Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS

Pierpaolo Esposito
Dipartimento di Matematica,
Università degli Studi "Roma Tre", Roma, Italy

Nassif Ghoussoub and Yujin Guo
Department of Mathematics,
University of British Columbia, Vancouver, Canada

March 2, 2007

Contents

Preface

Micro-Electromechanical Systems (MEMS) combine electronics with micro-size mechanical devices to design various types of microscopic machinery. Having been rapidly developed since 1980s, MEMS devices become the essential components of modern sensors in various areas, including commercial systems, biomedical industry, space exploration and telecommunications. The advent of MEMS has revolutionized numerous branches of science and industry. Mathematical modeling of MEMS devices has already turned out to be a very rich source of interesting mathematical phenomena.

This monograph is to present analysis and simulations of a partial differential equation (PDE) modeling a simple idealized electrostatic MEMS, which was firstly derived by John A. Pelesko four years ago. Even though numerics give lots of information and point to many conjectures, the existing arsenal of nonlinear analysis and PDE techniques can only tackle a precious few. However, its partial resolution has already led to a successful and exciting interdisciplinary scientific effort.

The present version of this monograph consists of seven Chapters. Chapter 1 is an introduction of MEMS, together with a brief derivation of a PDE modeling electrostatic MEMS. Chapters 2 to 5 are focussed on pull-in voltage and the analysis of an elliptic problem with singular nonlinearity, where it assumes some familiarity with basic elliptic PDE theory, measure theory and functional analysis. While, Chapters 6 and 7 are devoted to the corresponding parabolic problem with singular nonlinearity, where it is useful to have some prior knowledge of basic parabolic PDE theory. Some comments and many unsolved problems are also given in the final section of each Chapter.

The authors would like to thank Louis Nirenberg, Xavier Cabré, Tai-Peng Tsai, Stephen Gustafson, Daniele Cassani and Abbas Moameni for helpful discussions on the subject. The authors would also like to thank Zongming Guo and Juncheng Wei for sharing their research results before publication.

Chapter 1

Introduction

The roots of micro-system technology lie in the technological developments accompanying World War II. In particular, the development of radar stimulated research in the synthesis of pure semiconducting materials. These materials, especially pure silicon, have become the lifeblood of integrated and modern technology of MEMS: Micro-Electromechanical Systems, which combine electronics with micro-size mechanical devices to design various types of microscopic machinery. An overview of the rapidly developing field of MEMS technology is given in [86].

At present, the variety of MEMS devices and applications is continually increasing, and the advent of MEMS has revolutionized numerous branches of science and industry. Already firmly established as an essential component of modern sensors, such as those used for automobile airbag deployment, MEMS are making inroads into areas as diverse as the biomedical industry, space exploration, and telecommunications.

1.1 Electrostatic MEMS devices

Spurred by rapid advances in integrated circuit manufacturing, microsystems process technology is already well developed. As a result, researchers are increasingly focusing their attention on device engineering questions. Foremost among these is the question of how to provide accurate, controlled, stable locomotion for MEMS devices. Just as what has been recognized for some time, it is neither feasible nor desirable to attempt to reproduce modes of locomotion used in the macro world. In fact, the unfavorable scaling of force with device size prohibits this approach in many cases. For example, magnetic forces, which are often used for actuation in the macro world, scale poorly into the micro domain, decreasing in strength by a factor of ten thousand when linear dimensions are reduced by a factor of ten. This unfavorable scaling renders magnetic forces essentially useless. At the micro level, researchers have proposed a variety of new modes of locomotion based upon thermal, biological, and electrostatic forces. The use of electrostatic forces to provide locomotion for MEMS devices is the subject of this text.

Everyday micro-systems use the Coulomb force to grab, pimp, bend, spin, and even slide. Experimental work in this area dates back to 1967 and the work of Nathanson et. al. [83]. In their seminal paper, Nathanson and his coworkers describe the manufacture, experimentation with, and modeling of, a millimeter-sized resonant gate transistor. This early MEMS devices utilized both electrical and mechanical components on the same substrate resulting in improved efficiency, lowered cost, and reduced system size. Nathanson and his coworkers even introduced a simple lumped mass-spring model of electrostatic actuation. In an interesting parallel development, the prolific British scientist G.I. Taylor [92] investigated electrostatic actuation at about the same time as Nathanson. While Taylor was concerned with electrostatic deflection of soap films rather than the development of MEMS devices, his work spawned a small body of the literature with relevance to MEMS. Since Nathanson and Taylor's seminal work, numerous investigators have been continually exploring new uses of electrostatic actuation, such as Micropumps, Microswitches, Microvalves, Shuffle Motor and etc. See [86] for more details of these devices that use electrostatic forces for their operations.

1.2 PDEs modeling electrostatic MEMS

As mentioned in §1.1, many MEMS devices use electrostatic forces for their operation. A key component of some MEMS systems is the simple idealized electrostatic device shown in Figure 1.1. The upper part of this device consists of a thin and deformable elastic membrane that is held fixed along its boundary and which lies above a parallel rigid grounded plate. This elastic membrane is modeled as a dielectric with a small but finite thickness. The upper surface of the membrane is coated with a negligibly thin metallic conducting film. When a voltage V is applied to the conducting film, the thin dielectric membrane deflects towards the bottom plate. A similar deflection phenomenon, but on a macroscopic length scale, occurs in the field of electrohydrodynamics. In this context, Taylor [92] studied the electrostatic deflection of two oppositely charged soap films, and he predicted a critical voltage for which the two soap films would touch together.

A similar physical limitation on the applied voltage occurs for the MEMS device of Figure 1.1, in that there is a maximum voltage V^* –known as pull-in voltage– which can be safely applied to the system. More specifically, if the applied voltage V is increased beyond the critical value V^*, the steady-state of the elastic membrane is lost, and proceeds to snap through at a finite time creating the so-called pull-in instability (cfr. [51, 52, 64, 85]). The existence of such a pull-in voltage was first demonstrated for a lumped mass-spring model of electrostatic actuation in the pioneering study of [83], where the restoring force of the deflected membrane is modeled by a Hookean spring. In this lumped model the attractive inverse square law electrostatic force between the membrane and the ground plate dominates the restoring force of the spring for small gap sizes and large applied voltages. This leads to snap-through behavior whereby the membrane hits the ground plate when the applied voltage is large enough.

Following the analysis in [64, 86, 88], in the following we shall formulate some partial differential equations modeling dynamic deflection $\widehat{w} = \widehat{w}(x', y', t')$ of the membrane shown

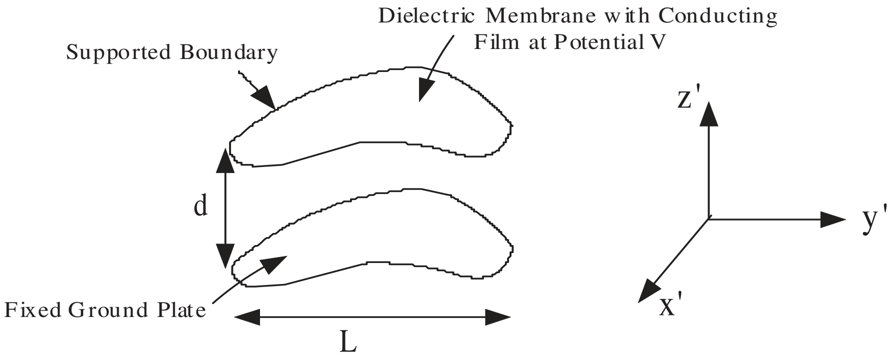

Figure 1.1: *The simple electrostatic MEMS device.*

in Figure 1.1.

1.2.1 Analysis of elastic problem

We first consider the elastic problem of the membrane in the dynamic case. We shall apply Hamilton's principle and minimize the *action* $\mathcal{S}$ of the system, which is referred to as the principle of least action. Here the action consists of kinetic energy, damping energy and potential energy in the system. The pointwise total of these energies is the *Lagrangian* for the system. Denoting the Lagrangian by $\mathcal{L}$ we have

$$\begin{aligned} \mathcal{S} = \int_{t_1}^{t_2} \int_{\Omega'} \mathcal{L} dX'dt' &= \text{Kinetic Energy} + \text{Damping Energy} + \text{Potential Energy} \\ &=: \ E_k + E_d + E_p \,, \end{aligned} \tag{1.2.1}$$

where Ω' is the domain of the membrane with respect to (x', y'). In this subsection, dX' denotes $dx'dy'$, and the gradient ∇' (and the Laplace operator Δ') denotes the differentiation only with respect to x' and y'.

For the dynamic deflection $\widehat{w} = \widehat{w}(x', y', t')$ of the membrane, the kinetic energy E_k satisfies

$$E_k = \frac{\rho A}{2} \int_{t_1}^{t_2} \int_{\Omega'} \widehat{w}_{t'}^2 dX'dt' \,, \tag{1.2.2}$$

where ρ is the mass density per unit volume of the membrane, and A is the thickness of the membrane. The damping energy E_d is assumed to satisfy

$$E_d = \frac{a}{2} \int_{t_1}^{t_2} \int_{\Omega'} \widehat{w}^2 dX'dt' \,, \tag{1.2.3}$$

where a is the damping constant.

For this model, the potential energy E_p is composed of

$$E_p = \text{Stretching Energy} + \text{Bending Energy}\,.$$

It is reasonable to assume that the stretching energy in the elastic membrane is proportional to the changes in the area of the membrane from its un-stretched configuration. Since we assume the membrane is held fixed at its boundary, we may write the stretching energy as

$$\text{Stretching Energy} := -\mu\Big(\int_{t_1}^{t_2}\int_{\Omega'}\sqrt{1+|\nabla'\widehat{w}|^2}dX'dt' - |\Omega'|(t_2-t_1)\Big)\,.$$

Here the constant of proportionality, μ, is simply the tension in the membrane. We linearize this expression to obtain

$$\text{Stretching Energy} := -\frac{\mu}{2}\int_{t_1}^{t_2}\int_{\Omega'}|\nabla'\widehat{w}|^2 dX'dt'\,.$$

The bending energy is assumed to be proportional to the linearized curvature of the membrane, that is

$$\text{Bending Energy} := -\frac{D}{2}\int_{t_1}^{t_2}\int_{\Omega'}\big(\Delta'\widehat{w}\big)^2\,dX'dt'\,.$$

Here the constant D is the flexural rigidity of the membrane. For the total potential energy E_p we now have

$$E_p = -\int_{t_1}^{t_2}\int_{\Omega'}\big(\frac{\mu}{2}|\nabla'\widehat{w}|^2 + \frac{D}{2}(\Delta'\widehat{w})^2\big)dX'dt'\,. \tag{1.2.4}$$

Now combining (1.2.1)-(1.2.3) and (1.2.4) yields that

$$\mathcal{L} = \frac{\rho A}{2}\widehat{w}_{t'}^2 + \frac{a}{2}\widehat{w}^2 - \frac{\mu}{2}|\nabla'\widehat{w}|^2 - \frac{D}{2}(\Delta'\widehat{w})^2\,.$$

According to Hamilton's principle, we minimize

$$\int_{t_1}^{t_2}\int_{\Omega'}\big(\frac{\rho A}{2}\widehat{w}_{t'}^2 + \frac{a}{2}\widehat{w}^2 - \frac{\mu}{2}|\nabla'\widehat{w}|^2 - \frac{D}{2}(\Delta'\widehat{w})^2\big)dX'dt'\,,$$

which implies that the elastic membrane's deflection $\widehat{w}$ satisfies

$$\rho A\frac{\partial^2\widehat{w}}{\partial t'^2} + a\frac{\partial\widehat{w}}{\partial t'} - \mu\Delta'\widehat{w} + D\Delta'^2\widehat{w} = 0\,. \tag{1.2.5}$$

1.2.2 Analysis of electrostatic problem

In this subsection, we analyze the electrostatic problem of Figure 1.1 and we assume the dielectric permittivity $\varepsilon_2 = \varepsilon_2(x',y')$ of the elastic membrane can exhibit a spatial variation characterizing the varying dielectric permittivity of the membrane. Therefore, in view of (1.2.5) we assume the membrane's deflection $\widehat{w}$ satisfying

$$\rho A\frac{\partial^2\widehat{w}}{\partial t'^2}+a\frac{\partial\widehat{w}}{\partial t'}-\mu\Delta'\widehat{w}+D\Delta'^2\widehat{w}=-\frac{\varepsilon_2}{2}\left(|\nabla'\phi|^2+(\frac{\partial\phi}{\partial z'})^2\right). \tag{1.2.6}$$

where the term on the right hand side of (1.2.6) denotes the force on the elastic membrane due to the electric field. We suppose that such force is proportional to the norm squared of the gradient of the electrostatic potential and couples the solution of the elastic problem to the solution of the electrostatic problem. A derivation of such source term may be found in [74].

