$\mathcal{P}$-kernel normal systems for $\mathcal{P}$-inversive semigroups

Hengwu Zheng

Abstract As a generalization of Preston's kernel normal systems, $\mathcal{P}$-kernel normal systems for $\mathcal{P}$-inversive semigroups are introduced, and strongly regular $\mathcal{P}$-congruences on $\mathcal{P}$-inversive semigroups in terms of their $\mathcal{P}$-kernel normal systems are characterized. These results generalize the corresponding results for $\mathcal{P}$-regular semigroups and $\mathcal{P}$-inversive semigroups.

Keywords E-inversive semigroup · $\mathcal{P}$-inversive semigroup · Strongly regular $\mathcal{P}$-congruence · $\mathcal{P}$-kernel normal system

1 Introduction

For standard terminology and notation in semigroup theory see Howie [11]. As usual, $E(S)$ is the set of idempotents of a semigroup S, $V(a) = \{a' \in S : aa'a = a, a'aa' = a'\}$ is the set of all *inverses* of $a \in S$.

A semigroup S is *E-inversive* if for every $a \in S$ there exists $x \in S$ such that ax is idempotent. This concept was introduced by Thierrin [25]. A semigroup is an *E-semigroup* if its idempotents form a subsemigroup. Basic properties of E-inversive semigroups were given by Blyth and Almeida Santos [1], Catino and Miccoli [2], Hayes [9, 10], Mitsch and Petrich [18] and Mitsch [15–17]. From Lemma 3.1 in [15], a semigroup S is E-inversive if and only if

$$(\forall a \in S) \quad W(a) = \{a' \in S : a' = a'aa'\} \neq \emptyset.$$

The elements of $W(a)$ are called *weak inverses* of a.

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Preston [21] introduced kernel normal systems to characterize congruences on inverse semigroups. Meakin [14] gave a generalization of kernel normal systems for orthodox semigroups. Imaoka [12] obtained a generalization of Preston's kernel normal systems for regular *-semigroups. A semigroup S is a *regular *-semigroup* if S with a unary operation * satisfies $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $aa^*a = a$ for any $a, b \in S$ (see [12, 19]). We have seen that orthodox semigroups and regular *-semigroups are within the class of $\mathcal{P}$-regular semigroups. A $\mathcal{P}$-*regular semigroup* $S(P)$ is a regular semigroup S whose set of idempotents $E(S)$ includes a subset P with properties $P^2 \subseteq E(S)$, $pPp \subseteq P$ for all $p \in P$, and $\{a' \in V(a) : aP^1a' \cup a'P^1a \subseteq P\} \neq \emptyset$ for all $a \in S$, where $P^1 = P \cup \{1\}$ (see [28, 29]). Sen [23] gave a characterization of $\mathcal{P}$-congruences (the terminology a $\mathcal{P}$-*congruence* for a usual congruence on a $\mathcal{P}$-regular semigroup [28]) on $\mathcal{P}$-regular semigroups in terms of their $\mathcal{P}$-kernel normal systems.

The existence of the least group congruence on an E-inversive semigroup was noted by Hall and Munn [8]. An explicit description of this congruence was given by Mitsch [15]. The least group congruence on an E-inversive E-semigroup was characterized by Reither [22]. Further characterizations of group congruences on an E-inversive semigroup has been obtained by the author [30]. An alternative description of this congruence on an E-inversive E-semigroup was given by Weipoltshammer [27]. Blyth and Almeida Santos [1] gave an alternative description of group congruences on an E-inversive semigroup.

Weipoltshammer [27] described some congruences on E-inversive E-semigroups. References [4, 5] and [3] refer to work by Fan and Chen in which $\mathcal{P}$-inversive semigroups where first studied (using the so called $\mathcal{P}$-kernel normal systems) and the regular $\mathcal{P}$-congruences on $\mathcal{P}$-inversive semigroups. A $\mathcal{P}$-*inversive semigroup* is an E-inversive semigroup whose set of idempotents includes a subset P with properties that make it suitable in a "kernel normal systems" that generalizes the systems of Preston [21] and Meakin [14] in their respective congruence theories for inverse and orthodox semigroups.

Definition 1.1 (see [3–5]) An E-inversive semigroup S is called a $\mathcal{P}$-inversive semigroup, if there exists a nonempty subset P of $E(S)$ such that

(1) $P^2 \subseteq E(S)$;
(2) $(\forall p \in P)\ pPp \subseteq P$;
(3) $(\forall a \in S)\ W_P(a) = \{a' \in W(a) : aP^1a' \cup a'P^1a \subseteq P\} \neq \emptyset$, where $P^1 = P \cup \{1\}$.

The subset P of $E(S)$ satisfying (1)–(3) above is called a *characteristic set (C-set,* for short) of S, each element in $W_P(a)$ is called a *weak $\mathcal{P}$-inverse* of a. Clearly, for any $p \in P$, we have $p \in W_P(p)$. Because of the central role of P in S we denote a $\mathcal{P}$-inversive semigroup S with the C-set P by $S(P)$. Throughout this paper, $S(P)$ is always an arbitrary $\mathcal{P}$-inversive semigroup.

