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$\mathcal{N} = 6$ superspace constraints, SUSY enhancement and monopole operators

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Abstract: We present a systematic analysis of the $\mathcal{N} = 6$ superspace constraints in three space-time dimensions. The general coupling between vector and scalar supermultiplets is encoded in an SU(4) tensor $W^i{}_j$ which is a function of the matter fields and subject to a set of algebraic and super-differential relations. We give a genuine $\mathcal{N} = 6$ classification for superconformal models with polynomial interactions and find the known ABJM and ABJ models.

We further study the issue of supersymmetry enhancement to $\mathcal{N} = 8$ and the role of monopole operators in this scenario. To this end we assume the existence of a composite monopole operator superfield which we use to formulate the additional supersymmetries as internal symmetries of the $\mathcal{N} = 6$ superspace constraints. From the invariance conditions of these constraints we derive a system of superspace constraints for the proposed monopole operator superfield. This constraint system defines the composite monopole operator superfield analogously to the original $\mathcal{N} = 6$ superspace constraints defining the dynamics of the elementary fields.

Keywords: Supersymmetric gauge theory, Superspaces, Chern-Simons Theories

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Contents

1 Introduction

The construction of three-dimensional $\mathcal{N} = 8$ superconformal field theories of Bagger-Lambert-Gustavsson (BLG) [1, 2] has triggered an immense body of work on its relation to the dynamics of multiple M2-branes. While manifest $\mathcal{N} = 8$ supersymmetry singles out a unique theory with gauge group SO(4), it has been proposed in [3] that N M2-branes located on a $\mathbb{C}^4/\mathbb{Z}_k$ singularity are described by a matter coupled $\mathrm{U}(N) \times \mathrm{U}(N)$ Chern-Simons theory of level $\pm k$ with manifest $\mathcal{N} = 6$ supersymmetry, the ABJM model. For level $k = 1, 2$ the expected enhancement to $\mathcal{N} = 8$ supersymmetry is proposed to rely on the existence of monopole or 't Hooft operators.

The purpose of this paper is to develop an $\mathcal{N} = 6$ superspace approach for the formulation and classification of three-dimensional $\mathcal{N} = 6$ theories, to recover the known models in this formalism and to analyze the structure of the $\mathcal{N} = 8$ supersymmetry enhancement of the ABJM model by monopole operators by deriving a system of superspace constraints for these operators.

Previous superspace formulations of the ABJM model include the off-shell formulations in $\mathcal{N}=2$ [4] and harmonic $\mathcal{N}=3$ superspace [5, 6], the $\mathcal{N}=6$ formulation of [7] based on pure spinors and some results in on-shell $\mathcal{N}=6$ superspace announced in [8]. In the work presented here, we formulate and analyze the $\mathcal{N}=6$ superspace constraints for general three-dimensional gauge theories. The matter sector is described by a complex scalar superfield Φ^i transforming in the fundamental representation of the $\mathrm{SU}(4) \sim \mathrm{SO}(6)$ R-symmetry group. The gauge sector is described by a vector superfield which is an SU(4) singlet. These superfields are subject to appropriate constraints to restrict the field content and we study the possible couplings of the gauge and matter superfields. In close analogy to the structure of $\mathcal{N}=8$ superspace constraints which we have worked out in [9], the set of consistent $\mathcal{N}=6$ theories can be parametrized by a hermitean SU(4) tensor $W^i{}_j$, which is a function of the matter superfields subject to the following concise SU(4)-projection conditions:

$$\nabla_{\alpha\, ij} W^k{}_l \Big|_{\mathbf{64}} = 0\,, \qquad W^i{}_j \cdot \Phi^k \Big|_{\mathbf{36}} = 0\;,$$

which will be explained in detail in the main text. The $\mathcal{N}=6$ superspace formulation implemented here is necessarily on-shell, so that pure superspace geometrical considerations of the multiplet structure determine the dynamics of the system in terms of superfield equations of motions. These can be expressed in terms of the tensor $W^k{}_l$ and its super-derivatives.

We give an explicit class of solutions to the above conditions which describe the superconformal model with gauge group $\mathrm{U}(N) \times \mathrm{U}(\widetilde{N})$. We work out the superfield and component field equations and show that the latter reproduce the results of [3, 10–12]. The superspace formulation that we present here provides a setting which allows the study of possible generalizations of these models and the determination of quantum corrections (to the e.o.m.) through symmetry considerations and by the rigidness of the $\mathcal{N}=6$ superspace, circumventing perturbation theory.

The $\mathcal{N}=6$ superspace formalism developed in this paper provides a suitable framework for a closer analysis of the proposed supersymmetry enhancement in the ABJM model in terms of monopole operators and a more explicit description of such operators. The basic idea is to formulate the enhanced supersymmetry as an internal $\mathcal{N}=2$ supersymmetry of the $\mathcal{N}=6$ superspace constraint equations rather than for the Lagrangian. The additional susy is thus an infinitesimal symmetry of the equations of motions, a typical situation for hidden symmetries. By starting from a general ansatz for the additional supersymmetry transformations in terms of monopole operators, we analyze their compatibility with the above superspace constraints and deduce the full set of superspace constraints for these operators. While we leave a detailed analysis of this system to future work let us stress that in contrast to previous approaches [14] this system does not involve any additional conditions on the elementary fields of the theory.

We analyze two different situations. First we study the case of a covariantly constant monopole operator, an assumption which was also made in the ordinary space-time approach of [13, 14]. We prove that under this assumptions monopole operators exist only in the case that the gauge group is $\mathrm{U}(2) \times \mathrm{U}(2)$ and recover the superspace version of

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the result given in [13]. We argue on generals reasons that susy enhancement based on this particular operator ceases to exist in the quantum theory. Next we relax the assumption of covariant constancy and derive a system of superspace constraints for the proposed monopole operator which does not impose any apparent restrictions on the dimension of the gauge group.

Eventually, this system of superspace constraints should lead to space-time equations of motion for the composite monopole superfield, in analogy to our superspace analysis for the elementary superfields. These space time e.o.m. for the composite monopole superfield might then describe the dynamics of a theory dual to the ABJM model, in the sense of the three-dimensional mirror symmetry [15]. This would finally be a nonabelian gauge theory analogon of the explicit duality relations for two-dimensional soliton models [16, 17].

The paper is organized as follows. In section 2 we review our conventions for $\mathcal{N}=6$ superspace and superfields. Section 3 is devoted to an analysis of the gauge sector of three-dimensional theories and presents the superfield constraints to be imposed on the super field strength in order to properly restrict the field content and dynamics. We introduce deformations of the free Chern-Simons constraint which are parametrized by a hermitean SU(4) tensor $W^i{}_j$. In section 4, we describe the free matter sector and the interactions induced by minimal coupling to the gauge sector which are parametrized by the tensor $W^i{}_j$. In particular, we derive the full set of consistency constraints which this tensor must satisfy. We derive the explicit component equations of motion for general $W^i{}_j$ and discuss their equivalence to the superfield constraints.

In section 5 we classify the conformal solutions to these consistency constraints. We work out the full set of field equations and show that they reproduce the ABJ model with gauge group $\mathrm{U}(N)\times\mathrm{U}(\widetilde{N})$. Section 6 finally addresses the issue of supersymmetry enhancement of the ABJM model by monopole operators. After a detailed discussion of the general properties of such operators we start from the general ansatz for the additional supersymmetry transformations in terms of the monopole operators and derive the consistency conditions that are implied by compatibility with the superspace constraints derived in the earlier sections. We show that for covariantly constant monopole operators these conditions necessarily imply $N=\widetilde{N}=2$. For covariantly non-constant monopole operators and general N we derive the full system of superspace constraints for these operators and their super-derivatives.

Note added: while finishing this work the paper [18] appeared on the arXiv which analyzes in a complementary approach some of the questions which are addressed also in the current investigations.

2 $\mathcal{N}=6$ superspace setup

We briefly introduce the basic setup and our conventions for the $\mathcal{N}=6$ superspace calculus. For more explicit details see the appendix. The R-symmetry group of the $\mathcal{N}=6$ susy algebra is $\mathrm{SO}(6)\sim\mathrm{SU}(4)$, where we will use the SU(4) notation throughout the paper. The $\mathcal{N}=6$ superspace $\mathbb{R}^{2,1|12}$ is parametrized by coordinates $(x^{\alpha\beta},\theta^{\alpha\, ij})$, $\alpha,\beta=1,2$

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and $i, j = 1, \ldots, 4$, where $x^{\alpha\beta}$ is a real symmetric matrix and the fermionic coordinates $\theta^{\alpha\, ij} = \theta^{\alpha\,[ij]}$ are in the real **6** of SU(4), i.e. they satisfy the reality property

$$(\theta^{\alpha\, ij})^* = \frac{1}{2}\epsilon_{ijkl}\theta^{\alpha\, kl} =: \theta^{\alpha}_{ij} \ . \tag{2.1}$$

With the fermionic derivative $\partial_{\alpha\, ij}\theta^{\beta}_{kl} = \frac{1}{2}\epsilon_{ijkl}\delta^{\beta}_{\alpha}$ we define the susy covariant derivatives and the susy generators as the operators,

$$D_{\alpha\, ij} = \partial_{\alpha\, ij} + i\theta^{\beta}_{ij}\partial_{\alpha\beta}\,, \qquad Q_{\alpha\, ij} = \partial_{\alpha\, ij} - i\theta^{\beta}_{ij}\partial_{\alpha\beta}\,, \tag{2.2}$$

respectively, such that $\{D_{\alpha\, ij}, Q_{\beta\, kl}\} = 0$ and $\{Q_{\alpha\, ij}, Q_{\beta\, kl}\} = -\{D_{\alpha\, ij}, D_{\beta\, kl}\} = -i\epsilon_{ijkl}\,\partial_{\alpha\beta}$.

For the formulation of gauge theories we introduce connections, i.e. covariant derivatives, on the considered superspace in the following way:

$$\begin{aligned} \nabla_{\alpha\beta} &= \partial_{\alpha\beta} + \mathcal{A}_{\alpha\beta} && \text{with} \quad (\mathcal{A}_{\alpha\beta})^\dagger = -\mathcal{A}_{\alpha\beta}\,, \\ \nabla_{\alpha\, ij} &= D_{\alpha\, ij} + \mathcal{A}_{\alpha\, ij} && \text{with} \quad (\mathcal{A}_{\alpha\, ij})^\dagger = \frac{1}{2}\epsilon^{ijkl}\mathcal{A}_{\alpha\, kl} \ . \end{aligned} \tag{2.3}$$

The connection one-forms live in a Lie algebra $\mathfrak{g}$ and are expanded[1] as $\mathcal{A}_{\alpha\beta} = \mathcal{A}^M_{\alpha\beta}T_M$ and $\mathcal{A}_{\alpha\, ij} = \mathcal{A}^M_{\alpha\, ij}iT_M$ in terms of the anti-hermitian generators $(T_M)^\dagger = -T_M$ of the yet unspecified gauge group G. The connection (2.3) defines a super field strength via (anti)commutators minus torsion terms, whose components are

$$\begin{aligned} \mathcal{F}_{\alpha\beta,\gamma\delta} &= [\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] \quad , \qquad \mathcal{F}_{\alpha\beta,\gamma\, ij} = [\nabla_{\alpha\beta}, \nabla_{\gamma\, ij}]\,, \\ \mathcal{F}_{\alpha\, ij,\beta\, kl} &= \{\nabla_{\alpha\, ij}, \nabla_{\beta\, kl}\} - i\epsilon_{ijkl}\nabla_{\alpha\beta} \ . \end{aligned} \tag{2.4}$$

A given field strength has to satisfy the Bianchi identities, which are simply obtained from the super-Jacobi identities for the covariant derivatives:[2]

$$\begin{aligned} &\sum_{\text{cyclic}} [\nabla_{\alpha\, ij}, \{\nabla_{\beta\, kl}, \nabla_{\gamma\, mn}\}] \equiv 0\,, \qquad \sum_{\text{cyclic}} (-1)^\pi \{\nabla_{\alpha\, ij}, [\nabla_{\beta\, kl}, \nabla_{\gamma\delta}]\} \equiv 0\,, \\ &\sum_{\text{cyclic}} [\nabla_{\rho\, ij}, [\nabla_{\alpha\beta}, \nabla_{\gamma\delta}]] \equiv 0\,, \qquad \sum_{\text{cyclic}} [\nabla_{\alpha\beta}, [\nabla_{\gamma\delta}, \nabla_{\rho\sigma}]] \equiv 0 \ . \end{aligned} \tag{2.5}$$

These identities will lead to consistency conditions for the constraints to be imposed on the super field strength.

In addition to super gauge fields we will need matter superfields. The $\mathcal{N} = 6$ matter component multiplet $(\phi^i, \psi_{\alpha\, i})$ consists of scalar and fermion fields in the **4** and $\bar{\mathbf{4}}$ of SU(4), respectively. Accordingly we introduce complex bosonic and fermionic matter superfields Φ^i, $\Psi_{\alpha\, i}$ transforming in the representation R of the gauge group which when indicate carry an upper index from the range $a, b, c, \ldots$. The complex conjugated fields

$$(\Phi^{i\, a})^* =: \bar{\Phi}_{i\, a} \quad , \quad (\Psi^a_{\alpha\, i})^* =: \bar{\Psi}^i_{\alpha\, a}\,, \tag{2.6}$$

[1] We include in general an extra i in such expansions for fermionic superfields in the gauge sector so that the coefficient superfields satisfy reality conditions corresponding to their SU(4) representations. See the appendix for more details.

[2] The exponent π in the second identity counts the cyclic permutations where (anti)commutators are distributed corresponding to the occurrence of bosonic/fermionic connections

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transform in the representation $\bar{R}$ and carry a lower gauge index, as indicated here. However, frequently we will not write the gauge indices explicitly.

In the following sections we will impose constraints on the superfields and investigate the resulting dynamics.

3 Gauge field constraints

We closely follow the methods developed in [9] for the $\mathcal{N} = 8$ case.[3] To eliminate unphysical degrees of freedom one imposes (partial) flatness conditions on the bi-spinor field strength [21–24], which is $\mathcal{F}_{\alpha\, ij,\beta\, kl}$ here. In many cases this corresponds to an underlying geometric structure of twistors and pure spinors [23–26]. The bi-spinor field strength in (2.4) contains the representations[4]

$$\mathcal{F}_{\alpha\, ij,\beta\, kl} \sim ((\mathbf{2},\mathbf{6}) \otimes (\mathbf{2},\mathbf{6}))_{\mathrm{sym}} = (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{15}) \oplus (\mathbf{3},\mathbf{20}), \tag{3.1}$$

where the first entry refers to the SO(2, 1) and the second entry to the SU(4) representation. The SU(4) representations $\mathbf{6}$ carried by $\mathcal{F}_{\alpha\, ij,\beta\, kl}$ are the real $\mathbf{6}$ and consequently also the SU(4) representations appearing on the r.h.s. of (3.1) are the real $\mathbf{15}$ and $\mathbf{20}$.[5]

The $(\mathbf{3},\mathbf{1})$ part in (3.1) corresponds to a second component vector field in the superfield expansion of $\mathcal{A}_{\alpha\, ij}$ with the same gauge-transformation as the lowest component of $\mathcal{A}_{\alpha\beta}$. Setting this part to zero imposes the so-called "conventional constraint" [29, 30] for the three-dimensional $\mathcal{N} = 6$ case and eliminates this additional component vector field. Putting further constraints on $\mathcal{F}_{\alpha\, ij,\beta\, kl}$, in contrast, will not only eliminate component fields, but also induce equations of motion for the remaining fields. We shall analyze this in more detail now.

In analogy to [9] we allow the $(\mathbf{1},\mathbf{15})$ in (3.1) to be non-vanishing but set the $(\mathbf{3},\mathbf{1}) \oplus (\mathbf{3},\mathbf{20})$ part to zero (a non-vanishing $(\mathbf{3},\mathbf{20})$ part may be taken into account for studying higher derivative corrections). Given the definition (2.4) this constraint writes as[6]

$$\{\nabla_{\alpha\, ij}, \nabla_{\beta\, kl}\} = i\left(\epsilon_{ijkl}\nabla_{\alpha\beta} + \varepsilon_{\alpha\beta}\, \epsilon_{mij[k}W^m{}_{l]}\right)\,, \tag{3.2}$$

where $W^i{}_j$ is an SU(4) tensor transforming in the real $\mathbf{15}$ of SU(4) and lives in the Lie algebra $\mathfrak{g}$ of the yet unspecified gauge group G. It is therefore traceless ($W^i{}_i = 0$) and satisfies the hermiticity conditions

$$(W^i{}_j)^\dagger = W^j{}_i\,, \tag{3.3}$$

[3] For other $\mathcal{N} = 8$ superfield approaches which specifically describe the BLG model using Nambu-brackets and pure spinors see [19, 20]

[4] Decompositions of tensor products of representations can be computed with the program LiE [27] or found in [28].

[5] Reality conditions can be implemented for representations whose Dynkin labels are self-conjugated under $(r, s, t) \to (t, s, r)$, i.e. if $r = t$. This is the case for the $\mathbf{15}$ and $\mathbf{20}$ (denoted $\mathbf{20'}$ in [28]) which have Dynkin labels $(1, 0, 1)$ and $(0, 2, 0)$. In the appendix we give the reality conditions for the tensors appearing in the following.