We now apply dimensionless analysis to equation (1.2.6). We scale the electrostatic potential with the applied voltage V, time with a damping timescale of the system, the x' and y' variables with a characteristic length L of the device, and z' $\widehat{w}$ with the size of the gap d between the ground plate and the undeflected elastic membrane. So we define

$$w=\frac{\widehat{w}}{d},\quad \psi=\frac{\phi}{V},\quad x=\frac{x'}{L},\quad y=\frac{y'}{L},\quad z=\frac{z'}{d},\quad t=\frac{\mu t'}{aL^2}, \tag{1.2.7}$$

and substitute these into equation (1.2.6) to find

$$\gamma^2\frac{\partial^2 w}{\partial t^2}+\frac{\partial w}{\partial t}-\Delta w+\delta\Delta^2 w=-\lambda\Big(\frac{\varepsilon_2}{\varepsilon_0}\Big)\Big[\varepsilon^2|\nabla\psi|^2+(\frac{\partial\psi}{\partial z})^2\Big]\quad \text{in}\quad \Omega, \tag{1.2.8}$$

where Ω is the dimensionless domain of the elastic membrane. Here the parameter γ satisfies

$$\gamma=\frac{\sqrt{\rho\mu A}}{aL}, \tag{1.2.9}$$

and the parameter δ measures the relative importance of tension and rigidity and is defined by

$$\delta=\frac{D}{\mu L^2}. \tag{1.2.10}$$

The parameter ε is the aspect ratio of the system

$$\varepsilon=\frac{d}{L},$$

and the parameter λ is a ratio of the reference electrostatic force to the reference elastic force and is defined by

$$\lambda=\frac{V^2L^2\varepsilon_0}{2\mu d^3}. \tag{1.2.11}$$

In the point of equation (1.2.6), in order to further understand membrane's deflection, we need to know more about the electrostatic potential ϕ inside the elastic membrane. In the actual design of a MEMS device there are several issues that must be considered. Typically, one of the primary device design goals is to achieve the maximum possible stable steady-state deflection, referred to as the pull-in distance, with a relatively small applied voltage V. Another consideration may be to increase the stable operating range of the device by increasing the pull-in voltage V^* subject to the constraint that the range of the applied voltage is limited by the available power supply. This increase in the stable operating range may be important for the design of microresonators. For other devices such as micropumps and microvalves, where snap-through (or called touchdown) behavior is explicitly exploited, it is of interest to decrease the time for touchdown, thereby increasing the switching speed. One way of achieving larger values of V^* while simultaneously increasing the pull-in distance, is to use a voltage control scheme imposed by an external circuit in which the device is placed (cfr. [88]). This approach leads to a nonlocal problem for the deflection of the membrane. A different approach is to introduce a spatial variation in the dielectric permittivity of the membrane, which was theoretically studied in [51, 52, 63, 64, 85].

In the following we discuss the electrostatic potential ϕ by introducing a spatial varying dielectric permittivity, from which we shall further formulate our simple MEMS model. The idea is to locate the region where the membrane deflection $\widehat{w}$ would normally be smallest ($\widehat{w} = 0$ corresponds to touch the fixed ground plate) under a spatially uniform permittivity, and then make sure that a new dielectric permittivity ε_2 is largest –and consequently the profile $f(x,y) = \frac{\varepsilon_0}{\varepsilon_2(Lx,Ly)}$ smallest (see (1.2.18))– in that region.

We assume that the ground plate, located at $z' = 0$, is a perfect conductor. The elastic membrane is assumed to have a uniform thickness $A = 2d\iota$. The deflection of the membrane at time t' is specified by the deflection of its center plane, located at $z' = \widehat{w}(x', y', t')$. Hence the top surface is located at $z' = \widehat{w}(x', y', t') + d\iota$, while the bottom of the membrane is located at $z' = \widehat{w}(x', y', t') - d\iota$. We also assume that the potential in the region between the membrane and ground plate, ϕ_1, satisfies

$$\begin{aligned} &\Delta\phi_1 = 0\,, \\ &\phi_1(x', y', 0) = 0 \quad \text{in} \quad \Omega'\,, \end{aligned}$$

where we assume that the fixed ground plate is held at zero potential. The potential inside the membrane, ϕ_2, satisfies

$$\begin{aligned} &\nabla \cdot (\varepsilon_2 \nabla\phi_2) = 0\,, \\ &\phi_2(x', y', \widehat{w} + d\iota) = V \quad \text{in} \quad \Omega'\,. \end{aligned}$$

Defining the dimensionless scaled potential

$$\psi = \frac{\phi_1}{V}\,, \quad \psi = \frac{\phi_2}{V}$$

together with (1.2.7), and applying the dimensionless analysis again, the electrostatic problem reduces to

$$\frac{\partial^2\psi}{\partial z^2} + \varepsilon^2\Big(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\Big) = 0\,, \quad 0 \le z \le w - \iota\,; \tag{1.2.12a}$$

$$\varepsilon_2\frac{\partial^2\psi}{\partial z^2} + \varepsilon^2\Big(\frac{\partial}{\partial x}\big(\varepsilon_2\frac{\partial\psi}{\partial x}\big) + \frac{\partial}{\partial y}\big(\varepsilon_2\frac{\partial\psi}{\partial y}\big)\Big) = 0\,, \quad w - \iota \le z \le w + \iota\,; \tag{1.2.12b}$$

$$\psi = 0\,,\ z = 0 \text{ (ground plate)}; \quad \psi = 1\,,\ z = w + \iota \text{ (upper membrane surface)}\,, \tag{1.2.13}$$

together with the continuity of the potential and the displacement field across $z = w - \iota$. Here, $\varepsilon \equiv d/L$ is the device aspect ratio.

In general, we note that one has little hope of finding an exact solution ψ from (1.2.12) and (1.2.13). However, we can simplify the system by examining a restricted parameter regime. In particular, we consider the small-aspect ratio limit $\varepsilon \equiv d/L \ll 1$. Physically, this means that the lateral dimensions of the device in Figure 1.1 are larger compared to the size of the gap between the undeflected membrane and ground plate. In the small-aspect ratio limit $\varepsilon \ll 1$, equation (1.2.12) gives $\frac{\partial^2\psi}{\partial z^2} = 0$. Further, the asymptotical solution of ψ which is continuous across $z = w - \iota$ is

$$\psi = \begin{cases} \psi_L \frac{z}{w-\iota}\,, & 0 \le z \le w - \iota\,, \\ 1 + \frac{(1-\psi_L)}{2\iota}\left(z - (w+\iota)\right)\,, & w - \iota \le z \le w + \iota\,. \end{cases} \tag{1.2.14}$$

To ensure that the displacement field is continuous across $z = w - \iota$ to leading order in ε, we impose that

$$\varepsilon_0\frac{\partial\psi}{\partial z}\Big|_- = \varepsilon_2\frac{\partial\psi}{\partial z}\Big|_+\,,$$

where the plus or minus signs indicate that $\frac{\partial\psi}{\partial z}$ is to be evaluated on the upper or lower side of the bottom surface $z = w - \iota$ of the membrane, respectively. This condition determines ψ_L in (1.2.14) as

$$\psi_L = \Big[1 + \frac{2\iota}{w-\iota}\Big(\frac{\varepsilon_0}{\varepsilon_2}\Big)\Big]^{-1}\,. \tag{1.2.15}$$

From (1.2.14) and (1.2.15), we observe that the electric field in the z-direction inside the membrane is independent of z, and is given by

$$\frac{\partial\psi}{\partial z} = \frac{\varepsilon_0}{\varepsilon_2(w-\iota)}\Big[1 + \frac{2\iota}{w-\iota}\frac{\varepsilon_0}{\varepsilon_2}\Big]^{-1} \sim \frac{\varepsilon_0}{\varepsilon_2 w} \quad \text{for} \quad \iota \ll 1\,. \tag{1.2.16}$$

In engineering parlance, this approximation is equivalent to ignoring fringing fields. Therefore, in the small-aspect ratio limit $\varepsilon \ll 1$, the governing equation (1.2.8) is simplified from (1.2.16) into

$$\gamma^2\frac{\partial^2 w}{\partial t^2} + \frac{\partial w}{\partial t} - \Delta w + \delta\Delta^2 w = -\frac{\lambda\varepsilon_0}{\varepsilon_2 w^2} \quad \text{in} \quad \Omega\,.$$

We now suppose the membrane is undeflected at the initial time, that is $w(x, y, 0) = 1$. Since the boundary of the membrane is held fixed, we have $w(x, y, t) = 1$ on the boundary of Ω at any time $t > 0$. We now also assume that the membrane's thickness $A = 2d\iota$ satisfies $A = 2d\iota \ll 1$ which gives $\gamma \ll 1$ in view of (1.2.9), and assume that the elastic membrane has no rigidity which gives $\delta = 0$ in view of (1.2.10). Therefore, by using further simplification, the dynamic deflection $w = w(x,t)$ of the membrane on a bounded domain Ω in $\mathbb{R}^2$, is found to satisfy the following parabolic problem

$$\frac{\partial w}{\partial t} - \Delta w = -\frac{\lambda f(x)}{w^2} \qquad \text{for} \quad x \in \Omega\,, \tag{1.2.17a}$$

$$w(x,t) = 1 \qquad \text{for} \quad x \in \partial\Omega\,, \tag{1.2.17b}$$

$$w(x,0) = 1 \qquad \text{for} \quad x \in \Omega\,, \tag{1.2.17c}$$

where the parameter $\lambda > 0$ is called the applied voltage in view of relation (1.2.11), while the nonnegative continuous function $f(x)$ characterizes the varying dielectric permittivity of the elastic membrane, in the point of the relation

$$f(x) = \frac{\varepsilon_0}{\varepsilon_2(Lx)}\,. \tag{1.2.18}$$

Therefore, understanding dynamic deflection of our MEMS model is equivalent to studying solutions of (1.2.17).

1.3 Overview and some comments

This book is divided into two major Parts.

Part I: Semilinear Elliptic Problems with Singular Nonlinearities

The first part of this monograph is focussed on the stationary case of problem (1.2.17). For convenience, by setting $w = 1 - u$ we study the following semilinear elliptic problem with a singular nonlinearity

$$\begin{cases} -\Delta u = \dfrac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{S}_\lambda$$

where $\lambda > 0$ denotes the applied voltage and the nonnegative continuous function $f(x)$ characterizes the varying dielectric permittivity of the elastic membrane. Mathematically, we consider the domain $\Omega \subset \mathbb{R}^N$ with any dimension $N \geq 1$. The elliptic problem $(S)_\lambda$ was first studied by Pelesko in [85], where the author focussed on lower dimension $N = 1$ or 2, and he considered the profiles $f(x) \geq C > 0$ or $f(x) = |x|^\alpha$. The study of $(S)_\lambda$ was then extended by Guo, Pan and Ward in [64], where the authors considered $(S)_\lambda$ for a more general profile $f(x)$ which can vanish at somewhere. In the past two years, the elliptic problem $(S)_\lambda$ was further extended and sharpened in [36, 38, 51, 67, 68] and the references therein.

Chapter 2 is a detailed study of pull-in voltage λ^* and minimal positive solutions of $(S)_\lambda$. We first investigate the existence of pull-in voltage λ^* defined as

$$\lambda^* = sup\{\lambda > 0 : (S)_\lambda \text{ possesses at least one solution}\}.$$

Numerically and analytically, we also discuss pull-in voltage's analytic bounds and dependence on the size and the shape of the domain, as well as on the permittivity profile of the elastic membrane. We then discuss linear stability of minimal solutions of $(S)_\lambda$ and, by means of energy estimates, we establish some properties of the minimal branch, such as compactness for $1 \leq N \leq 7$, uniqueness and comparison results. Compactness of minimal branch solutions of $(S)_\lambda$ is extended to higher dimension $N \geq 8$ with a power-law permittivity profile $f(x) = |x|^\alpha$ on the unit ball.
We establish also some uniqueness results of positive solutions of (S_λ) for small λ. In particular, we introduce Guo and Wei's works [67, 68], where such uniqueness is considered in the power-law case and established, by means of a monotonicity formula, under fairly general assumptions.

When compactness of the minimal branch holds, it is classical to show the existence of a second branch U_λ of solutions for (S_λ), for λ on the deleted left neighborhood of λ^*. For them, in Chapter 3 we provide a Mountain Pass variational characterization . The main goal now is to apply a blow-up analysis to our singular nonlinearity case, to establish compactness of solutions with Morse index-1 (lying along the second branch) by using spectral information. As far as we know, there are no such compactness results in the case of regular nonlinearities, marking a substantial difference with the singular situation. As a byproduct, we are able to follow the second branch of the bifurcation diagram and prove the existence of a second solution for λ in a natural range.