From Definition 1.1 it follows that all $\mathcal{P}$-regular semigroups and E-inversive E-semigroups are within the class of $\mathcal{P}$-inversive semigroups. We have seen that there exist $\mathcal{P}$-inversive semigroups which are neither $\mathcal{P}$-regular nor E-inversive E-semigroups (see [4] and [7]).

A *regular $\mathcal{P}$-congruence* ρ on $S(P)$ is a congruence ρ with property $a\rho aa'a$ for all $a \in S(P)$ and all $a' \in W_P(a)$ in the sense of [3].

In [7], the strong $\mathcal{P}$-congruences and some sublattices of the strong $\mathcal{P}$-congruence lattice on a $\mathcal{P}$-inversive semigroup were studied. A *strong $\mathcal{P}$-congruence* ρ on $S(P)$ is a congruence ρ with properties for all $a, b \in S(P)$, $a\rho b$ implies $a'\rho b'$ for all $a' \in W_P(a), b' \in W_P(b)$, and $a\rho aa'a$ for all $a \in S(P)$ and all $a' \in W_P(a)$ in the sense of [4]. Siripitukdet and Sattayaporn [24] characterized the maximum idempotent separating congruence on an E-inversive E-semigroup.

The article [13] by Lou, Fan and Li is a major comprehensive paper on this topic. In it, any regular congruence on an arbitrary E-inversive semigroup is uniquely specified in terms of its trace and kernel (generalizing results for regular semigroups by Feigenbaum [6], Trotter [26] and especially by Pastijn and Petrich [20]). In the final section of [13], the regular congruences on E-inversive semigroups are characterized in terms of kernel normal systems.

The purpose of this paper is to generalize Preston's kernel normal systems to $\mathcal{P}$-inversive semigroups, and to give a description of strongly regular $\mathcal{P}$-congruences on $\mathcal{P}$-inversive semigroups in terms of their $\mathcal{P}$-kernel normal systems. The type of technique used here is basically the one used in [23]. These results generalize the corresponding results for $\mathcal{P}$-regular semigroups [23] and $\mathcal{P}$-inversive semigroups [3, 5].

By using the weak inverses in semigroups, Lou etc. [13] described regular congruences on E-inversive semigroups in terms of their kernel normal systems. In this paper we show that weak $\mathcal{P}$-inverses indeed can replace weak inverses in a congruence theory for $\mathcal{P}$-inversive semigroups, in a neater manner than that obtained in [13].

Recall that a congruence ρ on a semigroup S is said to be *regular* if S/ρ is a regular semigroup. Obviously, a congruence satisfying the property

$$(\forall a \in S) \, (\exists a' \in W_P(a)) \quad a\rho aa'a \tag{P}$$

on $S(P)$ is regular. In general, regular congruences on $S(P)$ may not satisfy the property (P). For example, let $S = (\mathbb{N}, \cdot)$ be the multiplicative semigroup of all non-negative integers and ρ be the congruence determined by the partition:

$$\{0\}, \{m \in \mathbb{N} : m > 0\}.$$

Then S is an E-inversive semigroup with $E(S) = \{0, 1\}$, and $S/\rho = \{\overline{0}, \overline{1}\}$ is a band. Let $P = \{0\}$. Certainly, $S(P)$ is $\mathcal{P}$-inversive, and ρ is regular. But there is no weak $\mathcal{P}$-inverse x of 1 such that $1\rho 1x1$. In this paper we shall be interested in the regular congruences which possesses the property (P) and we call such congruences *strongly regular*. That is,

Definition 1.2 A congruence ρ on $S(P)$ is called a strongly regular $\mathcal{P}$-congruence, if it satisfies

$$(\forall a \in S) \, (\exists a' \in W_P(a)) \quad a\rho aa'a.$$

Example 1.3 Let S be the semigroup with Cayley table [27, Example 5.1]):

S	a	b	c	d	e	f	g
a	e	g	a	e	e	g	e
b	c	d	e	d	e	e	f
c	e	f	c	e	e	f	e
d	e	d	e	d	e	e	e
e	e	e	e	e	e	e	e
f	c	e	e	e	e	e	f
g	a	e	e	e	e	e	g

Then S is an E-inversive E-semigroup with $E(S) = \{c, d, e, g\}$. Let $P = E(S)$. Then $S(P)$ is $\mathcal{P}$-inversive. Let ρ be the congruence determined by the partition $\{a\}, \{b, f\}, \{c\}, \{d, e\}, \{g\}$. Notice that b is the only non-regular element of S, $a \in W_P(b)$, and $bab = f\rho b$, we have that ρ is a strongly regular $\mathcal{P}$-congruence on $S(P)$. Now $d \in W_P(b)$ and $bdb = d$, but d and b are not ρ-related. This implies that ρ is not a regular $\mathcal{P}$-congruence on $S(P)$. Certainly, ρ is not a strong $\mathcal{P}$-congruence on $S(P)$.

The previous example illustrates that the concept of strong regular $\mathcal{P}$-congruences on $\mathcal{P}$-inversive semigroups is a generalization of the concept of strong $\mathcal{P}$-congruences and regular $\mathcal{P}$-congruences on this class of semigroups.

