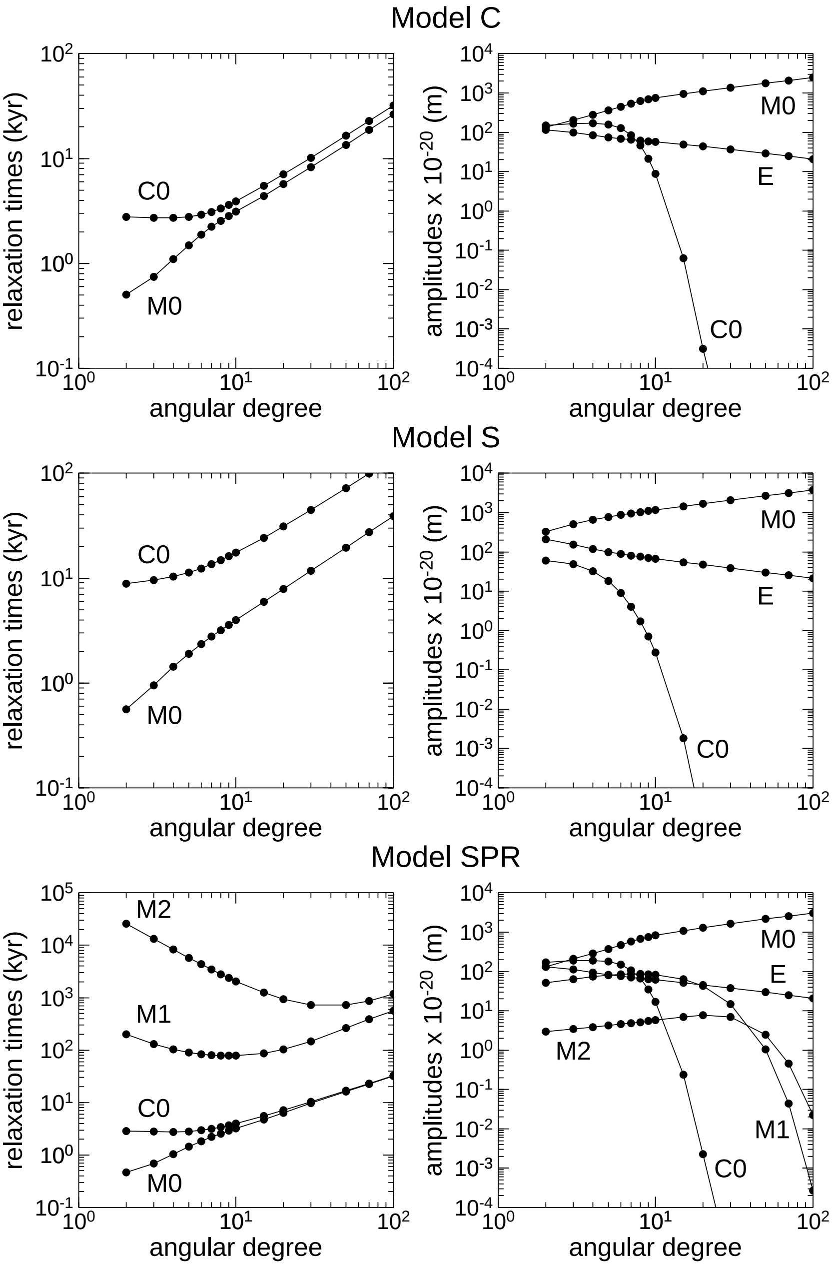

Figure 2. The relaxation times $-1/s_j$, the elastic amplitude $-y_1^{\mathrm{E}}(a)$ and the viscous amplitudes $-y_{1,j}^{\mathrm{V}}(a)/s_j$ of the radial surface displacement as functions of the angular degree n for earth models C, S and SPR. M0, C0, M1 and M2 denote relaxation modes and E the elastic amplitude. Results apply to Heaviside load forcing.

Figs 3–5 show the time evolution of the radial and horizontal surface displacements and the surface gravitational potential perturbation for earth models C, S and SPR in the time interval 0–10 kyr. The models are loaded at time $t=0+$ by the second, 10th, 20th and 50th zonal spherical harmonics of the unit point mass applied at the north pole of the earth's surface as the Heaviside step function in time. The solid and dashed lines show the time relaxation curves when the local incompressibility condition, div $(\varrho_0\mathbf{u})=0$, and the material incompressibility condition, div $\mathbf{u}=0$, are applied, respectively. It is seen that the relative differences between the locally incompressible and materially incompressible solutions can reach 15 per cent for the radial displacement $U_2(a)$ for models C and SPR, whereas they are about 50 per cent for models S and SPR. The differences in the gravitational potential perturbation $F_2(a)$ are only a few per cent for models C and SPR, whereas they may reach 100 per cent for models S and SPR. The difference in behaviour of the locally incompressible and the materially incompressible materials is even more significant in the time relaxation of the horizontal displacement $V_n(a)$; the relaxation curves differ by several orders of magnitude and may even have opposite signs.

It is further seen that the different behaviour of the locally incompressible and materially incompressible media diminishes with increasing angular degree n; Figs 3–5 show that there is hardly any difference in the time relaxation curves of $U_n(a)$ and $F_n(a)$ for angular degree $n=50$. This is consequence of the fact that higher-degree viscoelastic perturbations do not penetrate as deeply as

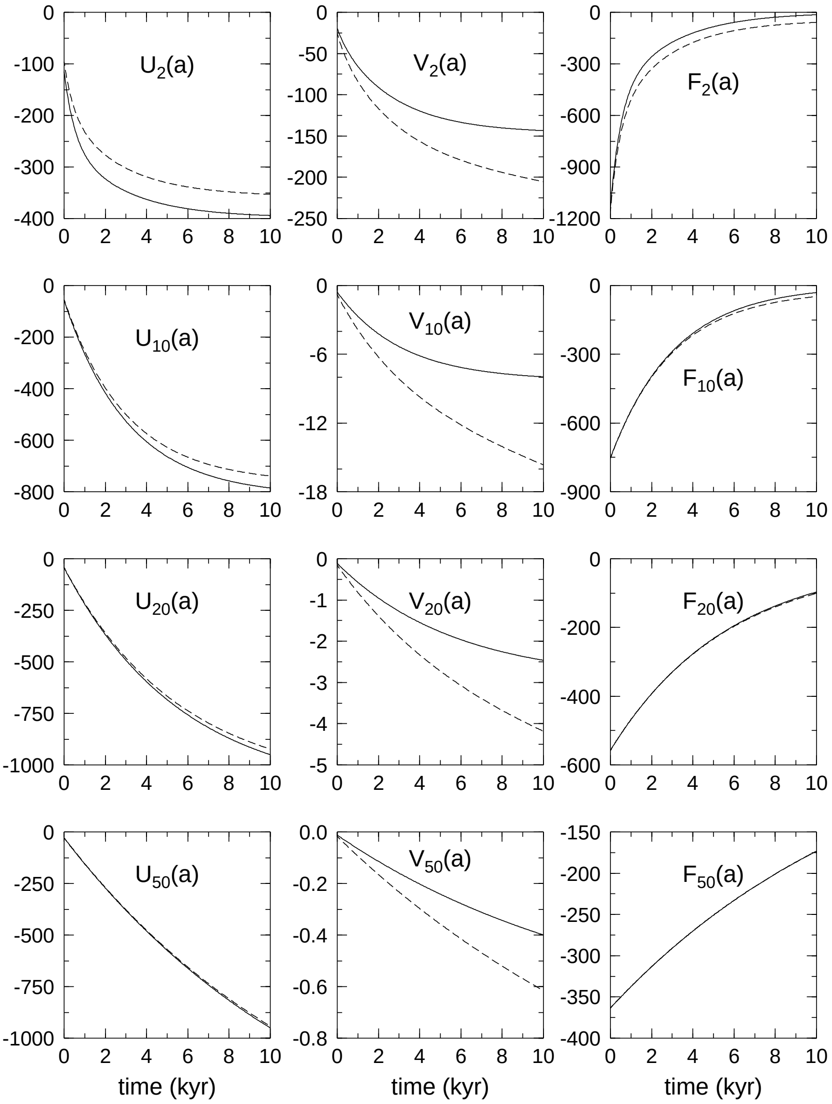

Figure 3. Comparison of the solutions for local incompressibility (solid lines) and material incompressibility (dashed lines) for the two-layer model C. Time relaxation of the radial surface displacement $U_n(a)$, the horizontal surface displacement $V_n(a)$ and the surface gravitational potential perturbation $F_n(a)$ are shown for various angular degrees n. Results apply to Heaviside load forcing. The physical units of $U_n(a)$ and $V_n(a)$ are 10^{-20} m and that of $F_n(a)$ is 10^{-20} m^2 s^{-2}.

low-degree perturbations and hence they are mainly controlled by the surface density jump. Consequently, the particular type of incompressibility becomes unimportant for higher-degree perturbations.