Chapter 4 is a continuation and a strong improvement of Chapters 2 & 3. §4.1 will be devoted to the compactness of any solution for $(S)_\lambda$ in terms of spectral information. One of the main results in Chapter 4 is the equivalence between compactness, energy bounds and Morse index of solutions for $(S)_\lambda$. Using this equivalence, in §4.2 we shall discuss the uniqueness of solutions for $(S)_\lambda$ in the class of finite Morse index for λ small or close to $\lambda*$. We also prove the existence of singular solutions for $(S)_\lambda$ with $2 \leq N \leq 7$.
In §4.3, following Guo and Wei's papers [67, 68], a general uniqueness result for $(S)_\lambda$ when λ is small is established in a symmetric setting for the power-law case. Radial solutions in a ball for any $N \geq 2$ or solutions in $2D$-symmetric domains for $f \equiv 1$ are considered. Moreover, infinite multiplicity holds as the branch of positive solutions is seen to undergo infinitely many bifurcations as the maximums of the solutions on the branch go to 1 (possibly only changes of direction). This gives an analytic proof for some bifurcation phenomena observed in §2.5. Central to this analysis are ODE estimates and a one-dimensional Sobolev inequality. One dimensional asymptotic analysis is also considered in §4.4.

The asymptotic analysis, introduced in Chapter 2 and deeply used in Chapters 3 & 4, leads to the following limiting problem:

$$\Delta U = \frac{|y|^\alpha}{U^2} \quad \text{in } \mathbb{R}^2, \;\; U(0) = 1, \; U(y) \geq 1.$$

In Chapter 5, we establish a general (linear) unstability property for entire solutions of such an equation. It is the key ingredient to relate spectral properties and compactness along any branch. In §5.2, we establish the radial symmetry in $2D$ of any enire solution U which arises from the blow up of a non compact sequence of solutions of $(S)_\lambda$ in symmetric domains (see [68]).

Part II: Semilinear Parabolic Problems with Singular Nonlinearities

The second part of this monograph is devoted to dynamic deflection of (1.2.17). When $f(x) \equiv 1$, there have already existed some results for touchdown (quenching) behavior of (1.2.17) since 1980s, see [61, 62, 79] and references therein. However, since the profile $f(x)$ is assumed to be varying and vanish somewhere for MEMS models, the dynamic behavior of (1.2.17) turns out to be a more rich source of interesting mathematical phenomena. So far the dynamic behavior of (1.2.17) with varying profile has been investigated in [46, 52, 63, 64].

In Chapter 6 we shall prove that the unique solution w of (1.2.17) must globally converge as $t \to +\infty$, monotonically and pointwise to its unique maximal steady-state when $\lambda \le \lambda^*$; on the other hand, if $\lambda > \lambda^*$ the unique solution w of (1.2.17) must touchdown at finite time T_λ in the sense that $w(x,t)$ reaches 0 at finite time T_λ, and any isolated touchdown point of w can not take place at a zero point of profile $f(x)$. For the case $\lambda > \lambda^*$, we shall analyze and compute finite touchdown time T_λ. In Chapter 6, we also use asymptotic analysis to discuss finite-time touchdown profiles. Pull-in distance of (1.2.17) will be also discussed, together with some interesting numerical phenomena.

The purpose of Chapter 7 is to discuss the refined touchdown behavior of (1.2.17) for the case $\lambda > \lambda^*$. Some a priori estimates of touchdown behavior will be established there, including lower bound estimates, gradient estimates and upper bound estimates. Then we shall obtain the refined touchdown profiles by adapting self-similar method and center manifold analysis. Applying various analytical and numerical techniques, some properties of touchdown set –such as compactness, location and shape– are also discussed in Chapter 7 for different classes of varying permittivity profiles.

Part I

Semilinear Elliptic Problems with Singular Nonlinearities

Chapter 2

Pull-In Voltage and Steady-States

In this Chapter we study pull-in voltage and stationary deflection of the elastic membrane satisfying (1.2.17), such that our discussion is centered on the following elliptic problem

$$\begin{cases} -\Delta u = \dfrac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (S)_\lambda$$

where $\lambda > 0$ characterizes the applied voltage, while nonnegative $f(x)$ describes the varying permittivity profile of the elastic membrane shown in Figure 1.1. Throughout this Chapter and unless mentioned otherwise, solutions for $(S)_\lambda$ are taken in the classical sense. The permittivity profile $f(x)$ will be allowed to vanish somewhere, and will be assumed to satisfy

$$f \in C^\beta(\bar{\Omega}) \text{ for some } \beta \in (0,1],\ 0 \le f \le 1 \text{ and } f \not\equiv 0. \qquad (2.0.1)$$

In §2.1 we mainly show the existence of a specific pull-in voltage in the sense

$$\lambda^*(\Omega, f) = \sup\{\lambda > 0 \mid (S)_\lambda \text{ possesses at least one solution}\},$$

and we also study its dependence on the size and shape of the domain, as well as on the permittivity profile.

These properties will help us in §2.2 to establish some lower and upper bounds on the pull-in voltage, and the main results are stated in Theorem 2.2.4.

In this Chapter, we also consider uniqueness issues for solutions of $(S)_\lambda$ with $0 < \lambda \le \lambda^*$. The bifurcation diagrams in Figure 2.2 of §2.3 show the complexity of the situation, even in the radially symmetric case. One can observe from Figure 2.2 that the number of branches –and of solutions– is closely connected to the space dimension, a fact that we analytically discuss in §2.3, by focussing on the very first branch of solutions considered to be "minimal" in the following way:

Definition 2.0.1. A solution $u_\lambda(x)$ of $(S)_\lambda$ is said to be minimal if for any other solution u of $(S)_\lambda$ we have $u_\lambda(x) \le u(x)$ for all $x \in \Omega$.

On the other hand, one can introduce for any solution u of $(S)_\lambda$, the linearized operator at u defined by $L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3}$ and its eigenvalues $\{\mu_{k,\lambda}(u); k = 1, 2, ...\}$. The first eigenvalue is then simple and is given by:

$$\mu_{1,\lambda}(u) = \inf \Big\{ \langle L_{u,\lambda}\phi, \phi \rangle_{H_0^1(\Omega)};\ \phi \in C_0^\infty(\Omega),\ \int_\Omega \phi^2(x)dx = 1 \Big\}.$$

Stable solutions (resp., *semi-stable solutions*) of $(S)_\lambda$ are those solutions u such that $\mu_{1,\lambda}(u) > 0$ (resp., $\mu_{1,\lambda}(u) \geq 0$). We note that there already exist in the literature many interesting results concerning the properties of the branch of semi-stable solutions for Dirichlet boundary value problems of the form $-\Delta u = \lambda h(u)$ where h is a regular nonlinearity (for example of the form e^u or $(1+u)^p$ for $p > 1$). See for example the seminal papers [31, 75, 76] and also [20] for a survey on the subject and an exhaustive list of related references. Let us mention also the recent developments in [22, 84] for quite general nonlinearities $h(u)$. The singular situation was considered in a very general context in [82], and the analysis of this Chapter is completed to allow for zeroes of the permittivity profile $f(x)$. In [25] the behavior of the stable branch is considered for singular nonlinearities in a larger class than $(1-u)^{-2}$ and for $p-$Laplace operator, $p > 1$.

Our main results in this direction are collected in Theorem 2.4.3, where fine properties of steady states –such as regularity, stability, uniqueness, energy estimates and comparison results– are shown there to depend on the dimension of the ambient space and on the permittivity profile. We discuss also the compactness of the minimal branch for power-like permittivity profiles on the unit ball. To this aim, we introduce a blow-up procedure which we will exploit deeper in Chapters 3 and 4.

In §2.4 we present some uniqueness results for $(S)_\lambda$ for small voltages λ. In particular, Guo and Wei's papers [67, 68] deals with the power-law case $f(x) = |x|^\alpha$ and show that the minimal solution u_λ is the unique solution of $(S)_\lambda$, for $0 < \lambda < \lambda_*$, in some suitable class. §2.4.1 is devoted to a monotonicity inequality, an essential ingredient to establish such an uniqueness property. All these results agree with numerical evidences observed in §2.5.

2.1 The pull-in voltage λ^*

In this section, we first establish the existence and some monotonicity properties for the pull-in voltage λ^*, which is defined as

$$\lambda^*(\Omega, f) = \sup\{\lambda > 0 \mid (S)_\lambda \text{ possesses at least one solution}\}.$$

In other words, λ^* is called pull-in voltage if there exist uncollapsed states for $0 < \lambda < \lambda^*$ while there are none for $\lambda > \lambda^*$. We then study how $\lambda^*(\Omega, f)$ varies with the domain Ω, the dimension N and the permittivity profile f.

2.1.1 Existence of the pull-in voltage

For any bounded domain Ω in $\mathbb{R}^N$, we denote by μ_Ω the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ and by ϕ_Ω the corresponding positive eigenfunction normalized with $\max\limits_{x\in\Omega} \phi_\Omega = 1$. We also associate with any domain Ω in $\mathbb{R}^N$ the following parameter:

$$\nu_\Omega = \sup \left\{\mu_\Gamma H(\inf_\Omega \phi_\Gamma);\ \Gamma \text{ domain of } \mathbb{R}^N,\ \Gamma \supset \bar{\Omega}\right\},$$

where H is the function $H(t) = \frac{t(t+1+2\sqrt{t})}{(t+1+\sqrt{t})^3}$.

Theorem 2.1.1. *Assume f is a function satisfying (2.0.1) on a bounded domain Ω in $\mathbb{R}^N$, then there exists a finite pull-in voltage $\lambda^* := \lambda^*(\Omega, f) > 0$ such that*

1. *if $\lambda < \lambda^*$, there exists at least one solution for $(S)_\lambda$;*
2. *if $\lambda > \lambda^*$, there is no solution for $(S)_\lambda$.*

Moreover, we have the lower bound

$$\lambda^*(\Omega, f) \geq \frac{\nu_\Omega}{\sup\limits_{x\in\Omega} f(x)}. \tag{2.1.1}$$

Proof: By the Implicit Function Theorem, equation $(S)_\lambda$ has a solution bifurcating from the trivial solution $u = 0$ at $\lambda = 0$. By the maximum principle, such a solution is positive. Hence, λ^* is a well defined positive number (possibly $+\infty$). To show the finiteness of λ^*, let $0 < u < 1$ be a solution of $(S)_\lambda$. Integrating the equation against the first (positive) eigenfunction ϕ_Ω, we get

$$\mu_\Omega \int_\Omega u\phi_\Omega = \lambda \int_\Omega \frac{f\phi_\Omega}{(1-u)^2} dx. \tag{2.1.2}$$

Since $0 < u < 1$, by (2.1.2) we deduce the following upper bound:

$$\lambda^* \leq \mu_\Omega \Big(\int_\Omega \phi_\Omega dx\Big)\Big(\int_\Omega f\phi_\Omega dx\Big)^{-1}, \tag{2.1.3}$$

and in particular, $\lambda^* < +\infty$. Once we know that λ^* is a finite positive number, pick $\lambda \in (0, \lambda^*)$ and use the definition of λ^* to find a $\bar{\lambda} \in (\lambda, \lambda^*)$ such that $(S)_{\bar{\lambda}}$ has a solution $u_{\bar{\lambda}}$,

$$-\Delta u_{\bar{\lambda}} = \frac{\bar{\lambda} f(x)}{(1-u_{\bar{\lambda}})^2} \text{ in } \Omega\,;\ \ 0 < u_{\bar{\lambda}} < 1 \text{ in } \Omega\,;\ \ u_{\bar{\lambda}} = 0 \text{ on } \partial\Omega\,,$$

and in particular $-\Delta u_{\bar{\lambda}} \geq \frac{\lambda f(x)}{(1-u_{\bar{\lambda}})^2}$ for $x \in \Omega$. It implies that $u_{\bar{\lambda}}$ is a supersolution of $(S)_\lambda$. Since $u \equiv 0$ is a subsolution of $(S)_\lambda$, then by the sub/super solutions method we can conclude that there is a solution u_λ of $(S)_\lambda$ for every $\lambda \in (0, \lambda^*)$. The definition of λ^* implies that there is no solution of $(S)_\lambda$ for any $\lambda > \lambda^*$.