[6] For the symmetry properties of the second term on the r.h.s. see (B.15).

where $W^i{}_j$ is expanded in the gauge algebra as $W^i{}_j = W^{M\,i}{}_j T_M$, see the appendix for more details. A priori $W^i{}_j$ is an independent superfield but we will see that eventually it will be a function of the matter superfields (2.6), i.e. $W^i{}_j = W^i{}_j(\Phi^k, \Psi_{\alpha\, l})$. In this regard we will call $W^i{}_j$ the *deformation potential* since it represents a deformation of the constraint $\mathcal{F}_{\alpha\, ij,\beta\, kl} = 0$ for which the resulting multiplet contains exclusively a free Chern-Simons component gauge field, i.e. a flat connection on $\mathbb{R}^{2,1}$, see below.

Bianchi identities. For the constraint (3.2) to be consistent $W^i{}_j$ cannot be chosen arbitrarily but is itself subjected to certain conditions so that the Bianchi identities (2.5) are satisfied. The immediate nontrivial conditions on the superfields are given by the first two Bianchi identities in (2.5), which involve the constrained bi-spinor field strength $\mathcal{F}_{\alpha\, ij,\beta\, kl}$.

Using the constraint (3.2) the first Bianchi identity imposes the condition

$$\begin{aligned} \epsilon_{ijkl}\mathcal{F}_{\alpha\beta,\gamma\, mn} + \epsilon_{mnij}\mathcal{F}_{\gamma\alpha,\beta\, kl} + \epsilon_{klmn}\mathcal{F}_{\beta\gamma,\alpha\, ij} = \\ \varepsilon_{\beta\gamma}\nabla_{\alpha\, ij} W^p{}_{[m}\,\epsilon_{n]klp} + \varepsilon_{\gamma\alpha}\nabla_{\beta\, kl} W^p{}_{[i}\,\epsilon_{j]mnp} + \varepsilon_{\alpha\beta}\nabla_{\gamma\, mn} W^p{}_{[k}\,\epsilon_{l]ijp}\ . \end{aligned} \tag{3.4}$$

Decomposing l.h.s. and r.h.s. of this equation according to their SU(4) representation content, one deduces that solvability requires the representation **64** to vanish within the the tensor product $\nabla_{\alpha\, ij} W^k{}_l \sim \mathbf{6} \otimes \mathbf{15} = \mathbf{6} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{64}$. It also implies the existence of superfields $\lambda_{\alpha\, ij} = \lambda_{\alpha\,[ij]}$ in the real **6**, i.e. $(\lambda_{\alpha\, ij})^\dagger = \frac{1}{2}\epsilon^{ijkl}\lambda_{\alpha\, kl}$, and $\rho_{\alpha\, ij} = \rho_{\alpha(ij)}$ in the complex $\overline{\mathbf{10}}$, i.e. $(\rho_{\alpha\, ij})^\dagger =: \bar\rho_\alpha^{ij} \sim \mathbf{10}$, such that the superderivative $\nabla_{\alpha\, ij} W^k{}_l$ satisfies the condition

$$\begin{aligned} &\nabla_{\alpha\, ij} W^k{}_l \Big|_{\mathbf{64}} = 0 \\ \Longrightarrow\quad &\nabla_{\alpha\, ij} W^k{}_l = \delta^k_{[i}\,\lambda_{\alpha\, j]l} + \frac{1}{4}\delta^k_l\,\lambda_{\alpha\, ij} + \delta^k_{[i}\,\rho_{\alpha\, j]l} - \frac{1}{2}\epsilon_{ijln}\,\bar\rho_\alpha^{kn}\ . \end{aligned} \tag{3.5}$$

This constraint will play a central role in the following. If we consider $W^i{}_j$ as a function of the matter fields of the theory, this composite superfield must satisfy (3.5) in order for the system (3.2) to be consistent. The Bianchi identity (3.4) then fixes the fermionic field strength $\mathcal{F}_{\alpha\beta,\gamma\, ij}$ to

$$\mathcal{F}_{\alpha\beta,\gamma\, ij} = -\frac{1}{2}\varepsilon_{\gamma(\alpha}\lambda_{\beta)\, ij}\ . \tag{3.6}$$

Before investigating the residual Bianchi identities let us assume that we have picked a $W^i{}_j(\Phi^k, \Psi_{\alpha\, l})$ satisfying (3.5). The integrability condition of (3.5) is then identically fulfilled and determines the superderivatives of the composite fields $\lambda_{\alpha\, ij}$ and $\rho_{\alpha\, ij}$,

$$\begin{aligned} \nabla_{\alpha\, ij}\lambda_{\beta\, kl} &= i\epsilon_{ijkl}\mathcal{F}_{\alpha\beta} + 2i\nabla_{\alpha\beta} W^m{}_{[k}\epsilon_{l]ijm} + 2i\varepsilon_{\alpha\beta} V^m{}_{[k}\epsilon_{l]ijm}\,, \\ \nabla_{\alpha\, ij}\rho_{\beta\, kl} &= -i\nabla_{\alpha\beta} W^m{}_{(k}\epsilon_{l)ijm} + i\varepsilon_{\alpha\beta} V^m{}_{(k}\epsilon_{l)ijm} + U_{\alpha\beta}{}^m{}_{(k}\epsilon_{l)ijm} \\ &\quad + i\,\varepsilon_{\alpha\beta}\left(\frac{1}{2}\epsilon_{ijmn}\,[W^m{}_k, W^n{}_l] + \frac{1}{2}\epsilon_{mij(k}\,[W^n{}_{l)}, W^m{}_n]\right)\,, \end{aligned} \tag{3.7}$$

up to the space-time vectors and scalar $\mathcal{F}_{\alpha\beta}$, $U_{\alpha\beta}{}^i{}_j$ and $V^i{}_j$. The vector $(\mathcal{F}_{\alpha\beta})^\dagger = -\mathcal{F}_{\alpha\beta}$ is an anti-hermitian SU(4) singlet, whereas the other two tensors transform in the real

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15 of SU(4) thus satisfying a hermiticity condition as given in (3.3). Comparing the equations (3.7) with the second Bianchi identity

$$\nabla_{\alpha\, ij}\mathcal{F}_{\gamma\delta,\beta\, kl} + \nabla_{\beta\, kl}\mathcal{F}_{\gamma\delta,\alpha\, ij} = i(\epsilon_{ijkl}\mathcal{F}_{\gamma\delta,\alpha\beta} + \varepsilon_{\alpha\beta}\nabla_{\gamma\delta}W^{m}{}_{[k}\,\epsilon_{l]ijm})\,, \tag{3.8}$$

and using (3.6), shows that it is identically fulfilled upon setting the vector $\mathcal{F}_{\alpha\beta} = \varepsilon^{\gamma\delta}\mathcal{F}_{\alpha\gamma,\beta\delta}$, which thus equals the dual bosonic field strength. With this identification one obtains that the integrability condition of the first equation in (3.7), besides determining $\nabla_{\alpha\, ij}V^{k}{}_{l}$, equals the third Bianchi identity in (2.5).

Super Chern-Simons e.o.m. With a chosen deformation potential $W^{i}{}_{j}(\Phi^{k}, \Psi_{\alpha\, l})$ the derived superfields $\lambda_{\alpha\, ij}$, $\rho_{\alpha\, ij}$, etc. are given functions of the matter superfields. In particular, the first equation of (3.7) together with (3.5) gives the following Chern-Simons equations of motion for the bosonic field strength:

$$\mathcal{F}_{\alpha\beta} = \frac{1}{24\, i}\,\epsilon^{ijkl}\nabla_{ij\,(\alpha}\lambda_{\beta)\, kl} = \frac{1}{15\, i}\,\nabla^{ik}_{(\alpha}\nabla_{\beta)\, jk}W^{j}{}_{i}, \tag{3.9}$$

which for $W^{i}{}_{j} = 0$ reduces to the free CS-e.o.m. A priori, with (3.9) the fourth Bianchi identity in (2.5), which takes the form

$$\nabla^{\alpha\beta}\mathcal{F}_{\alpha\beta} = 0\,, \tag{3.10}$$

might give rise to yet another condition. Evaluating the l.h.s. of (3.10) with $\mathcal{F}_{\alpha\beta}$ given by (3.9) shows that this Bianchi identity is also identically satisfied without any further conditions and thus (3.5) automatically defines a covariantly conserved current, the r.h.s. of (3.9).

Consequently, with a given choice for the deformation potential $W^{i}{}_{j}$ which satisfies the constraint (3.5) all Bianchi identities are identically fulfilled upon imposing the CS equations of motion (3.9), the fermionic field strength is given by (3.6). Therefore (3.5) represents the only restriction on the choice of the deformation potential $W^{i}{}_{j}(\Phi^{k}, \Psi_{\alpha\, l})$ for the gauge field constraint (3.2) to be consistent. We will address the issue of component field equations and their equivalence to the constraint (3.2) at a later point when we have discussed the matter sector which couples non-trivially to the gauge sector.

4 Matter field constraints

The $\mathcal{N} = 6$ matter multiplet $(\phi^{i}, \psi_{\alpha\, i})$ consists of scalar and fermion component fields in the complex **4** and $\bar{\mathbf{4}}$ of SU(4), respectively. It is therefore natural to encode this multiplet in a scalar superfield Φ^{i} in the **4** to be subjected to appropriate constraints. At first order in $\theta^{\alpha\, jk}$ this superfield contains a fermionic component $\chi^{i}_{\alpha\, kl}$ which decomposes into irreps as $\mathbf{4} \otimes \bar{\mathbf{6}} = \bar{\mathbf{4}} \oplus \overline{\mathbf{20}}$. The super- and gauge-covariant way to project out the $\overline{\mathbf{20}}$ in accordance with the field content is to impose the condition

$$\nabla_{\alpha\, ij}\Phi^{k}\Big|_{\overline{\mathbf{20}}} = 0 \quad\Longrightarrow\quad \nabla_{\alpha\, ij}\Phi^{k} = i\,\delta^{k}_{[i}\Psi_{j]\,\alpha}\,, \tag{4.1}$$

with a fermionic superfield $\Psi_{\alpha\, i}$ in the $\bar{\mathbf{4}}$, which is defined by this equation. Explicitly this gives[7] $\chi^{\,k}_{\alpha\, ij} = i\, \delta^k_{[i} \Psi_{j]\,\alpha}|_{\theta=0} =: i\, \delta^k_{[i}\psi_{j]\,\alpha}$.

In the following we derive the consequences implied by combining (4.1) with the vector superfield constraint (3.2). Using the latter, the integrability condition of (4.1) takes the form

$$\epsilon_{ijkl}\nabla_{\alpha\beta}\Phi^m + \varepsilon_{\alpha\beta}\,\epsilon_{nij[k}W^n{}_{l]}\cdot\Phi^m = -\nabla_{\alpha\, ij}\Psi_{\beta\,[k}\delta^m_{l]} - \nabla_{\beta\, kl}\Psi_{\alpha\,[i}\delta^m_{j]}\ , \tag{4.2}$$

where $W^i{}_j\cdot\Phi^k$ denotes the action of the algebra valued $W^i{}_j$ onto the scalar superfield. This poses an algebraic constraint on the deformation potential $W^i{}_j$ since the unpaired $\mathbf{36}$ in

$$W^i{}_j\cdot\Phi^k \sim \mathbf{15}\otimes\mathbf{4} = \mathbf{4}\oplus\mathbf{20}\oplus\mathbf{36} \tag{4.3}$$

does not drop out of equation (4.2). Explicitly this constraint writes as

$$W^i{}_j\cdot\Phi^k\Big|_{\mathbf{36}} = 0 \qquad \Longleftrightarrow \qquad W^{(i}{}_j\cdot\Phi^{k)} = \frac{1}{5}\,\delta^{(i}_j\, W^{k)}{}_n\cdot\Phi^n\ , \tag{4.4}$$

and thus the deformation potential has to be a function of the matter fields, as mentioned before. In addition to the super-differential constraint (3.5) this algebraic constraint will be the main restriction on the possible choices for the deformation potential $W^i{}_j(\Phi^k,\Psi_{\alpha\, l})$, which fixes the details of the dynamics. In the following we will refer to these two constraints (3.5) and (4.4), which determine the set of possible models, as the *W-constraints*, as in the $\mathcal{N}=8$ case [9].

With the condition (4.4) the integrability condition (4.2) can be resolved to determine the superderivative $\nabla_{\alpha\, ij}\Psi_{\beta\, k}$:

$$\nabla_{\alpha\, ij}\Psi_{\beta\, k} \;=\; -2\epsilon_{ijkl}\nabla_{\alpha\beta}\Phi^l + \frac{1}{2}\varepsilon_{\alpha\beta}\left(\epsilon_{ijmn}\, W^m{}_k\cdot\Phi^n + \frac{3}{5}\,\epsilon_{ijkl}\, W^l{}_n\cdot\Phi^n\right)\ , \tag{4.5}$$

The analogous equations of this subsection for the complex conjugated fields $(\Phi^i)^* =: \bar{\Phi}_i$ and $(\Psi_{\alpha\, i})^* =: \bar{\Psi}^i_\alpha$ are obtained by complex conjugation. The necessary relations and conventions are given in the appendix.

Superfield e.o.m. The integrability conditions of (4.5) together with the gauge field constraint (3.2) and the various constraint relations and Bianchi identities yield the fermion superfield e.o.m. for $\Psi_{\alpha\, i}$,

$$\varepsilon^{\beta\gamma}\nabla_{\alpha\beta}\Psi_{\gamma\, i} = \frac{i}{2}\,\lambda_{\alpha ij}\cdot\Phi^j - \frac{i}{10}\,\rho_{\alpha ij}\cdot\Phi^j - \frac{1}{5}\,W^j{}_i\cdot\Psi_{\alpha\, j}\ . \tag{4.6}$$

Its superderivative gives the bosonic superfield e.o.m. for Φ^i:

$$\begin{aligned}\nabla^2\Phi^i \;&=\; \frac{1}{4}\,\varepsilon^{\alpha\beta}\left(\lambda^{ij}_\alpha\cdot\Psi_{\beta\, j} + \frac{1}{5}\,\bar{\rho}^{ij}_\alpha\cdot\Psi_{\beta\, j}\right)\\ &\qquad + \frac{2}{5}\,V^i{}_j\cdot\Phi^j + \frac{1}{25}\,W^i{}_j\cdot(W^j{}_k\cdot\Phi^k) - \frac{1}{20}\,W^j{}_k\cdot(W^k{}_j\cdot\Phi^i)\ .\end{aligned} \tag{4.7}$$

[7]Here we anticipate that $\mathcal{A}_{\alpha\, ij}|_{\theta=0}=0$, i.e. we omit the term $\epsilon^{ijlm}\mathcal{A}_{\alpha\, lm}\cdot\Phi^k|_{\theta=0}$. This will be justified below.

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The check that no other integrability relations descend from (4.6) requires the double superderivative of the algebraic constraint (4.4). By virtue of the same constraint one can recast the scalar self-interaction involving $W^i{}_j$ in different forms. Equations (4.6), (4.7) together with the Chern-Simons equation (3.9) for the field strength constitute the complete set of superfield e.o.m.

Superfield expansion and equivalence to component e.o.m. We briefly discuss the component content of the so far developed superspace expressions. This was explained in detail in [9] for the $\mathcal{N}=8$ case and we can be rather short here and refer to [9] for more details. The very same structure and methods as in [9] apply mutatis mutandis to the $\mathcal{N}=6$ case considered here.

An essential ingredient in the derivation of component expressions and the proof of the equivalence between the on-shell component multiplet and the superspace constraints is the so called "transverse gauge". It allows one to formulate recursion relations for the superfield expansion of various expressions, explicitly

$$\theta^{\alpha\, ij}\mathcal{A}_{\alpha\, ij} = 0 \quad \Rightarrow \quad \mathcal{R} := \theta^{\alpha\, ij}\nabla_{\alpha\, ij} = \theta^{\alpha\, ij}\partial_{\alpha\, ij}\,, \tag{4.8}$$

so that the recursion operator $\mathcal{R}$ satisfies $\mathcal{R}(\theta^{\alpha_1 i_1 j_1}\dots\theta^{\alpha_n i_n j_n}) = n\ \theta^{\alpha_1 i_1 j_1}\dots\theta^{\alpha_n i_n j_n}$.

This method was developed in [31, 32] for super Yang-Mills theories, where all superfields have a geometric origin in a (higher dimensional) vector superfield. The cases considered here and in [9] lead to modifications due to that fact that the matter superfields are not in this sense geometrically related to the vector superfields and especially due to the constraints on the deformation potential $W^i{}_j$, which is otherwise not specified at this point.

Contracting the constraints (4.1), (4.5), (3.2) with $\theta^{\alpha\, ij}$ and the Bianchi identity (3.6) with $\theta^{\gamma\, ij}$ one obtains the recursion relations

$$\begin{aligned}
\mathcal{R}\,\Phi^i &= i\theta^{\alpha ij}\Psi_{\alpha\, j}\ , \\
\mathcal{R}\,\Psi_{\beta\, k} &= -4\theta^{\alpha}_{kl}\nabla_{\alpha\beta}\Phi^l + \left(\theta_{\beta\, mn}W^m{}_k\cdot\Phi^n + \frac{3}{5}\theta_{\beta\, kl}W^l{}_n\cdot\Phi^n\right)\ , \\
(1+\mathcal{R})\,\mathcal{A}_{\beta\, kl} &= 2i(\theta^{\alpha}_{kl}\mathcal{A}_{\alpha\beta} + \theta_{\alpha\, m[k}W^m{}_{l]})\ , \\
\mathcal{R}\,\mathcal{A}_{\alpha\beta} &= \frac{1}{2}\theta^{\gamma\, ij}\varepsilon_{\gamma(\alpha}\lambda_{\beta)ij}\ .
\end{aligned} \tag{4.9}$$

These recursion relations define the order $n+1$ in θ of the l.h.s. expressions in terms of the order n in θ of the expressions on the r.h.s. Therefore all superfields are expended in terms of the lowest component fields[8]

$$\phi^i := \Phi^i|_{\theta=0}\ ,\quad \psi_{\alpha\, i} := \Psi_{\alpha\, i}|_{\theta=0} \quad A_{\alpha\beta} := \mathcal{A}_{\alpha\beta}|_{\theta=0}\,, \tag{4.10}$$

which represent the $\mathcal{N}=6$ CS-matter component multiplet (note that (4.9) implies that $\mathcal{A}_{\alpha\, ij}$ has no component at $\theta=0$). Consequently, the lowest component of the superfield e.o.m. (3.9) and (4.6), (4.7) gives automatically the e.o.m. for these component fields.