2 $\mathcal{P}$-kernel normal systems

The set $\mathcal{B} = \{B_i : i \in I\}$ of subsemigroups of $S(P)$ is said to be a $\mathcal{P}$-*kernel normal system* for $S(P)$ if the following conditions hold:

(K1) $B_i \cap B_j = \emptyset$ if $i \neq j \in I$.
(K2) Each B_i contains an element of P and each element of P is contained in some B_i.
(K3) For any $a \in S(P)$ there exists $a' \in W_P(a)$ such that

$$(\forall x, y \in S^1) \, (\forall B \in \mathcal{B}) \, xay \in B \Leftrightarrow xaa'ay \in B.$$

(K4)

$$(\forall x, y \in S^1) \, (\forall i_1, i_2, \ldots, i_n, j \in I)$$
$$x B_{i_1} B_{i_2} \cdots B_{i_n} y \cap B_j \neq \emptyset \Rightarrow x B_{i_1} B_{i_2} \cdots B_{i_n} y \subseteq B_j.$$

The weak $\mathcal{P}$-inverse a' of a satisfying (K3) above is called a weak $\mathcal{B}$-inverse of a, and $W_\mathcal{B}(a)$ denotes the set of weak $\mathcal{B}$-inverses of a. Now $W_\mathcal{B}(a) \neq \emptyset$ for any $a \in S(P)$.

Let ρ be a strongly regular $\mathcal{P}$-congruence on $S(P)$. The set $\{p\rho : p \in P\}$ is denoted by $\mathcal{B}_\rho$. The following lemma also gives the examples of $\mathcal{P}$-kernel normal systems for $S(P)$.

Lemma 2.1 *If ρ is a strongly regular $\mathcal{P}$-congruence on $S(P)$, then $\mathcal{B}_\rho$ is a $\mathcal{P}$-kernel normal system for $S(P)$.*

Proof (K1) and (K2) are clear.

(K3) Since ρ is a strongly regular $\mathcal{P}$-congruence, for any $a \in S(P)$ there exists $a' \in W_P(a)$ such that $a\rho aa'a$. Obviously, $a' \in W_B(a)$.

(K4) Let $b = xb_{i_1}b_{i_2}\cdots b_{i_n}y \in xB_{i_1}B_{i_2}\cdots B_{i_n}y \cap B_j$, where $b_{i_t} \in B_{i_t}, t = 1, 2, \ldots, n$, and let $B_j = p\rho, p \in P$. Then $b\rho p$. For any $c = xc_{i_1}c_{i_2}\cdots c_{i_n}y \in xB_{i_1}B_{i_2}\cdots B_{i_n}y$, where $c_{i_t} \in B_{i_t}, t = 1, 2, \ldots, n$, we have that $b_{i_t}, c_{i_t} \in B_{i_t}$ implies $b_{i_t}\rho = c_{i_t}\rho$, and so that $c\rho = x\rho\, c_{i_1}\rho\, c_{i_2}\rho \cdots c_{i_n}\rho\, y\rho = x\rho\, b_{i_1}\rho\, b_{i_2}\rho \cdots b_{i_n}\rho\, y\rho = b\rho = p\rho$. Hence $c \in B_j$. $\qquad\square$

We have the following results on $\mathcal{P}$-kernel normal systems which will be useful in the sequel.

Lemma 2.2 *Let $\mathcal{B} = \{B_i : i \in I\}$ be a $\mathcal{P}$-kernel normal system for $S(P)$.*

(1) *If $a' \in W_B(a), b' \in W_B(b)$, then $aa'a \in W_B(a'), b'a' \in W_B(ab)$.*
(2) *Let $x, y \in S^1, i_1, i_2, \ldots, i_n \in I$. If $xB_{i_1}B_{i_2}\cdots B_{i_n}y \cap P \neq \emptyset$, then there exists $j \in I$ such that $xB_{i_1}B_{i_2}\cdots B_{i_n}y \subseteq B_j$.*
(3) *For $a \in S(P), a' \in W_B(a)$ and $i \in I$, there exist $j, k \in I$ such that $aB_ia' \subseteq B_j$, and $a'B_ia \subseteq B_k$.*
(4) *If $a \in B_i, a' \in W_B(a)$ then there exist $j, k \in I$ such that $a'B_i \subseteq B_j$ and $B_ia' \subseteq B_k$.*
(5) *If $a, ab \in B_i$ and $ab', bb' \in B_j$ for some $b' \in W_B(b), i, j \in I$, then $b \in B_i$.*
(6) *If $a, ba \in B_i$ and $b'a, b'b \in B_j$ for some $b' \in W_B(b), i, j \in I$, then $b \in B_i$.*

Proof (1) Let $a' \in W_B(a), b' \in W_B(b)$. It is easy to see that $aa'a \in V(a')$ and $aa'a \in W_P(a')$. Thus $aa'a \in W_B(a')$.

Now $a' \in W_P(a), b' \in W_P(b)$. It follows from [4, Lemma 1.4] that $b'a' \in W_P(ab)$. Let $x, y \in S^1$. Then

$$xabb'a'aby \in B_i \Leftrightarrow xaa'abb'a'abb'by \in B_i \quad \text{(since } a' \in W_B(a) \text{ and } b' \in W_B(b)\text{)}$$

$$\Leftrightarrow xa(a'abb')^2by \in B_i$$

$$\Leftrightarrow xaa'abb'by \in B_i \quad \text{(since } a'abb' \in P^2 \subseteq E(S)\text{)}$$

$$\Leftrightarrow xaby \in B_i \quad \text{(since } a' \in W_B(a) \text{ and } b' \in W_B(b)\text{)}.$$

Hence $b'a' \in W_B(ab)$.