It is also seen that the locally incompressible material relaxes faster in time than the materially incompressible material. The physical explanation is that the perturbation of the initial density distribution at a particular point vanishes during the viscoelastic relaxation process for a locally incompressible medium. Thus, there are no internal buoyancy forces associated with the continuous radial density gradient and the buoyancy forces are generated only by the internal density jumps. On the other hand, slowly decaying internal buoyancy forces in a materially incompressible medium cause this medium to reach hydrostatic equilibrium after a considerably longer time than a locally incompressible medium.

The final hydrostatic equilibrium for the Heaviside loading history of an earth model is the sum of the elastic amplitude and the amplitudes of all viscoelastic modes. Since there is no elastic layer in our earth models that would partly store the shear energy, the radial displacement of the earth's surface once the final hydrostatic equilibrium is attained is governed by the density jump on the earth's surface. In addition, when the equilibrium between the external gravitational forcing and the internal buoyancy is restored, the gravitational potential perturbation vanishes. In summary, the hydrostatic equilibrium is characterized by the equations

$$U_n(a,t)|_{t\to\infty} = -\sqrt{\frac{2n+1}{4\pi}}\,\frac{1}{a^2\varrho_0(a)}\,,\quad F_n(a,t)|_{t\to\infty}=0 \tag{53}$$

(Wu & Peltier 1982). Fig. 6 shows the time relaxation behaviour of model C for the time interval 0–10^4 kyr. It is seen that a locally incompressible material attains hydrostatic equilibrium much more quickly than a materially incompressible material.

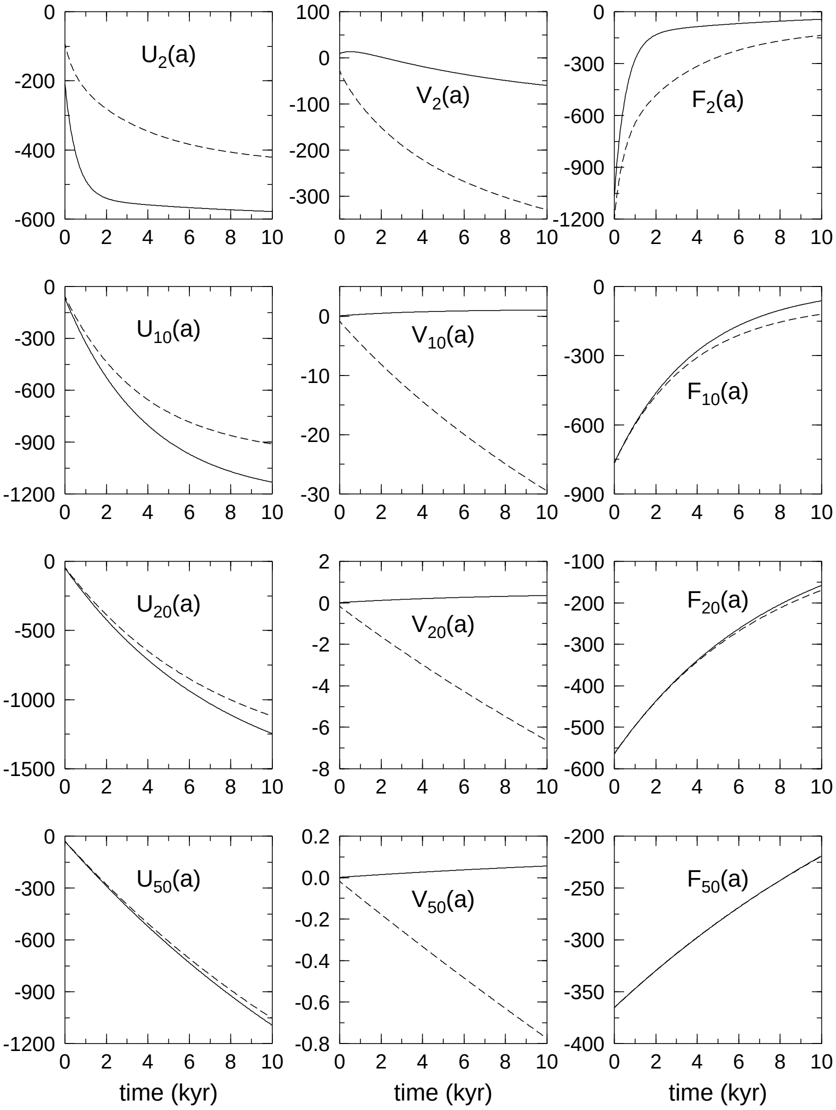

Figure 4. As for Fig. 3 but for the two-layer model S.

So far, we have considered earth models that are homogeneous with respect to the viscosity and the elastic shear modulus. We now compare the responses of locally incompressible and materially incompressible models that have viscosities and elastic shear modulus stratifications. The initial density distribution of model SPR′ is the same as that of model SPR but the viscosity and the elastic shear moduli of the lower and upper mantle are chosen as follows: $\nu_{\rm LM}=5.9\times10^{21}$ Pa s, $\nu_{\rm UM}=4.8\times10^{20}$ Pa s, $\mu^e_{\rm LM}=2\times10^{11}$ Pa and $\mu^e_{\rm UM}=1.45\times10^{11}$ Pa. Model SPR′ also contains an elastic lithosphere of 95 km thickness with an elastic shear modulus of $\mu^e=0.67\times10^{11}$ Pa. We note that this model belongs to the set of two- and three-layer models preferred by Lambeck *et al.* (1990) on the basis of sea-level curves from northwest Europe.

The relaxation of model SPR′ with locally incompressible stratification is characterized by a set of seven discrete relaxation modes. In addition to the buoyancy modes M0, M1, M2 and C0 that characterize the response of model SPR, the viscoelastic modes V1, V2 and L0 appear. They are associated with jumps of the viscosity and the elastic shear modulus at 670 km depth and at the base of the lithosphere of model SPR′ (e.g. Wu & Peltier 1982; Wolf 1985). As for model SPR, the condition of material incompressibility imposed on the response of model SPR′ causes the Laplace spectrum to be continuous, which complicates the inversion of the solution in the Laplace transform domain. We therefore compute the response of model SPR′ for material incompressibility directly in the time domain using the spectral–finite element approach (Martinec 2000).