[8]The composite fields $W^i{}_j$ and $\lambda_{\alpha\, ij}$ are given functions of the matter fields Φ^i, $\Psi_{\alpha\, i}$ and thus their lowest components are functions of the here given component multiplet.

It is a general feature of this approach, that many equations can be considered as component field expressions but are at the same time superfield expressions and thus automatically susy covariant. There is no need to check supersymmetry via component field susy transformations. Nevertheless, the component susy transformations are easily obtained by acting with the susy operators $Q_{\alpha\, ij}$ (2.2) on the superfields expanded to order $\mathcal{O}(\theta)$ (modulo restoring supergauge transformations, see [9] for details). As a matter of fact the situation is even simpler. The recursions (4.9) actually resemble the component susy transformations. By replacing in (4.9) $\mathcal{R}\,\Phi^i$, $\mathcal{R}\,\Psi_{\alpha\, i}$, $\mathcal{R}\,\mathcal{A}_{\alpha\beta}$ with the susy transformations of the component fields, i.e. $\delta\phi^i$, $\delta\psi_{\alpha\, i}$, $\delta A_{\alpha\beta}$, and all fermionic coordinates $\theta^{\alpha\, ij}$ on the r.h.s. with susy-parameters $\epsilon^{\alpha\, ij}$ one directly obtains the susy transformations for the component multiplet (4.10).

This shows that the basic constraints (4.1), (3.2) imply the on-shell component multiplet (4.10) with respective susy transformations. To prove that these two descriptions are actually equivalent, i.e. that the superspace constraints do not imply any further conditions, one has to show that the on-shell component multiplet implies superfields satisfying the given constraints. We will not outline this here but refer the reader to [9] where the proof was given for the $\mathcal{N}=8$ case, to convince oneself along the same lines that the same is true for the here considered $\mathcal{N}=6$ case.

5 Conformal gauge theories, ABJM

We start by reviewing the structure of superspace constraints identified so far. The matter sector of these three-dimensional gauge theories is described by a scalar superfield subject to the constraint (4.1)

$$\nabla_{\alpha\, ij}\Phi^k\Big|_{\mathbf{20}} = 0\,. \tag{5.1}$$

The full theory is then identified by specifying their gauge algebra $\mathfrak{g}$ and by choosing $W^i{}_j(\Phi,\Psi)$ in (3.2) as a function of the matter superfields of the theory. This choice of the deformation potential $W^i{}_j$ is not arbitrary but must satisfy two independent superfield conditions, the W-constraints (3.5) and (4.4):

$$\nabla_{\alpha\, ij}W^k{}_l\Big|_{\mathbf{64}} = 0\,, \tag{5.2}$$

$$W^i{}_j\cdot\Phi^k\Big|_{\mathbf{36}} = 0\,. \tag{5.3}$$

The first equation requires that the deformation potential $W^i{}_j$ depends on the matter fields in such a way that (5.2) is satisfied as a consequence of (5.1). In contrast, equation (5.3) also explicitly contains the action of the gauge group on the matter fields and will thus put further restrictions on the possible gauge groups.

5.1 Conformal deformation potentials

In the following we explicitly indicate the gauge group structure. The matter superfields Φ^i, $\Psi_{\alpha\, i}$ are taken to transform in some representation R (which we denote by an upper index

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from the range $a, b, \ldots$) of the to be specified gauge group G. They are thus denoted by $\Phi^{i\,a}$, $\Psi^a_{\alpha\,i}$, whereas the superfields of the gauge sector, which are in the adjoint representation, are represented as matrices $\mathcal{A}_{\alpha\beta}{}^a{}_b$ etc. and act by matrix multiplication on the matter fields. The complex conjugated matter fields transform in the conjugate representation $\bar{R}$ and thus carry a lower index, see (B.10).

The constraint (3.2) implies that the composite field $W^i{}_j$ has canonical dimension one. Given that the scalar fields in three dimensions have canonical dimension $\frac{1}{2}$, scale invariance implies that with a polynomial ansatz $W^i{}_j$ is bilinear in the scalar superfields[9] $\Phi^{i\,a}$. The most general ansatz in the **15** of SU(4) is given by

$$(W^i{}_j)^a{}_b := f^{ac}_{\ \ bd} \left(\Phi^{i\,d} \bar{\Phi}_{j\,c} - \frac{1}{4} \delta^i_j \, \Phi^{k\,d} \bar{\Phi}_{k\,c} \right) \tag{5.4}$$

with dimensionless constants $f^{ac}_{\ \ bd}$. The potential $(W^i{}_j)^a{}_b$ is supposed to be the matrix that acts on fields in the representation R. The hermiticity condition (3.3) requires that $(f^{ac}_{\ \ bd})^* = f^{bd}_{\ \ ac}$. By construction $W^i{}_j$ has to be an element of the Lie algebra $\mathfrak{g}$ of G, therefore gauge covariance requires that $f^{ac}_{\ \ bd}$ is an invariant tensor of the gauge group. Before specifying further the allowed gauge groups G and representations R we check the the W-constraints (5.4), (5.2).

It is straightforward to see that (5.4) is a solution to (5.2) as a consequence of (5.1): since $\nabla_{\alpha\,ij} W^k{}_l$ is composed of a single Φ^i and a single $\bar{\Psi}^j_\alpha$ (or their complex conjugates) it transforms in the tensor product $\mathbf{4} \otimes \mathbf{4} = \mathbf{6} \oplus \mathbf{10}$ and the c.c. thereof, which does not contain a **64**. To solve the algebraic constraint (5.3) we evaluate the action of (5.4) on a scalar field and extract the contribution **36** (4.4):

$$(W^i{}_j \cdot \Phi^k)^a \big|_{\mathbf{36}} = f^{ac}_{\ \ bd} \left(\delta^l_j \Phi^{i(b} \Phi^{d)k} - \frac{2}{5} \delta^{(i}_j \Phi^{k)(b} \Phi^{d)l} \right) \bar{\Phi}_{l\,c} \,. \tag{5.5}$$

This shows that the tensor $f^{ac}_{\ \ bd}$ has to be antisymmetric in its indices $[bd]$ and thus by complex conjugation also in its upper indices $[ac]$, i.e.

$$f^{ac}_{\ \ bd} = f^{[ac]}_{\ \ [bd]} \,. \tag{5.6}$$

Interestingly, this solution to the W-constraints occupies all of the allowed SU(4) representation content, i.e. $W^i{}_j \cdot \Phi^k \sim \mathbf{4} \oplus \mathbf{20}$. This differs from the conformal solution in the $\mathcal{N} = 8$ case, where $W \cdot \Phi$ has no component in the allowed **8** of SO(8) [9].

What are the restrictions on the gauge group G and the representation R? Since $W^i{}_j \in \mathfrak{g}$, gauge covariance translates into the quadratic condition [33]

$$(f^{ac}_{\ \ bd} f^{dg}_{\ \ eh} - f^{gc}_{\ \ hd} f^{da}_{\ \ eb}) + (f^{ca}_{\ \ ed} f^{dg}_{\ \ bh} - f^{ga}_{\ \ hd} f^{dc}_{\ \ be}) = 0 \,, \tag{5.7}$$

which can be obtained by comparing $W^i{}_j \cdot W^k{}_l$ evaluated as commutator and by the action of $W^i{}_j$ on the scalars in $W^k{}_l$, respectively. This condition is identical with the *fundamental*

[9] This dimensional analysis excludes a possible dependence on the fermionic superfields $\Psi_{\alpha\,i}$, which have dimension one, for a polynomial ansatz. Higher order corrections with non-polynomial deformation potentials allow for more possibilities.

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identity of (hermitean) three-algebras [34, 35] and shows how our result is related to this structure. We will not elaborate on this point, but rather make a more definite statement: Writing the Lie algebra valued deformation potential as $(W^i{}_j)^a{}_b = W^{M\,i}{}_j(T_M)^a{}_b$ and using the hermiticity condition for f^{ac}_{bd} and that it is an invariant tensor one obtains

$$f^{ac}_{\ bd} \;\to\; \bigoplus_{l=1}^{L} \frac{1}{g_{(l)}}\, \kappa^{MN}_{(l)} T^{(l)a}_{M\ \ b}\, T^{(l)c}_{N\ \ d}\,, \tag{5.8}$$

where we assumed a product $G = G_1 \times \ldots \times G_L$ for the gauge group. The quantities $\kappa^{MN}_{(l)}$ are the inverse of $\kappa^{(l)}_{MN} = \mathrm{Tr}_R(T^{(l)}_M T^{(l)}_N)$ for each factor in the gauge group and $g_{(l)}$ are so far arbitrary constants.

This very same expression with the antisymmetry conditions given in (5.6) was classified in a beautiful analysis in [12]. The allowed groups are $\mathrm{U}(N)\times\mathrm{U}(\widetilde{N})$, $\mathrm{SU}(N)\times\mathrm{SU}(N)$ with matter fields in the bifundamental representation and $g_{(1)} = -g_{(2)} =: g$, as well as $\mathrm{U}(N)$, $\mathrm{Sp}(N)$ with matter in the fundamental representation. most of the time we will express our results in a more compact notation using the symbol f^{ac}_{bd}. Of particular interest will be the $\mathrm{U}(N)\times\mathrm{U}(\widetilde{N})$ case for which we give here the solution [12] and we also indicate how to translate the compact notation into the explicit $\mathrm{U}(N)\times\mathrm{U}(\widetilde{N})$ notation. Every gauge index encountered so far becomes a double index of the bifundamental representation $R = (\mathbf{N}, \overline{\widetilde{\mathbf{N}}})$ as follows:

$$\begin{aligned}
\Phi^{i\,a} \;&\to\; \Phi^{i\,a}{}_{\tilde a}\;, \quad \bar\Phi^i{}_b \;\to\; \bar\Phi^{i\,\tilde b}{}_b, \quad \text{etc.}\,,\\
f^{ac}_{\ bd} \;&\to\; \frac{1}{g}\left(\delta^a_d\,\delta^c_b - \frac{1}{N}\delta^a_b\,\delta^c_d\right)\delta^{\tilde b}_{\tilde a}\,\delta^{\tilde d}_{\tilde c} - \frac{1}{g}\,\delta^a_b\,\delta^c_d\left(\delta^{\tilde b}_{\tilde c}\,\delta^{\tilde d}_{\tilde a} - \frac{1}{\widetilde N}\delta^{\tilde b}_{\tilde a}\,\delta^{\tilde d}_{\tilde c}\right)\\
&\qquad + \frac{1}{g}\left(\frac{1}{N} - \frac{1}{\widetilde N}\right)\delta^a_b\,\delta^c_d\,\delta^{\tilde b}_{\tilde a}\,\delta^{\tilde d}_{\tilde c} \;=\; \frac{1}{g}\big(\delta^a_d\,\delta^c_b\,\delta^{\tilde b}_{\tilde a}\,\delta^{\tilde d}_{\tilde c} - \delta^a_b\,\delta^c_d\,\delta^{\tilde b}_{\tilde c}\,\delta^{\tilde d}_{\tilde a}\big)\,.
\end{aligned} \tag{5.9}$$

The first expression for f^{ac}_{bd} gives the explicit decomposition according to the contributions of the different group factors. The first two terms are $su(N)\otimes\mathbb{1}$ and $\mathbb{1}\otimes su(\widetilde N)$ whereas the third term depicts the respective $u(1)$ contributions. As one can see, for $\widetilde N = N$ the $u(1)$ factors cancel and $\mathrm{SU}(N)\times\mathrm{SU}(N)$ is contained as a special case in (5.9) and cannot be distinguished from $\mathrm{U}(N)\times\mathrm{U}(N)$ on the basis of the tensor f^{ac}_{bd}. Also the case $\mathrm{U}(N)$ is contained as a special case by setting the range for the tilded indices $\widetilde N = 1$. In the last equality we collect the terms in a more compact form.

We have thus classified all conformal $\mathcal{N} = 6$ gauge theories. We want to emphasize that this classification is more complete than previous ones in the sense that we did not have to assume that the theory descends from a particular $\mathcal{N} = 2$ theory [3]. This is a genuine $\mathcal{N} = 6$ classification. Our result is in accordance with a complementary approach [36], where possible scattering amplitudes for conformal $\mathcal{N} = 6$ theories where studied, also without any reference to a particular theory with less susy. We would also like to mention that parts of the structure developed here were already discussed in [8].

We will now evaluate the general expressions derived in the previous sections for the conformal deformation potential (5.4).

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5.2 Superfield equations and lagrangian

To obtain the explicit form of the superfield e.o.m. one first needs the composite fields appearing in (4.6), (4.7) and (3.9) for the deformation potential (5.4). They can be extracted from (3.5) and the first equation in (3.7):

$$\lambda_{\alpha\,ij} = \frac{4}{5}\,\nabla_{\alpha\,k[i}W^k{}_{j]}\;,\quad \rho_{\alpha\,ij} = \frac{2}{3}\,\nabla_{\alpha\,k(i}W^k{}_{j)}\;,\quad V^i{}_j = -\frac{i}{8}\epsilon^{iklm}\nabla_{\alpha\,kl}\lambda^\alpha_{mj}\;. \tag{5.10}$$

The explicit expressions, evaluated for (5.4) are given in the appendix (B.18). Inserting these in (4.6), (4.7) and (3.9) gives the superfield e.o.m. for the superconformal theories classified above. For the Chern-Simons (3.9), the fermionic (4.6) and the bosonic (4.7) equations of motions one obtains:

$$\mathcal{F}_{\alpha\beta}{}^a{}_b = \frac{1}{2}\,f^{ac}{}_{bd}\left(\Phi^{k\,d}\nabla_{\alpha\beta}\bar\Phi_{k\,c} - \nabla_{\alpha\beta}\Phi^{k\,d}\bar\Phi_{k\,c} + \frac{i}{2}\Psi^d_{k\,(\alpha}\bar\Psi^k_{\beta)\,c}\right)\,, \tag{5.11}$$

$$\begin{aligned}
\varepsilon^{\beta\gamma}\nabla_{\alpha\beta}\Psi^a_{\gamma\,i} &= \frac{1}{4}f^{ac}{}_{bd}\left(\Psi^b_{\alpha\,i}(\Phi\bar\Phi)^d{}_c - 2(\Psi_\alpha\Phi)^{bd}\,\bar\Phi_{i\,c} - \epsilon_{ijkl}\Phi^{j\,b}\Phi^{k\,d}\bar\Psi^l_{\alpha\,c}\right)\,,\\
\nabla^2\Phi^{i\,a} &= \mathcal{V}^{i\,a}_{\rm bos} + \\
&\quad + \frac{i}{4}\,\varepsilon^{\alpha\gamma}f^{ac}{}_{bd}\left((\Phi\Psi_\alpha)^{bd}\,\bar\Psi^i_{\gamma\,c} - \frac{1}{2}\Phi^{i\,b}(\Psi_\alpha\bar\Psi_\gamma)^d{}_c + \frac{1}{2}\epsilon^{ijkl}\bar\Phi_{j\,c}\Psi^b_{\alpha\,k}\Psi^d_{\gamma\,l}\right),
\end{aligned} \tag{5.12}$$

where $(\Phi\bar\Phi)^d{}_c = \Phi^{d\,k}\bar\Phi_{k\,c}$ etc. is short for contracted SU(4) indices. The dual field strength $\mathcal{F}_{\alpha\beta}{}^a{}_b$ is the matrix in the representation R acting on fields $\Phi^{i\,a}$, for example. Under the identification (5.9) with $R = (\mathbf{N}, \overline{\tilde{\mathbf{N}}})$ it thus decomposes as

$$\mathcal{F}_{\alpha\beta}{}^a{}_b \to \mathcal{F}_{\alpha\beta}{}^a{}_b\,\delta^{\tilde b}{}_{\tilde a} - \delta^a{}_b\,\tilde{\mathcal{F}}_{\alpha\beta}{}^{\tilde a}{}_{\tilde b}\;. \tag{5.13}$$

and the covariant derivative, also acting in the bifundamental representation, is given by

$$\nabla_{\alpha\beta}\Phi^{i\,a}{}_{\tilde a} = \partial_{\alpha\beta}\Phi^{i\,a}{}_{\tilde a} + \mathcal{A}_{\alpha\beta}{}^a{}_b\Phi^{i\,b}{}_{\tilde a} - \Phi^{i\,a}{}_{\tilde b}\,\tilde{\mathcal{A}}_{\alpha\beta}{}^{\tilde b}{}_{\tilde a}\;. \tag{5.14}$$