(2) By (K2), each element of P is contained in some B_j. The result follows from (K4).

(3) Suppose that B_i contains some element $p \in P$. Then $apa' \in P$. Hence $aB_ia' \cap P \neq \emptyset$. Now it follows from (2) that there exists $j \in I$ such that $aB_ia' \subseteq B_j$. Similarly, we can show the other result.

(4) Since $a \in B_i$ and $a' \in W_B(a), a'a \in a'B_i \cap P$. It follows from (2) that there exists $B_j \in \mathcal{B}$ such that $a'B_i \subseteq B_j$. Similarly, we can prove the other result.

(5) Since $ab \in B_i, abb'b \in B_i$. Now $b'b \in P$, by (K2), there exists $B_k \in \mathcal{B}$ such that $b'b \in B_k$. Hence $ab'b, abb'b \in B_i B_k$, and so $B_i B_k \cap B_i \neq \emptyset$. By (K4), $B_i B_k \subseteq B_i$, so that $ab'b \in B_i$. Now $ab'b \in B_j b$. Thus $B_j b \cap B_i \neq \emptyset$. By (K4), $B_j b \subseteq B_i$. Since $bb' \in B_j, bb'b \in B_i$, that is, $b \in B_i$.

(6) Similar to the proof of (5), we can obtain (6). □

We now introduce the following notation. Let $\mathcal{B} = \{B_i : i \in I\}$ be a $\mathcal{P}$-kernel normal system for $S(P)$ and $a, b \in S(P)$. By $a \sim b$ we mean that a and b are contained in a same B_i. Define a relation $\rho_{\mathcal{B}}$ on $S(P)$ by

$$a\rho_{\mathcal{B}} b \Leftrightarrow (\exists a' \in W_B(a))(\exists b' \in W_B(b))ba' \sim aa' \ \& \ b'a \sim b'b.$$

Clearly, $\rho_{\mathcal{B}}$ is a reflexive relation on $S(P)$.

In order to obtain the main result in this paper, we shall set up a series of lemmas.

Lemma 2.3 $\rho_{\mathcal{B}}$ *is a symmetric relation on* $S(P)$.

Proof Let $a\rho_{\mathcal{B}} b$. Then there exist $a' \in W_B(a), b' \in W_B(b)$ such that $ba' \sim aa'$ and $b'a \sim b'b$. Then there exist $B_{i_1}, B_{i_2} \in \mathcal{B}$ such that $ba', aa' \in B_{i_1}$ and $b'a, b'b \in B_{i_2}$. Let $a^+ \in W_B(a)$ and $b^+ \in W_B(b)$. Now by Lemma 2.2(1), $aa'ab^+ \in W_B(ba')$. Then by Lemma 2.2(4), $B_{i_1}(aa'ab^+) \subseteq B_{i_3}$ for some $i_3 \in I$. Now $aa'ab^+ = (aa')aa'ab^+ \in B_{i_1}(aa'ab^+) \subseteq B_{i_3}$. By (K3), $ab^+ \in B_{i_3}$. Also, $ba'ab^+ = (ba')(aa')ab^+ \in B_{i_3}$. Then by $b' \in W_B(b), b(b'b)(a'a)b^+ \in B_{i_3}$. Hence $bB_{i_2}B_{i_4}b^+ \cap B_{i_3} \neq \emptyset$, where $a'a \in B_{i_4}$. By (K4), $bB_{i_2}B_{i_4}b^+ \subseteq B_{i_3}$. Hence $bb'aa'ab^+ \in B_{i_3}$, that is, $bb'ab^+ \in B_{i_3}$. Hence $ab^+, bb'ab^+ \in B_{i_3}$. Now $b'a \sim b'b$ implies that $b(b'b)b^+ \sim b(b'a)b^+$ by Lemma 2.2(3). Therefore $ab^+ \sim bb'ab^+ \sim bb'bb^+ \sim bb^+$, and so $ab^+ \sim bb^+$. Similarly, we can prove that $a^+b \sim a^+a$. Hence $b\rho_{\mathcal{B}} a$, and so that $\rho_{\mathcal{B}}$ is symmetric. □

Remark In the proof of Lemma 2.3, we also prove that if there exist $a' \in W_B(a), b' \in W_B(b)$ such that $ba' \sim aa'$ and $b'a \sim b'b$, then $ab^+ \sim bb^+$ and $a^+b \sim a^+a$ for all $a^+ \in W_B(a), b^+ \in W_B(b)$. In the similar way, we can prove that if there exist $a^+ \in W_B(a), b^+ \in W_B(b)$ such that $ab^+ \sim bb^+$ and $a^+b \sim a^+a$, then $ba' \sim aa'$ and $b'a \sim b'b$ for all $a' \in W_B(a), b' \in W_B(b)$. In the previous definition of $\rho_{\mathcal{B}}$ we can substitute "there exists" by "for all".

Lemma 2.4 *If* $(a, b), (b, c) \in \rho_{\mathcal{B}}$, *then* $a(c'c)a' \sim aa', c(a'a)c' \sim cc', a'(cc')a \sim a'a, b'(aa')b \sim b'b$ *for all* $a' \in W_B(a), b' \in W_B(b), c' \in W_B(c)$.