Fig. 7 shows the time relaxation of the radial and horizontal surface displacements and the surface gravitational potential perturbation for model SPR′ during the time interval 0–10 kyr. Model SPR′ is forced by the same load as models C, S and SPR, whose relaxation is shown in Figs 3–5. Comparing Figs 5 and 7, we notice that models SPR and SPR′ behave similarly for the radial surface displacement and surface gravitational potential perturbation. The largest differences in these components occur for angular degree $n=2$ and decrease with increasing degree n. The relative difference between the local incompressibility and material incompressibility cases can reach 20 per cent for $U_2(a)$ and $F_2(a)$ and is about 2 per cent for $U_{50}(a)$ and $F_{50}(a)$. Because of the existence

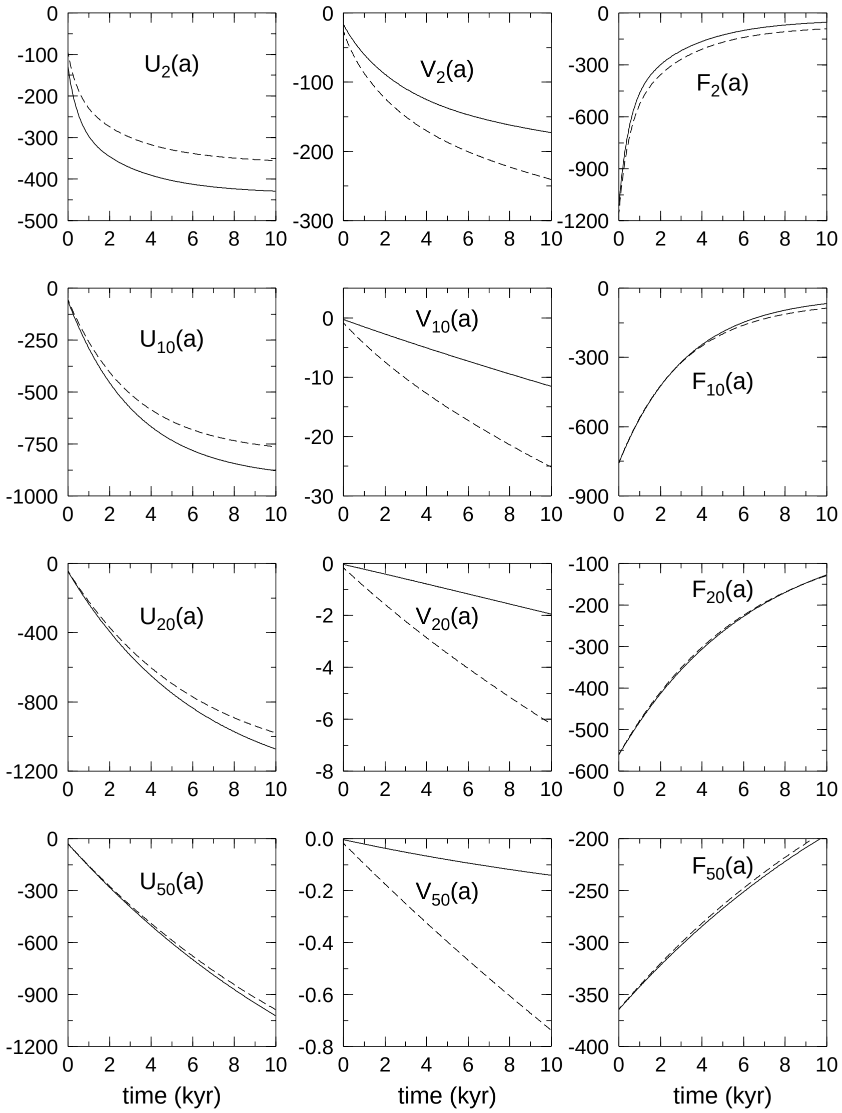

Figure 5. As for Fig. 3 but for the four-layer model SPR.

of an elastic lithosphere in model SPR, the spectral amplitudes do not reach the hydrostatic limit (53). The most significant difference between the responses of models SPR and SPR′ appears in the horizontal displacement $V_n(a)$. The relaxation curves $V_n(a)$ of model SPR′ have opposite signs compared to those of model SPR and they change the sign of the curvature with increasing time. In addition, the relative difference between local incompressibility and material incompressibility is much smaller for model SPR′ than for model SPR. As is seen in the panels of the middle columns of Figs 5 and 7, the relaxation curves for the horizontal displacements of model SPR may differ by several orders of magnitude, while these differences may reach 50 per cent at most for model SPR′, for example, for the spherical component $V_{10}(a)$. We conclude that the viscosity and elastic shear modulus stratifications significantly reduce the difference between the horizontal displacements of the locally incompressible and materially incompressible earth models. The relaxation spectrum of an earth model with viscosity and elastic shear modulus stratifications contains both the buoyancy and the viscoelastic modes, while a model that is homogeneous with respect to viscosity and elastic shear modulus contains only the buoyancy modes. Since the viscoelastic modes control the horizontal displacements more strongly than the buoyancy modes, the horizontal displacement in model SPR′ is not influenced by the particular type of incompressibility condition as significantly as it is for model SPR.

10 CONCLUSIONS

This paper was motivated by the question raised by Wolf & Kaufmann (2000) whether the neglect of sphericity and self-gravitation in planar viscoelastic earth modelling may affect the long-wavelength behaviour of their solution. They showed in particular that, for local incompressibility, the elastic limit of the radial surface displacement may overshoot the hydrostatic limit for very long-wavelength perturbations. Our modelling based on a gravitationally self-consistent spherical earth model has not confirmed the

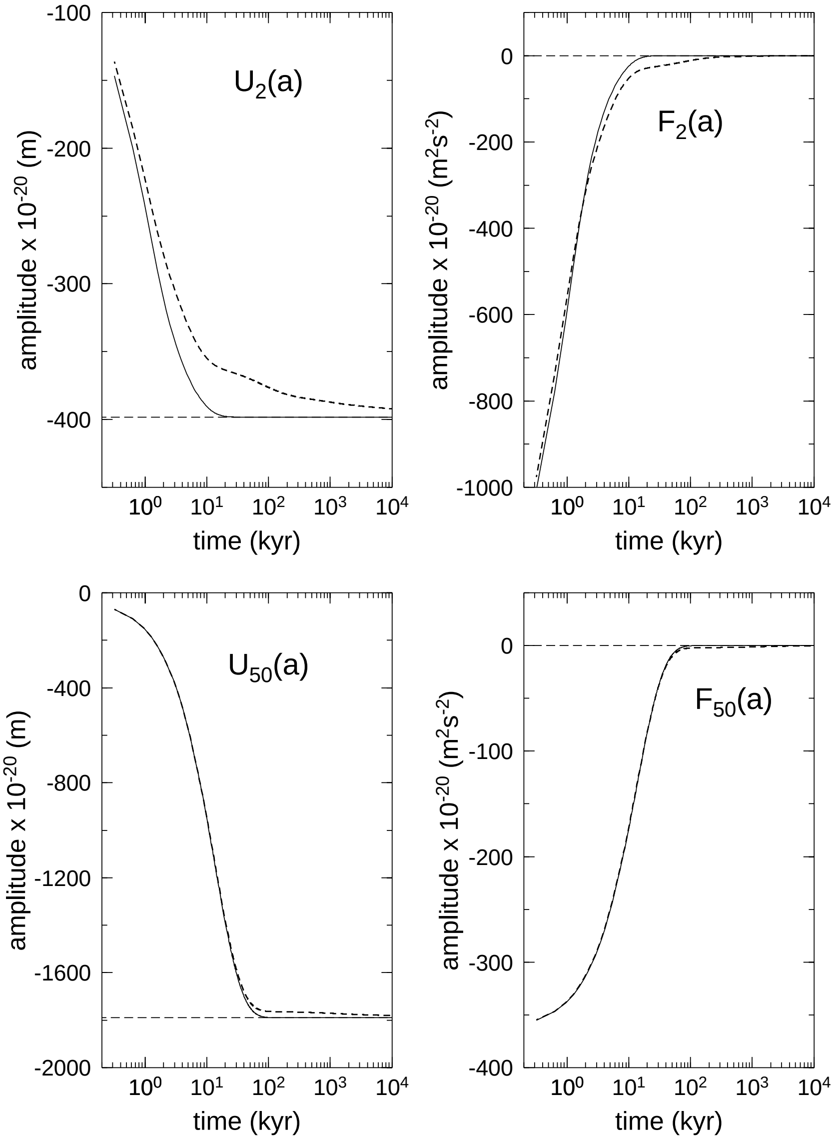

Figure 6. Time relaxation of the radial surface displacement $U_n(a)$ and the surface gravitational potential perturbation $F_n(a)$, $n=2$ and 50, for the locally incompressible (solid lines) and materially incompressible (dashed lines) model C over the interval 0–10^4 kyr. The horizontal dashed lines denote the hydrostatic limits expressed by eq. (53). Results apply to Heaviside load forcing.

overshoot for long-wavelength perturbations. The time relaxation curves smoothly increase in magnitude and reach hydrostatic equilibrium without overshoots during relaxation. This indicates that sphericity and self-gravitation cannot be neglected when modelling long-wavelength viscoelastic perturbations for the case of local incompressibility.