To conclude, we need to show that $(S)_\lambda$ has at least one solution when $\lambda < \nu_\Omega(\sup_\Omega f(x))^{-1}$. To construct a supersolution of $(S)_\lambda$, we consider a bounded domain $\Gamma \supset \bar{\Omega}$ with smooth boundary, and let $(\mu_\Gamma, \phi_\Gamma)$ be its first eigenpair normalized in such a way that

$$\max_{x\in\Gamma} \phi_\Gamma(x) = 1 \text{ and } \inf_{x\in\Omega} \phi_\Gamma(x) := s_1 > 0.$$

We construct a supersolution in the form $\phi = A\phi_\Gamma$ where A is a scalar to be chosen later. First, we must have $A\phi_\Gamma \geq 0$ on $\partial\Omega$ and $0 < A\phi_\Gamma < 1$ in Ω, which requires that $0 < A < 1$. We also require

$$-\Delta\phi - \frac{\lambda f(x)}{(1-A\phi)^2} \geq 0 \quad \text{in} \quad \Omega\,,$$

which can be satisfied as long as:

$$\mu_\Gamma A\ \phi_\Gamma \geq \frac{\lambda \sup_\Omega f(x)}{(1-A\ \phi_\Gamma)^2} \quad \text{in} \quad \Omega\,,$$

or

$$\lambda \sup_\Omega f(x) < \beta(A,\Gamma) := \mu_\Gamma \min\big\{g(sA);\ s\in[s_1(\Gamma),1]\big\}\,,$$

where $g(s) = s(1-s)^2$. In other words, $\lambda^* \sup_\Omega f(x) \geq \sup\{\beta(A,\Gamma);\ 0<A<1, \Gamma\supset\bar{\Omega}\}$, and therefore it remains to show that

$$\nu_\Omega = \sup\big\{\beta(A,\Gamma);\ 0<A<1,\Gamma\supset\bar{\Omega}\big\}.$$

For that, since $g(s) = (s-1)(3s-1)$ we note first that

$$\min_{s\in[s_1,1]} g(As) = \min\big\{g(As_1), g(A)\big\}.$$

Observe that $g(As_1) \leq g(A)$ rewrites as $A^2(s_1^3-1) - 2A(s_1^2-1) + (s_1-1) \leq 0$ or equivalently $A^2(s_1^2+s_1+1) - 2A(s_1+1) + 1 \geq 0$. The last inequality is true if either $A \leq A_-$ or $A \geq A_+$, where

$$A_+ = \frac{s_1+1+\sqrt{s_1}}{s_1^2+1+s_1} = \frac{1}{s_1+1-\sqrt{s_1}}\,, \qquad A_- = \frac{s_1+1-\sqrt{s_1}}{s_1^2+1+s_1} = \frac{1}{s_1+1+\sqrt{s_1}}\,.$$

Since $A_- < 1 < A_+$, we get that

$$G(A) = \min_{s\in[s_1,1]} g(As) = \begin{cases} g(As_1) & \text{if} \quad 0\leq A\leq A_-\,, \\ g(A) & \text{if} \quad A_-\leq A\leq 1\,. \end{cases}$$

Since $A_- s_1 \leq \frac{1}{3}$, we now have that $\frac{dG}{dA} = g'(As_1)s_1 \geq 0$ for all $0 \leq A \leq A_-$. And since $A_- \geq \frac{1}{3}$, we have $\frac{dG}{dA} = g'(A) \leq 0$ for all $A_- \leq A \leq 1$. It follows that

$$\begin{aligned}\sup_{0<A<1} \min_{s\in[s_1,1]} g(As) &= \sup_{0<A<1} G(A) = G(A_-) = g(A_-) = \frac{1}{s_1+1+\sqrt{s_1}}\Big(1-\frac{1}{s_1+1+\sqrt{s_1}}\Big)^2 \\ &= \frac{s_1(s_1+1+2\sqrt{s_1})}{(s_1+1+\sqrt{s_1})^3} = H\big(\inf_\Omega \phi_\Gamma\big)\,,\end{aligned}$$

which proves our lower estimate. ∎

2.1.2 Monotonicity results for the pull-in voltage

In this subsection, we give a more precise characterization of λ^*, namely as the endpoint for the branch of minimal solutions. This will allow us to establish various monotonicity properties for λ^* that will help in the estimates given in next subsections. First we give a recursive scheme for the construction of minimal solutions.

Theorem 2.1.2. *Assume f is a function satisfying (2.0.1) on a bounded domain Ω in $\mathbb{R}^N$, then for any $0 < \lambda < \lambda^*(\Omega, f)$ there exists a (unique) minimal positive solution u_λ for $(S)_\lambda$. It is obtained as the limit of the increasing sequence $\{u_n(\lambda; x)\}$ constructed recursively as follows: $u_0 \equiv 0$ in Ω and, for each $n \geq 1$,*

$$\begin{aligned} -\Delta u_n &= \frac{\lambda f(x)}{(1-u_{n-1})^2}\,, \quad x \in \Omega\,; \\ 0 \leq u_n < 1\,, &\quad x \in \Omega\,; \quad u_n = 0\,, \quad x \in \partial\Omega\,. \end{aligned} \tag{2.1.4}$$

Proof: Let u be any positive solution for $(S)_\lambda$, and consider the sequence $\{u_n(\lambda; x)\}$ defined in (2.1.4). Clearly $u(x) > u_0 \equiv 0$ in Ω, and whenever $u(x) \geq u_{n-1}$ in Ω, then

$$\begin{aligned} -\Delta(u-u_n) &= \lambda f(x)\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_{n-1})^2}\Big] \geq 0\,, \quad x \in \Omega\,, \\ u - u_n &= 0\,, \qquad x \in \partial\Omega\,. \end{aligned}$$

The maximum principle and an immediate induction yield to $1 > u(x) \geq u_n$ in Ω for all $n \geq 0$. In a similar way, the maximum principle implies that the sequence $\{u_n(\lambda; x)\}$ is monotone increasing. Therefore, $\{u_n(\lambda; x)\}$ converges uniformly to a positive solution $u_\lambda(x)$, satisfying $u(x) \geq u_\lambda(x)$ in Ω. Since this inequality holds for any solution u of $(S)_\lambda$, then u_λ is a minimal positive solution of $(S)_\lambda$ and is clearly unique in this minimal class. ∎

Remark 2.1.1. Let $g(x, \xi, \Omega)$ be the Green's function of the Laplace operator, with $g(x, \xi, \Omega) = 0$ on $\partial\Omega$. Then the iteration in (2.1.4) can be replaced by $u_0 \equiv 0$ in Ω, and for each $n \geq 1$,

$$u_n(\lambda; x) = \lambda \int_\Omega \frac{f(\xi) g(x, \xi, \Omega)}{(1 - u_{n-1}(\lambda; \xi))^2} d\xi\,, \quad x \in \Omega. \tag{2.1.5}$$

The same reasoning as above yields to $\lim_{n\to\infty} u_n(\lambda; x) = u_\lambda(x)$ for all $x \in \Omega$.

The above construction of solutions yields to the following monotonicity result for the pull-in voltage.

Proposition 2.1.3. *If $\Omega_1 \subset \Omega_2$ and if f is a function satisfying (2.0.1) on Ω_2, then $\lambda^*(\Omega_1, f) \geq \lambda^*(\Omega_2, f)$ and the corresponding minimal solutions satisfy $u_{\Omega_1}(\lambda, x) \leq u_{\Omega_2}(\lambda, x)$ on Ω_1 for every $0 < \lambda < \lambda^*(\Omega_2, f)$.*

Proof: Again the sub/super solutions method immediately yields to $\lambda^*(\Omega_1, f) \geq \lambda^*(\Omega_2, f)$. Now consider, for $i = 1, 2$, the sequences $\{u_n(\lambda, x, \Omega_i)\}$ on Ω_i defined by (2.1.5) where $g(x, \xi, \Omega_i)$ are the corresponding Green's functions on Ω_i. Since $\Omega_1 \subset \Omega_2$, we have that $g(x, \xi, \Omega_1) \leq g(x, \xi, \Omega_2)$ on Ω_1. Hence, it follows that

$$u_1(\lambda, x, \Omega_2) = \lambda \int_{\Omega_2} f(\xi) g(x, \xi, \Omega_2) d\xi \geq \lambda \int_{\Omega_1} f(\xi) g(x, \xi, \Omega_1) d\xi = u_1(\lambda, x, \Omega_1)$$

on Ω_1. By induction we conclude that $u_n(\lambda, x, \Omega_2) \geq u_n(\lambda, x, \Omega_1)$ on Ω_1 for all n. On the other hand, since $u_n(\lambda, x, \Omega_2) \leq u_{\Omega_2}(\lambda, x)$ on Ω_2 for any n, we get that $u_n(\lambda, x, \Omega_1) \leq u_{\Omega_2}(\lambda, x)$ on Ω_1, and we are done. ■

We also note the following easy comparison result, whose details are omitted.

Corollary 2.1.4. *Suppose f_1, f_2 are two functions satisfying (2.0.1) such that $f_1(x) \leq f_2(x)$ on Ω, then $\lambda^*(\Omega, f_1) \geq \lambda^*(\Omega, f_2)$, and for $0 < \lambda < \lambda^*(\Omega, f_2)$ we have $u_1(\lambda, x) \leq u_2(\lambda, x)$ on Ω, where $u_1(\lambda, x)$ (resp., $u_2(\lambda, x)$) is the unique minimal positive solution of*

$$-\Delta u = \tfrac{\lambda f_1(x)}{(1-u)^2} \ \textit{(resp., } -\Delta u = \tfrac{\lambda f_2(x)}{(1-u)^2}\textit{)} \text{ on } \Omega \text{ and } u = 0 \text{ on } \partial\Omega.$$

Moreover, if $f_2(x) \neq f_1(x)$ then $u_1(\lambda, x) < u_2(\lambda, x)$ for all $x \in \Omega$.

We shall also need the following result taken and adapted from [9] (Theorem 4.10).

Proposition 2.1.5. *For any bounded domain Ω in $\mathbb{R}^N$ and any function f satisfying (2.0.1), we have*

$$\lambda^*(\Omega, f) \geq \lambda^*(B_R, f^*)$$

where f^* is the Schwarz symmetrization of f and $B_R = B_R(0)$ is the Euclidean ball in $\mathbb{R}^N$, with radius $R > 0$ so that $|B_R| = |\Omega|$.

Proof: If u is a real-valued function on Ω, we define its symmetrized function $u^* : \Omega^* = B_R \to \mathbb{R}$ by

$$u^*(x) = u^*(|x|) = \sup\{\mu : x \in B_R(\mu)\},$$

where $B_R(\mu)$ is the symmetrization of the superlevel set $\Omega(\mu) = \{x \in \Omega : \mu \leq u(x)\}$ (i.e., $B_R(\mu) = \Omega(\mu)^*$). The key propriety of symmetrization is the following inequality (see Lemma 2.4 of [9])

$$\int_\Omega hg dx \leq \int_{B_R} h^* g^* dx \,, \tag{2.1.6}$$

for any h, g continuous functions on Ω. As in Theorem 4.10 of [9], we consider for any $\lambda \in (0, \lambda^*(B_R, f^*))$ the minimal sequence $\{u_n\}$ for (S_λ) as defined in (2.1.4), and let $\{v_n\}$ be the (radial) minimal sequence for the corresponding Schwarz symmetrized problem:

$$-\Delta v = \frac{\lambda f^*(|x|)}{(1-v)^2} \qquad x \in B_R \,, \tag{2.1.7a}$$

$$v = 0 \qquad x \in \partial B_R. \tag{2.1.7b}$$

Since $\lambda \in (0, \lambda^*(B_R, f^*))$, we can consider the corresponding minimal solution $v_\lambda(x) = v_\lambda(|x|)$ for (2.1.7). As in Theorem 2.1.2 we have $0 \le v_n \le v_\lambda < 1$ on B_R for all $n \ge 1$. We shall show that $\{u_n\}$ also satisfies $0 \le u_n^* \le v_\lambda < 1$ on B_R for all $n \ge 1$.

Applying (2.1.6) and the argument for (4.9) in [9], we have that

$$\frac{du_n^*}{dr} + \frac{\lambda}{r^{N-1}} \int_0^r \frac{f^*}{(1-u_{n-1}^*)^2} ds \ge 0 \quad \text{in } (0,R)\,, \tag{2.1.8}$$

and

$$\frac{dv_n}{dr} + \frac{\lambda}{r^{N-1}} \int_0^r \frac{f^*}{(1-v_{n-1})^2} ds = 0 \quad \text{in } (0,R). \tag{2.1.9}$$

We claim that for any $n \ge 1$, we have

$$u_n^*(r) \le v_n(r) \quad r \in (0,R)\,. \tag{2.1.10}$$

In fact, since $u_0^* = v_0 = 0$ in $(0,R)$, we have $du_1^*/dr \ge dv_1/dr$ which, by integration, yields to

$$u_1^*(r) = u_1^*(r) - u_1^*(R) \le v_1(r) - v_1(R) = v_1(R)\,,$$

for any $r \in [0,R]$. (2.1.10) is now proved by induction. If it holds for $n-1$, from (2.1.8) and (2.1.9) then one gets that $du_n^*/dr \ge dv_n/dr$ and, by integration, $u_n^*(r) \le v_n(r)$ for any $r \in [0,R]$. Hence, (2.1.10) is established for all $n \ge 1$.
Since $\max_\Omega u_n = \max_{B_R} u_n^*$, the minimal sequence $\{u_n(x)\}$ on Ω is bounded by $\max_{B_R} v_\lambda(x) < 1$. As in the proof of Theorem 2.1.2, the sequence u_n converges monotonically to a (classical) solution u_λ of $(S)_\lambda$ which is minimal on Ω. This means $\lambda^*(\Omega, f) \ge \lambda^*(B_R, f^*)$. ■

2.2 Estimates for the pull-in voltage

In this section, analytically and numerically we shall discuss estimates of pull-in voltage λ^*. For that we shall write $|\Omega|$ for the volume of a domain Ω in $\mathbb{R}^N$ and $P(\Omega) := \int_{\partial\Omega} dS$ for its "perimeter", with ω_N referring to the volume of the unit ball $B_1(0)$ in $\mathbb{R}^N$. We denote by μ_Ω the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ and by ϕ_Ω the corresponding positive eigenfunction normalized with $\max_\Omega \phi_\Omega = 1$.