Proof Since $(a, b), (b, c) \in \rho_{\mathcal{B}}, ba' \sim aa', b'a \sim b'b, a'b \sim a'a, ab' \sim bb', cb' \sim bb', c'b \sim c'c, b'c \sim b'b, bc' \sim cc'$ for all $a' \in W_B(a), b' \in W_B(b)$ and $c' \in W_B(c)$ (from the remark after Lemma 2.3). So there exist $B_{i_1}, B_{i_2}, B_{i_3}, B_{i_4}, B_{i_5}, B_{i_6}, B_{i_7}$ and B_{i_8} in $\mathcal{B}$ such that $ba', aa' \in B_{i_1}, b'a, b'b \in B_{i_2}, a'b, a'a \in B_{i_3}, ab', bb' \in B_{i_4}, cb', bb' \in B_{i_5}, c'b, c'c \in B_{i_6}, b'c, b'b \in B_{i_7}$ and $bc', cc' \in B_{i_8}$. From (K1), $B_{i_2} = B_{i_7}$ and $B_{i_4} = B_{i_5}$.

We first show that $a(c'c)a' \sim aa'$. Now $a(a'b)(b'b)(c'c)a'$, $a(a'b)(b'c)(c'c)a'$, $a(a'a)(b'c)(c'b)a' \in aB_{i_3}B_{i_2}B_{i_6}a'$. Since $a(b'c)(c'b)a' \in P$, by (K2), there exists $i_9 \in I$ such that $a(b'c)(c'b)a' \in B_{i_9}$, and so $a(a'a)(b'c)(c'b)a' \in B_{i_9}$ by (K3). Thus $aB_{i_3}B_{i_2}B_{i_6}a' \cap B_{i_9} \neq \emptyset$. Hence from (K4), $aB_{i_3}B_{i_2}B_{i_6}a' \subseteq B_{i_9}$, so that $a(a'b)(b'b)(c'c)a'$, $a(a'b)(b'c)(c'c)a' \in B_{i_9}$, that is, $a(a'b)(c'c)a'$, $a(a'b)(b'c)a' \in B_{i_9}$.

Now $a(a'a)(c'c)a', a(a'b)(c'c)a' \in aB_{i_3}B_{i_6}a'$. Since $a(c'c)a' \in P$, there exists $i_{10} \in I$ such that $a(c'c)a' \in B_{i_{10}}$. Then by (K3), $a(a'a)(c'c)a' \in B_{i_{10}}$. Hence $aB_{i_3}B_{i_6}a' \cap B_{i_{10}} \neq \emptyset$. Then by (K4), $aB_{i_3}B_{i_6}a' \subseteq B_{i_{10}}$. Thus $a(a'b)(c'c)a' \in B_{i_{10}}$. By (K1), $B_{i_{10}} = B_{i_9}$. Hence $a(a'a)(c'c)a' \in B_{i_9}$, and so $a(c'c)a' \in B_{i_9}$.

Now $a(a'b)(b'b)a', a(a'b)(b'c)a', a(a'a)(b'b)a' \in aB_{i_3}B_{i_7}a'$. Note $a(b'b)a' \in P$, there exists $i_{11} \in I$ such that $a(b'b)a' \in B_{i_{11}}$. By (K3), $a(a'a)(b'b)a' \in B_{i_{11}}$, and so $aB_{i_3}B_{i_7}a' \cap B_{i_{11}} \neq \emptyset$. Then by (K4), $aB_{i_3}B_{i_7}a' \subseteq B_{i_{11}}$. Hence $a(a'b)(b'c)a' \in B_{i_{11}}$. By (K1), $B_{i_{11}} = B_{i_9}$. Then $a(c'c)a' \sim a(a'b)(b'b)a' \sim a(a'b)a'$. Now $a'b \sim a'a$ implies (by Lemma 2.2(3)) that $a(a'b)a' \sim a(a'a)a' = aa'$. Hence $a(c'c)a' \sim aa'$.

Similarly, we can prove the other results. □

Lemma 2.5 $\rho_\mathcal{B}$ *is a transitive relation on* $S(P)$.

Proof Here we use all the notation of Lemma 2.4. Let $(a, b) \in \rho_\mathcal{B}$ and $(b, c) \in \rho_\mathcal{B}$. Let $x = bc'cb'aa'$. Now $(cb')(aa'), (bb')(ba') \in B_{i_5}B_{i_1}$ and $(bb')(ba') \sim ba' \in B_{i_1}$. Hence $B_{i_5}B_{i_1} \cap B_{i_1} \neq \emptyset$. Then by (K4), $B_{i_5}B_{i_1} \subseteq B_{i_1}$. Hence $(cb')(aa') \in B_{i_1}$. This implies that $x = (bc')(cb'aa'), (cc')(cb'aa') \in B_{i_8}B_{i_1}$. But $cb'aa' \sim (cc')(cb'aa')$. Hence $B_{i_8}B_{i_1} \cap B_{i_1} \neq \emptyset$. Then by (K4), $B_{i_8}B_{i_1} \subseteq B_{i_1}$. Hence $x \in B_{i_1}$.