Furthermore, we have considered two types of incompressibility conditions: the local form of incompressibility, which is based on the principle that the local (Eulerian) perturbations of the initial density vanish during the viscoelastic relaxation process, and the material (Lagrangian) form of incompressibility, which assumes that an elementary volume associated with a particle does not change during its displacement. We have demonstrated that the particular type of incompressibility may influence the radial surface displacement and surface gravitational perturbation by more than 20 per cent depending on the radial gradient of the mantle density. The larger the density gradient, the more significant the type of incompressibility condition. This applies in particular to the horizontal surface displacement, for which the differences range from 1–3 orders of magnitude. However, taking into account viscosity and shear modulus stratifications, the difference in horizontal displacements is significantly reduced. This reduction is effected in particular by the existence of an elastic lithosphere.

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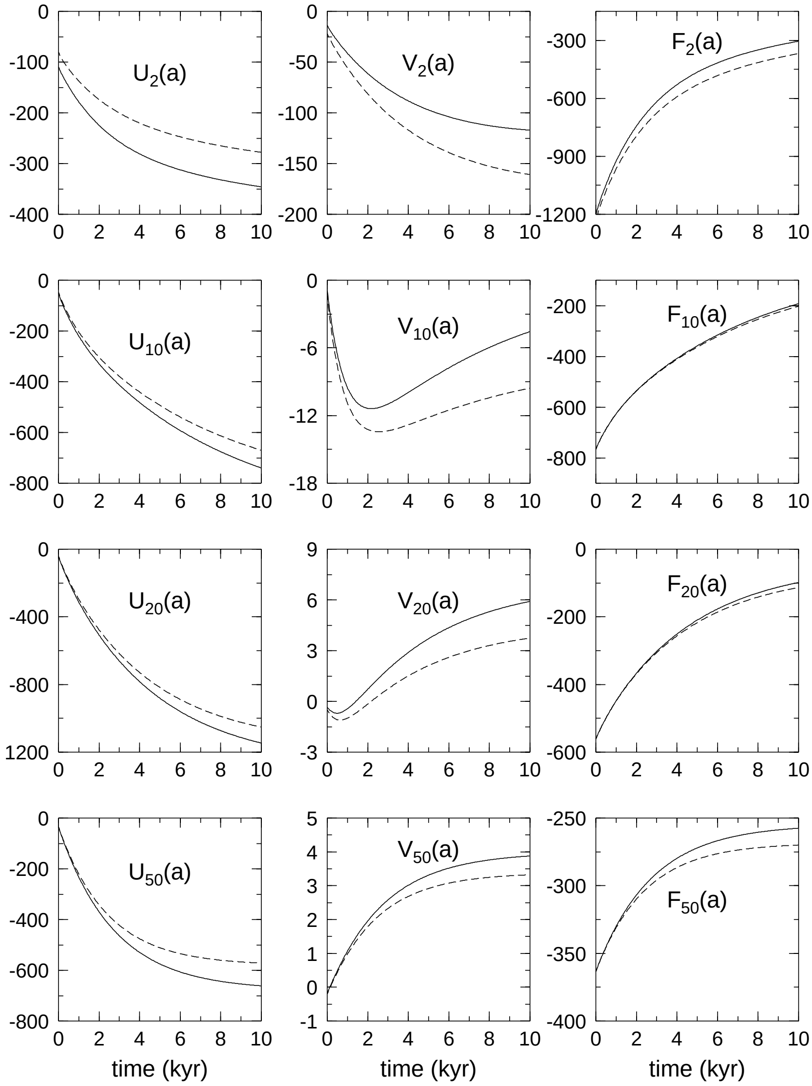

Figure 7. Comparison of the solutions for local incompressibility (solid lines) and material incompressibility (dashed lines) for the five-layer model SPR′. The initial density distribution of model SPR′ is the same as that of model SPR but the viscosities and the elastic shear moduli of the lower and upper mantle are as follows: $\nu_{\rm LM}=5.9\times10^{21}$ Pa s, $\nu_{\rm UM}=4.8\times10^{20}$ Pa s, $\mu^e_{\rm LM}=2\times10^{11}$ Pa and $\mu^e_{\rm UM}=1.45\times10^{11}$ Pa. Model SPR′ contains an elastic lithosphere of 95 km thickness with an elastic shear modulus of $\mu^e=0.67\times10^{11}$ Pa. Time relaxation of the radial surface displacement $U_n(a)$, the horizontal surface displacement $V_n(a)$ and the surface gravitational potential perturbation $F_n(a)$ are shown for various angular degrees n. Results apply to Heaviside load forcing. The physical units of $U_n(a)$ and $V_n(a)$ are 10^{-20} m and that of $F_n(a)$ is 10^{-20} m^2 s^{-2}.

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APPENDIX A: VECTOR SPHERICAL HARMONICS

We define the vector spherical harmonics and present the results of applying the invariant differential operators on scalar and vector spherical harmonics.

The vector spherical harmonics $\mathbf{S}^{(\lambda)}_{nm}(\Omega)$, $n=0, 1, \ldots, m=-n, -n+1, \ldots, n, \lambda=-1, 0, 1$, can be defined as follows (e.g. Phinney & Burridge 1973):

$$\mathbf{S}^{(-1)}_{nm}(\Omega) := Y_{nm}(\Omega)\mathbf{e}_r\,,$$

$$\mathbf{S}^{(1)}_{nm}(\Omega) := \mathrm{grad}_{\Omega}\, Y_{nm}(\Omega)\,, \tag{A1}$$

$$\mathbf{S}^{(0)}_{nm}(\Omega) := \mathbf{e}_r \times \mathrm{grad}_{\Omega}\, Y_{nm}(\Omega)\,,$$

where grad_{Ω} is the angular part of the gradient operator,

$$\mathrm{grad}_{\Omega} := \mathbf{e}_{\vartheta}\frac{\partial}{\partial\vartheta} + \mathbf{e}_{\varphi}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\,. \tag{A2}$$

$Y_{nm}(\Omega)$ are scalar spherical harmonics normalized according to Edmonds (1957) or Varshalovich *et al.* (1989, Chapter 5), $\mathbf{e}_r$, $\mathbf{e}_\vartheta$ and $\mathbf{e}_\varphi$ are spherical unit base vectors, ϑ and φ are the co-latitude and longitude, respectively, and $\Omega := (\vartheta, \varphi)$. The vector functions $\mathbf{S}^{(\pm 1)}_{nm}(\Omega)$ and $\mathbf{S}^{(0)}_{nm}(\Omega)$ are called spheroidal and toroidal vector spherical harmonics, respectively. To abbreviate the notation, we follow the usual convention in postglacial rebound studies and replace the pair of subscripts nm by subscript n only and the double summation over n and m by summation over n (e.g. Wu & Peltier 1982).