2.2.1 Lower bounds for λ^*

While the lower bound in (2.1.1) is useful to prove existence, it is not easy to compute. We will provide below more computationally accessible lower estimates on λ^*.

First of all, we have to enlarge the class of solutions to $(S)_\lambda$ we are interested in.

Definition 2.2.1. A function $u \in L^1(\Omega)$ is a weak solution (resp., supersolution) of $(S)_\lambda$ if

$$\int_\Omega u(-\Delta\phi)dx = \int_\Omega \frac{\lambda f \phi}{(1-u)^2} dx \ \text{(resp., } \ge) \quad \forall\, 0 \le \phi \in C^2(\bar{\Omega}), \phi = 0 \text{ on } \partial\Omega$$

(with the convention $\frac{f}{(1-u)^2}=0$ when $f=0$).

Let us remark that a $H_0^1(\Omega)-$weak solution (resp., supersolution) u of $(S)_\lambda$ simply satisfies

$$\int_\Omega \nabla u \nabla \phi dx = \int_\Omega \frac{\lambda f \phi}{(1-u)^2} dx \text{ (resp., } \geq) \quad \forall\, 0 \leq \phi \in C^2(\bar{\Omega}),\ \phi = 0 \text{ on } \partial\Omega. \tag{2.2.1}$$

By density of $C_0^\infty \cap \{\phi \geq 0\}$ in $H_0^1(\Omega) \cap \{\phi \geq 0\}$ and Levi Theorem, such an equality (resp., inequality) holds for any $0 \leq \phi \in H_0^1(\Omega)$. Observe that we are not assuming any a priori integrability on $\frac{f}{(1-u)^2}$. For weak solutions $u \in H_0^1(\Omega)$, then (2.2.1) rewrites in the more familiar way:

$$\int_\Omega \nabla u \nabla \phi dx = \int_\Omega \frac{\lambda f \phi}{(1-u)^2} dx \quad \forall\, \phi \in H_0^1(\Omega). \tag{2.2.2}$$

The following Proposition does hold:

Proposition 2.2.1. *Assume f is a function satisfying (2.0.1) on a bounded domain Ω in $\mathbb{R}^N$, then we have the following lower bound:*

$$\lambda^*(\Omega, f) \geq \max\Big\{\frac{8N}{27}, \frac{2(3N-4)}{9}\Big\} \frac{1}{\sup_\Omega f} \Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{2}{N}}.$$

Moreover, if $f(x) \equiv |x|^\alpha$ with $\alpha \geq 0$ and Ω is a ball of radius R, then we have

$$\lambda^*(B_R, |x|^\alpha) \geq \max\Big\{\frac{4(2+\alpha)(N+\alpha)}{27}, \frac{(2+\alpha)(3N+\alpha-4)}{9}\Big\} R^{-(2+\alpha)}. \tag{2.2.3}$$

Finally, if $N \geq 8$ and $0 \leq \alpha \leq \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, we have

$$\lambda^*(B_1, |x|^\alpha) = \frac{(2+\alpha)(3N+\alpha-4)}{9}.$$

Proof: Setting $R = \Big(\frac{|\Omega|}{\omega_N}\Big)^{\frac{1}{N}}$, it suffices –in view of Proposition 2.1.5– and since $\sup\limits_{B_R} f^* = \sup\limits_\Omega f$, to show that

$$\lambda^*(B_R, f^*) \geq \max\Big\{\frac{8N}{27R^2 \sup_\Omega f^*}, \frac{2(3N-4)}{9R^2 \sup_\Omega f^*}\Big\} \tag{2.2.4}$$

for the case where $\Omega = B_R$. In fact, the function $w(x) = \frac{1}{3}(1 - \frac{|x|^2}{R^2})$ satisfies on B_R

$$\begin{aligned} -\Delta w &= \frac{2N}{3R^2} = \frac{2N(1-\frac{1}{3})^2}{3R^2} \frac{1}{(1-\frac{1}{3})^2} \geq \frac{8N}{27R^2 \sup_\Omega f} \frac{f(x)}{[1-\frac{1}{3}(1-\frac{|x|^2}{R^2})]^2} \\ &= \frac{8N}{27R^2 \sup_\Omega f} \frac{f(x)}{(1-w)^2}. \end{aligned}$$

So, for $\lambda \leq \frac{8N}{27R^2 \sup_\Omega f}$ w is a supersolution of $(S)_\lambda$ in B_R. Since on the other hand $w_0 \equiv 0$ is a subsolution of $(S)_\lambda$ and $w_0 \leq w$ in B_R, then there exists a solution of $(S)_\lambda$ in B_R which proves a part of (2.2.4). A similar computation applied to the function $v(x) = 1 - (\frac{|x|}{R})^{\frac{2}{3}}$ shows that v is also a supersolution in $B_R \setminus \{0\}$ as long as $0 < \lambda \leq \frac{2(3N-4)}{9R^2 \sup_\Omega f}$. Observe that $3N - 4 > 0$ iff $N \geq 2$.
Since for $N \geq 2$ $v \in H_0^1(B_R)$ and $\frac{1}{(1-v)^2} \in L^1(B_R)$, v is a $H_0^1(B_R)-$weak supersolution of $(S)_\lambda$ (see (2.2.1)). By Proposition 2.4.2 there exists a classical solution of $(S)_\lambda$ in B_R for any $0 < \lambda < \frac{2(3N-4)}{9R^2 \sup_\Omega f}$.

In order to prove (2.2.3), it suffices to note that $w(x) = \frac{1}{3}\big(1 - \frac{|x|^{2+\alpha}}{R^{2+\alpha}}\big)$ is a supersolution for $(S)_\lambda$ on B_R provided $\lambda \leq \frac{4(2+\alpha)(N+\alpha)}{27R^{2+\alpha}}$, and that $v(x) = 1 - (\frac{|x|}{R})^{\frac{2+\alpha}{3}}$ is a $H_0^1(B_R)-$weak supersolution for $(S)_\lambda$ on B_R, provided $\lambda \leq \frac{(2+\alpha)(3N+\alpha-4)}{9R^{2+\alpha}}$ and $\alpha > 4 - 3N$.

In order to complete the proof of Proposition 2.2.1, we need to establish that the function $u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ is the extremal solution on B_1 as long as $N \geq 8$ and $0 \leq \alpha \leq \alpha_N$. This will then yield that, for such dimensions and these values of α, the voltage $\lambda = \frac{(2+\alpha)(3N+\alpha-4)}{9}$ is exactly the pull-in voltage λ^*.
As before, u^* is a $H_0^1(B_1)$-weak solution of $(S)_\lambda$ for any $\alpha > 4 - 3N$, according to (2.2.2). Since $\|u^*\|_\infty = 1$, and by the characterization of Theorem 2.4.1 below, we need only to prove that

$$\int_{B_1} |\nabla\phi|^2 \geq \int_{B_1} \frac{2\lambda|x|^\alpha}{(1-u^*)^3}\phi^2 \quad \forall\phi \in H_0^1(B_1). \tag{2.2.5}$$

However, Hardy's inequality gives for $N \geq 2$:

$$\int_{B_1} |\nabla\phi|^2 \geq \frac{(N-2)^2}{4} \int_{B_1} \frac{\phi^2}{|x|^2}$$

for any $\phi \in H_0^1(B_1)$, which means that (2.2.5) holds whenever $2\lambda \leq \frac{(N-2)^2}{4}$ or, equivalently, if $N \geq 8$ and $0 \leq \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. ■

Remark 2.2.1. First, when $N > \frac{12+\alpha}{5}$ note that $\lambda_2 = \frac{(2+\alpha)(3N+\alpha-4)}{9}$ is the better lower bound in (2.2.3) and is actually sharp on the ball as soon as $N \geq 8$ and $\alpha \leq \alpha_N$.
For lower dimensions, the above lower bounds can be improved by considering supersolutions of the form $v(x) = a(1 - (\frac{|x|}{R})^k)$ and optimizing $\lambda(a, k, R)$ over a and k. For example, in the case where $\alpha = 0$, $N = 2$ and $R = 1$, one can see that a better lower bound $\lambda^* \geq \frac{64\sqrt[4]{5}}{135}$ can be obtained via the supersolution $v(x) = \frac{5}{12}(1 - |x|^{\frac{8}{5}})$.

2.2.2 Upper bounds for λ^*

We note that (2.1.3) is already an upper bound for λ^*. However, other upper bounds can be established according to [64] and [85]:

Proposition 2.2.2. *(1) Assume f is a function satisfying (2.0.1) on a bounded domain Ω in $\mathbb{R}^N$ such that $\inf_\Omega f > 0$, then*

$$\lambda^*(\Omega, f) \le \bar\lambda_1 \equiv \frac{4\mu_\Omega}{27}\big(\inf_\Omega f\big)^{-1}. \tag{2.2.6}$$

(2) More generally, it holds

$$\lambda^*(\Omega, f) \le \bar\lambda_2 \equiv \frac{\mu_\Omega}{3}\Big(\int_\Omega \phi_\Omega dx\Big)\Big(\int_\Omega f\phi_\Omega\, dx\Big)^{-1}. \tag{2.2.7}$$

Proof: (1). Since $u(1-u)^2 \le \frac{4}{27}$ for any $u \in [0,1]$, by (2.1.2) we deduce (2.2.6).

As we will see below, the bound (2.2.6) on λ^* is rather good when applied to the constant permittivity profile $f(x) \equiv 1$ but is useless when $\inf_\Omega f = 0$. To allow power-law permittivity profiles $f(x) = |x|^\alpha$, $\alpha > 0$, it is desirable to obtain a bound on λ^* that depends more on the global properties of f. Such a bound was established in [64] and here is a sketch of its proof.

(2). Multiply now $(S)_\lambda$ by $\phi_\Omega(1-u)^2$, and integrate the resulting equation over Ω to get

$$\int_\Omega \lambda f\phi_\Omega\, dx = -\int_\Omega \phi_\Omega(1-u)^2\Delta u\, dx\,.$$

Using the identity $\nabla\cdot(Hg) = g\nabla\cdot H + H\cdot\nabla g$ for any smooth scalar field g and vector field H, together with the Divergence Theorem, we calculate

$$\int_\Omega \lambda f\phi_\Omega\, dx = -\int_{\partial\Omega}(1-u)^2\phi_\Omega\nabla u\cdot\nu\, dS + \int_\Omega \nabla u\cdot\nabla\left[\phi_\Omega(1-u)^2\right]\, dx\,, \tag{2.2.8}$$

where ν is the unit outward normal to $\partial\Omega$. Since $\phi_\Omega = 0$ on $\partial\Omega$, the first term on the right-hand side of (2.2.8) vanishes. By calculating the second term on the right-hand side of (2.2.8) we get:

$$\begin{aligned}\textstyle\int_\Omega \lambda f\phi_\Omega\, dx &= -\textstyle\int_\Omega 2(1-u)\phi_\Omega|\nabla u|^2\, dx + \int_\Omega(1-u)^2\nabla u\cdot\nabla\phi_\Omega\, dx\\ &\le -\textstyle\int_\Omega \frac{1}{3}\nabla\phi_\Omega\cdot\nabla\left[(1-u)^3\right]\, dx\,.\end{aligned} \tag{2.2.9}$$

The right-hand side of (2.2.9) is evaluated explicitly by

$$\int_\Omega \lambda f\phi_\Omega\, dx \le -\frac{1}{3}\int_{\partial\Omega}(1-u)^3\nabla\phi_\Omega\cdot\nu\, dS - \frac{\mu_\Omega}{3}\int_\Omega(1-u)^3\phi_\Omega\, dx\,. \tag{2.2.10}$$

For $0 \le u < 1$, the last term on the right-hand side of (2.2.10) is negative. Moreover, $u = 0$ on $\partial\Omega$ so that $\int_{\partial\Omega}(1-u)^3\nabla\phi_\Omega\cdot\nu\, dS = -\mu_\Omega\int_\Omega\phi_\Omega$. Therefore, if (S_λ) has a solution, then (2.2.10) yields to

$$\lambda\int_\Omega f\phi_\Omega\, dx \le \frac{\mu_\Omega}{3}\int_\Omega\phi_\Omega\,.$$

This proves that there is no solution for $\lambda > \bar\lambda_2$, which gives (2.2.7). ■

Remark 2.2.2. The above estimate is not sharp, at least in dimensions $1 \leq N \leq 7$, as one can show that there exists $1 > \alpha(\Omega, N, f) > 0$ such that

$$\lambda \leq \frac{\mu_\Omega}{3}(1-\alpha(\Omega, N))\Big(\int_\Omega f\phi_\Omega dx\Big)^{-1}.$$

Indeed, this follows from inequality (2.2.10) above and Theorem 2.3.4 below where it will be shown that in these dimensions, there exists $0 < C(\Omega, N, f) < 1$ independent of λ such that $\|u_\lambda\|_\infty \leq C(\Omega, N, f)$ for any minimal solution u_λ. It is now easy to see that $\alpha(\Omega, N, f)$ can be taken to be

$$\alpha(\Omega, N, f) := \big(1 - C(\Omega, N, f)\big)^3\Big(\int_\Omega \phi_\Omega dx\Big)\Big(\int_\Omega f\phi_\Omega dx\Big).$$