Let $y = ca'$. By Lemma 2.2(1), $aa'ac' \in W_B(y)$. Let $y' = aa'ac'$. Now $a'b \sim a'a$ implies (by Lemma 2.2(3)) $c(a'b)c' \sim c(a'a)c'$. From Lemma 2.4, $ca'ac' \sim cc'$. Hence $c(a'b)c', cc' \in B_{i_8}$. Now $(cb')(aa'), (bb')(ba') \in B_{i_5}B_{i_1}$. But $(bb')(ba') \sim ba' \in B_{i_1}$. Hence $B_{i_5}B_{i_1} \cap B_{i_1} \neq \emptyset$, and so $(cb')(aa') \in B_{i_1}$. Then $(ca'bc')(cb')(aa')$, $(cc')(cb')(aa') \in B_{i_8}B_{i_1}$. But $(cc')(cb')(aa') \sim (cb')(aa') \in B_{i_1}$. Thus $B_{i_8}B_{i_1} \cap B_{i_1} \neq \emptyset$, so that $B_{i_8}B_{i_1} \subseteq B_{i_1}$. Hence

$$yx = ca'bc'cb'aa' \in B_{i_1}.$$

Now $x, yx \in B_{i_1}$ implies that

$$yx \sim x. \tag{2.1}$$

To show that $y'y \sim y'x$, we notice that

$$a(c'b)(c'c)(b'c)a', a(c'b)(c'c)(b'a)a', a(c'c)(c'b)(b'b)a' \in aB_{i_6}B_{i_6}B_{i_7}a'.$$

Here $a(c'b)(c'c)(b'c)a' = a(c'(b(c'c)b')c)a' \in P$. Hence by Lemma 2.2(2), there exists i_{12} such that $aB_{i_6}B_{i_6}B_{i_7}a' \subseteq B_{i_{12}}$. Hence $a(c'c)(c'b)(b'b)a' \in B_{i_{12}}$, that is, $ac'ba' \in B_{i_{12}}$.

Now $c'c \sim c'b$. Hence by Lemma 2.2(3), $a(c'c)a' \sim a(c'b)a'$. But $a(c'c)a', aa' \in B_{i_1}$ by Lemma 2.4. Hence $a(c'b)a' \in B_{i_1}$. By (K1), $B_{i_1} = B_{i_{12}}$. Consequently, $ac'bc'cb'aa' \in B_{i_1}$, and so, $aa'ac'bc'cb'aa' \in B_{i_1}$. Hence $y'x, x \in B_{i_1}$. Again

$ac'ca' \sim aa'$. Then $y'y = aa'ac'ca' \sim ac'ca' \sim aa' \sim x$. Therefore

$$y'x \sim y'y. \tag{2.2}$$

From (2.1), (2.2), by Lemma 2.2(6), we have $y \sim x$. Then

$$ca' \sim x \sim aa'.$$

Next we show that $c'a \sim c'c$. Let $x_1 = c'cb'aa'b$, $y_1 = c'a$. Now $a'cc'c \in W_B(y_1)$. Let $y_1' = a'cc'c$. From Lemma 2.4, $b'aa'b \sim b'b \in B_{i_7}$. Hence $(c'c)(b'aa'b)$, $(c'c)(b'a), (c'b)(b'b) \in B_{i_6} B_{i_7}$. But $(c'b)(b'b) \sim c'b \in B_{i_6}$, and so $B_{i_6} B_{i_7} \cap B_{i_6} \neq \emptyset$. Then (K4) implies that $B_{i_6} B_{i_7} \subseteq B_{i_6}$, so that $x_1 = (cc')(b'aa'b) \in B_{i_6}$.

Now $x_1 y_1 = (c'cb'a)(a'b)(c'a) = (c'c)(b'a)(a'bc'a)$. Here $a'(bc')a \sim a'(cc')a$ (by Lemma 2.2(3)) $\sim a'a$ (by Lemma 2.4) $\in B_{i_3}$. Notice that $(c'c)(b'a) \in B_{i_6}$ and so $x_1 y_1 \in B_{i_6} B_{i_3}$. Also, $(c'c)(b'a)(a'a) \in B_{i_6} B_{i_3}$, and $(c'c)(b'a)(a'a) \sim (c'c)(b'a) \in B_{i_6}$. Hence $B_{i_6} B_{i_3} \subseteq B_{i_6}$, so that $x_1 y_1 \in B_{i_6}$. Now $x_1, x_1 y_1 \in B_{i_6}$ implies that

$$x_1 y_1 \sim x_1. \tag{2.3}$$

To show that $x_1 y_1' \sim y_1 y_1'$, notice that $x_1 y_1' = c'cb'aa'ba'cc'c$. Now $(cb')(aa')$, $(bb')(ba') \in B_{i_5} B_{i_1}$. But $(bb')(ba') \sim ba' \in B_{i_1}$. Hence $(cb')(aa'), ba' \in B_{i_1}$. So $cb'aa'ba' = (cb'aa')(ba') \in B_{i_1} B_{i_1} \subseteq B_{i_1}$ (since B_{i_1} is a subsemigroup). Then $cb'aa'ba' \sim aa'$. It follows from Lemma 2.2(3) that $c'(cb'aa'ba')c \sim c'(aa')c$. Hence $c'(cb'aa'ba')cc'c \sim c'(aa')cc'c$, that is,

$$x_1 y_1' \sim y_1 y_1'. \tag{2.4}$$

From (2.3), (2.4), by Lemma 2.2(5), we have that $x_1 \sim y_1$, and so $c'c \sim x_1 \sim c'a$. Hence $(a, c) \in \rho_\mathcal{B}$. $\square$

Lemma 2.6 $\rho_\mathcal{B}$ is a strongly regular $\mathcal{P}$-congruence on $S(P)$.

