Transactions on Engineering Sciences vol 15, © 1997 WIT Press, www.witpress.com, ISSN 1743-3533

MATHEMATICA-aided study of Lie algebras and their cohomology - from supergravity to ballbearings and magnetic hydrodynamics

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Abstract

We describe applications of a MATHEMATICA-based package for the study of Lie algebras and their cohomology such as (1) the possibility to write Supergravity Equations for any N-extended Minkowski superspace and to find out the possible models for these superspaces; (2) the possibility of studying stability of nonholonomic systems (ballbearings, gyroscopes, electromechanical devices like a rotor collector with a gliding contact; waves in plasma, etc.); (3) description of the analogue of the curvature tensor for nonlinear nonholonomic constraints and the fields of solids or their surfaces, e.g., cones, as in optimal control; (4) a new method for the study of integrability of dynamical systems.

The above problems are particular instances of the general problem to compute cohomology or homology of the given Lie algebra or superalgebra with various coefficients. The package **SuperLie** makes it possible to determine (1) Lie algebras via defining relations, from the Cartan matrix, realized via vector fields, as polynomials with Poisson or contact (Legendre) bracket, etc., (2) various modules over these Lie algebras (tensors, with vacuum vector, etc.), (3) list central extensions and deformations and even (4) back up the Leites conjecture (an analog of Kostrikin–Shafarevich conjecture) classifying simple Lie algebras over the algebraically closed field of characteristic 2 with new examples. For the details see references.

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1 History. Formulation of problems

1.1 Supergravity

In 1972 Leites introduced [L1] the what is now called *superscheme* or *algebraic supervariety.* This was an answer to a question of Berezin ([BL], [L2]) together with whom he was slightly ahead of the time, see Minlos' recollections in [D]. Now supermanifold theory is fully recognized thanks to its part, *supersymmetry,* considered in theoretical physics as the language of the future unified theory of all fundamental forces. The discovery of supersymmetry, hence, the realization that we live on *Minkowski superspace* is one of the most fundamental steps in physics.

One of the leading modern theoretical physicists, Witten wrote: "*Direct experimental confirmation of supersymmetry is one of the prime missions of the proposed Superconducting supercollider.*" And then added: "*More fundamentally, I believe that the main obstacle is that the core geometrical ideas — which must underline string theory the way Riemannian geometry underlines general relativity — have not yet been unearthed.*"

We believe that Leites was lucky to have discovered these ideas. They are related to (a) *nonholonomic* mechanics and (b) the presentation of the curvature tensor via cohomology. The first cirumstance is occasioned by the fact that Minkowski *super*spaces (whatever it is, see [WB], [SS]; there is a hierarchy of models, labeled by an inner parameter N running 1 to 8) are **nonholonomic**. The importance of Lie algebra cohomology for the study of supergravity was felt and is clearly stated by several authors (e.g., [CDF]) but the description of the curvature tensor in nonholonomic case was never given so far.

1.2 Mechanics

In XIX century Hertz [H] divided the dynamical systems into *holonomic* (no constraints on velocities) and *nonholonomic* (with (nonintegrable) constraints on velocities). Analytical study of holonomic systems has progressed much further than that of nonholonomic.

Though nonholonomic systems are important in various branches of applied physics end engineering, they have not been sufficiently studied in mathematics, even the best text books on analytical mechanics just mention several examples and pass to holonomic systems, cf. [A1], [A2]. One of the reasons: the qualitative study, such as stability problems, require the notion of curvature tensor which is not defined for the nonholonomic systems in the literature, cf. [S], the best for this purpose.

Examples of nonholonomic systems. 1) In **mechanics.** A ball on a rough plane; ballbearings and gyroscopes; any vehicle with wheels (the point at which the moving body is tangent to the surface has zero velocity). A car with a cruise control switched ON is an example of a nonholonomic

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system with a nonlinear constraint. For further (numerous) examples we refer the reader to Netscape, especially to papers by H. Kleinerts as well as [G] and refs. therein.