We now consider problem $(S)_\lambda$ in the case where $\Omega \subset \mathbb{R}^N$ is a strictly star-shaped domain with respect to 0, meaning that Ω satisfies:

$$x\cdot\nu \geq a > 0 \quad \text{for all} \quad x \in \partial\Omega\,, \tag{2.2.11}$$

where ν is the unit outward normal to $\partial\Omega$. For such domains, Pohozev-type arguments provide more computable upper bounds:

Proposition 2.2.3. *Suppose $f \equiv 1$ and $\Omega \subset \mathbb{R}^N$ satisfies* (2.2.11)*. Then the pull-in voltage $\lambda^*(\Omega)$ satisfies:*

$$\lambda^*(\Omega) \leq \bar\lambda_3 = \frac{(N+2)^2 P(\Omega)}{8aN|\Omega|}\,, \tag{2.2.12}$$

where $|\Omega|$ is the volume and $P(\Omega)$ is the perimeter of Ω.
In particular, if Ω is the Euclidean unit ball in $\mathbb{R}^N$, then we have the bound

$$\lambda^*(B_1(0)) \leq \frac{(N+2)^2}{8}\,.$$

Proof: For later purposes, let us recall the Pohozaev identity in a suitable general form. Let u is a solution of $\Delta u + \lambda|x|^\alpha g(u) = 0$ in Ω. Then, it holds

$$\begin{aligned}&\lambda(N+\alpha)\int_\Omega |x|^\alpha G(u)dx - \frac{N-2}{2}\lambda\int_\Omega |x|^\alpha u g(u)dx\\ &= \int_{\partial\Omega}\Big((\nabla u\cdot x)\frac{\partial u}{\partial\nu} - \frac{1}{2}|\nabla u|^2 x\cdot\nu + \frac{N-2}{2}u\frac{\partial u}{\partial\nu} + \lambda|x|^\alpha G(u)x\cdot\nu\Big)dS\,,\end{aligned} \tag{2.2.13}$$

where $G(u) = \int g(s)ds$.

Since $u = 0$ on $\partial\Omega$, by (2.2.13) with $g(u) = \frac{1}{(1-u)^2}$ and $G(u) = \frac{u}{1-u}$ we get that

$$\begin{aligned} \frac{\lambda}{2}\int_\Omega \frac{u(N+2-2Nu)}{(1-u)^2}dx &= \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\big(\frac{\partial u}{\partial\nu}\big)^2 dS \\ &\geq \frac{a}{2P(\Omega)}\Big(\int_{\partial\Omega}\frac{\partial u}{\partial\nu}dS\Big)^2 \\ &= \frac{a}{2P(\Omega)}\Big(-\int_\Omega \Delta u dx\Big)^2 \\ &= \frac{a\lambda^2}{2P(\Omega)}\Big(\int_\Omega \frac{dx}{(1-u)^2}\Big)^2, \end{aligned} \tag{2.2.14}$$

where we have used the Divergence Theorem and Hölder's inequality:

$$0 < -\int_{\partial\Omega}\frac{\partial u}{\partial\nu}dS \leq \Big(\int_{\partial\Omega}\big(\frac{\partial u}{\partial\nu}\big)^2 dS\Big)^{1/2}\Big(\int_{\partial\Omega}dS\Big)^{1/2}.$$

Since

$$\begin{aligned} \int_\Omega \frac{u(N+2-2Nu)}{(1-u)^2}dx &= \int_\Omega \Big[-2N\big(u-\frac{N+2}{4N}\big)^2 + \frac{(N+2)^2}{8N}\Big]\frac{1}{(1-u)^2}dx \\ &\leq \frac{(N+2)^2}{8N}\int_\Omega \frac{dx}{(1-u)^2}, \end{aligned}$$

we deduce from (2.2.14) that

$$\frac{(N+2)^2}{8N} \geq \frac{a\lambda}{P(\Omega)}\int_\Omega \frac{dx}{(1-u)^2} \geq \frac{a\lambda|\Omega|}{P(\Omega)},$$

which implies the upper bound (2.2.12) for λ^*. Finally, for the special case where $\Omega = B_1(0) \subset \mathbb{R}^N$, we have $a = 1$ and $\frac{P(B_1(0))}{|B_1(0)|} = N$ and hence the bound $\lambda^*(B_1(0)) \leq \bar\lambda_3 = \frac{(N+2)^2}{8}$ holds. ■

Applying above results of §2.1 and §2.2, we now collect the main results concerning the pull-in voltage λ^*:

Theorem 2.2.4. *Assume f is a function satisfying (2.0.1) on a bounded domain Ω in $\mathbb{R}^N$, then there exists a finite pull-in voltage $\lambda^* := \lambda^*(\Omega, f) > 0$ such that:*

1. *If $0 \leq \lambda < \lambda^*$, there exists at least one solution for $(S)_\lambda$.*
2. *If $\lambda > \lambda^*$, there is no solution for $(S)_\lambda$.*
3. *The following bounds on λ^* hold for any bounded domain Ω:*

$$\underline{\lambda} := \max\Big\{\frac{8N}{27}, \frac{2(3N-4)}{9}\Big\}\frac{1}{\sup_\Omega f}\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{2}{N}} \leq \lambda^*(\Omega, f), \tag{2.2.15a}$$

$$\min\Big\{\bar\lambda_1 := \frac{4\mu_\Omega}{27\inf_{x\in\Omega} f(x)}, \bar\lambda_2 := \frac{\mu_\Omega\int_\Omega \phi_\Omega}{3\int_\Omega f\phi_\Omega\, dx}\Big\} \geq \lambda^*(\Omega, f). \tag{2.2.15b}$$

4. *If Ω is a strictly star-shaped domain with respect to 0, that is $x \cdot \nu(x) \geq a > 0$ for all $x \in \partial\Omega$ ($\nu(x)$ is the unit outward normal at $x \in \partial\Omega$), and if $f \equiv 1$, then*

$$\lambda^*(\Omega) \leq \bar{\lambda}_3 = \frac{(N+2)^2 P(\Omega)}{8aN|\Omega|}.$$

In particular, if $\Omega = B_1(0) \subset \mathbb{R}^N$ then we have the bound $\lambda^*(B_1(0)) \leq \frac{(N+2)^2}{8}$.

5. *If $f(x) \equiv |x|^\alpha$ with $\alpha \geq 0$ and Ω is a ball of radius R, then we have*

$$\lambda^*(B_R, |x|^\alpha) \geq \lambda_c(\alpha) := \max\{\frac{4(2+\alpha)(N+\alpha)}{27}, \frac{(2+\alpha)(3N+\alpha-4)}{9}\}R^{-(2+\alpha)}. \quad (2.2.16)$$

Moreover, if $N \geq 8$ and $0 \leq \alpha \leq \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, we have

$$\lambda^*(B_1, |x|^\alpha) = \frac{(2+\alpha)(3N+\alpha-4)}{9}. \quad (2.2.17)$$

Note that the upper bound $\bar{\lambda}_2$ is valid for all permittivity profiles. However, the order between the two upper bounds in (2.2.15b) can vary in general. For example, in the case of exponential permittivity profiles of the form $f(x) = e^{\alpha(|x|^2-1)}$ on the unit disc, one can see that $\bar{\lambda}_1$ is a better upper bound than $\bar{\lambda}_2$ for small α while the reverse holds true for larger values of α. The lower bounds in (2.2.15a) and (2.2.16) can be improved in small dimensions, but they are optimal –at least for the ball– in dimension no less than 8.

2.2.3 Numerical estimates for λ^*

In this subsection, we apply numerical methods to discuss the bounds of λ^*. In the computations below we shall consider two choices for the domain Ω,

$$\Omega = [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R} \quad \text{(slab)}; \quad \Omega = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2 \quad \text{(unit disk)}.$$

For the permittivity profile, we consider

$$\text{slab}: \quad f(x) = |2x|^\alpha \quad \text{(power-law)}; \quad f(x) = e^{\alpha(x^2-1/4)} \quad \text{(exponential)}, \quad (2.2.18a)$$

$$\text{unit disk}: \quad f(x) = |x|^\alpha \quad \text{(power-law)}; \quad f(x) = e^{\alpha(|x|^2-1)} \quad \text{(exponential)}, \quad (2.2.18b)$$

with $\alpha \geq 0$. To compute the bounds $\bar{\lambda}_1$ and $\bar{\lambda}_2$, we must calculate the first eigenpair μ_Ω and ϕ_Ω of $-\Delta$ on Ω, normalized by $\max_\Omega \phi_\Omega = 1$, for each of these domains. A simple calculation yields to

$$\mu_\Omega = \pi^2, \qquad \phi_\Omega = \sin\left[\pi\left(x + \frac{1}{2}\right)\right] \quad \text{(slab)}; \quad (2.2.19a)$$

$$\mu_\Omega = z_0^2 \approx 5.783, \qquad \phi_\Omega = \frac{z_0}{J_1(z_0)} J_0(z_0|x|) \quad \text{(unit disk)}. \quad (2.2.19b)$$

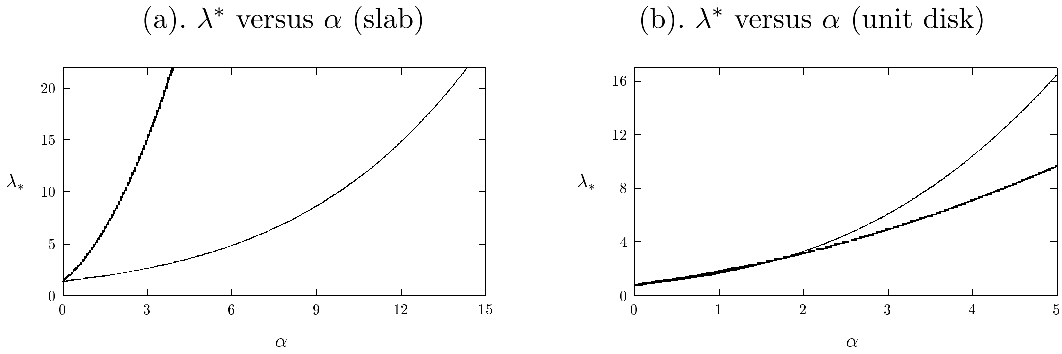

Figure 2.1: *Plots of λ^* versus α for a power-law profile (heavy solid curve) and the exponential profile (solid curve). The left figure corresponds to the slab domain, while the right figure corresponds to the unit disk.*

Exponential Profiles:

Ω	α	$\underline{\lambda}$	λ^*	$\bar{\lambda}_1$	$\bar{\lambda}_2$
slab	0	1.185	1.401	1.462	3.290
slab	1.0	1.185	1.733	1.878	4.023
slab	3.0	1.185	2.637	3.095	5.965
slab	6.0	1.185	4.848	6.553	10.50
unit disk	0	0.593	0.789	0.857	1.928
unit disk	0.5	0.593	1.153	1.413	2.706
unit disk	1.0	0.593	1.661	2.329	3.746
unit disk	3.0	0.593	6.091	17.21	11.86

Table 2.1: *Numerical values for pull-in voltage λ^* with the bounds given in Theorem 2.2.4. Here the exponential permittivity profile is chosen as (2.2.18).*

Power-Law Profiles:

Ω	α	$\lambda_c(\alpha)$	λ^*	$\bar\lambda_1$	$\bar\lambda_2$
slab	0	1.185	1.401	1.462	3.290
slab	1.0	3.556	4.388	∞	9.044
slab	3.0	11.851	15.189	∞	28.247
slab	6.0	33.185	43.087	∞	76.608
unit disk	0	0.593	0.789	0.857	1.928
unit disk	1.0	1.333	1.775	∞	3.019
unit disk	5.0	7.259	9.676	∞	15.82
unit disk	20	71.70	95.66	∞	161.54

Table 2.2: *Numerical values for pull-in voltage λ^* with the bounds given in Theorem 2.2.4. Here the power-law permittivity profile is chosen as (2.2.18).*

Here J_0 and J_1 are Bessel functions of the first kind, and $z_0 \approx 2.4048$ is the first zero of $J_0(z)$. The bounds $\bar\lambda_1$ and $\bar\lambda_2$ can be evaluated by substituting (2.2.19) into (2.2.15b). Notice that $\bar\lambda_2$ is, in general, determined only up to a numerical quadrature.