The self-interaction in the scalar field e.o.m., $\mathcal{V}^{i\,a}_{\rm bos}$, is a derivative of the bosonic potential

$$\mathcal{V}^{i\,a}_{\rm bos} = -\frac{1}{4}\left(f^{ac}{}_{hd}f^{dg}{}_{eb} - \frac{1}{2}f^{ca}{}_{ed}f^{dg}{}_{bh}\right)(\Phi\bar\Phi)^e{}_c(\Phi\bar\Phi)^h{}_g\Phi^{i\,b}\;. \tag{5.15}$$

We can now integrate back these e.o.m. to obtain the Lagrangian from which they can be derived. To make contact with the existing literature we write space-time vectors for the bosonic part in the vector notation, see appendix D. The result is:

$$\begin{aligned}
\mathcal{L}_{\rm ABJM} = &-{\rm tr}\,\nabla_\mu\Phi^i\nabla^\mu\bar\Phi_i - \frac{i}{4}{\rm tr}\,\bar\Psi^{\alpha\,i}\nabla_{\alpha\beta}\Psi^\alpha_i - \mathcal{V}_{\rm bos} + \mathcal{L}_{\rm Yuk}\\
&- 2\,g\,\varepsilon^{\mu\nu\lambda}\,{\rm tr}\Big(\mathcal{A}_\mu\partial_\nu\mathcal{A}_\lambda + \frac{2}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\lambda - \tilde{\mathcal{A}}_\mu\partial_\nu\tilde{\mathcal{A}}_\lambda - \frac{2}{3}\tilde{\mathcal{A}}_\mu\tilde{\mathcal{A}}_\nu\tilde{\mathcal{A}}_\lambda\Big)\,,
\end{aligned} \tag{5.16}$$

where $\mathcal{A}_\mu$, $\tilde{\mathcal{A}}_\mu$ are the gauge fields (in vector notation) as given below (5.11). The sextic bosonic potential writes as

$$\begin{aligned}\mathcal{V}_{\rm bos} &= \frac{1}{24}\left(f^{ac}_{\ hd}f^{dg}_{\ eb} - \frac{1}{2}f^{ca}_{\ ed}f^{dg}_{\ bh}\right)(\Phi\bar{\Phi})^e{}_c(\Phi\bar{\Phi})^h{}_g(\Phi\bar{\Phi})^b{}_a \\ &= -\frac{1}{48g^2}\mathrm{tr}\,(\Phi^i\bar{\Phi}_i\Phi^j\bar{\Phi}_j\Phi^k\bar{\Phi}_k + \bar{\Phi}_i\Phi^i\bar{\Phi}_j\Phi^j\bar{\Phi}_k\Phi^k \\ &\qquad + 4\,\Phi^i\bar{\Phi}_j\Phi^k\bar{\Phi}_i\Phi^j\bar{\Phi}_k - 6\,\Phi^k\bar{\Phi}_i\Phi^i\bar{\Phi}_k\Phi^j\bar{\Phi}_j)\,,\end{aligned} \tag{5.17}$$

and the Yukawa interaction written in the compact notation is

$$\begin{aligned}\mathcal{L}_{\rm Yuk} = \frac{i}{8}\,\varepsilon^{\alpha\gamma}f^{ac}_{\ bd}\Big\{&(\Phi\Psi_\alpha)^{bd}\,(\bar{\Phi}\bar{\Psi}_\gamma)_{ac} - \frac{1}{2}(\Phi\bar{\Phi})^b{}_a(\Psi_\alpha\bar{\Psi}_\gamma)^d{}_c \\ &+ \left(\frac{1}{4}\,\epsilon_{ijkl}\,\Phi^{i\,b}\Phi^{j\,d}\bar{\Psi}^k_{\alpha\,a}\bar{\Psi}^l_{\gamma\,c} - \,c.c.\right)\Big\}\,,\end{aligned} \tag{5.18}$$

where we again indicated summed SU(4) indices by parentheses. In lowest order in the θ expansion these expressions reproduce the component Lagrangians of [3, 10, 11].

Consistency of the quantum theory requires the gauge invariance of $\exp(iS_{CS})$ and thus determines the coupling constant to be

$$g = \frac{k}{8\pi} \quad \text{with} \quad k = 1, 2, \dots\,, \tag{5.19}$$

where we used that our anti-hermitian Lie-algebra generators, given below (5.13), are normalized as $\mathrm{tr}(T_M T_N) = -\frac{1}{2}\,\delta_{MN}$ and $\mathrm{tr}(\tilde{T}_{\tilde{M}}\tilde{T}_{\tilde{N}}) = -\frac{1}{2}\,\delta_{\tilde{M}\tilde{N}}$. Here we also assumed that there are no contributions from boundary terms in the gauge transformation of the CS-action [37]. We will comment on this in the following section.

The Lagrangian (5.16) is formally written in superspace. If the superfield equations imply the constraint equations this Lagrangian would provide an off-shell superspace formulation at least in the sense that on-shell the field content describes the ABJM model. Independently of this formal observation, the superfield expansions (4.9) implies that the lowest component in the θ-expansion of this Lagrangian gives directly the ABJM Lagrangian and that the lowest component of the superfield equations (5.11), (5.12) give the associated component equations or vice versa, respectively. We emphasized already that in our formalism the superfield expressions are often formally identical to their component field counterparts. This fact will be convenient also in the following sections.

6 $\mathcal{N} = 8$ enhancement, monopole operators

6.1 General properties of monopole operators

With the formulation of the ABJM model and its proposed AdS/CFT duality relation to $M2$ branes sitting at the singularity of $\mathbb{C}^4/\mathbb{Z}_k$ it was also argued that for CS-coupling $k = 1, 2$, supersymmetry should be enhanced to $\mathcal{N} = 8$ and that monopole operators might play a crucial role in this enhancement [3]. Before we implement this structure into our formalism we discuss some general properties of these monopole operators.

Monopole operators were first introduced by 't Hooft for $3D$- and $4D$- gauge theories to define an alternative criterion for confinement [38]. Therefore monopole operators are often called 't Hooft operators.[10] We rephrase here some of the illustrative arguments given in [38]. The basic idea of 't Hooft was to introduce an operator that creates/annihilates topological quantum numbers, i.e. magnetic fluxes. For solitons (monopoles, vortices) it is known that the associated fields can be written as pure gauge configurations which are singular at the position of the soliton. Following this observation one can define an operator that acts as a gauge transformation $\Omega^{[x_0]}(\varphi) \in G$ which is singular at the insertion point x_0, φ is the spatial angular coordinate with center x_0. Such gauge transformations have a nontrivial monodromy (winding), given by a center element of the gauge group G, for example if $G = \mathrm{SU}(N)$:

$$\Omega^{[x_0]}(2\pi) = e^{2\pi i\, n/N}\, \Omega^{[x_0]}(0) \ , \quad e^{2\pi i\, n/N} \in Z(G)\,, \tag{6.1}$$

with n being an integer. Such a prescription defines an operator whose action can be described in a Hilbert space basis $|A_i(\vec{x}), H(\vec{x})\rangle$, which are eigenstates of the (spatial) gauge field operator $\hat{A}_i$ and matter field operators $\hat{H}$ (e.g. Higgs field, etc.) with given eigenvalues $A_i(x)$ and $H(x)$. The so defined monopole operator is then given by

$$\mathcal{M}(x_0)|A_i(\vec{x}), H(\vec{x})\rangle = |A_i^{\Omega^{[x_0]}}(\vec{x}), H^{\Omega^{[x_0]}}(\vec{x})\rangle\,, \tag{6.2}$$

where $A_i^{\Omega^{[x_0]}}(\vec{x})$, $H^{\Omega^{[x_0]}}(\vec{x})$ are the fields obtained by the gauge transformation described in (6.1). From this representation it is clear that that the monodromy (6.1) requires the matter fields denoted by H to be invariant under the center $Z(G)$ of the gauge group, for example to transform in the adjoint representation. Otherwise the states obtained by the action of the monopole operator (6.2) are multi-valued and the monopole operator cannot be defined directly in this way. For $Z(G)$-invariant matter it was shown in [38] that this prescription defines a local operator and that the class of gauge transformations which have nontrivial monodromy (6.1) in case that the insertion point is encircled and trivial monodromy otherwise is generated by the Cartan subalgebra. Further it was shown in [38] that operator insertion of such kind can be described as the prescription of appropriate singularities in the elementary fields when integrated in the path integral (also for $Z(G)$ variant matter).

It turns out to be a general method to define *local* operators by requiring the elementary fields in the path integral to have certain singularities at the insertion point, a point of view very much appreciated in [39]. In a CFT, such as the ABJM model, one has in addition the operator-state correspondence, so that local operators can be described (in radial quantization) by states corresponding to the specified boundary conditions of the elementary fields. Monopole operators are understood as singularity prescriptions in the gauge field[11] which create magnetic U(1) flux embedded in the gauge group under consideration. With this understanding one can compute perturbatively quantum numbers (expectation values) for these operators by simply expanding the quantum fields around the

[10] Though the concept of 't Hooft operators is more general, as they might also be non-topological [39].
[11] For symmetry reasons also the matter fields might have corresponding singularity prescriptions.

specified singularities. This was done for different models in [40–42], whereas considerable effort was necessary to circumvent the problem of strong coupling in the ABJM model [43, 44]. This procedure is in complete analogy to quantum computations for solitons, where the theory under consideration is quantized in a soliton background and posses surprising effects, see [45–47] for example.

The explicit prescription for monopole operator insertion in a three-dimensional gauge theory is as follows. The gauge field is supposed to have a Dirac monopole singularity at the insertion point, i.e

$$A_{N/S} = \frac{H}{2}\frac{\pm 1 - \cos\theta}{r} d\varphi\,, \tag{6.3}$$

such that it produces magnetic flux in a U(1) subgroup through a sphere surrounding the insertion point in three dimensions.[12] Hence such an operator prescription creates/annihilates topological quantum numbers. It was shown in [48] (for static monopoles in $4D$ gauge theories which is equivalent to the situation considered here) that H has to satisfy the quantization condition $e^{2\pi\, i\, H} = \mathbb{1}_G$ and therefore is an element of the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ of the form $H = \mathrm{diag}(q_1, \ldots, q_N)$. The integers q_i are the fluxes in the U(1) subgroups. The GNO charges ${}^L w = (q_1, \ldots, q_N)$, form a highest weight state of the GNO - or Langlands dual group ${}^L G$, where Weyl-reflection can be used to choose $q_1 \geq \ldots \geq q_N$, see also [39, 49]. Therefore monopole operators are classified by irreps of the dual gauge group ${}^L G$ and come in representations specified by the highest weight ${}^L w$.

In the presence of a Chern-Simons term with CS-level k the GNO charges are of the form $q_i = k\, n_i$, with integers n_i, as will be seen in a specific situation below. Taking also into account that ${}^L(G_1 \times G_2) = {}^L G_1 \times {}^L G_2$ and that $\mathrm{U}(N)$ is selfdual, i.e. ${}^L\mathrm{U}(N) = \mathrm{U}(N)$ one finds that for the $\mathrm{U}(N) \times U(\tilde{N})$ ABJM model monopole operators are in the representation $({}^L w, {}^L \tilde{w})$ with $q_i = k\, n_i$ and $\tilde{q}_i = -k\, \tilde{n}_i$. In the following, monopole operators of the form $\mathcal{M}^{ab}_{\tilde{a}\tilde{b}}$ are of particular interest, i.e. we choose[13] ${}^L \tilde{w} = \overline{{}^L w}$, the conjugate representation. Then one has the possible weights

$${}^L w = (2, 0, \ldots, 0) \quad \text{and} \quad {}^L w = (1, 1, 0, \ldots, 0)\,, \tag{6.4}$$

the associated representations have Young tableaux $\square\!\square$ and $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ [51, 52]. The first weight in (6.4) is possible for $k = 1, 2$ with $n_1 = 2, 1$ and the associated monopole operator is in the $(\mathbf{N}^2_{\mathrm{sym}}, \bar{\mathbf{N}}^2_{\mathrm{sym}})$ representation, whereas the second weight allows only for $k = 1$ with $n_1 = n_2 = 1$ and the monopole operator is in the $(\mathbf{N}^2_{\mathrm{asym}}, \bar{\mathbf{N}}^2_{\mathrm{asym}})$ representation.

The prescription given above defines operator insertions which create topological quantum numbers, i.e. magnetic fluxes. Such a prescription is very different from the definition of a local (composite) operator as a polynomial local function of the elementary fields and in general such a description will not be available. Nevertheless it would be of great interest to have a more explicit formulation for such operators, especially with regard to the

[12] The signs correspond to N-/S-pole of the surrounding sphere. The surrounding sphere and the associated flux quantization, see below, may be best understood in the euclidean setting or in the radial quantization picture, where $\mathbb{R}^3 \;\to\; S^2 \times \mathbb{R}$.

[13] Due to the dynamics of the ABJM model the fluxes ${}^L w$, ${}^L \tilde{w}$ have to satisfy certain constraints, see e.g. [50] which is the case with the here given choice which also implies $\tilde{N} = N$.

conjectured dualities between different gauge theories. It is believed that such dualities have the same origin as for example the well understood duality between the sine-Gordon and massive Thirring model [16, 17] which arises from reformulating the model in terms of local operators which are non-polynomial non-local functionals of the elementary fields and create topological quantum numbers. So far such an explicit formulation for higher dimensional ($D > 2$) (nonabelian) gauge theories has not been found. In three dimensions such a duality is mirror symmetry, which was first proposed in [15].[14] In the following we give a step in the direction of an explicit description of such dual degrees of freedom in terms of monopole operators. The reason why this is possible is susy enhancement which is supposed to be triggered by monopole operators and thus results in very specific conditions which allow to specify these operators to a certain extent.

Before implementing monopole operators into the structure of superspace constraints we mention an example where monopole operators can be constructed explicitly [37, 53, 54], without going into details. For this we come back to idea of singular gauge transformations as described in the beginning of this section. In the case of pure Chern-Simons theory (perhaps with matter invariant under the center $Z(G)$) one can define an operator insertion by defining a singular gauge transformation at the (spacial) insertion point. This carves out a tube of space time and as mentioned below (5.19), the CS-action has boundary contributions from a gauge transformation which gives a nonvanishing contribution for singular gauge transformations even if the tube shrinks to zero. This change in the action can be interpreted as an operator insertion. For $G = \mathrm{U}(N)$ this operator is a Wilson line of the form,

$$\mathcal{M}(x) = \frac{1}{d(R)}\,\mathscr{P}\exp\left(-\int_\infty^x A^R\right). \tag{6.5}$$

Here $d(R)$ is the dimension of the representation R in which the connection A is given and the highest weight of R is $w_R = kw_{\mathrm{fund}}$ with w_{fund} being the highest weight of the fundamental representation $\mathbf{N}$ of $\mathrm{U}(N)$. Under gauge transformations $\mathcal{M}$ transforms as $R(g)(x)\mathcal{M}R(g^{-1})(\infty)$ and thus w.r.t. genuine gauge transformations ($g(\infty) = \mathbb{1}$) the monopole operator transforms in the ($\mathbf{N}^k_{\mathrm{sym}}$) representation. Due to the special dynamics of pure CS-theory this operator is local, i.e. it depends only on the endpoint of the path, but this construction does not work in the case charged matter is present. However, the principal structure is an appealing guideline.

Monopole operator superfields. In the following we investigate the possible supersymmetry enhancement of the $\mathcal{N} = 6$ ABJM model to $\mathcal{N} = 8$ supersymmetry for the gauge groups $\mathrm{U}(N) \times \mathrm{U}(\widetilde{N})$ with a priori arbitrary $N, \widetilde{N} \geq 2$. We start from our basic description of the model through the superspace constraints (4.1), (3.2) with the deformation potential (5.4). The principle idea is to realize the additional $\mathcal{N} = 2$ susy as infinitesimal internal fermionic symmetry of the superspace constraints. The superspace description is on-shell, and thus any additional supersymmetry obtained in this way will be, a priori, an (infinitesimal) symmetry of the classical e.o.m. only. This is a situation well known

[14] Another example is S-duality for $\mathcal{N} = 4$ SYM and it was shown recently that 't Hooft and Wilson operators are related under under S-duality [39, 49].

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from two dimensional models with hidden symmetries, or even in four dimensions [55]. Nevertheless, there should exist currents associated with the hidden symmetry which are conserved on-shell, i.e. dynamically. We assume the existence of a (composite) monopole operator superfield to be able to formulate these additional symmetries and derive superspace constraints which this operator has to satisfy in order to obtain susy enhancement.

In the following it will be convenient to use the more compact notation as employed before, with a single index labelling the matter field representation R. Only when necessary translate the expressions to the explicit notation as described in (5.9). The starting point is the transformation behaviour for the bosonic superfield under the additional supersymmetry, which we define as follows:

$$\delta\Phi^{i\,a} := \epsilon^{\alpha}\mathcal{M}^{ab}\,\bar{\Psi}^{i}_{\alpha\,b} \ \ , \quad \delta\bar{\Phi}_{\,i\,a} := -\bar{\epsilon}^{\,\alpha}\bar{\mathcal{M}}_{ab}\,\Psi^{b}_{\alpha\,i} \tag{6.6}$$

where ϵ^{α} is a complex anti-commuting constant spinor and $\mathcal{M}^{ab} =: (\bar{\mathcal{M}}_{ab})^*$ is the proposed monopole operator superfield. It is supposed to be a *local* composite superfield but in general non-polynomial in and a nonlocal functional of the elementary superfields. It is necessary that the superfields Φ^i transform into superfields $\bar{\Psi}^i_{\alpha}$ to obtain a transformation different from the original $\mathcal{N}=6$ susy. Therefore one has to assume the existence of a monopole operator so that the transformation (6.6) is gauge covariant.