Related problems: general description of the movement and stability of a rapidly rotating hole body with a liquid or gyroscope inside it; orbital stability of a missile with liquid inside it. (The linear approximation does not work here.)

For several interesting practical applications (e.g., how a cat, when dropped, uses nonholonomicity to fall on its paws) see [M].

2) In **electromechanics** (see, e.g., the book by Nejmark and Fufaev [NF] or [BF]) an early example was provided by Gaponov who, in 1952, found that a conductor connected to a gliding contact (as in a rotor collector) is equivalent to imposing nonholonomic constraints on the distribution of the electric current.

For comprehensive reviews of nonholonomic problems, see the books [NF] and [VG]. They list numerous problems which go back to Euler, Gauss, Carnot, Hertz, Appell, Caratheodory, Schouten, Synge, et al. In the review, and in references therein (see also Agrachev's plenary talk at the 1994 International Congress of Mathematicians [Ag]), there are mentioned several problems of **optimal control**, which lead to nonlinear constraints on generalized velocities (fields of cones, spheres, etc.). Numerous works on hypoelliptic differential equations (by Hörmander, Melin, Malliavin, Bismut, Bell; for a review see, e.g., [Ar]) also lead to nonholonomic distributions.

3) the **magneto-hydrodynamics**: waves in plasma. Study of stability of such waves is extremely important for the development of thermonuclear power plants.

4) Recent studies of Komech, Spohn and Kunze [KS] show that electron represents a nonholonomic system with a (Coulomb, Maxwell, etc.) gauge as a nonholonomic constraint.

1.3 Related problems

• Supersymmetry appears even in seemingly non-super questions, like the study of the spectrum of the Schrödinger operator [L2] or relation between the Schrödinger operator and the KdV (Korteveg de Vries) operator [LX].

• Recent studies of Gelfand–Dickey bracket lead us to the discovery of a new class of simple Lie algebras of polynomial growth — generalization of Lie algebras of matrices of "complex size" [GL1], [GL2]. Simultaneously, together with Shchepochkina, we announced two classifications ([GLS], [LS]) of simple Lie superalgebras: (1) of vector fields and (2) stringy or superconformal superalgebras. In November 1996 V. Kac wrote about problem (1) "*The problem of classification of all primitive (simple) Lie superalgebras is a very interesting problem in itself (in view if recent progress in representation theory of finite dimensional simple Lie superalgebras, this is the* *last open problem from my Advances in Math. 1977 paper). However, my feeling is that it is too difficult to be solved in this millenium."*

Amazingly, a solution of these problems was considerably speedified by means of a computer-aided study. Accordingly, our research splits into several topics united by the usage of a common MATHEMATICA-based package SuperLie written by Grozman.

1. Study of nonholonomic mechanics: from supergravity to ballbearings. Applications to stochastic analysis, media with dislocations and disclinations and optimal control. Study of stability via cohomology.

2. Study of representation theory; in particular, classification simple Lie superalgebras of vector fields, stringy or superconformal and generalizing Lie algebra of matrices of complex size. Description of defining relations via cohomology. Applications to Lie algebras and superalgebras and their representations over fields of prime characteristic and over integers.

3. Application of representation theory to the study of integrability of differential equations (KdV, KP, Schrödinger, Liouville, Toda lattice, etc. and their superizations). Criteria of integrability expressed in terms of Lie algebra cohomology.

2 Problem Formulation

According to Arnold-Kozlov-Neishtadt [A2], the behavior of nonholonomic systems is often "surprising", and its quantitative study is handicapped by the lack of adequate tools. Let us recall some examples from [A2].

Consider a skate on the inclined plane. Where do you think it will move if pushed not directly downwards, but sideways? If we ignore friction it will never reach the floor but will oscillate between certain horizontal lines on the plane. Similarly, consider a ball rolling along the wall inside a vertical tube. It seems natural to expect the ball to descend on a spiral trajectory with increasing steepness. In reality, however, the ball will perform harmonic oscillations between two fixed vertical planes. Though individual solutions of nonholonomic systems are usually known, the stability questions are solved *ad hoc.*

2.1 Stability problems

The stability of a holonomic system can be studied in terms of the Riemann curvature tensor. The sign of the curvature indicates whether the geodesics converge or diverge: compare meridians on a sphere (positive curvature) with those on a trumpet (negative curvature).