Proof It follows from Lemma 2.3 and Lemma 2.5 that $\rho_\mathcal{B}$ is an equivalence relation on $S(P)$. Let us show that $\rho_\mathcal{B}$ is left compatible. Let $(a, b) \in \rho_\mathcal{B}$, $c \in S$, $a' \in W_B(a)$, $b' \in W_B(b)$ and $c' \in W_B(c)$. Then $ba' \sim aa'$, $b'a \sim b'b, a'b \sim a'a$, $ab' \sim bb'$. So there exist $B_{i_1}, B_{i_2}, B_{i_3}, B_{i_4} \in \mathcal{B}$ such that $ba', aa' \in B_{i_1}, b'a, b'b \in B_{i_2}, a'b, a'a \in B_{i_3}, ab', bb' \in B_{i_4}$. Now $ba' \sim aa'$ implies that $c(ba')c' \sim c(aa')c'$ (by Lemma 2.2(3)).

We show that $b'c'ca \sim b'c'cb$ as follows. Let $x = b'c'cba'a$, $y = b'c'ca$, $y' = a'c'cbb'b$. Then $y' \in W_B(y)$. Notice that $b'c'cb \in P$. Let $b'c'cb \in B_{i_5} \in \mathcal{B}$. Now $(b'c'cb)(b'a), (b'c'cb)(b'b) \in B_{i_5} B_{i_2}$. Since $(b'c'cb)(b'b) \sim b'c'cb \in B_{i_5}$, $B_{i_5} B_{i_2} \cap B_{i_5} \neq \emptyset$. Hence by (K4), $B_{i_5} B_{i_2} \subseteq B_{i_5}$. Then $(b'c'cb)(b'a) \in B_{i_5}$. Now $(b'c'cb)(b'b)(a'a), (b'c'cb)(b'a)(a'a) \in B_{i_5} B_{i_3}$. But $(b'c'cb)(b'a)(a'a) \sim (b'c'cb)(b'a) \in B_{i_5}$ shows that $B_{i_5} B_{i_3} \cap B_{i_5} \neq \emptyset$. Then $B_{i_5} B_{i_3} \subseteq B_{i_5}$ and $(b'c'cb)(b'b)(a'a) \in B_{i_5}$, that is, $x = b'c'cba'a \in B_{i_5}$.

Now $ab' \sim bb'$, and $b'c'c \in W_B(c'cb)$. Hence by Lemma 2.2(3), $b'(c'c(ab')c'c)b \sim b'(c'c(bb')c'c)b$. Since $b'c'cb \in P \subseteq E_S$, $b'(c'c(bb')c'c)b = (b'c'cb)^2 = b'c'cb$.

Now $b'c'cb \in B_{i_5}$, hence $b'(c'c(ab')c'c)b \in B_{i_5}$. Then $yx = (b'c'ca)(b'c'cba'a) = (b'(c'cab'c'c)b)(a'a) \in B_{i_5}B_{i_3}$. But $B_{i_5}B_{i_3} \subseteq B_{i_5}$ and so $yx \in B_{i_5}$. Then

$$yx \sim x \sim b'c'cb. \tag{2.5}$$

To show that $y'x \sim y'y$, notice that $(c'c)(bb')(c'c) \in P$. Then there exists $B_{i_6} \in \mathcal{B}$ such that $(c'c)(bb')(c'c) \in B_{i_6}$. Now $y'x = a'c'cbb'bb'c'cba'a = a'(c'cbb'c'c)(ba')a$, $a'(c'cbb'c'c)(aa')a \in a'B_{i_6}B_{i_1}a$. Notice that $y'y = a'c'cbb'bb'c'cca = a'(c'cbb'c'c)a \in P$. Then there exists $B_{i_7} \in \mathcal{B}$ such that $y'y \in B_{i_7}$, and so $a'(c'cbb'c'c)(aa')a \in B_{i_7}$. Hence $a'B_{i_6}B_{i_1}a \subseteq B_{i_7}$. Then

$$y'x \sim y'y. \tag{2.6}$$

From (2.5), (2.6), by Lemma 2.2(6), $y \sim x \sim b'c'cb$, that is, $b'c'ca \sim b'c'cb$.

By Lemma 2.2(1), $a'c' \in W_B(ca), b'c' \in W_B(cb)$. Hence $(ca, cb) \in \rho_\mathcal{B}$, and so $\rho_\mathcal{B}$ is left compatible. A similar argument shows that $\rho_\mathcal{B}$ is right compatible.