In Figure 2.1(a) we plot the saddle-node value λ^* versus α for the slab domain. A similar plot is shown in Figure 2.1(b) for the unit disk. The numerical computations are done using BVP solver COLSYS [6] to solve the boundary value problem $(S)_\lambda$ and Newton's method to determine the saddle-node point. Theorem 2.2.4 guarantees a finite pull-in voltage for any $\alpha > 0$, while λ^* is seen to increase rapidly with α. Therefore, by increasing α, or equivalently by increasing the spatial extent where $f(x) \ll 1$, one can increase the stable operating range of the MEMS capacitor. In Table 2.1 we give numerical results for λ^*, together with the bounds given by Theorem 2.2.4, in the case of exponential permittivity profiles, while Table 2.2 deals with power-law profiles. From Table 2.1, we observe that the bound $\bar\lambda_1$ for λ^* is better than $\bar\lambda_2$ for small values of α. However, for $\alpha \gg 1$, we can use Laplace's method on the integral defining $\bar\lambda_2$ to obtain for the exponential permittivity profile that

$$\bar\lambda_1 = \frac{4b_1}{27} e^{c_1\alpha}\,, \qquad \bar\lambda_2 \sim c_2\alpha^2\,. \tag{2.2.20}$$

Here $b_1 = \pi^2$, $c_1 = 1/4$, $c_2 = 1/3$ for the slab domain, and $b_1 = z_0^2$, $c_1 = 1$, $c_2 = 4/3$ for the unit disk. Therefore, for $\alpha \gg 1$, the bound $\bar\lambda_2$ is better than $\bar\lambda_1$. A similar calculation can be done for the power-law profile, see Table 2.2. For this case, it is clear that the lower bound $\lambda_c(\alpha)$ in (2.2.16) is better than $\underline{\lambda}$ in (2.2.15a), and the upper bound $\bar\lambda_1$ is undefined. However, by using Laplace's method, we readily obtain for $\alpha \gg 1$ that $\bar\lambda_2 \sim \alpha^2/3$ for the unit disk and $\bar\lambda_2 \sim 4\alpha^2/3$ for the slab domain.

Therefore, what is remarkable is that $\bar\lambda_1$ and $\bar\lambda_2$ are not comparable even when f is bounded away from 0 and that neither one of them provides the optimal value for λ^*. This leads us to conjecture that there should be a better estimate for λ^*, one involving the distribution of f in Ω, as opposed to the infimum or its average against the first eigenfunction.

2.3 The branch of minimal solutions

In the rest of this Chapter, we consider issues of uniqueness and multiplicity of solutions for $(S)_\lambda$ with $0 < \lambda \leq \lambda^*$. The following bifurcation diagrams in Figure 2.2 show the complexity of the situation, even in the radially symmetric case. One can see that the number of branches –and of solutions– is closely connected to the space dimension. In this section, we focus on the very first branch of solutions considered to be "minimal".

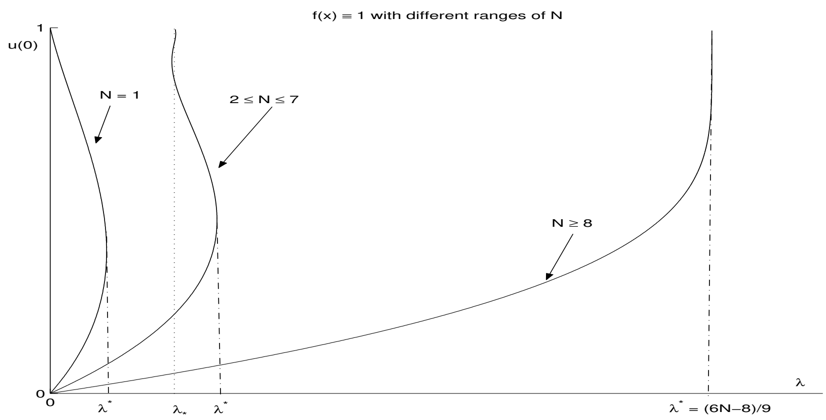

Figure 2.2: *Plots of* $u(0)$ *versus* λ *for profile* $f(x) \equiv 1$ *defined in the unit ball* $B_1(0) \subset \mathbb{R}^N$ *with different ranges of* N. *In the case* $N \geq 8$, *we have* $\lambda^* = 2(3N-4)/9$.

The branch of minimal solutions corresponds to the lowest branch in the bifurcation diagram, the one connecting the origin point $\lambda = 0$ to the first fold at $\lambda = \lambda^*$. To analyze further the properties of this branch, we consider for each solution u of $(S)_\lambda$, the operator

$$L_{u,\lambda} = -\Delta - \frac{2\lambda f}{(1-u)^3}$$

associated with the linearized problem around u. We denote by $\mu_1(\lambda, u)$ the smallest eigenvalue of $L_{u,\lambda}$, that is, the least μ corresponding to the following Dirichlet eigenvalue problem:

$$-\Delta\phi - \frac{2\lambda f(x)}{(1-u)^3}\phi = \mu\phi \quad x \in \Omega\,, \quad \phi = 0 \quad x \in \partial\Omega\,.$$

In other words,

$$\mu_1(\lambda, u) = \inf_{\phi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left\{|\nabla\phi|^2 - 2\lambda f(1-u)^{-3}\phi^2\right\}dx}{\int_\Omega \phi^2 dx}\,.$$

A solution u for $(S)_\lambda$ is said to be *stable* (resp., *semi- stable*) if $\mu_1(\lambda, u) > 0$ (resp., $\mu_1(\lambda, u) \geq 0$).

2.3.1 Spectral properties of minimal solutions

We start with the following crucial Lemma, which shows among other things that semi-stable solutions are necessarily minimal solutions.

Lemma 2.3.1. *Assume f is a function satisfying (2.0.1) on a bounded domain Ω in $\mathbb{R}^N$. Let u, v be a $H_0^1(\Omega)$-weak solution, supersolution of $(S)_\lambda$, respectively. If $\mu_1(\lambda, u) > 0$, then $u \le v$ a.e. in Ω. If $\mu_1(\lambda, u) = 0$, then $u = v$ a.e. in Ω.*

Proof: For any $\theta \in [0, 1]$ and $0 \le \phi \in H_0^1(\Omega)$, we have that

$$\begin{aligned} I_{\theta,\phi} := &\int_\Omega \nabla\big(\theta u + (1-\theta)v\big)\nabla\phi - \int_\Omega \frac{\lambda f(x)}{\big(1-\theta u-(1-\theta)v\big)^2}\phi \\ = &\lambda \int_\Omega f(x)\Big(\frac{\theta}{(1-u)^2} + \frac{1-\theta}{(1-v)^2} - \frac{1}{\big(1-\theta u-(1-\theta)v\big)^2}\Big)\phi \ge 0 \end{aligned}$$

due to the convexity of $s \to 1/(1-s)^2$. Since $I_{1,\phi} = 0$, the derivative of $I_{\theta,\phi}$ at $\theta = 1$ is non positive:

$$\int_\Omega \nabla(u-v)\nabla\phi - \int_\Omega \frac{2\lambda f(x)}{\big(1-u\big)^3}(u-v)\phi \le 0 \tag{2.3.2}$$

for any $0 \le \phi \in H_0^1(\Omega)$. Testing on $(u-v)^+$, we get that

$$\int_\Omega \Big[|\nabla(u-v)^+|^2 - \frac{2\lambda f(x)}{(1-u)^3}\big((u-v)^+\big)^2\Big] \le 0\,.$$

When $\mu_{1,\lambda}(u) \ge 0$, then we have $u \le v$ a.e. in Ω. It is clearly true if $\mu_{1,\lambda}(u) > 0$ but it holds in general. Indeed, we have that:

$$\int_\Omega \nabla(u-v)\nabla\bar\phi - \int_\Omega \frac{2\lambda f(x)}{\big(1-u\big)^3}(u-v)\bar\phi = 0, \tag{2.3.3}$$

where $\bar\phi = (u-v)^+$. Since $I_{\theta,\bar\phi} \ge 0$ for any $\theta \in [0, 1]$ and $I_{1,\bar\phi} = \partial_\theta I_{1,\bar\phi} = 0$, we get that:

$$\partial^2_{\theta\theta} I_{1,\bar\phi} = -\int_\Omega \frac{6\lambda f(x)}{(1-u)^4}(u-v)^2\bar\phi \ge 0\,.$$

Let $\Omega_0 = \{x \in \Omega :\ f(x) = 0\}$. Since $\bar\phi = (u-v)^+$, clearly $(u-v)^+$ a.e. in $\Omega \setminus \Omega_0$ and, by (2.3.3) we get:

$$\int_\Omega |\nabla(u-v)^+|^2 = 0.$$

Hence, $u \le v$ a.e. in Ω as claimed.

When $\mu_{1,\lambda}(u) = 0$, we can push the analysis beyond. Let $\phi_{1,\lambda}$ be the first eigenfunction. We want to establish now the following scheme: if $u < v - \bar t\phi_{1,\lambda}$ on a set A of positive measure, then there exists $\epsilon_0 > 0$ small such that $u < v - t\phi_{1,\lambda}$ a.e. in Ω, for any $\bar{t} \le t < \bar{t} + \epsilon_0$. Indeed, since $\phi_{1,\lambda}$ is in the kernel of the linearized operator $L_{u,\lambda}$, (2.3.2) is still valid when we replace $u - v$ with $u - v - t\phi_{1,\lambda}$. As before, we can get that

$$\int_\Omega |\nabla (u - v - t\phi_{1,\lambda})^+|^2 - \int_\Omega \frac{2\lambda f(x)}{(1-u)^3} \big((u - v - t\phi_{1,\lambda})^+\big)^2 = 0. \tag{2.3.4}$$

By the variational characterization of $\phi_{1,\lambda}$, we get that $(u - v - t\phi_{1,\lambda})^+ = \beta\phi_{1,\lambda}$ a.e. in Ω, for some β. By assumption, we can find a set $A' \subset\subset A$ of positive measure so that $u < v - \bar{t}\phi_{1,\lambda} - \delta$, $\delta > 0$, and hence, $\epsilon_0 > 0$ small so that $u < v - t\phi_{1,\lambda}$ in A', for any $\bar{t} \le t \le \bar{t} + \epsilon_0$. Hence, $\beta\phi_{1,\lambda} = 0$ a.e. in A'. Since $\phi_{1,\lambda} > 0$ in Ω, we have $\beta = 0$ and $u < v + t\phi_{1,\lambda}$ a.e. in Ω, for any $\bar{t} \le t < \bar{t} + \epsilon_0$.
We use now the scheme in the following way. Assume by contradiction that $u = v$ a.e. in Ω does not hold. Since $u \le v$, we find a set A of positive measure so that $u < v$ in A. Applying the sheme with $\bar{t} = 0$, $u < v - t\phi_{1,\lambda}$ a.e. in Ω, for any $0 \le t < \epsilon_0$ and $\epsilon_0 > 0$ small. Set now $t_0 = \sup\{t > 0 : u < v - t\phi_{1,\lambda} \text{ a.e. in } \Omega\}$. Clearly, t_0 is a finite well defined number and $u \le v - t_0\phi_{1,\lambda}$ a.e. in Ω. The scheme above and the maximal property of t_0 imply that necessarily $u = v - t_0\phi_{1,\lambda}$ a.e. in Ω. Hence, (2.3.3) holds for any $0 \le \bar{\phi} \in H_0^1(\Omega)$. Taking $\bar{\phi} = v - u$ and arguing as before, finally we get that

$$\int_\Omega |\nabla(u - v)|^2 = 0.$$

This in contradiction with the assumption: $u < v$ on a set of positive measure. The proof is done. ∎

Theorem 2.3.2. *Assume f is a function satisfying (2.0.1) on a bounded domain Ω in $\mathbb{R}^N$, and consider the branch $\lambda \to u_\lambda$ of minimal solutions on $(0, \lambda^*)$. Then the following properties hold:*

1. *for each $x \in \Omega$, the function $\lambda \to u_\lambda(x)$ is differentiable and strictly increasing on $(0, \lambda^*)$;*
2. *for each $\lambda \in (0, \lambda^*)$, the minimal solution u_λ is stable and the function $\lambda \to \mu_{1,\lambda} := \mu_1(\lambda, u_\lambda)$ is decreasing on $(0, \lambda^*)$.*

Proof: Let $\lambda_1 < \lambda_2 < \lambda^*$ and $f_1 = \frac{\lambda_1}{\lambda_2} f$. Note that u_{λ_1} is the minimal solution of $(S)_{\lambda_2}$ corresponding to f_1. Since $f_1 \le f$ and $f_1 \ne f$, by Corollary 2.1.4 we get that $\lambda^* = \lambda^*(\Omega, f) \le \lambda^*(\Omega, f_1)$ and $u_{\lambda_1} < u_{\lambda_2}$ in Ω for $\lambda_2 < \lambda^*$.
That $\lambda \to \mu_{1,\lambda}$ is decreasing follows easily from the variational characterization of $\mu_{1,\lambda}$, the monotonicity of $\lambda \to u_\lambda$, as well as the monotonicity of $(1-u)^{-3}$ with respect to u. Since $\mu_{1,\lambda} \to \mu_1(-\Delta) > 0$ as $\lambda \to 0^+$, it is well defined

$$\lambda^{**} = \sup\big\{\lambda > 0 :\ u_\lambda \text{ is a stable solution for } (S)_\lambda\big\}$$

and clearly satisfies $\lambda^{**} \le \lambda^*$. If $\lambda^{**} < \lambda^*$, $u_{\lambda^{**}}$ is a minimal solution of $(S)_{\lambda^{**}}$:. Since $u_\lambda \le u_{\lambda^{**}}$ for any $\lambda \le \lambda^{**}$ and $\max_\Omega u_{\lambda^{**}} < 1$, by elliptic regularity theory it follows that the limit $\lim_{\lambda \nearrow \lambda^{**}} u_\lambda \le u_{\lambda^{**}}$ exists in $C^2(\bar\Omega)$ and is a solution of $(S)_{\lambda^{**}}$. By minimality of $u_{\lambda^{**}}$, we get that $\lim_{\lambda \nearrow \lambda^{**}} u_\lambda = u_{\lambda^{**}}$ and $\mu_{1,\lambda^{**}} \ge 0$. By the Implicit Function Theorem, it follows $\mu_{1,\lambda^{**}} = 0$ and by Lemma 2.3.1, $u_{\lambda^{**}} = u_\lambda$ for any $\lambda^{**} < \lambda < \lambda^*$. A contradiction which proves that $\lambda^{**} = \lambda^*$.