Let us summarize the basic differential operations with vector spherical harmonics (e.g. Varshalovich *et al.* 1989, Chapter 7). The gradient of the scalar function $f(r)\,Y_n(\Omega)$, where $f(r)$ is an arbitrary differentiable function of r, can be written in terms of spheroidal vector spherical harmonics as

$$\mathrm{grad}\,[\,f(r)\,Y_n(\Omega)] = \frac{df(r)}{dr}\,\mathbf{S}^{(-1)}_n(\Omega) + \frac{f(r)}{r}\,\mathbf{S}^{(1)}_n(\Omega)\,; \tag{A3}$$

the rotations of the vector spherical harmonics have the forms

$$\operatorname{rot}[f(r)\mathbf{S}_n^{(-1)}(\Omega)] = -\frac{f(r)}{r}\mathbf{S}_n^{(0)}(\Omega)\,,$$
$$\operatorname{rot}[f(r)\mathbf{S}_n^{(1)}(\Omega)] = \left(\frac{d}{dr}+\frac{1}{r}\right)f(r)\mathbf{S}_n^{(0)}(\Omega)\,, \tag{A4}$$
$$\operatorname{rot}[f(r)\mathbf{S}_n^{(0)}(\Omega)] = -n(n+1)\frac{f(r)}{r}\mathbf{S}_n^{(-1)}(\Omega) - \left(\frac{d}{dr}+\frac{1}{r}\right)f(r)\mathbf{S}_n^{(1)}(\Omega)\,;$$

and the divergences of the vector spherical harmonics are

$$\operatorname{div}[f(r)\mathbf{S}_n^{(-1)}(\Omega)] = \left(\frac{d}{dr}+\frac{2}{r}\right)f(r)Y_n(\Omega)\,,$$
$$\operatorname{div}[f(r)\mathbf{S}_n^{(1)}(\Omega)] = -n(n+1)\frac{f(r)}{r}Y_n(\Omega)\,, \tag{A5}$$
$$\operatorname{div}[f(r)\mathbf{S}_n^{(0)}(\Omega)] = 0\,.$$

The cross-products of the spherical base vector $\mathbf{e}_r$ with the vector spherical harmonics are

$$\mathbf{e}_r\times\mathbf{S}_n^{(-1)}(\Omega)=0\,,\quad \mathbf{e}_r\times\mathbf{S}_n^{(1)}(\Omega)=\mathbf{S}_n^{(0)}(\Omega)\,,\quad \mathbf{e}_r\times\mathbf{S}_n^{(0)}(\Omega)=-\mathbf{S}_n^{(1)}(\Omega)\,. \tag{A6}$$

APPENDIX B: FUNDAMENTAL SOLUTION FOR A LOCALLY INCOMPRESSIBLE MODEL WITH INITIAL DENSITY ACCORDING TO DARWIN'S LAW

We derive the analytical form of the fundamental solution to the system of differential equations (9) and (10) for the special case when the s-dependent shear modulus μ is constant and the initial density ϱ_0 is described by Darwin's law (27).

To satisfy the local incompressibility condition (9), it is convenient to introduce the toroidal vector potential $\mathbf{A}_\mathrm{T}=\mathbf{A}_\mathrm{T}(r,\,\Omega)$ that generates the spheroidal displacement $\mathbf{u}_\mathrm{S}$. Since the rotation of a toroidal vector is a spheroidal vector and since the initial density varies only radially, we may write

$$\varrho_0\mathbf{u}_\mathrm{S} = \operatorname{rot}\mathbf{A}_\mathrm{T}\,. \tag{B1}$$

To construct the fundamental solution for $\mathbf{A}_\mathrm{T}$, we take the rotation of eq. (10), consider the fact that μ is constant and substitute for $\mathbf{u}_\mathrm{S}$ from eq. (B1). Then, using the differential identity

$$\operatorname{rot}(\varrho_0\operatorname{grad}\phi_1) = \frac{d\varrho_0}{dr}(\mathbf{e}_r\times\operatorname{grad}\phi_1)\,, \tag{B2}$$

where $\times$ denotes the cross-product of vectors, we obtain a fourth-order differential equation for $\mathbf{A}_\mathrm{T}$,

$$\mu\operatorname{rot}\operatorname{rot}\operatorname{rot}\left(\frac{1}{\varrho_0}\operatorname{rot}\mathbf{A}_\mathrm{T}\right) = -\frac{d\varrho_0}{dr}(\mathbf{e}_r\times\operatorname{grad}\phi_1)\,. \tag{B3}$$

Here, the gravitational potential perturbation ϕ_1 is assumed to be a known function of r and Ω that is given by eqs (19) and (20).

The general solution of the differential equation (B3) can be expressed as the sum of the solution $\mathbf{A}_\mathrm{T}^\mathrm{h}$ to the homogeneous equation

$$\operatorname{rot}\operatorname{rot}\operatorname{rot}\left(\frac{1}{\varrho_0}\operatorname{rot}\mathbf{A}_\mathrm{T}^\mathrm{h}\right) = 0 \tag{B4}$$

and a particular solution $\mathbf{A}_\mathrm{T}^\mathrm{p}$ to the non-homogeneous equation (B3),

$$\boxed{\mathbf{A}_\mathrm{T} = \mathbf{A}_\mathrm{T}^\mathrm{h} + \mathbf{A}_\mathrm{T}^\mathrm{p}\,.} \tag{B5}$$

It is straightforward to show that the solution to eq. (B4) for Darwin's model of the initial density, $\varrho_0(r)=\alpha r^{-\beta}$, is

$$\boxed{\mathbf{A}_\mathrm{T}^\mathrm{h}(r,\,\Omega) = \alpha r^{-\beta+1}\sum_n (A_{1,n}r^\sigma + A_{2,n}r^\tau + A_{3,n}r^{n+1} + A_{4,n}r^{-n})\mathbf{S}_n^{(0)}(\Omega)\,,} \tag{B6}$$

where the coefficients $A_{i,n}$, $i=1,\,\ldots,\,4$, are constants and

$$\left\{\begin{matrix}\sigma\\ \tau\end{matrix}\right\} := \frac{1}{2}\left[\beta-3\pm\sqrt{(\beta-1)^2+4n(n+1)}\right]. \tag{B7}$$

Applying the operator grad to eq. (19), taking the cross-product of $\mathbf{e}_r$ and the result, and employing eqs (A3) and (A6) for the gradient of a scalar function and the cross-products of vector $\mathbf{e}_r$ and vector spherical harmonics, respectively, the vector occurring on the right-hand side of eq. (B3) can be expressed as

$$\mathbf{e}_r \times \operatorname{grad} \phi_1 = \sum_n (\phi_{1,n} r^{n-1} + \phi_{2,n} r^{-n-2}) \mathbf{S}_n^{(0)}(\Omega) . \tag{B8}$$

This enables us to express the two particular solutions to eq. (B3) in the form

$$\boxed{\mathbf{A}_{\mathrm{T}}^{\mathrm{p}}(r, \Omega) = \frac{\alpha^2}{\mu} r^{-2\beta+1} \sum_n \left(\frac{\phi_{1,n}}{d_1} r^{n+1} + \frac{\phi_{2,n}}{d_2} r^{-n} \right) \mathbf{S}_n^{(0)}(\Omega) ,} \tag{B9}$$

where

$$\begin{aligned} d_1 &:= (\beta - 2n - 1) e_1 , \quad e_1 := 2(\beta-1)^2 + (n+1)(4-3\beta) , \quad f_1 := \beta(n+2) - 2(2n+3) , \\ d_2 &:= (\beta + 2n + 1) e_2 , \quad e_2 := 2(\beta-1)^2 - n(4-3\beta) , \quad f_2 := \beta(n-1) - 2(2n-1) . \end{aligned} \tag{B10}$$