For nonholonomic manifolds there was no such tool: in the literature the definition of the analogue of the Riemann tensor is only given in a few particular cases of little practical value even in pure mathematics, cf. [VG], [T].

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2.2 Difficulties

In the last decades, dozens of publications on dynamical problems in non-holonomic mechanics appeared each year in technical journals, but very few in mathematical journals. The main reason is the lack of adequate language needed to even formulate the problem of calculating the local curvature tensor (the analogue of the Riemann tensor) in intrinsic terms. The bulky coordinate expressions used by classics in the 1920s hinder further progress. Vershik and Gerschkovich gloomily stated ([VG]) that there is no language to even formulate the integrability problems for nonholonomic systems with linear constraints (to say nothing about nonlinear constraints).

This as one of the reasons why this important field was abandoned by mathematicians, and even the interest and occasional works of such leaders as Faddeev, Griffiths, Chern and Arnold (see [VG], [Br], [A2]) did not attract any followers.

The only examples of the Riemann tensor calculated in nonholonomic case is for 3-dimensional manifold (Martinet, see [VG]), whereas according to É. Cartan the first interesting case is in dimension 5.

3 What is done: a formula and a package

We know the importance of the notion of curvature in drawing very exact maps; application to stability were already mentioned. How to compute the curvature? In text books on geometry this is done in terms of Spencer cohomology that are difficult to calculate. Same cohomology are used in Goldschmidt's criteria for (formal) integrability of differential equations. Observe that these criteria [Br] are only given for "a half of" the cases. The point is that by a theorem of Cartan the symmetries of each differential equation are induced by either point or contact transformations [KLV]. The latter case belongs to the realm of the simplest nonholonomic manifold — a contact one — and, apart from partial results of [T], nothing was known in this case.

3.1 A formula: Lie Algebra Cohomology

In lectures at ICTP, in 1990, Leites reformulated the definition of the Riemann tensor in terms of *Lie algebra cohomology.* In these terms, the problems discussed above can be posed in precise terms and, in principle, solved. Most important, it becomes possible to generalize the description of the local curvature tensor for nonholonomic manifolds with any constraints, like the fields of surfaces.

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3.2 Pilot Package

During 1992-94, Grozman, working as a guest researcher at Stockholm University, developed a MATHEMATICA-based pilot package SuperLie for computing Lie algebras and their cohomology. The package embraces determination of various types of Lie algebras: matrix algebras, algebras of vector fields, algebras defined via generators and relations, via Cartan matrices, via Poisson brackets, etc.; the modules over them, and their cohomology *per se.*

The mentioned above reformulation of Spencer cohomology in terms of Lie algebra cohomology means that much of the time-consuming calculations become considerably simplified and new ways to diminish the amount of calculations appear. However, the computation effort is still overwhelming: to compute the curvature tensor even for the simplest practical examples of, e.g., a ballbearing, or a bike, we need to develop the package further and complete its documentation to make available to engineers. Besides, only version 2 of MATHEMATICA was available to us.

In spite of these drawbacks of the existing pilot package, we used it to correct several results found analytically by mathematicians (certain cohomology from [FL], [F]) and by physicists (certain Wess–Zumino constraint in supergravity [WB] turned out to be redundant). The last example is of particular interest since the scene for supergravity is a nonholonomic supermanifold.

The results, obtained completely or patly with the help of the package are described in [GL], [GKLP], [GLS] and used also in [LS], [LX]. Several more are in preparation.

3.3 Rival and not so rival teams

An alternative package (not so overwhelming but much faster) is now being developed in JINR, Dubna, by V.Kornyak, our coworker, in C. For results of REDUCE fans from Twente Univ. see [LP], [PH], refs. in [GKLP].

Acknowledgments.
We are thankful to the Swedish Institute and NFR for financial support.

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