Taking the canonical dimension for the new susy parameter to be the standard one, i.e. $[\epsilon^{\alpha}] = [\theta^{\alpha\,ij}] = -\frac{1}{2}$, one sees that the monopole operator has canonical dimension zero, i.e. $[\mathcal{M}^{ab}] = 0$. Given that the monopole operator should have canonical dimension zero and that the extra susy should commute with the SU(4) R-symmetry (the susy parameter ϵ is an SU(4) singlet), the transformation rule (6.6) is the only conceivable one. As a matter of fact, this is the only *ansatz* that we make, all other relations will be derived. This clearly shows the effectiveness of the formalism developed here.

As mentioned, the monopole operator has to compensate the different gauge transformation properties of the elementary fields in (6.6), which for the non-abelian part of the gauge group determines the representation (indices) it carries. In addition, for possible gauged U(1) factors the monopole operator has to carry appropriate U(1) charges so that (6.6) is gauge covariant. We will discuss this point in more detail below.

6.2 U(2) × U(2): fake monopole operators

Having defined the transformations for the bosonic superfields we have to determine the transformations of the residual elementary superfields such that the superfield constraints (3.2), (4.1) are invariant under these transformations. This will necessarily impose also conditions on the monopole operator superfield. In this subsection we will assume that the monopole operator is covariantly constant, an assumption which has been considered also in [13] and [14]. We will show that this eventually restricts the gauge group to be U(2)×U(2) (or SU(2)×SU(2)). Later we will relax this condition and derive superspace constraints for the monopole operator without any obvious restriction on the dimension/rank of the gauge group. The main point is that we do not allow any additional condition on the elementary component or superfields of the theory, i.e. for $\Phi^i, \Psi_{\alpha\,i}, \mathcal{A}_{\alpha\,ij}, \mathcal{A}_{\alpha\beta}$, since this would change the theory. This is a major difference to the considerations in [14] where numerous nontrivial conditions (called "identities") are imposed on the elementary fields. While finding nontrivial solutions for these conditions represents a formidable task the very presence of such conditions inevitably changes the original model.

Matter constraint. We first consider the matter constraint (4.1) which for the transformation of the fields reads

$$\left\{ \nabla_{\alpha\, ij} \delta\Phi^k + \delta\mathcal{A}_{\alpha\, ij} \cdot \Phi^k \right\} |_{\overline{\mathbf{20}}} \overset{!}{=} 0 \; . \tag{6.7}$$

Thus one has to define $\delta\mathcal{A}_{\alpha\, ij}$ such that for the given transformation $\delta\Phi^k$ (6.6) the $\overline{\mathbf{20}}$ contribution in the product $\bar{\mathbf{6}} \otimes \mathbf{4} = \bar{\mathbf{4}} \oplus \overline{\mathbf{20}}$ in (6.7) vanishes. The remaining part, transforming in the $\bar{\mathbf{4}}$ defines the enhanced susy transformation of the fermionic superfield $\Psi_{\alpha\, i}$, see (4.1). Explicitly one finds with the complex conjugate of (4.5),

$$\delta\mathcal{A}_{\alpha\, ij}{}^a{}_d \, \Phi^{k\, d} - \epsilon^\beta \left(\nabla_{\alpha\, ij} \mathcal{M}^{ab} \, \bar{\Psi}^k_{\beta\, b} - \epsilon_{\alpha\beta} \, \mathcal{M}^{ab} f^{ce}_{bd} \, \bar{\Phi}_{i\, c} \bar{\Phi}_{j\, e} \Phi^{k\, d} \right) \overset{!}{\sim} \bar{\mathbf{4}} \, , \tag{6.8}$$

where we have used that for the conformal deformation potential (5.4) $(W^k{}_{[i} \cdot \bar{\Phi}_{j]})_b = f^{ce}_{bd} \, \bar{\Phi}_{i\, c} \bar{\Phi}_{j\, e} \Phi^{k\, d}$, up to terms transforming in the $\bar{\mathbf{4}}$.

At this point we impose the condition that the monopole operator is covariantly constant,

$$\nabla_{\alpha\, ij} \mathcal{M}^{ab} = 0 \; . \tag{6.9}$$

Covariant constancy is defined here with respect to the fermionic covariant derivative. We will discuss in a moment the implications of the integrability conditions of (6.9) and show that as a consequence the monopole operator is also covariantly constant w.r.t. the bosonic covariant (super) derivative. Imposing (6.9) it seems to be trivial to choose $\delta\mathcal{A}_{\alpha\, ij}$ such that the invariance condition for the matter constraint (6.8) is satisfied, but there is an additional obstacle. The transformation of the gauge superfield $\mathcal{A}_{\alpha\, ij}$ has to conserve the reality condition (2.3), so that

$$\delta\mathcal{A}_{\alpha\, ij}{}^a{}_d = \epsilon_\alpha \, (\mathcal{M} \cdot f)^{a,ce}{}_d \bar{\Phi}_{i\, c} \bar{\Phi}_{j\, e} + \frac{1}{2} \, \bar{\epsilon}_\alpha \, \varepsilon_{ijkl} (\bar{\mathcal{M}} \cdot f)_{d,ce}{}^a \Phi^{k\, c} \Phi^{l\, e} \, , \tag{6.10}$$

where we introduced the abbreviation $(\mathcal{M} \cdot f)^{a,ce}{}_d := \mathcal{M}^{ab} f^{ce}_{bd}$ and analogously for the complex conjugate expression. The first term in (6.10) is determined by the invariance condition (6.8) whereas the second term is necessary to obey the reality condition (2.3).[15] Inserting the transformation (6.10) back into the invariance condition (6.8) results in the following condition:

$$(\bar{\mathcal{M}} \cdot f)_{d,ce}{}^a \Phi^{k\, c} \Phi^{l\, e} \Phi^{k\, d} \overset{!}{\sim} \bar{\mathbf{4}} \qquad \Leftrightarrow \qquad (\bar{\mathcal{M}} \cdot f)_{d,ce}{}^a \overset{!}{=} (\bar{\mathcal{M}} \cdot f)_{[dce]}{}^a \; . \tag{6.11}$$

With the explicit solution (5.9) for f^{ab}_{ce} and employing the explicit notation of (5.9) for the monopole operator one finds a unique solution to this condition:

$$N = \widetilde{N} = 2 \quad \text{and} \quad \bar{\mathcal{M}}^{\tilde{a}\tilde{b}}_{ab} \sim \varepsilon^{\tilde{a}\tilde{b}} \varepsilon_{ab} \; . \tag{6.12}$$

[15] There is still the freedom to add a term $\delta\tilde{\mathcal{A}}_{\alpha\, ij}$ for which one has to assume that it satisfies $\delta\tilde{\mathcal{A}}_{\alpha\, ij} \cdot \Phi^k |_{\overline{\mathbf{20}}} = 0$, but such a term does not play a role in the following considerations. We will comment on this in the succeeding section when we consider covariant non-constant monopole operators.

The only other possibility is the trivial case $N = \widetilde{N} = 1$. Before we discuss the possible proportionality factor for the monopole operator in (6.12) we investigate the invariance of gauge field constraint (3.2). Here we just notice, that the monopole operator (6.12) transforms in the $(\mathbf{N}^2_{\text{asym}}, \mathbf{N}^2_{\text{asym}})$.

Gauge field constraint. We have now defined the enhanced susy transformation for the scalar and fermionic vector superfields $\Phi^i, \mathcal{A}_{\alpha\, ij}$ (and for the fermionic superfield $\Psi_{\alpha\, i}$, though we did not write it explicitly), such that the matter constraint is invariant with a covariantly constant monopole operator. This imposed the restrictions (6.12) on gauge group and the monopole operator. The invariance of the gauge field constraint (3.2) requires

$$\nabla_{\alpha\, ij}\, \delta\mathcal{A}_{\beta\, kl} + \nabla_{\beta\, kl}\, \delta\mathcal{A}_{\alpha\, ij} \overset{!}{=} i\,(\varepsilon_{ijkl}\, \delta\mathcal{A}_{\alpha\beta} + \varepsilon_{\alpha\beta}\varepsilon_{mij[k}\, \delta W^m{}_{l]})\,, \tag{6.13}$$

where $\delta W^m{}_l$ is the susy transformation of the conformal deformation potential (5.4) obtained from the transformation of the scalar superfields (6.6). Inserting the transformation (6.10) and using the matter constraint (4.1) one obtains for the transformation of the bosonic super vector field

$$\delta\mathcal{A}_{\alpha\beta}{}^a{}_d = \frac{1}{2}\,(\mathcal{M}\cdot f)^{a,ce}{}_d\, \bar{\Phi}_{k\,c}\, \epsilon_{(\alpha}\bar{\Psi}^k_{\beta)e} + \frac{1}{2}\,(\bar{\mathcal{M}}\cdot f)_{d,ce}{}^a\, \Phi^{k\,c}\, \bar{\epsilon}_{(\alpha}\Psi^e_{\beta)k}\,, \tag{6.14}$$

where we have used the conditions (6.9), (6.12) for the monopole operator and did not encounter any further restrictions.

SU(2) × SU(2): the assumption that the monopole operator is covariant constant resulted in the restriction[16] $N = \widetilde{N} = 2$ and the particular form for the monopole operator as given in (6.12). For the gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)$ there is no local U(1) transformation which has to be compensated by the operator $\mathcal{M}^{ab}$ in the transformation rules (6.6), (6.10) and (6.14) so that the proportionality factor in (6.12) is an arbitrary number which can be absorbed in the susy parameter ϵ. Thus one has

$$\mathcal{M}^{ab}_{\tilde{a}\tilde{b}} = \varepsilon^{ab}\varepsilon_{\tilde{a}\tilde{b}}\,. \tag{6.15}$$

Clearly this "operator" is covariantly constant, i.e. satisfies (6.9), and the gauge field constraint (3.2) is consistent with the fact that this $\mathrm{SU}(2)\times\mathrm{SU}(2)$ invariant is also covariantly constant w.r.t. the bosonic covariant derivative. This is not a monopole operator in the sense discussed above (it does not create any magnetic flux), but rather shows that for the gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)$ the supersymmetry is "kinematically" enhanced to $\mathcal{N} = 8$. This case actually describes the BLG model [1–3]. In [56] a $\mathrm{SU}(2)\times\mathrm{SU}(2)$ formulation of the BLG model with manifest SO(8) R-symmetry was given but at the cost of an additional condition on the matter fields, which reads for the scalar fields as $\varepsilon_{\tilde{a}\tilde{b}}\, \bar{X}^{I\,\tilde{b}}{}_a\, \varepsilon^{ab} = X^{I\,b}{}_{\tilde{a}}$, $I = 1, \ldots 8$, and similar for the fermions. In the equivalent formulation of ABJM, as given

[16] The mentioned trivial solution $N = \widetilde{N} = 1$ refers to the $\mathrm{U}(1)\times\mathrm{U}(1)$ case which was considered in detail in [3] and [13]. The considerations become become rather trivial in this case since $f^{ac}_{\tilde{b}d} = 0$ for $\mathrm{U}(1)\times\mathrm{U}(1)$.

here, no such constraint is present, but therefore the manifest R-symmetry is only SU(4). The fields in the two different descriptions are related as follows,

$$\Phi^{i\,a}{}_{\tilde{b}} = X^{i\,a}{}_{\tilde{b}} + i\,X^{i+4\,a}{}_{\tilde{b}} \;\;,\;\; \bar{\Phi}_i{}^{\tilde{b}}{}_a = \bar{X}_i{}^{\tilde{b}}{}_a - i\,\bar{X}_{i+4}{}^{\tilde{b}}{}_a \,, \quad i = 1,\ldots,4\,, \tag{6.16}$$

which shows that the ABJM formulation resolves the mentioned condition at the price of loosing manifest SO(8) R-symmetry according to

$$\mathrm{SO}(8) \;\; \rightarrow \;\; \mathrm{U}(1)\times\mathrm{SU}(4) \tag{6.17}$$

with remaining manifest SU(4) R-symmetry. Furthermore, in the case of an U(2) × U(2) gauge group the U(1) factor in (6.17) is also gauged. This is the case that we discuss next.

U(2) × U(2): the U(1) sectors of the the Lagrangian (5.16) are of a very particular form. The bifundamental action of the covariant derivative (5.14) implies that the matter fields couple exclusively to the difference of the two U(1) gauge fields,[17]

$$\mathcal{A}^{\mathrm{bar}}_{\alpha\beta} = \frac{1}{2}(\mathrm{tr}\mathcal{A}_{\alpha\beta} - \mathrm{tr}\tilde{\mathcal{A}}_{\alpha\beta})\,, \tag{6.18}$$

where the superscript "bar" stands for baryonic. Thus the U(1) factor described in (6.17) is the baryonic $\mathrm{U}(1)_b$ gauge symmetry. The matter fields (Φ^i, Ψ_i) have $\mathrm{U}(1)_b$ charge +1 while the c.c. thereof have $\mathrm{U}(1)_b$ charge −1. Denoting the opposite combination of the U(1) gauge fields, i.e. the diagonal U(1) in U(2) × U(2), by $\mathcal{A}^{\mathrm{diag}}_{\alpha\beta} = \frac{1}{2}(\mathrm{tr}\mathcal{A}_{\alpha\beta} + \mathrm{tr}\tilde{\mathcal{A}}_{\alpha\beta})$, the U(1) sector of the CS-Lagrangian in (5.16) writes as

$$d\mathrm{Vol}\,\mathcal{L}^{\mathrm{U}(1)}_{CS} = -\frac{k}{4\pi}(\mathcal{A}^{\mathrm{bar}}\wedge d\mathcal{A}^{\mathrm{diag}} + \mathcal{A}^{\mathrm{diag}}\wedge d\mathcal{A}^{\mathrm{bar}})\;. \tag{6.19}$$

The gauge field $\mathcal{A}^{\mathrm{diag}}$ appears only at this place and its variation imposes a flatness condition for the baryonic U(1) connection. Explicitly, the e.o.m. (5.11) for the U(1) sectors write as

$$*\,\mathcal{F}^{\mathrm{bar}} = 0 \;\;,\;\; \frac{k}{2\pi}*\mathcal{F}^{\mathrm{diag}} = -j^{\mathrm{bar}}\,, \tag{6.20}$$

where j^{bar} is the current associated with the $\mathrm{U}(1)_b$ symmetry. It has been shown in several places, e.g. [3], that the diagonal U(1) field strength $\mathcal{F}^{\mathrm{diag}} = d\mathcal{A}^{\mathrm{diag}}$ which appears as a Lagrange multiplier (up to a surface term) in (6.19) can be considered as a fundamental field and treated in for a dual 2π-periodic scalar τ, such that the flatness condition for the baryonic connection is expressed as $\mathcal{A}^{\mathrm{bar}} = \frac{1}{k}d\tau$.

From the point of view of the classical dynamics the U(1) factors are irrelevant, only the baryonic connection $\mathcal{A}^{\mathrm{bar}}$ couples to the matter but due to its flatness it can be gauged away locally. The "dynamics" of the diagonal field strength (6.20), appearing as a Lagrange multiplier, is just the attachment of magnetic "flux" in the diagonal U(1) factor to the "electric" $\mathrm{U}(1)_b$ current, an effect well known in the presence of a CS-field [57]. This e.o.m.

[17] In accordance with our normalization of the generators, see below (5.19), the U(1) generators are given by $T^0 = \frac{i}{\sqrt{2N}}\mathbb{1}$ with $N = 2$ in the considered case.

implies that magnetic flux and $\mathrm{U}(1)_b$ charge obey a Dirac-quantization condition according to (see also footnote 12)

$$\frac{k}{2\pi}\int_{S^2}\mathcal{F}^{\rm diag} = k\,n = \int_{S^2} *j^{\rm bar} = q_{\rm bar}\,, \tag{6.21}$$

which is understood to be read for the lowest component of the superfield expressions. Though dynamically irrelevant the gauged U(1) factors have an important impact on the moduli space of the theory, in particular on the possible vacua in the quantum theory. The additional identification due to the U(1) gauge symmetry reduces the naive moduli space $\mathbb{C}^4$ to the orbifold $\mathbb{C}^4/\mathbb{Z}_k$ [3].

We now describe the effect for the monopole operator of the just discussed U(1) factors. In the case that the gauge group is $\mathrm{U}(2)\times\mathrm{U}(2)$ the operator (6.15) cannot be the whole answer, it is neither covariantly constant nor does it carry the right $\mathrm{U}(1)_b$ charge. One therefore sets $\mathcal{M}^{ab}_{\tilde a\tilde b} = T\varepsilon^{ab}\varepsilon_{\tilde a\tilde b}$, so that covariant constance implies for the allowed (6.12) pre-factor

$$\nabla_{\alpha\,ij}\mathcal{M}^{ab}_{\tilde a\tilde b} = 0 \quad\Rightarrow\quad D_{\alpha\,ij}T + 2\,\mathcal{A}^{\rm bar}_{\alpha\,ij}\,T = 0\;. \tag{6.22}$$

The integrability condition for this equation gives with the gauge constraint (3.2)

$$\nabla_{\alpha\beta}\mathcal{M}^{ab}_{\tilde a\tilde b} = 0 \quad\Rightarrow\quad \partial_{\alpha\beta}T + 2\,\mathcal{A}^{\rm bar}_{\alpha\beta}\,T = 0\,, \tag{6.23}$$

where for the particular form of $\mathcal{M}^{ab}_{\tilde a\tilde b}$ the conformal deformation potential (5.4) does not give any contribution. Thus also in the $\mathrm{U}(2)\times\mathrm{U}(2)$ case covariant constance w.r.t to the fermionic connection implies covariant constance in the usual sense. This in fact results in yet another integrability condition, i.e. the existence of non-trivial solutions to (6.23) requires $\mathcal{F}^{\rm bar} = 0$, which is satisfied due to the e.o.m. (6.20).