Having found the toroidal vector potential $\mathbf{A}_{\mathrm{T}}$ and using the relation (A4) for the rotation of the toroidal vector spherical harmonics, the spheroidal displacement $\mathbf{u}_{\mathrm{S}}$ can be determined from the rotation of $\mathbf{A}_{\mathrm{T}}$ divided by ϱ_0:

$$\mathbf{u}_{\mathrm{S}}(r, \Omega) = \sum_n [U_n(r) \mathbf{S}_n^{(-1)}(\Omega) + V_n(r) \mathbf{S}_n^{(1)}(\Omega)] , \tag{B11}$$

where

$$U_n(r) := -n(n+1)(A_{1,n} r^{\sigma} + A_{2,n} r^{\tau} + A_{3,n} r^{n+1} + A_{4,n} r^{-n}) - n(n+1) \frac{\varrho_0}{\mu} \left(\frac{\phi_{1,n}}{d_1} r^{n+1} + \frac{\phi_{2,n}}{d_2} r^{-n} \right) , \tag{B12}$$

$$\begin{aligned} V_n(r) := {} & (\tau+1) A_{1,n} r^{\sigma} + (\sigma+1) A_{2,n} r^{\tau} - (n - \beta + 3) A_{3,n} r^{n+1} + (n + \beta - 2) A_{4,n} r^{-n} \\ & - \frac{\varrho_0}{\mu} \left[(n - 2\beta + 3) \frac{\phi_{1,n}}{d_1} r^{n+1} - (n + 2\beta - 2) \frac{\phi_{2,n}}{d_2} r^{-n} \right] . \end{aligned} \tag{B13}$$

The determination of the fundamental solution for the pressure Π is rather laborious. Introducing the notation

$$\pi := \Pi + 2\mu \operatorname{div} \mathbf{u} - \varrho_0 (\mathbf{u} \cdot \operatorname{grad} \phi_0) , \tag{B14}$$

eq. (10) for constant μ reads

$$\operatorname{grad} \pi = \varrho_0 \operatorname{grad} \phi_1 + \mu \operatorname{rot} \operatorname{rot} \mathbf{u} . \tag{B15}$$

Taking the divergence of the last equation, making use of the differential identity

$$\operatorname{div} (\varrho_0 \operatorname{grad} \phi_1) = \varrho_0 \nabla^2 \phi_1 + \operatorname{grad} \varrho_0 \cdot \operatorname{grad} \phi_1 \tag{B16}$$

and using the Laplace equation (11) for ϕ_1, we obtain the Poisson differential equation for the pressure π,

$$\nabla^2 \pi = \frac{d\varrho_0}{dr} (\mathbf{e}_r \cdot \operatorname{grad} \phi_1) , \tag{B17}$$

where the right-hand side is a known function of r and Ω.

The general solution to the differential equation (B17) can be expressed as the sum of the solution π^{h} to the homogeneous equation $\nabla^2 \pi^{\mathrm{h}} = 0$ and a particular solution π^{p} to the non-homogeneous equation (B17):

$$\pi = \pi^{\mathrm{h}} + \pi^{\mathrm{p}} . \tag{B18}$$

The fundamental solution to the homogeneous equation is given by the solid spherical harmonics,

$$\pi^{\mathrm{h}}(r, \Omega) = \mu \sum_n (C_{1,n} r^n + C_{2,n} r^{-n-1}) Y_n(\Omega) , \tag{B19}$$

where the s-dependent shear modulus μ has been introduced for normalization. In order to derive the analytical form of the right-hand side of eq. (B17), we take the gradient of eq. (19) and consider the scalar product of $\mathbf{e}_r$ and the result. We obtain

$$\mathbf{e}_r \cdot \operatorname{grad} \phi_1 = \sum_n (n\, \phi_{1,n} r^{n-1} - (n+1)\, \phi_{2,n} r^{-n-2}) Y_n(\Omega) , \tag{B20}$$

which enables us to express two particular solutions of eq. (B17) in the form

$$\pi^{\mathrm{p}}(r, \Omega) = \alpha r^{-\beta} \sum_n \left(\frac{n}{2n+1-\beta} \phi_{1,n} r^n + \frac{n+1}{2n+1+\beta} \phi_{2,n} r^{-n-1} \right) Y_n(\Omega) . \tag{B21}$$

With the help of eqs (B14), (B18), (B19) and (B21), we now have

$$\Pi(r,\Omega)=-2\mu\,\mathrm{div}\,\mathbf{u}+\varrho_0(\mathbf{u}\cdot\mathrm{grad}\,\phi_0)+\mu\sum_n\left(C_{1,n}r^n+C_{2,n}r^{-n-1}\right)Y_n(\Omega)$$
$$+\varrho_0\sum_n\left(\frac{n}{2n+1-\beta}\,\phi_{1,n}r^n+\frac{n+1}{2n+1+\beta}\,\phi_{2,n}r^{-n-1}\right)Y_n(\Omega)\,, \tag{B22}$$

where the coefficients $C_{1,n}$ and $C_{2,n}$ depend on the constants $A_{i,n}$, $i=1,\ldots,4$, via eq. (B15). After considerable algebraic manipulation, we arrive at

$$C_{1,n}=(n+1)[2(2n+3)-\beta(n+2)]A_{3,n}\,,\quad C_{2,n}=n[2(2n-1)-\beta(n-1)]A_{4,n}\,. \tag{B23}$$

Finally, the spherical harmonic coefficients $\Pi_n(r)$ of the series expansion (21) of the pressure Π can be expressed in the form

$$\Pi_n(r)=\left(\varrho_0 g_0-\frac{2\mu\beta}{r}\right)U_n(r)+\mu(n+1)[2(2n+3)-\beta(n+2)]r^nA_{3,n}+\mu\,n[2(2n-1)-\beta(n-1)]r^{-n-1}A_{4,n}$$
$$+\varrho_0\left(\frac{n}{2n+1-\beta}\,\phi_{1,n}r^n+\frac{n+1}{2n+1+\beta}\,\phi_{2,n}r^{-n-1}\right). \tag{B24}$$

For completeness, let us state, without detailed derivation, the formulae for the spheroidal vector components of the stress vector $\mathbf{T}_r(r,\Omega):=\mathbf{e}_r\cdot\boldsymbol{\tau}$, where the stress tensor $\boldsymbol{\tau}$ is defined by eq. (7):

$$\mathbf{T}_r(r,\Omega)=\sum_n\left[T_{rn}(r)\mathbf{S}_n^{(-1)}(\Omega)+T_{\vartheta n}(r)\mathbf{S}_n^{(1)}(\Omega)+T_{\varphi n}(r)\mathbf{S}_n^{(0)}(\Omega)\right]. \tag{B25}$$

The coefficients of the spheroidal vector spherical harmonics in this equation have the form

$$T_{rn}(r):=\Pi_n+2\mu\,\frac{dU_n}{dr} \tag{B26}$$
$$=-n(n+1)\left[\frac{2\mu}{r}(\sigma-\beta)+\varrho_0g_0\right]r^\sigma A_{1,n}-n(n+1)\left[\frac{2\mu}{r}(\tau-\beta)+\varrho_0g_0\right]r^\tau A_{2,n}$$
$$-(n+1)\left[\frac{2\mu}{r}\left(n^2-n-3-\frac{n-2}{2}\,\beta\right)+n\varrho_0g_0\right]r^{n+1}A_{3,n}+n\left[\frac{2\mu}{r}\left(n^2+3n-1+\frac{n+3}{2}\,\beta\right)-(n+1)\varrho_0g_0\right]r^{-n}A_{4,n}$$
$$-\frac{n\varrho_0}{d_1}\left\{\frac{1}{r}\,[e_1+2(n+1)(n-2\beta+1)]+\frac{n+1}{\mu}\,\varrho_0g_0\right\}r^{n+1}\phi_{1,n}+\frac{(n+1)\varrho_0}{d_2}\left\{\frac{1}{r}\,[e_2+2n(n+2\beta)]-\frac{n}{\mu}\,\varrho_0g_0\right\}r^{-n}\phi_{2,n}\,,$$

where e_1 and e_2 are defined by eq. (B10) and

$$T_{\vartheta n}(r):=\mu\left(\frac{dV_n}{dr}-\frac{V_n}{r}+\frac{U_n}{r}\right)$$
$$=-\mu\{[n(n+1)-(\sigma-1)(\tau+1)]r^{\sigma-1}A_{1,n}+[n(n+1)-(\sigma+1)(\tau-1)]r^{\tau-1}A_{2,n}+n(2n-\beta+4)r^nA_{3,n}+(n+1)(2n+\beta-2)r^{-n-1}A_{4,n}\}$$
$$-\varrho_0\left\{[n(n+1)+(n-2\beta+3)(n-\beta)]r^n\,\frac{\phi_{1,n}}{d_1}+[n(n+1)+(n+2\beta-2)(n+\beta+1)]r^{-n-1}\,\frac{\phi_{2,n}}{d_2}\right\}. \tag{B27}$$