It is easy to verify that $a' \in W_B(aa'a)$ and $(a, aa'a) \in \rho_\mathcal{B}$. Hence $\rho_\mathcal{B}$ is a strongly regular $\mathcal{P}$-congruence on $S(P)$. $\qquad\square$

Lemma 2.7 *If $p \in B_i \cap P$, then $p\rho_\mathcal{B} = B_i$.*

Proof Let $a \in p\rho_\mathcal{B}$. Then $p\rho_\mathcal{B}a$. Since $p \in W_B(p), ap \sim pp = p$ and $a'p \sim a'a$ for any $a' \in W_B(a)$. Hence by Lemma 2.2(6), $a \in B_i$.

Conversely, let $b \in B_i$. Since B_i is a subsemigroup, $pb \in B_i$. Then $pb \sim pp$. Let $b' \in W_B(b)$. By Lemma 2.2(4), there exists $B_j \in \mathcal{B}$ such that $B_ib' \subseteq B_j$. Hence $pb', bb' \in B_j$, and so $pb' \sim bb'$. By the definition of $\rho_\mathcal{B}, (b, p) \in \rho_\mathcal{B}$. Thus $b \in p\rho_\mathcal{B}$. $\qquad\square$

Lemma 2.8 *Let ρ and σ be two strongly regular $\mathcal{P}$-congruences on $S(P)$. Then $\rho = \sigma$ if and only if $\mathcal{B}_\rho = \mathcal{B}_\sigma$.*

Proof It suffices to show the "if" part. Let $a\rho b$. Since σ is a strongly regular $\mathcal{P}$-congruence, there exists $a' \in W_P(a)$ such that $a\sigma aa'a$. Now $aa'\rho ba'$. Note $aa' \in P$. By the assumption, there exists $p \in P$ such that $aa'\rho = p\sigma$. Hence $aa'\rho = aa'\sigma$, and so $ba' \in aa'\sigma$, that is, $aa'\sigma ba'$. Similarly, we have that $b\sigma bb'b$ and $b'a\sigma b'b$ for some $b' \in W_P(b)$. Thus

$$a\sigma aa'a\sigma ba'a\sigma bb'ba'a\sigma bb'aa'a\sigma bb'a\sigma bb'b\sigma b.$$

Hence $a\sigma b$, and so $\rho \subseteq \sigma$. Similarly, we have $\sigma \subseteq \rho$. Thus $\rho = \sigma$. $\qquad\square$

The following theorem is the main result in this paper.

Theorem 2.9 *If $\mathcal{B}$ is a $\mathcal{P}$-kernel normal system for $S(P)$, then $\rho_\mathcal{B}$ is a strongly regular $\mathcal{P}$-congruence on $S(P)$ and $\mathcal{B}_{\rho_\mathcal{B}} = \mathcal{B}$. Conversely, if ρ is a strongly regular $\mathcal{P}$-congruence on $S(P)$, then $\mathcal{B}_\rho$ is a $\mathcal{P}$-kernel normal system for $S(P)$ and $\rho_{\mathcal{B}_\rho} = \rho$.*

Proof Direct part. Let $\mathcal{B} = \{B_i : i \in I\}$ be a $\mathcal{P}$-kernel normal system for $S(P)$. By Lemma 2.6, $\rho_{\mathcal{B}}$ is a strongly regular $\mathcal{P}$-congruence on $S(P)$.

Let $p \in P$. Then there exists $B_i \in \mathcal{B}$ such that $p \in B_i$. Hence by Lemma 2.7, $p\rho_{\mathcal{B}} = B_i$, and so $\mathcal{B}_{\rho_{\mathcal{B}}} \subseteq \mathcal{B}$. Conversely, suppose that $B_i \in \mathcal{B}$. Then B_i contains some element $p \in P$. By Lemma 2.7 again, $B_i = p\rho_{\mathcal{B}}$, so that $\mathcal{B}_{\rho_{\mathcal{B}}} = \mathcal{B}$ as required.

Converse part. Let ρ be a strongly regular $\mathcal{P}$-congruence on $S(P)$. Then it follows from Lemma 2.1 that $\mathcal{B}_{\rho}$ is a $\mathcal{P}$-kernel normal system for $S(P)$. From the direct part of this theorem, we have that $\rho_{\mathcal{B}_{\rho}}$ is also a strongly regular $\mathcal{P}$-congruence on $S(P)$ and $\mathcal{B}_{\rho_{\mathcal{B}_{\rho}}} = \mathcal{B}_{\rho}$. Hence by Lemma 2.8, $\rho_{\mathcal{B}_{\rho}} = \rho$. $\qquad\square$

The set of all strongly regular $\mathcal{P}$-congruences on $S(P)$ and the set of all $\mathcal{P}$-kernel normal systems for $S(P)$ are denoted by $SRC_P(S)$ and $KNS_P(S)$, respectively.

Define a relation $\leq$ on $KNS_P(S)$ by

$$\mathcal{A} \leq \mathcal{B} \Leftrightarrow (\forall A \in \mathcal{A})\, (\exists B \in \mathcal{B})\, A \subseteq B.$$

It is clear that $\leq$ is a partial order on $KNS_P(S)$.

Now the next result follows immediately.

Corollary 2.10 *The mappings*

$$\varphi : \quad SRC_P(S) \to KNS_P(S)$$

$$\rho \mapsto \mathcal{B}_{\rho}$$

and

$$\psi : \quad KNS_P(S) \to SRC_P(S)$$

$$\mathcal{B} \mapsto \rho_{\mathcal{B}}$$

are mutually inverse order preserving bijections.

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