Solutions to equations of the form (6.22), (6.23) are given by local, i.e. path independent, Wilson lines which are abelian here. To implement this in superspace an unimportant but necessary little formality has to be introduced. The super connections $\mathcal{A}_A = (\mathcal{A}_{\alpha\beta}, \mathcal{A}_{\alpha\,ij})$, as usual in supersymmetric theories, is defined w.r.t to the non-holonomic basis $D_A = (\partial_{\alpha\beta}, D_{\alpha\,ij})$. The relation to the holonomic coordinate basis $\partial_M = (\partial_{\alpha\beta}, \partial_{\alpha\,ij})$ is given by a super vielbein, i.e. $\partial_M = e_M{}^A D_A$. For more details on superspace Wilson lines see [58]. For the purpose of the present consideration, there is no need to go into further detail.[18] The solution to (6.22), (6.23) is then given by

$$T(x,\theta) = \exp\left\{-2\int_{\tau_\infty}^{\tau_x} d\tau\,\dot z^M(\tau)\,e_M{}^A\mathcal{A}^{\rm bar}_A(z(\tau))\right\}, \tag{6.24}$$

where $z^M(\tau) = (x^{\alpha\beta}(\tau), \theta^{\alpha\,ij}(\tau))$ is a path in superspace with $z^M(\tau_x) = (x^{\alpha\beta}, \theta^{\alpha\,ij})$ being the insertion point and otherwise to be specified in a moment. The lowest component of the superfield T is just an ordinary Wilson line, i.e. $T|_{\theta=0} = e^{-2\int_{\mathcal{C}}A}$. In this form the superfield T satisfies the equation

$$\dot z^M\,e_M{}^A(D_A\,T + 2\,\mathcal{A}^{\rm bar}_A\,T)_{\tau=\tau_x} = 0\,, \tag{6.25}$$

[18] For completeness, the super vielbein is given by $e_M{}^A = \begin{bmatrix} \delta^\gamma_{(\alpha}\delta^\delta_{\beta)} & 0 \\ -i\theta^{(\delta}_{ij}\delta^{\gamma)}_\alpha & \delta^\gamma_\alpha\delta^{kl}_{ij}\end{bmatrix}$.

and under a $\mathrm{U}(1)_b$ super-gauge transformation $\delta\mathcal{A}_A^{\rm bar} = -\frac{1}{2}D_A(\mathrm{tr}\,\Omega - \mathrm{tr}\,\tilde{\Omega}) =: -D_A\Omega^{\rm bar}$ behaves as

$$T \quad\to\quad e^{\,2\,(\Omega^{\rm bar}(\tau_x)-\Omega^{\rm bar}(\tau_\infty))}\;T\;. \tag{6.26}$$

The requirement that equation (6.25) implies (6.22), (6.23) is equivalent to the requirement that the operator T is local, i.e. $T=1$ for any closed path and thus $\dot{z}^M$ can be arbitrary. This is guaranteed by the e.o.m. (6.20) which implies the already below (6.23) mentioned integrability condition. The requirement that the monopole operator has $\mathrm{U}(1)_b$ charge 2, so that the transformation (6.6) is gauge covariant, requires that $x^{\alpha\beta}(\tau_\infty)\to\infty$ so that the monopole operator has charge $+2$ under genuine gauge transformations for which then $\Omega^{\rm bar}(\tau_\infty)=0$, a structure similar to (6.5).

We thus have for the $\mathrm{U}(2)\times\mathrm{U}(2)$ model the monopole operator superfield

$$\mathcal{M}^{ab}_{\tilde{a}\tilde{b}} = T(x,\theta)\,\varepsilon^{ab}\varepsilon_{\tilde{a}\tilde{b}}\,, \tag{6.27}$$

with T given in (6.24). At zero order in θ this agrees with the result of [13]. The obtained monopole operator is in the $(\mathbf{N}^2_{\rm asym},\mathbf{N}^2_{\rm asym})$ representation, a puzzling fact as we discuss in a moment. In [13] it is also argued that this monopole operator exists for $k=1,2$, but our general discussion below (6.4) indicates that it exists actually only for $k=1$. The criterion given in [13] is perhaps not specific enough.

One may ask if covariant constancy w.r.t. the super-derivative (6.9) is a stronger (or too strong) requirement compared to covariant constancy w.r.t. the bosonic connection, as it was implemented in an ordinary space-time approach in [13]. To answer this question we look at the proposed enhanced R-symmetry current [44, 52]. Employing the compact notation the enhancement current of [44, 52] can be promoted to a superfield expression in the form

$$\mathcal{J}^{ij}_{\alpha\beta} = i\,\bar{\mathcal{M}}_{ab}\left\{2\,\Phi^{a[i}\,\nabla_{\alpha\beta}\,\Phi^{j]b} - \frac{1}{4}\varepsilon^{ijkl}\;\Psi^a_{k(\alpha}\Psi^b_{\beta)l}\right\}\,. \tag{6.28}$$

This current can be understood as the enhanced current superfield, with the enhanced R- and susy-current at the lowest and first order in the θ expansion. For this current actually representing new symmetries it has to be conserved (on-shell). Under the assumption that the monopole operator is covariantly constant w.r.t. to the bosonic (or ordinary space-time) connection this requirement leads to the same condition as given in (6.11) for two independent reasons. First, using the e.o.m. (5.12) one finds that the current is conserved only if (6.11) is satisfied. Second, $\nabla_{\alpha\beta}\mathcal{M}^{ab}=0$ implies an integrability condition for the bosonic field strength, which together with the e.o.m. (5.11) again leads to the condition (6.11). Therefore it is clear that the assumption of a covariantly constant monopole operator is consistent only for $\mathrm{U}(2)\times\mathrm{U}(2)$ gauge group.

However, as mentioned before the monopole operator (6.27) gives rise to a puzzle. The current (6.28) is a dimension 2 conformal primary and thus as a vector has to be conserved in a unitary CFT. For this to be true the monopole operator has to have dimension zero, also in the quantized theory. This was proved in [44] for the case of a monopole operator in the $(\mathbf{N}^2_{\rm sym},\mathbf{N}^2_{\rm sym})$, i.e. corresponding to the first weight in (6.4). Further it was argued in [52] that with such a monopole operator one can form the 20 missing dimension 1 operators to match the SUGRA spectrum on S^7 for $k=1$. Given that there is an monopole operator in the $(\mathbf{N}^2_{\mathrm{asym}}, \mathbf{N}^2_{\mathrm{asym}})$ representation (6.27), this would mean that one would double the number of enhancement currents (6.28). Further, one could build additional dimension 1 operators of the form $\bar\Phi_i \mathcal{M} \bar\Phi_j$ and their complex conjugates, which would give additional (too many) $10+10$ states. We therefore assume that the dimension of the monopole operator (6.27) is not protected and therefore is not connected to susy enhancement in the full theory. We therefore call this operator in the context of susy enhancement "fake monopole operator".[19]

6.3 U(N) $\times$ U(N): monopole operator superspace constraints

We have shown in the previous section, that the assumption that the monopole operator is covariantly constant is too restrictive. We now relax this assumption and derive superspace constraints for the (composite) monopole operator superfield, analogously to the formulation of the constraint equations for the matter and vector superfields (4.1), (3.2), such that the supersymmetry is enhanced to $\mathcal{N}=8$ without any restriction on the rank of the gauge group factors.

Matter constraint. We again start from our basic and only assumption, the transformation rule for the bosonic superfield (6.6) which leads to the invariance condition (6.7), (6.8) for the matter constraint. Invariance of the matter constraint, equ. (6.7), implies that the $\bar\epsilon$-part of enhanced susy transformation for the fermionic connection $\delta\mathcal{A}_{\alpha\, ij}$ has to satisfy an algebraic condition whereas the ϵ-part can now be absorbed in the super-derivative of the monopole operator. Explicitly we have the algebraic condition

$$\delta\mathcal{A}_{\alpha\, ij}{}^a{}_d = \delta_\epsilon \mathcal{A}_{\alpha\, ij}{}^a{}_d + \frac{1}{2}\varepsilon_{ijkl}[\delta_\epsilon \mathcal{A}_{\alpha\, ij}{}^d{}_a]^* \quad \text{with} \quad \frac{1}{2}\varepsilon_{ijkl}[\delta_\epsilon \mathcal{A}_{\alpha\, ij}{}^d{}_a]^*\ \Phi^{k\, d} \overset{!}{\sim} \bar{\mathbf{4}}\,, \tag{6.29}$$

where the first equation implements the reality condition (2.3).

We now solve the algebraic condition (6.29) and then determine the super-derivative of the monopole operator superfield. To this end we introduce a structure, which will be extremely useful following, in terms of the determinant of *mesonic* operators:

$$X^i{}_j := \operatorname{tr} \Phi^i \bar\Phi_j = \Phi^{i\, a} \bar\Phi_{j\, a} \quad , \quad |X| := \det(X^i{}_j)\ . \tag{6.30}$$

The mesonic operators $X^i{}_j$ form a hermitian matrix and transform in the $(\mathbf{4}, \bar{\mathbf{4}})$ under the R-symmetry SU(4). We collect a number of curious relations which are needed in the following in the appendix C. For the understanding of the main text we introduce here a part of this structure, where we use the abbreviations $\partial_{i\, a} := \frac{\partial}{\partial \Phi^{i\, a}}$ and $\bar\partial^{i\, a} := \frac{\partial}{\partial \bar\Phi_{i\, a}}$ in the following. First, with the help of the determinant of the mesonic operators one can translate an SU(4) index of *any* quantity into the SU(4) index of a scalar superfield, i.e.

$$\mathcal{O}^i |X| = \Phi^{i\, a}\, \mathcal{O}^j\, \partial_{j\, a} |X| \qquad \text{and} \qquad \mathcal{O}_i |X| = \bar\Phi_{i\, a}\, \mathcal{O}_j\, \bar\partial^{j\, a} |X|\ . \tag{6.31}$$

[19] We want to mention however, that monopole operators in the $(\mathbf{N}^2_{\mathrm{asym}}, \mathbf{N}^2_{\mathrm{asym}})$ representation were considered in the context of mass deformations of the ABJM model [51]

Second, we introduce the hermitian *projection operators*

$$\mathcal{F}^a{}_b := \frac{1}{|X|}\,\Phi^{i\,a}\,\partial_{i\,b}|X| = \frac{1}{|X|}\,\bar\Phi_{i\,b}\,\bar\partial^{i\,a}|X| \quad \text{and} \quad \mathcal{P}^a{}_b := \delta^a{}_b - \mathcal{F}^a{}_b\,, \tag{6.32}$$

which have the properties

$$\mathcal{F}\cdot\mathcal{F} = \mathcal{F}\,,\quad \mathcal{P}\cdot\mathcal{P} = \mathcal{P}\,,\quad \mathcal{P}\cdot\mathcal{F} = 0 \quad \text{and} \quad \mathcal{F}\cdot\Phi^i = \Phi^i\,,\quad \mathcal{P}\cdot\Phi^i = 0\,, \tag{6.33}$$

where the gauge indices are contracted in an obvious way, indicated by a dot. The last two identities have a hermitian conjugated counterpart for the fields $\bar\Phi_i$. In these definitions appears the inverse of the mesonic determinant $|X|$. For generic configurations this is a well defined superfield, only for isolated points in the field configuration space it might be singular. In general we allow such operators in our considerations. The field configurations where these operators become singular might be of special interest but we leave this question for future studies.

With the introduced structure we can write down the general solution for the algebraic constraint (6.29). According to (6.29) there exists a (composite) superfield $G_{\alpha\beta\,m}^{\;\;a}$ in the $\bar{\mathbf{4}}$ of SU(4) such that

$$[\delta_\epsilon \mathcal{A}_{\alpha\,ij}{}^d{}_a]^*\,\Phi^{k\,d} = \bar\epsilon^\beta\,\varepsilon^{ijkm}\,|X|\,G_{\alpha\beta\,m}^{\;\;a}\,, \tag{6.34}$$

where for convenience we pulled out a factor $|X|$, which is pure convention (c.f. the remarks above regarding the existence of the inverse $|X|^{-1}$). Transforming the index k on the r.h.s. onto a scalar superfield according to (6.31) and using the projection operators (6.33) we can write down the general solution to (6.34). Consequently, the enhanced susy transformation of the fermionic connection is of the form:

$$\begin{aligned}\delta\mathcal{A}_{\alpha\,ij}{}^a{}_d = \epsilon^\beta\left(\varepsilon_{ijmn}\,\bar\partial^{m\,a}|X|\,\bar G^{\;\;n}_{\alpha\beta\,d} + \mathcal{P}^a{}_c\,H_{\alpha\beta\,ij}{}^c{}_d\right)&\\ +\frac{1}{2}\epsilon_{ijkl}\,\bar\epsilon^\beta\left(\varepsilon^{klmn}\partial_{m\,d}|X|G_{\alpha\beta\,n}^{\;\;a} + \mathcal{P}^c{}_d\,\bar H^{kl\;a}_{\alpha\beta\;\;c}\right)&\,,\end{aligned} \tag{6.35}$$

where following our general conventions $\bar G^{\;\;n}_{\alpha\beta\,d} = (G_{\alpha\beta\,n}^{\;\;d})^*$ and $H_{\alpha\beta\,ij}{}^c{}_d$ is a (composite) superfield in the $\bar{\mathbf{6}}$.

With this general solution to the algebraic part of the invariance condition (6.8) for the matter constraint one obtains the following constraint for the monopole operator superfield:

$$\begin{aligned}\nabla_{\alpha\,ij}\mathcal{M}^{ab}\,\bar\Psi^k_{\beta\,b}|_{\overline{\mathbf{20}}} = \Big(\varepsilon_{ijmn}\,\bar\partial^{m\,a}|X|\,\bar G^{\;\;n}_{\alpha\beta\,d}\Phi^{k\,d} + \mathcal{P}^a{}_c\,H_{\alpha\beta\,ij}{}^c{}_d\Phi^{k\,d}&\\ +\,\varepsilon_{\alpha\beta}(\mathcal{M}f)^{a,ce}{}_d\,\bar\Phi_{i\,c}\bar\Phi_{j\,e}\Phi^{k\,d}\Big)\,|_{\overline{\mathbf{20}}}&\,,\end{aligned} \tag{6.36}$$

which defines the super-derivative monopole operator field (though contracted with the fermionic superfield) up to a contribution in the $\bar{\mathbf{4}}$, in terms of the yet unconstrained composite superfields G and H. The unspecified $\bar{\mathbf{4}}$ contribution combines with all the other $\bar{\mathbf{4}}$ contributions which were not written explicitly to the enhanced susy variation of the fermionic superfield, see (4.1).

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A few comments are in order:

i.) Contrary to the previous case with a covariantly constant monopole operator now there is no algebraic condition on the monopole operator at this stage. One might think that the more general procedure presented here would also help for the case of a covariantly constant monopole operator. In this case, equation (6.36) imposes an algebraic condition on the superfields G, H and $\mathcal{M}$ which has no other obvious solution then the one considered in the previous subsection. In particular the invariance of the gauge constraint cause further problems.

ii.) The composite superfields G and H are not further specified yet. The requirement that the also the gauge field constraint (3.2) is invariant under the enhanced susy will determine also the super-derivatives of these fields, and thus in combination with (6.36) form a system of superspace constraints for the monopole operator superfield. This is what we consider next.

iii.) To obtain the super-derivative for the monopole operator without being contracted with the fermionic field one has to factor out a fermionic superfield in the whole equation. This can be done in an obvious way for the composite fields G and H but not for the monopole operator field itself without any further assumption/derivation of the composite structure of the monopole operator. We leave this question for a followup investigation, where we analyze the complete monopole operator constraint system, which we develop here.

Gauge field constraint. To complete the invariance of our original constraint system, (4.1), (3.2), the enhanced susy variations determined so far have to satisfy the invariance condition (6.13). In deriving the susy variation of the fermionic connection (6.35) we demonstrated explicitly the principle methods which are needed. The derivations of the following expressions, though more lengthy, work out in a similar way. All needed identities involving the above introduced mesonic operators are given in appendix C.