Note that we do not give the formula for the coefficients $T_{\varphi n}(r)$ of the toroidal vector spherical harmonics, since they are not excited in our problem. Finally, the perturbation of the gravitational potential gradient defined by

$$Q_n(r):=\frac{dF_n}{dr}+\frac{n+1}{r}\,F_n+4\pi G\varrho_0U_n \tag{B28}$$

takes the form

$$Q_n(r)=-4\pi G\varrho_0\,n(n+1)(A_{1,n}r^\sigma+A_{2,n}r^\tau+A_{3,n}r^{n+1}+A_{4,n}r^{-n})+(2n+1)r^{n-1}\phi_{1,n}-n(n+1)\,\frac{4\pi G\varrho_0^2}{\mu}\left(\frac{\phi_{1,n}}{d_1}\,r^{n+1}+\frac{\phi_{2,n}}{d_2}\,r^{-n}\right). \tag{B29}$$

APPENDIX C: MATRIX OF THE FUNDAMENTAL SOLUTION AND ITS INVERSE

We present the analytical forms of the component matrices $\mathsf{U}(r,\mu)$, V and $\mathsf{R}(r,\mu)$ of the fundamental matrix $\mathsf{M}(r,s)$ and of their inverses $\mathsf{U}^{-1}(r,\mu)$, V^{-1} and $\mathsf{R}^{-1}(r,\mu)$ as functions of the parameters of each layer. The parameters have the following meanings: n = angular degree, $N:=n(n+1)$, r = radial distance from the centre of the earth, $\mu(s):=\mu^e s/(s+\mu^e/\nu)$ = s-dependent shear modulus of the layer, μ^e = elastic shear modulus of the layer, ν = dynamic viscosity of the layer, $\varrho_0(r) = \alpha r^{-\beta}$ = initial density of the layer, α, β = density parameters, $g_0(r)$ = initial gravitation and G = Newton's gravitational constant. The symbols σ, τ and $d_1, d_2, e_1, e_2, f_1, f_2$ are defined by eqs (B7) and (B10), respectively.

The explicit forms of the component matrices are

$$\mathbf{U}(r,\mu) := \frac{1}{N}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \varrho_0 g_0 - \frac{4\mu}{r} & \frac{2\mu N}{r} & \frac{2\mu}{r} & 0 & 0 & \frac{2\mu N}{r} \\ \frac{2\mu}{r} & -\frac{2\mu}{r} & 0 & \frac{\mu}{r} & 0 & -\frac{2\mu}{r} \\ 0 & 0 & 0 & 0 & -\frac{\mu}{r\varrho_0} & 0 \\ 4\pi G\varrho_0 & 0 & 0 & 0 & 0 & -\frac{\mu d_1}{r^2 \varrho_0} \end{pmatrix}, \tag{C1}$$

$$\mathbf{V} := \begin{pmatrix} N & N & N & N & N & N \\ -\tau-1 & -\sigma-1 & n-\beta+3 & -n-\beta+2 & -n-2\beta+2 & -n-2\beta+2 \\ 0 & 0 & \frac{1}{2}(n+1)f_1 & \frac{1}{2}nf_2 & \frac{1}{2}ne_1 & -\frac{1}{2}(n+1)e_2 \\ 0 & 0 & -f_1 & f_2 & e_1 & e_2 \\ 0 & 0 & 0 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & 0 & 2n+1 & 0 \end{pmatrix}, \tag{C2}$$

and $\mathbf{R}$ is a diagonal matrix of the form

$$\mathbf{R}(r,\mu) := \mathrm{diag}\left(r^{\sigma}, r^{\tau}, r^{n+1}, r^{-n}, \frac{\varrho_0}{\mu} r^{n+1}, \frac{\varrho_0}{\mu} r^{-n}\right). \tag{C3}$$

The explicit forms of the inverses of the component matrices are

$$\mathbf{U}^{-1}(r,\mu) = N\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{4\pi G\varrho_0^2 r^2}{\mu d_1} & 1 & 0 & 0 & 0 & \frac{\varrho_0 r^2}{\mu d_1} \\ 2 - \frac{\varrho_0 g_0 r}{2\mu} & -N & \frac{r}{2\mu} & 0 & 0 & 0 \\ -2 & 2 & 0 & \frac{r}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\varrho_0 r}{\mu} & 0 \\ \frac{4\pi G\varrho_0^2 r^2}{\mu d_1} & 0 & 0 & 0 & 0 & -\frac{\varrho_0 r^2}{\mu d_1} \end{pmatrix}, \tag{C4}$$

$$\mathbf{V}^{-1} = \mathbf{D}\begin{pmatrix} \frac{\sigma+1}{N} & 1 & V^{-1}_{13} & V^{-1}_{14} & V^{-1}_{15} & V^{-1}_{16} \\ \frac{\tau+1}{N} & 1 & V^{-1}_{23} & V^{-1}_{24} & V^{-1}_{25} & V^{-1}_{26} \\ 0 & 0 & 2 & -n & \frac{2n+1}{2n+1+\beta} & -\frac{d_1}{2n+1+\beta} \\ 0 & 0 & 2 & n+1 & 0 & -e_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2n+1 & -d_1 \end{pmatrix}, \tag{C5}$$

where we have introduced a diagonal matrix of the form

$$\mathbf{D} := \operatorname{diag}\left(\frac{1}{\sigma-\tau}, \frac{1}{\tau-\sigma}, \frac{1}{(2n+1)f_1}, \frac{1}{(2n+1)f_2}, \frac{1}{2n+1}, \frac{1}{(2n+1)d_2}\right) \tag{C6}$$

and used the abbreviations

$$\begin{aligned}
V_{13}^{-1} &:= \frac{2}{f_1 f_2}\,[f_1 + (n+4+\sigma-\beta)(4-\beta)]\,,\\
V_{23}^{-1} &:= \frac{2}{f_1 f_2}\,[f_1 + (n+4+\tau-\beta)(4-\beta)]\,,\\
V_{14}^{-1} &:= \frac{1}{f_1 f_2}\,[(n+1)f_1 + 2(n+4+\sigma-\beta)(3-\beta)]\,,\\
V_{24}^{-1} &:= \frac{1}{f_1 f_2}\,[(n+1)f_1 + 2(n+4+\tau-\beta)(3-\beta)]\,,\\
V_{15}^{-1} &:= \frac{1}{e_2 f_1}\,[f_1 + 2(n+4+\sigma-\beta)(2-\beta)]\,,\\
V_{25}^{-1} &:= \frac{1}{e_2 f_1}\,[f_1 + 2(n+4+\tau-\beta)(2-\beta)]\,,\\
V_{16}^{-1} &:= \frac{1}{e_2 f_1 f_2}\,[f_1 v_{16} + 2(n+4+\sigma-\beta)(2-\beta)v_{26}]\,,\\
V_{26}^{-1} &:= \frac{1}{e_2 f_1 f_2}\,[f_1 v_{16} + 2(n+4+\tau-\beta)(2-\beta)v_{26}]\,,\\
v_{16} &:= -e_1 e_2 + (\beta-1)(7\beta-8)f_2\,,\\
v_{26} &:= -(\beta^2-7\beta+8)e_2 + (\beta-1)(7\beta-8)f_2\,.
\end{aligned} \tag{C7}$$

The evaluation of the inverse of the diagonal matrix $\mathbf{R}$ yields

$$\mathbf{R}^{-1}(r,\mu) = \operatorname{diag}\left(r^{-\sigma}, r^{-\tau}, r^{-n-1}, r^{n}, \frac{\mu}{\varrho_0}\,r^{-n-1}, \frac{\mu}{\varrho_0}\,r^{n}\right). \tag{C8}$$

To compute the viscous amplitudes of the solution, the μ-derivative of the matrices $\mathbf{U}(r,\mu)$, $\mathbf{U}^{-1}(r,\mu)$ and $\mathbf{R}(r,\mu)$ is required. This derivative can be determined analytically by differentiating eqs (C1), (C3) and (C4) with respect to μ. The result is

$$\mathbf{N}^{(1)}(r) := \frac{d\mathbf{U}(r,\mu)}{d\mu} = \frac{1}{N\,r}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ -4 & 2N & 2 & 0 & 0 & 2N\\ 2 & -2 & 0 & 1 & 0 & -2\\ 0 & 0 & 0 & 0 & -\dfrac{1}{\varrho_0} & 0\\ 0 & 0 & 0 & 0 & 0 & -\dfrac{d_1}{r\varrho_0} \end{pmatrix}, \tag{C9}$$

$$\mathbf{N}^{(-1)}(r) := -\mu^2\,\frac{d\mathbf{U}^{-1}(r,\mu)}{d\mu} = N\,r\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ -\dfrac{4\pi G\varrho_0^2 r}{d_1} & 0 & 0 & 0 & 0 & \dfrac{\varrho_0 r}{d_1}\\ -\dfrac{\varrho_0 g_0}{2} & 0 & \dfrac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -\varrho_0 & 0\\ \dfrac{4\pi G\varrho_0^2 r}{d_1} & 0 & 0 & 0 & 0 & -\dfrac{\varrho_0 r}{d_1} \end{pmatrix}, \tag{C10}$$

$$\mathbf{N}^{(0)}(r) := -\mu^2 \frac{d\mathbf{R}(r, \mu)}{d\mu} = \mathrm{diag}(0, 0, 0, 0, \varrho_0 r^{n+1}, \varrho_0 r^{-n}) . \tag{C11}$$

APPENDIX D: *S*-DERIVATIVE OF THE SECULAR DETERMINANT

The analytical form of the layer propagator matrix allows us to compute the s-derivative of the secular determinant det $\mathbf{T}(s)$ in eq. (52) analytically. We have

$$\frac{d}{ds}[\det \mathbf{T}(s)] = \sum_{i,j=1}^{3} \frac{\partial[\det \mathbf{T}(s)]}{\partial T_{ij}} \frac{dT_{ij}(s)}{ds} = \sum_{i,j=1}^{3} T_{ij}^{\dagger} \frac{dT_{ij}(s)}{ds} , \tag{D1}$$

where $T_{ij}^{\dagger}$ is the co-factor of T_{ij}. The problem of finding an analytical s-derivative of the secular determinant thus reduces to that of finding analytical s-derivatives of the matrix elements T_{ij}. Taking into account eq. (46), we can write

$$\frac{d\mathbf{T}(s)}{ds} = \frac{d\mathbf{\Lambda}(a, b, s)}{ds} \mathbf{M}_c(b, s) + \mathbf{\Lambda}(a, b, s) \frac{d\mathbf{M}_c(b, s)}{ds} . \tag{D2}$$

The derivative in the first term on the right-hand side of eq. (D2) can be obtained by differentiating matrix $\mathbf{L}$ with respect to s and deleting rows one, two and five. Differentiating eq. (42) with respect to s yields

$$\begin{aligned} \frac{d\mathbf{L}(a, b, s)}{ds} &= \frac{d\mathbf{P}(r_N, r_{N-1})}{ds} \mathbf{P}(r_{N-1}, r_{N-2}) \dots \mathbf{P}(r_2, r_1) \\ &\quad + \mathbf{P}(r_N, r_{N-1}) \frac{d\mathbf{P}(r_{N-1}, r_{N-2})}{ds} \dots \mathbf{P}(r_2, r_1) \\ &\quad + \dots \\ &\quad + \mathbf{P}(r_N, r_{N-1}) \mathbf{P}(r_{N-1}, r_{N-2}) \dots \frac{d\mathbf{P}(r_2, r_1)}{ds} \\ &= \sum_{k=2}^{N} \mathbf{L}(r_N, r_k, s) \frac{d\mathbf{P}(r_k, r_{k-1})}{ds} \mathbf{L}(r_{k-1}, r_1, s) . \end{aligned} \tag{D3}$$

Note that, for the last step, we have used definition (42) again and considered that $\mathbf{L}(r_k, r_k, s)$ is equal to the 6×6 unit matrix. Differentiating also eq. (38) with respect to s yields

$$\frac{d\mathbf{P}(r_k, r_{k-1}, s)}{ds} = \left[\frac{d\mathbf{U}(r_k, \mu_k)}{d\mu_k} \mathbf{V}\mathbf{G}(r_k, r_{k-1}) \mathbf{V}^{-1} \mathbf{U}^{-1}(r_{k-1}, \mu_k) + \mathbf{U}(r_k, \mu_k) \mathbf{V}\mathbf{G}(r_k, r_{k-1}) \mathbf{V}^{-1} \frac{d\mathbf{U}^{-1}(r_{k-1}, \mu_k)}{d\mu_k} \right] \frac{d\mu_k}{ds} , \tag{D4}$$

where the s-derivative of the shear modulus can be obtained from eq. (8):

$$\frac{d\mu_k}{ds} = \frac{\mu_k^2}{v_k s^2} . \tag{D5}$$

Substituting for the μ-derivatives of matrices $\mathbf{U}(r, \mu)$ and $\mathbf{U}^{-1}(r, \mu)$ from eqs (C9) and (C10), eq. (D4) takes the form

$$\frac{d\mathbf{P}(r_k, r_{k-1}, s)}{ds} = \frac{1}{v_k s^2} [\mu_k^2 \mathbf{N}^{(1)}(r_k) \mathbf{V}\mathbf{G}(r_k, r_{k-1}) \mathbf{V}^{-1} \mathbf{U}^{-1}(r_{k-1}, \mu_k) - \mathbf{U}(r_k, \mu_k) \mathbf{V}\mathbf{G}(r_k, r_{k-1}) \mathbf{V}^{-1} \mathbf{N}^{(-1)}(r_{k-1})] . \tag{D6}$$

The derivative in the second term on the right-hand side of eq. (D2) is similarly obtained by differentiating eq. (34) with respect to s:

$$\frac{d\mathbf{M}_c(b, s)}{ds} = \frac{1}{v_1 s^2} [\mu_1^2 \mathbf{N}^{(1)}(b) \mathbf{V}\mathbf{R}(b, \mu_1) - \mathbf{U}(b, \mu_1) \mathbf{V}\mathbf{N}^{(0)}(b)] , \tag{D7}$$

where $\mathbf{N}^{(0)}(b)$ is given by eq. (C11) and the second, fourth and sixth columns of the matrix on the right-hand side have been deleted. Substituting eqs (D3) and (D6) into eq. (D2), we finally obtain

$$\begin{aligned} \frac{d\mathbf{T}(s)}{ds} = \frac{1}{s^2} \sum_{k=2}^{N} \Bigg\{ & \mathbf{\Lambda}(r_N, r_k, s) \frac{1}{v_k} [\mu_k^2 \mathbf{N}^{(1)}(r_k) \mathbf{V}\mathbf{G}(r_k, r_{k-1}) \mathbf{V}^{-1} \mathbf{U}^{-1}(r_{k-1}, \mu_k) \\ & - \mathbf{U}(r_k, \mu_k) \mathbf{V}\mathbf{G}(r_k, r_{k-1}) \mathbf{V}^{-1} \mathbf{N}^{(-1)}(r_{k-1})] \mathbf{L}(r_{k-1}, r_1, s) \Bigg\} \mathbf{M}_c(b, s) + \mathbf{\Lambda}(a, b, s) \frac{d\mathbf{M}_c(b, s)}{ds} . \end{aligned} \tag{D8}$$

Substituting the ij element of the last expression into eq. (49) provides the analytical form of the s-derivative of the secular determinant.