Due to the properties under complex conjugation it is sufficient to consider the part of the invariance condition (6.13) proportional to ϵ, the $\bar{\epsilon}$-part is then automatically satisfied. Writing the enhanced susy variation of the bosonic connection as

$$\delta\mathcal{A}_{\alpha\beta}{}^{a}{}_{b} := \epsilon^{\gamma} B_{\alpha\beta,\gamma}{}^{a}{}_{b} - \bar{\epsilon}^{\gamma} (B_{\alpha\beta,\gamma}{}^{b}{}_{a})^{*} \,, \tag{6.37}$$

so that it respects the reality condition (2.3) and $B_{\alpha\beta,\gamma} = B_{(\alpha\beta),\gamma}$, the ϵ-part of the invariance condition (6.13) writes as

$$\begin{aligned}\nabla_{\alpha\, ij}\delta_{\epsilon}\mathcal{A}_{\beta\, kl}{}^{a}{}_{b} + \nabla_{\beta\, kl}\delta_{\epsilon}\mathcal{A}_{\alpha\, ij}{}^{a}{}_{b} =& \\ i\,\epsilon^{\gamma}\Big\{\varepsilon_{ijkl} B_{\alpha\beta,\gamma}{}^{a}{}_{b} + \frac{1}{2}\epsilon_{\alpha\beta}(\mathcal{M}f)^{d,ca}{}_{d}\,&(\varepsilon_{m,ij[k}\bar{\Phi}_{l]c} - \varepsilon_{m,kl[i}\bar{\Phi}_{j]c})\,\bar{\Psi}^{m}_{\gamma\, d}\Big\}\,,\end{aligned} \tag{6.38}$$

where $\delta_\epsilon \mathcal{A}_{\alpha\, ij}{}^a{}_b$ is the ϵ-part in the enhanced susy variation given in (6.35). Explicitly, in terms of the composite superfields G and H one finds

$$\begin{aligned}
&\nabla_{\alpha\, ij}\delta_\epsilon \mathcal{A}_{\beta\, kl}{}^a{}_b + \nabla_{\beta\, kl}\delta_\epsilon \mathcal{A}_{\alpha\, ij}{}^a{}_b = \\
&\quad -\epsilon^\gamma \Big\{ \Big(\varepsilon_{ijmn} \bar\partial^{m\, a}|X| \nabla_{\beta\, kl} \bar G^n_{\alpha\gamma\, b} - i\, \varepsilon_{ijmn} \bar G^n_{\alpha\gamma\, b} \Psi^c_{\beta[k} \partial_{l]c} \bar\partial^{m\, a}|X| + i\, \nabla_{\alpha\, ij}(\mathcal{P}^a{}_c\, H_{\beta\gamma\, kl}{}^c{}_b) \\
&\qquad + \big(\begin{smallmatrix}\alpha\\ ij\end{smallmatrix}\big) \leftrightarrow \big(\begin{smallmatrix}\beta\\ kl\end{smallmatrix}\big) \Big) - \frac{i}{2}\, \varepsilon_{ijmn}\, \varepsilon_{klrs}\, (\bar G^n_{\alpha\gamma\, b} \bar\Psi^r_{\beta\, c} - \bar G^r_{\beta\gamma\, b} \bar\Psi^n_{\alpha\, c})\, \bar\partial^{s\, c} \bar\partial^{m\, a}|X| \Big\}\ . \qquad (6.39)
\end{aligned}$$

The invariance condition (6.38), with the l.h.s. given by (6.39), has to be satisfied so that also the gauge field constraint is invariant under the enhanced supersymmetry. This defines superspace constraints for the composite fields G and H which together with (6.36) form a constraint system for the monopole operator superfield $\mathcal{M}^{ab}$. We now extract the constraints for the composite superfields G and H.

Contracting (6.39) with a scalar superfield $\bar\Phi_{p\, a}$ one obtains an equation which contains only the super-derivative of the G-field, without any (non-invertible) field dependent pre-factor. To achieve this one uses $\bar\Phi_{p\, a} \bar\partial^{m\, a}|X| = \delta^m{}_p |X|$, see appendix C, and further, that with (6.33) and the matter constraint (4.1)

$$\bar\Phi_{p\, a} \nabla_{\alpha\, ij}(\mathcal{P}^a{}_c\, H_{\beta\gamma\, kl}{}^c{}_b) = -\frac{i}{2}\, \varepsilon_{ijpq} \bar\Psi^q_{\alpha\, a} (\mathcal{P}^a{}_c\, H_{\beta\gamma\, kl}{}^c{}_b)\ . \qquad (6.40)$$

The resulting equation has a unique solution for the super-derivative of the G-field, which is given by

$$\begin{aligned}
\nabla_{\alpha\, ij} \bar G^n_{\beta\gamma\, b} = \frac{1}{|X|} \Big\{ & 2\, i\, \delta^n{}_{[i} \bar\Phi_{j]a}\, B_{\alpha\beta,\gamma}{}^a{}_b + \frac{i}{2} \varepsilon_{\alpha\beta} (\mathcal{M} f)^{d,ce}{}_b\, (\delta^n{}_{[i} \bar\Phi_{j]c} \bar\Phi_{m\, e} \bar\Psi^m_{\gamma\, d} + \bar\Phi_{ic} \bar\Phi_{je} \bar\Psi^n_{\gamma\, d}) \\
& + i\, \bar G^n_{\beta\gamma\, b} \Psi^c_{\alpha[i}\, \partial_{j]c}|X| + \frac{i}{2}\, \bar\Psi^n_{\beta\, c}\, \mathcal{P}^c{}_a H_{\alpha\gamma\, ij}{}^a{}_b \\
& + \frac{i}{2}\, \varepsilon_{ijrs}\, (\bar G^n_{\beta\gamma\, b} \bar\Psi^r_{\alpha\, c} - \bar G^r_{\alpha\gamma\, b} \bar\Psi^n_{\beta\, c})\, \bar\partial^{s\, c}|X| \Big\}\ . \qquad (6.41)
\end{aligned}$$

Inserting this solution back into the original invariance condition (6.38) gives a superspace constraint equation for the H-field:

$$\begin{aligned}
&\mathcal{P}^e{}_a \nabla_{\alpha\, ij} H_{\beta\gamma\, kl}{}^a{}_b |_{(\mathbf{3},\mathbf{1}\oplus\mathbf{20})\oplus(\mathbf{1},\mathbf{15})} = \\
&\quad \mathcal{P}^e{}_a \Big\{ i\, \varepsilon_{ijmn}\, \bar G^n_{\alpha\gamma\, b} \Psi^c_{\beta[k}\, \partial_{l]c}\, \bar\partial^{m\, a}|X| - \frac{1}{|X|}\, \Psi^a_{\alpha[i}\, \partial_{j]c}|X|\, H_{\beta\gamma\, kl}{}^c{}_b \\
&\qquad -\frac{i}{2} \Big(\varepsilon_{ijkl}\, B_{\alpha\beta,\gamma}{}^a{}_b + \varepsilon_{\alpha\beta} (\mathcal{M} f)^{d,ca}{}_b\, \varepsilon_{mij[k} \bar\Phi_{l]c} \bar\Psi^m_{\gamma\, d} \Big) \Big\} |_{(\mathbf{3},\mathbf{1}\oplus\mathbf{20})\oplus(\mathbf{1},\mathbf{15})}, \qquad (6.42)
\end{aligned}$$

which fixes the $(\mathbf{3}, \mathbf{1} \oplus \mathbf{20}) \oplus (\mathbf{1}, \mathbf{15})$ content of the super-derivative of the H-field, but only the projection onto the eigenspace of the projector $\mathcal{P}^e{}_a$ defined in (6.32).

The constraint equations for the G- and H- field (6.41), (6.42) together with the constraint equation for the monopole operator $\mathcal{M}$ (6.36) define the constraint system for the monopole operator superfield. This constraint system was derived starting from a single assumption, namely the transformation (6.6), and the requirement that this transformation is part of the enhanced supersymmetry implemented as an internal fermionic ($\mathcal{N} = 2$ super) symmetry of our matter-gauge constraint system (4.1), (3.2).

At this stage we did not encounter any algebraic condition on the monopole operator field $\mathcal{M}$, as it is the case for a covariantly constant monopole operator and which allowed only for the gauge group $\mathrm{U}(2) \times \mathrm{U}(2)$. Actually, we did not encounter any condition on the gauge group yet, so in principle all gauge groups of the classification below (5.8) are allowed. We expect that a further study of the monopole constraint system will select the gauge group to be $\mathrm{U}(N) \times \mathrm{U}(N)$. The allowed CS-levels are in principle given by the general analysis following (6.4). To see this condition explicitly one might have to consider the operators, at least in principle, as quantum fields, as for example in the case for the quantization condition of the CS-level itself. See also the discussion below (6.27). In this regard we just want to mention that in the equations of the constraint system for the monopole operator superfield, i.e. (6.41), (6.42) and (6.36), an (abelian)[20] covariantly constant operator can be factored out.
A number of things remains to be done. First one has to analyze the monopole superfield constraint system (6.41), (6.42), (6.36) analogously to the procedure applied to the matter-gauge constraint system as done in sections 4 and 5 and as it was presented in greater detail in [9] for the $\mathcal{N} = 8$ case. One has to see if it will be necessary to strip off the fermionic field in equ. (6.36) and if so, if the analysis of the constraint system provides enough information about the composite structure of the monopole operator field $\mathcal{M}$ to do so or if further assumptions are necessary.

Second, we did not write down explicitly the enhanced susy transformation for the fermionic superfield, which is given by all the $\bar{\mathbf{4}}$ contributions in the invariance condition of the matter constraint (6.7). Given the transformation rule for the fermionic superfield one can check the algebra of the enhanced symmetry transformations which might give additional information to specify the monopole operator superfield further.

We leave these points for a follow up investigation to the structure developed here. Eventually, a detailed analysis of the here developed constraint system will lead also to space-time e.o.m. for the composite monopole superfield, as we obtained the superfield e.o.m (5.12) from the matter- and gauge constraint (4.1), (3.2). These space-time e.o.m. for the composite monopole superfield might then describe the dynamics of a theory dual to the ABJM model, in the sense of the three-dimensional mirror symmetry [15]. This would finally be a nonabelian gauge theory analogon of the explicit duality relations for two-dimensional soliton models [16, 17]. We will address these issues in future work.

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[20] Or a covariantly constant operator living in the eigenspace of the projector $\mathcal{P}$

A SO(6) ∼ SU(4) conventions

SO(6) Γ-matrices. Dirac spinors of SO(6) have eight components. The irreducible representations are given by four component Weyl spinors (though SO(6) provides also Majorana spinors). The Weyl representation of the hermitian SO(6) Γ-matrices is of the form

$$\hat{\Gamma}^I = \begin{bmatrix} 0 & \Gamma^I \\ \bar{\Gamma}^I & 0 \end{bmatrix} , \tag{A.1}$$

where hermiticity implies $\bar{\Gamma}^I = (\Gamma^I)^\dagger$ and these matrices satisfy $\Gamma^{(I}\bar{\Gamma}^{J)} = \delta^{IJ}\mathbb{1}$. We denote the components of these matrices by

$$\Gamma^{Iij} \quad , \quad \bar{\Gamma}^I_{ij} \quad \text{with} \quad i,j = 1,\ldots,4 \,. \tag{A.2}$$

A particular representation which makes the relation to SU(4) transparent is given by [59]

$$\begin{aligned} &\Gamma^1 = \sigma_2 \otimes \mathbb{1}_2 && \Gamma^2 = -i\sigma_2 \otimes \sigma_3 && \Gamma^3 = i\sigma_2 \otimes \sigma_1 \\ &\Gamma^4 = -\sigma_1 \otimes \sigma_2 && \Gamma^5 = \sigma_3 \otimes \sigma_2 && \Gamma^6 = -i\mathbb{1}_2 \otimes \sigma_2 \end{aligned} \tag{A.3}$$

These matrices are anti-symmetric, i.e. $\Gamma^{Iij} = -\Gamma^{Iji}$ and satisfy the reality condition

$$(\Gamma^{Iij})^* = \frac{1}{2}\epsilon_{ijkl}\Gamma^{Ikl} = -\bar{\Gamma}^I_{ij} \,, \tag{A.4}$$

further, the contractions satisfy

$$\Gamma^{ij}_I \bar{\Gamma}^I_{kl} = -4\delta^{ij}_{kl} \quad , \quad \Gamma^{ij}_I \Gamma^{Ikl} = 2\epsilon^{ijkl} \,, \tag{A.5}$$

where $\delta^{i_1\ldots i_n}_{k_1\ldots k_n} = \delta^{[i_1}_{k_1} \ldots \delta^{i_n]}_{k_n}$ and the totally antisymmetric epsilon symbol is defined as $\epsilon^{1234} = \epsilon_{1234} = 1$ so that

$$\epsilon^{i_1\ldots i_n j_1\ldots j_{D-n}} \epsilon_{i_1\ldots i_n k_1\ldots k_{D-n}} = (D-n)!\; n!\; \delta^{j_1\ldots j_{D-n}}_{k_1\ldots k_{D-n}} \,, \tag{A.6}$$

with $D = 4$ in this case. This is all that wee need of Γ-matrix relations to show how SO(6) representations are related to SU(4) representations. The main text is formulated in terms of SU(4) representations thus avoiding any explicit Γ-matrix relations.

SU(4) representations. The SO(6) generators write with (A.1) as

$$\hat{\Sigma}^{IJ} := \frac{1}{2}[\hat{\Gamma}^I, \hat{\Gamma}^J] = \begin{bmatrix} \Gamma^{IJ} & 0 \\ 0 & \bar{\Gamma}^{IJ} \end{bmatrix} , \tag{A.7}$$

where the irreducible 4 times 4 blocks

$$\Gamma^{IJi}{}_j := \Gamma^{[Iik}\bar{\Gamma}^{J]}_{kj} \quad , \quad \bar{\Gamma}^{IJ}{}_i{}^j := \bar{\Gamma}^{[I}_{ik}\Gamma^{J]kj} , \tag{A.8}$$

satisfy $\operatorname{tr}\Gamma^{IJ} = 0$ and $(\Gamma^{IJ})^\dagger = -\Gamma^{IJ} = \Gamma^{JI}$, with the same relations for $\bar{\Gamma}^{IJ}$. These matrices are therefore the 15 anti-hermitian generators of SU(4) and SU($\bar{4}$) transformations, respectively (the SU(4) algebra is implied by the SO(6) algebra). The above given relations for Γ^I and $\bar{\Gamma}^I$ imply the conjugation properties

$$(\Gamma^{IJi}{}_j)^* = \bar{\Gamma}^{IJ}{}_i{}^j = -\Gamma^{IJj}{}_i \,. \tag{A.9}$$

We give here the notation for some SU(4) representations which occur frequently in the main text but without taking the gauge group structure into account. Modifications for fields in the adjoint representation of the gauge group are given at the end of appendix B.

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Representations 4 and $\bar{\mathbf{4}}$: according to the conventions as given in (A.8) a field in the $\mathbf{4}$ carries an upper index, e.g. Φ^i, and a field in the $\bar{\mathbf{4}}$ carries a lower index, e.g. Ψ_i. Complex conjugation maps one representation into the other, see (A.9), which we denote by

$$(\Phi^i)^* =: \bar{\Phi}_i \quad , \quad (\Psi_i)^* =: \bar{\Psi}^i \ . \tag{A.10}$$

Representations 6, $\bar{\mathbf{6}}$ and their real form: the $\mathbf{6}$ appears in the product $\mathbf{4} \otimes \mathbf{4}$ (the $\bar{\mathbf{6}}$ in the complex conjugated thereof) and it is therefore natural to use the conventions

$$\mathbf{6} : v^{ij} \quad \text{with} \quad v^{ij} = v^{[ij]} \quad , \quad \bar{\mathbf{6}} : u_{ij} \quad \text{with} \quad u_{ij} = u_{[ij]} \ . \tag{A.11}$$

These are 6 dimensional complex representations. Complex conjugation relates this representation to each other and thus we have the convention $(v^{ij})^* =: \bar{v}_{ij}$ and accordingly for u_{ij}. SO(6) has a real representation $\mathbf{6}$ which translates into a real $\mathbf{6}$ or $\bar{\mathbf{6}}$ of SU(4) by

$$v^{ij} := \frac{1}{2}\Gamma^{I\ ij} v_I \quad \text{or} \quad v_{ij} := -\frac{1}{2}\bar{\Gamma}^{I}{}_{ij} v_I = \frac{1}{2}\epsilon_{ijkl} v^{kl} \,, \tag{A.12}$$

where we have used (A.4). This implies the reality condition

$$\bar{v}_{ij} := (v^{ij})^* = \frac{1}{2}\epsilon_{ijkl} v^{kl} = v_{ij} \ . \tag{A.13}$$

Since ϵ_{ijkl} is an invariant SU(4) tensor this reality condition is invariant. In fact it is the natural condition to impose on representations $\mathbf{6}$ of SU(4). Due to the identification (A.12), (A.13) of the real forms of the $\mathbf{6}$ and $\bar{\mathbf{6}}$ it is unnecessary to differentiate between them and we will generally speak if the real $\mathbf{6}$ in either case.

Representations 15, $\overline{\mathbf{15}}$ and their real form: the $\mathbf{15}$ appears in the product $\mathbf{4}\otimes\bar{\mathbf{4}}$ (the $\overline{\mathbf{15}}$ in the complex conjugated thereof) and it is therefore natural to use the conventions

$$\mathbf{15} : W^i{}_j \quad \text{with} \quad W^i{}_i = 0 \quad , \quad \overline{\mathbf{15}} : U_i{}^j \quad \text{with} \quad U_i{}^i = 0 \ . \tag{A.14}$$

These are $4 \times 4 - 1 = 15$ dimensional complex representations. Complex conjugation we denote according to (A.9) by $(W^i{}_j)^* =: \bar{W}_i{}^j$ and similar for $U_i{}^j$. SO(6) has also a representation $\mathbf{15}$ which is real, $W_{IJ} = W_{[IJ]}$. This is translated into a real $\mathbf{15}$ or $\overline{\mathbf{15}}$ of SU(4) by

$$W^i{}_j := -\frac{1}{2}\Gamma^{IJi}{}_j W_{IJ} \quad \text{or} \quad W_i{}^j = -\frac{1}{2}\bar{\Gamma}^{IJ}{}_i{}^j W_{IJ}. \tag{A.15}$$

This then implies with (A.9) the reality condition

$$\overline{W}_i{}^j := (W^i{}_j)^* = -W^j{}_i = W_i{}^j \ . \tag{A.16}$$

It is easy to see that this reality condition is invariant under SU(4) transformations. Again there is no reason to differentiate between this two real forms and we will express everything in terms of the real form of the $\mathbf{15}$.

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B Superspace and connections

The $\mathcal{N}=6$ susy algebra without (non)-central charges is given by six 3D-Majorana spinors transforming in the **6** of SO(6) and satisfying the algebra[21]

$$\{Q^I_\alpha, Q^J_\beta\} = 2\delta^{IJ}P_{\alpha\beta} \ . \tag{B.1}$$

Accordingly, the susy parameters $\epsilon^{I\alpha}$ and the superspace coordinates $\theta^{I\alpha}$ are also given by 3D-Majorana spinors in the **6** ($I=1,\dots,6$). Our 3D conventions are such that Majorana spinors are real (see appendix D). To represent the the algebra (B.1) and to form susy covariant expressions one introduces the following differential operator,

$$D_{\alpha\, I} := \partial_{\alpha\, I} + i\theta^\beta_I\partial_{\alpha\beta} \quad , \quad Q_{\alpha\, I} := \partial_{\alpha\, I} - i\theta^\beta_I\partial_{\alpha\beta} \, , \tag{B.2}$$

such that $\{D_{\alpha\, I}, Q_{\beta\, J}\}=0$ and

$$\{Q_{\alpha\, I}, Q_{\beta\, J}\} = -\{D_{\alpha\, I}, D_{\beta\, J}\} = -2i\delta_{IJ}\partial_{\alpha\beta} \, , \tag{B.3}$$

where $\partial_{\alpha I}\theta^{\beta J} = \delta^\alpha_\beta\delta^J_I$. These operators are hermitian in the Hilbert space of superfields and have the following definite properties under complex conjugation (therefore the i in the definition):

$$(D_{\alpha\, I}X)^* = (-)^{|X|+1}D_{\alpha\, I}\bar{X} \quad , \quad (\partial_{\alpha\beta})^* = \partial_{\alpha\beta} \, , \tag{B.4}$$

where $|X|$ is the fermion parity of a superfield X ($=0,1$ for bosonic/fermionic).

We now write the **6** of SO(6) in these definitions as real **6** of SU(4) according to (A.12):

$$\theta^{\alpha\, ij} = \frac{1}{2}\Gamma^{ij}_I\theta^{\alpha I} \quad , \quad D_{\alpha\, ij} = -\frac{1}{2}\bar{\Gamma}^I_{ij}D_{\alpha I} \ . \tag{B.5}$$

With this definitions $\theta^{\alpha\, ij}$ satisfies the conjugation property (A.13). The relations (A.9) and (B.4) imply for the superderivative

$$(D_{\alpha\, ij}X)^* = (-)^{|X|+1}\,\frac{1}{2}\epsilon^{ijkl}D_{\alpha\, kl}\bar{X} =: (-)^{|X|+1}D^{ij}_\alpha\bar{X} \ . \tag{B.6}$$

For the fermionic derivative one has now $\partial_{\alpha\, ij}\theta^\beta_{kl} = \frac{1}{2}\epsilon_{ijkl}\delta^\beta_\alpha$ and the anti-commutator (B.3) writes as

$$\{D_{\alpha\, ij}, D_{\beta\, kl}\} = -\{Q_{\alpha\, ij}, Q_{\beta\, kl}\} = i\epsilon_{ijkl}\partial_{\alpha\beta} \ . \tag{B.7}$$

Covariant derivatives. We introduce connections on superspace in the following way

$$\begin{aligned} \nabla_{\alpha\beta} &:= \partial_{\alpha\beta} + \mathcal{A}_{\alpha\beta} && \text{with} \quad \mathcal{A}_{\alpha\beta} = \mathcal{A}^M_{\alpha\beta}T_M \\ \nabla_{\alpha\, I} &:= D_{\alpha\, I} + \mathcal{A}_{\alpha\, I} && \text{with} \quad \mathcal{A}_{\alpha\, I} = \mathcal{A}^M_{\alpha\, I}iT_M \, , \end{aligned} \tag{B.8}$$

with anti-hermitian generators $(T_M)^\dagger = -T_M$. With these definitions and taking the coefficient fields $\mathcal{A}^M_{\alpha\beta}$, $\mathcal{A}^M_{\alpha\, I}$ to be real superfields, the connection one-forms are (anti)hermitian, i.e. $(\mathcal{A}_{\alpha\beta})^\dagger = -\mathcal{A}_{\alpha\beta}$ and $(\mathcal{A}_{\alpha\, I})^\dagger = \mathcal{A}_{\alpha\, I}$ and so are the resulting field strengths $(\mathcal{F}_{\alpha\beta,\gamma\delta})^\dagger =$

[21] SO(6) vector indices $I,\dots$ are raised and lowered with the Kronecker delta and therefore one does not have to pay any attention to their position.

$-\mathcal{F}_{\alpha\beta,\gamma\delta}$ and $(\mathcal{F}_{\alpha\,I,\beta\,J})^\dagger = \mathcal{F}_{\alpha\,I,\beta\,J}$. The **6** of SO(6) we again write as real **6** or $\bar{\mathbf{6}}$ of SU(4) according to (A.12) as

$$\nabla_{\alpha\,ij} := -\frac{1}{2}\bar{\Gamma}^I_{ij}\nabla_{\alpha\,I} \quad , \quad \nabla^{ij}_\alpha := \frac{1}{2}\Gamma^{I\,ij}\nabla_{\alpha\,I} \,. \tag{B.9}$$

For the action of the generators of the gauge group in a representation R and $\bar{R}$, respectively, we introduce the following convention:

$$T_M \cdot X^a := T_M{}^a{}_b X^b \quad , \quad T_M \cdot Y_b := -T_M{}^a{}_b Y_a \,, \tag{B.10}$$

where we denote a field in the representation $R/\bar{R}$ with an upper/lower index from the range $a, b, \ldots$. The action on $\bar{R}$ follows from the action on R by complex conjugation where $(X^a)^* =: \bar{X}_a$. The action on tensor products of these representations is defined accordingly. The covariant derivatives of a field in the R and its complex conjugate thus write as (suppressing space-time and SU(4) indices)

$$\begin{aligned} \nabla\Phi^a &= D\Phi^a + \mathcal{A}\cdot\Phi^a = D\Phi^a + \mathcal{A}^a{}_b\Phi^b \,, \\ \nabla\bar{\Phi}_b &= D\bar{\Phi}_b + \mathcal{A}\cdot\bar{\Phi}_b = D\bar{\Phi}_b - \mathcal{A}^a{}_b\bar{\Phi}_a \,, \end{aligned} \tag{B.11}$$

where D stands either for $D_{\alpha\,ij}$ or $\partial_{\alpha\beta}$.

With the definitions (B.9) we thus obtain the following properties under complex conjugation:

$$(\nabla_{\alpha\beta}X^a)^* = \nabla_{\alpha\beta}\bar{X}_a = \partial_{\alpha\beta}\bar{X}_a - \mathcal{A}_{\alpha\beta}{}^b{}_a\bar{X}_b \tag{B.12}$$

$$(\nabla_{\alpha\,ij}X^a)^* = (-)^{|X|+1}\frac{1}{2}\,\epsilon^{ijkl}\nabla_{\alpha\,kl}\bar{X}_a = (-)^{|X|+1}\nabla^{ij}_\alpha\bar{X}_a \,. \tag{B.13}$$

The fermionic field strength in SU(4) notation has therefore the conjugation property $(\mathcal{F}_{\alpha\beta,\gamma\,ij})^\dagger = \frac{1}{2}\epsilon^{ijkl}\mathcal{F}_{\alpha\beta,\gamma\,kl}$.

Gauge field constraint. Analogous to the $\mathcal{N}=8$ case [9] we impose the constraint

$$\{\nabla_{\alpha\,I}, \nabla_{\beta\,J}\} = 2i(\delta_{IJ}\nabla_{\alpha\beta} + \varepsilon_{\alpha\beta}W_{IJ}) \,. \tag{B.14}$$

Translating this into SU(4) representations we multiply this equation with $(-\frac{1}{2}\bar{\Gamma}^I_{ij})(-\frac{1}{2}\bar{\Gamma}^J_{kl})$. Using above relations one has

$$\bar{\Gamma}^I_{ij}\bar{\Gamma}^J_{kl}W_{IJ} = 2\epsilon_{mij[k}W^m{}_{l]} = -2\epsilon_{mkl[i}W^m{}_{j]} \,, \tag{B.15}$$

where we chose the real **15** of SU(4), a similar relation holds for the real $\overline{\mathbf{15}}$. The constraint (B.14) thus writes as

$$\{\nabla_{\alpha\,ij}, \nabla_{\beta\,kl}\} = i(\epsilon_{ijkl}\nabla_{\alpha\beta} + \varepsilon_{\alpha\beta}\epsilon_{mij[k}W^m{}_{l]}) \tag{B.16}$$

Fields in the gauge sector like $W^i{}_j$ and $\lambda_{\alpha\,ij}$, $\rho_{\alpha\,ij}$ etc. (see the main text) live in the Lie algebra $\mathfrak{g}$. We expand bosonic superfields like $W^i{}_j$ as $W^i{}_j = W^{Mi}{}_j T_M$ and fermionic ones with an extra i, i.e. as $\lambda_{\alpha\,ij} = \lambda^M_{\alpha\,ij} i T_M$ etc., with anti-hermitian generators $(T_M)^\dagger = -T_M$.

This means in replacing complex conjugation by hermitian conjugation the conditions due to SU(4) representations as given in appendix A are unchanged for fermionic fields and receive an extra minus for bosonic fields. For example

$$(W^i{}_j)^\dagger = W^j{}_i \quad \text{and} \quad (\lambda_{\alpha\, ij})^\dagger = \frac{1}{2}\epsilon^{ijkl}\lambda_{\alpha\, kl}\,, \tag{B.17}$$

for a bosonic field in the real **15** and a fermionic field in the real **6**. For conformal theories the potential $W^i{}_j$ is explicitly given in (5.4). For this potential the derived composite fields in the gauge sector, see section 3, are obtained according to (5.10) as follows:

$$\begin{aligned}
(\lambda_{\alpha\, ij})^a{}_b &= i f^{ac}_{\ bd}\left(\Psi^d_{\alpha\,[i}\bar\Phi_{j]\,c} + \frac{1}{2}\epsilon_{ijkl}\Phi^{k\,d}\bar\Psi^l_{\alpha\, c}\right)\,, \quad (\rho_{\alpha\, ij})^a{}_b = i f^{ac}_{\ bd}\;\bar\Phi_{c\,(i}\Psi^d_{j)\,\alpha}\,,\\
(V^i{}_j)^a{}_b &= \frac{i}{4}\varepsilon^{\alpha\beta} f^{ac}_{\ bd}\left(\Psi^d_{\alpha\, j}\bar\Psi^i_{\beta\, c} - \frac{1}{4}\delta^i{}_j(\Psi_\alpha\bar\Psi_\beta)^d{}_c\right) - \frac{1}{4} f^{gc}_{\ hd} f^{da}_{\ eb}\left(\Phi^{i\,h}\bar\Phi_{j\,g}(\Phi\bar\Phi)^e{}_c\right)\\
&\quad - \frac{1}{4} f^{ac}_{\ bd} f^{dg}_{\ eh}\left(\Phi^{i\,h}\bar\Phi_{j\,g}(\Phi\bar\Phi)^e{}_c - \frac{1}{2}\delta^i{}_j(\Phi\bar\Phi)^h{}_g(\Phi\bar\Phi)^e{}_c\right)\,,
\end{aligned} \tag{B.18}$$

where $(\Phi\bar\Phi)^d{}_c = \Phi^{d\,k}\bar\Phi_{k\,c}$ etc. is short for contracted SU(4) indices.

C Mesonic operators

We develop and collect here in some detail the structure of mesonic operators, in particular the SU(4) determinant thereof, which were introduced in the main text in (6.30). We define the mesonic operators as the gauge invariant quantities

$$X^i{}_j := \Phi^{i\,a}\bar\Phi_{j\,a} \quad \Rightarrow \quad (X^i{}_j)^* = X^j{}_i\,, \tag{C.1}$$

which transform in the $(\mathbf{4},\bar{\mathbf{4}})$ under SU(4). With this definition the determinant of the mesonic operators can be written as

$$\det(X^i{}_j) =: |X| = \frac{1}{24}\,\varepsilon_{ijkl}(\Phi^{i\,a}\Phi^{j\,b}\Phi^{k\,c}\Phi^{l\,d})\,\varepsilon^{mnpq}(\bar\Phi_{m\,a}\bar\Phi_{n\,b}\bar\Phi_{p\,c}\bar\Phi_{q\,d})\,, \tag{C.2}$$

which is a real SU(4)-invariant. This way of writing the determinant is very useful in the derivation the following identities, since it heavily uses the completeness of SU(4) indices, i.e. total anti-symmetrization in five/four indices gives zero/the epsilon tensor. In the main text we introduced already the hermitian projection operators $\mathcal{P}$ and $\mathcal{F}$ and associated identities (6.32), (6.33). Other useful relations are:

Basic relations.

$$\begin{aligned}
\Phi^{i\,a}\partial_{a,j}|X| &= \delta^i{}_j|X|\,,\\
\partial_{[i|a}|X|\,\partial_{j]b}|X| &= \frac{|X|}{2}\,\partial_{i\,a}\partial_{j\,b}|X|\,,\\
\frac{1}{|X|}\,\partial_{i\,a}\bar\partial^{j\,b}|X| &= \mathcal{F}^a{}_c\,\partial_{i\,a}\bar\partial^{j\,c}|X| = \mathcal{F}^c{}_b\,\partial_{i\,c}\bar\partial^{j\,a}|X|\,,
\end{aligned} \tag{C.3}$$

where $\partial_{i\,a} := \frac{\partial}{\partial\Phi^{i\,a}}$ and $\bar\partial^{i\,a} := \frac{\partial}{\partial\bar\Phi_{i\,a}} = (\partial_{i\,a})^*$. By complex conjugation of these relations one obtains a similar set of identities.

Super-derivatives.

$$\begin{aligned}\nabla_{\alpha\, ij}\, \partial_{k\, a}|X| &= \partial_{k\, a}\nabla_{\alpha\, ij}|X| \\ \nabla_{\alpha\, ij}|X| &= -i\left(\Psi^c_{\alpha[i}\,\partial_{j]c}|X| + \frac{1}{2}\varepsilon_{ijkl}\,\bar\Psi^k_{\alpha\, c}\,\bar\partial^{l\, c}\bar\partial^{k\, a}|X|\right) \\ \nabla_{\alpha\, ij}\mathcal{P}^a{}_b &= -\nabla_{\alpha\, ij}\mathcal{F}^a{}_b \\ &= \frac{i}{|X|}\left(\mathcal{P}^a{}_c\Psi^c_{\alpha[i}\,\partial_{j]b}|X| + \frac{1}{2}\,\varepsilon_{ijkl}\bar\Psi^k_{\alpha\, c}\mathcal{P}^c{}_b\bar\partial^{l\, a}|X|\right).\end{aligned} \tag{C.4}$$

These are basically the relations used to derive the expressions given in the main text.

D SO(2,1) spinor conventions

All spinors appearing in the main text, superspace coordinates or fields, are Majorana spinors in $2+1$-dimensional space-time. Our metric convention is $\eta_{\mu\nu} = (-,+,+)$ and we choose a Majorana representation for the gamma-matrices[22]

$$\{\gamma^\mu, \gamma^\nu\}^\alpha{}_\beta = 2\eta^{\mu\nu}\delta^\alpha{}_\beta \quad . \tag{D.1}$$

Thus the matrices $\gamma^{\mu\ \alpha}{}_\beta$ are real and the Majorana condition on spinors imply that they are real two component spinors. Spinor indices are raised/lowered by the epsilon symbols with $\varepsilon^{12} = \varepsilon_{12} = 1$ and choosing NW-SE conventions

$$\varepsilon^{\alpha\gamma}\varepsilon_{\beta\gamma} = \delta^\alpha_{\ \beta}\ , \quad \lambda^\alpha := \varepsilon^{\alpha\beta}\lambda_\beta \Leftrightarrow \lambda_\beta = \lambda^\alpha\varepsilon_{\alpha\beta}\ . \tag{D.2}$$

Introducing the real symmetric matrices $\sigma^\mu_{\alpha\beta} := \gamma^{\mu\ \rho}{}_\beta\ \varepsilon_{\rho\alpha}$ and $\bar\sigma^{\mu\ \alpha\beta} := \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\sigma^\mu_{\gamma\delta} = \varepsilon^{\beta\rho}\ \gamma^{\mu\ \alpha}{}_\rho$ a three vector in spinor notation writes as a symmetric real matrix as

$$v_{\alpha\beta} := \sigma^\mu_{\alpha\beta}\ v_\mu \quad\Rightarrow\quad v^\mu = -\frac{1}{2}\bar\sigma^{\mu\ \alpha\beta}\ v_{\alpha\beta}\ , \quad v^{\alpha\beta}w_{\alpha\beta} = -2\ v^\mu w_\mu\ . \tag{D.3}$$

Another useful formula is

$$\varepsilon^{\mu\nu\lambda}A_\mu B_\nu C_\lambda = \frac{1}{2}\ \varepsilon^{\alpha\beta}A^{\gamma\delta}B_{\alpha\gamma}C_{\delta\beta}\ . \tag{D.4}$$

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