Since u_λ is stable, the linearized operator $L_{u_\lambda,\lambda}$ at u_λ is invertible for any $0 < \lambda < \lambda^*$. By the Implicit Function Theorem, $\lambda \to u_\lambda(x)$ is differentiable in λ and by monotonicity, $\frac{du_\lambda}{d\lambda}(x) \ge 0$ for all $x \in \Omega$.

Finally, by differentiating $(S)_\lambda$ with respect to λ, we get

$$-\Delta \frac{du_\lambda}{d\lambda} - \frac{2\lambda f(x)}{(1-u_\lambda)^3}\frac{du_\lambda}{d\lambda} = \frac{f(x)}{(1-u_\lambda)^2} \ge 0\,, \qquad x \in \Omega$$

$$\frac{du_\lambda}{d\lambda} \ge 0\,, \qquad x \in \partial\Omega\,.$$

Applying the strong maximum principle, we conclude that $\frac{du_\lambda}{d\lambda} > 0$ on Ω for all $0 < \lambda < \lambda^*$, and the Theorem is proved. ■

Remark 2.3.1. Lemma 3 of [31] yields to $\mu_1(1,0)$ as an upper bound for λ^{**} – at least in the case where $\inf_\Omega f > 0$. Since $\lambda^{**} = \lambda^*$, this gives another upper bound for λ^* in our setting. It is worth noting that the upper bound in Theorem 2.2.4 gives a better estimate, since in the case $f \equiv 1$ we have $\mu_1(1,0) = \mu_\Omega/2$, while the estimate in Theorem 2.2.4 gives $\frac{4\mu_\Omega}{27}$ for an upper bound.

2.3.2 Energy estimates and regularity

For later purposes, we establish now a basic regularity result for a general boundary value problem:

$$\begin{cases} -\Delta u = \dfrac{f(x)}{(1-u)^2} & \text{in } \Omega, \\ u = \bar u & \text{on } \partial\Omega, \end{cases} \tag{2.3.5}$$

where $0 \le \bar u \in C^1(\bar\Omega)$ is so that $\|\bar u\|_\infty < 1$. According to Definition 2.2.1 and (2.2.2), solutions of (2.3.5) are considered in the following $H^1(\Omega)-$weak sense:

$$\int_\Omega \nabla u \nabla \phi dx = \int_\Omega \frac{f\phi}{(1-u)^2} dx \quad \forall\, \phi \in H_0^1(\Omega)\,, \quad u - \bar u \in H_0^1(\Omega) \tag{2.3.6}$$

(with the convention $\frac{f}{(1-u)^2} = 0$ when $f = 0$). The regularity result we have is the following:

Theorem 2.3.3. *Let f be a function satisfying (2.0.1) on a bounded domain Ω in $\mathbb{R}^N$ and $A > 0$. Let u be a weak solution of (2.3.5) so that*

$$\begin{aligned} &\textit{either } N = 1 \textit{ and } \|\frac{f}{(1-u)^3}\|_{L^1(\Omega)} \le A, \\ &\textit{or } N \ge 2 \textit{ and } \|\frac{f}{(1-u)^3}\|_{L^{N/2}(\Omega)} \le A. \end{aligned} \tag{2.3.7}$$

Then,

1. *if* $\Omega \setminus \Omega_0$ *is connected, then* $u \le 1$ *a.e. in* Ω*;*
2. *if* $u \le 1$ *a.e. in* Ω*, then* $u \in C^1(\bar\Omega)$ *and there exists a constant* $C < 1$ *so that*

$$0 < u \le C \qquad in\ \Omega,$$

where C *depends on* A*,* N*,* $\bar u$*,* Ω *and* f*.*

Here, $\Omega_0 = \{x \in \Omega : f(x) = 0\}$*.*

Proof: First of all, for $N \ge 2$ our assumption (2.3.7) gives that the right hand side of (2.3.5) is in $L^{\frac{3N}{4}}(\Omega)$. Standard regularity theory then implies that $u \in C^{0,\frac{2}{3}}(\bar\Omega) \cap H^1(\Omega)$. If $u(x_0) = 1$ at some $x_0 \in \Omega \setminus \Omega_0$, the Hölder continuity of u implies $|1-u(x)| = |u(x_0)-u(x)| \le C|x-x_0|^{\frac{2}{3}}$. Choosing small $\delta > 0$ so that $\inf\limits_{B_\delta(x_0)} f > 0$, for $N \ge 2$ then we have:

$$(\inf_{B_\delta(x_0)} f)^{-\frac{N}{2}} \int_{B_\delta(x_0)} \Big|\frac{f}{(1-u)^3}\Big|^{\frac{N}{2}} \ge \int_{B_\delta(x_0)} \frac{1}{|1-u|^{\frac{3N}{2}}} \ge \frac{1}{C} \int_{B_\delta(x_0)} \frac{1}{|x-x_0|^N} = +\infty,$$

in contradiction with our integrability assumption (2.3.7). When $N = 1$, the right hand side of (2.3.5) is in $L^{\frac{3}{2}}(\Omega)$, $u \in C^1(\bar\Omega)$ and the above argument works as well. In conclusion, $\{x \in \Omega : u(x) = 1\} \subset \Omega_0$. Let us observe that, if $\inf\limits_{\Omega} f > 0$, then $\Omega_0 = \emptyset$ and Theorem 2.3.3 already holds true in such a case.

Assume now $\Omega \setminus \Omega_0$ to be a connected set. Since $u \ne 1$ on $\Omega \setminus \Omega_0$ and $u = \bar u < 1$ on $\partial\Omega$, by continuity of u we deduce that $u < 1$ in $\Omega \setminus \Omega_0$, and in turn, $u \le 1$ on $\partial\Omega_0$. By convention $\frac{f}{(1-u)^2} = 0$ on Ω_0 and $(1-u)^- = 0$ on $\partial\Omega_0$. Take $(1-u)^-\chi_{\Omega_0} \in H_0^1(\Omega)$ as a test function in (2.3.6) to get $\int_{\Omega_0} |\nabla(1-u)^-|^2 = 0$. Hence, $(1-u)^- = 0$ a.e. in Ω_0 and $u \le 1$ in Ω.
To complete the proof, let us assume $u \le 1$ in Ω. Introduce $T_k u = \min\{u, 1-k\}$ as the truncated function of u at level $1-k$, $0 < k < 1$.

Let us first discuss the case $N = 1, 2$. For k small, take $(1-T_k u)^{-1} - (1-\bar u)^{-1} \in H_0^1(\Omega)$ as a test function in (2.3.6). Since $u \in H^1(\Omega)$ and $0 \le \bar u \in C^1(\bar\Omega)$ with $\|\bar u\|_\infty < 1$, it yields to

$$\begin{aligned} \int_\Omega \frac{|\nabla T_k u|^2}{(1-T_k u)^2} &= \int_\Omega \frac{\nabla u \nabla \bar u}{(1-\bar u)^2} + \int_\Omega \frac{f(x)}{(1-u)^2}\Big((1-T_k u)^{-1} - (1-\bar u)^{-1}\Big) \\ &\le C + \int_\Omega \frac{f(x)}{(1-u)^3} \le C \end{aligned} \tag{2.3.8}$$

in view of (2.3.7), because of $(1-T_k u)^{-1} \le (1-u)^{-1}$ for $u \le 1$.
When $N = 1$, let S be the Sobolev constant of the embedding $H_0^1(I) \hookrightarrow L^\infty(I)$. Since $\log\left(\frac{1-\bar u}{1-T_k u}\right) \in H_0^1(I)$ for k small, by (2.3.8) we get:

$$S \,\Big\| \log\left(\frac{1-\bar u}{1-T_k u}\right) \Big\|_\infty^2 \le \int_I \Big|\nabla \log\left(\frac{1-\bar u}{1-T_k u}\right)\Big|^2 \le C\left(1 + \int_I \frac{|\nabla T_k u|^2}{(1-T_k u)^2}\right) \le C,$$

where C does not depend on k. Taking the limit as $k \to 0$, we get that $\log\left(\frac{1-\bar{u}}{1-u}\right) \in L^\infty(\Omega)$ and then $u \le C < 1$.
When $N = 2$, let us recall a classical consequence of the Moser-Trudinger inequality: there exists $S > 0$ so that

$$\int_\Omega e^{pv} \le S \exp\left(\frac{p^2}{16\pi}\|v\|^2_{H^1_0(\Omega)}\right) \quad \forall\, v \in H^1_0(\Omega)\,,\ p > 1. \tag{2.3.9}$$

Since $\log\left(\frac{1-\bar{u}}{1-T_k u}\right) \in H^1_0(\Omega)$ for k small, by (2.3.8) and (2.3.9) we deduce that for any $p > 1$:

$$\int_\Omega (1 - T_k u)^{-p} \le C \int_\Omega \left(\frac{1-\bar{u}}{1-T_k u}\right)^p \le S \exp\left(\frac{p^2}{16\pi}\int_\Omega |\nabla \log(\frac{1-\bar{u}}{1-T_k u})|^2\right) \le C,$$

where C does not depend on k. Taking the limit as $k \to 0$, we get the validity of

$$\|(1-u)^{-1}\|_{L^p(\Omega)} \le C_p \quad \forall\, p > 1, \tag{2.3.10}$$

where C_p depends on A, N, $\bar{u}$, Ω and f.

Estimate (2.3.10) holds also when $N \ge 3$ but the proof is more involved. For k small, take $(1 - T_k u)^{-p-1} - (1-\bar{u})^{-p-1} \in H^1_0(\Omega)$ as a test function in (2.3.6), and obtain that

$$\begin{aligned} & (p+1)\int_\Omega \frac{|\nabla T_k u|^2}{(1-T_k u)^{p+2}} \\ = \ & (p+1)\int_\Omega \frac{\nabla u \nabla \bar{u}}{(1-\bar{u})^{p+2}} + \int_\Omega \frac{f(x)}{(1-u)^2}\Big((1-T_k u)^{-p-1} - (1-\bar{u})^{-p-1}\Big) \\ \le \ & C + \int_\Omega \frac{f(x)}{(1-u)^3}(1-T_k u)^{-p}. \end{aligned} \tag{2.3.11}$$

Using the relation

$$(a+b)^2 = a^2 + b^2 + 2ab \le (1+\delta)a^2 + \frac{1+\delta}{\delta}b^2$$

for $a, b \in \mathbb{R}$ and $\delta > 0$, we deduce the following estimate:

$$(1-T_k u)^{-p} \le (1+\delta)\left((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}}\right)^2 + \frac{1+\delta}{\delta}(1-\bar{u})^{-p}\,, \quad \delta > 0. \tag{2.3.12}$$

Inserting (2.3.12) with $\delta = 1$ into (2.3.11), we get:

$$(p+1)\int_\Omega \frac{|\nabla T_k u|^2}{(1-T_k u)^{p+2}} \le C + 2\int_\Omega \frac{f(x)}{(1-u)^3}\left((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}}\right)^2. \tag{2.3.13}$$

By (2.3.13) we get that:

$$
\begin{aligned}
&\int_\Omega |\nabla \left((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}}\right)|^2 \\
&\leq 2\int_\Omega |\nabla \left((1-T_k u)^{-\frac{p}{2}}\right)|^2 + C \\
&= \frac{p^2}{2}\int_\Omega \frac{|\nabla T_k u|^2}{(1-T_k u)^{p+2}} + C \\
&\leq \frac{p^2}{p+1}\int_\Omega \frac{f(x)}{(1-u)^3}\left((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}}\right)^2 + C \\
&\leq \frac{p^2}{p+1}\int_{\{1-u\leq\varepsilon\}} \frac{f(x)}{(1-u)^3}\left((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}}\right)^2 + C,
\end{aligned}
$$

where C does not depend on k. Since $\{u(x)=1\}\subset\Omega_0$ and by convention $\frac{f}{(1-u)^3}=0$ on Ω_0, by Hölder inequality and the Sobolev embedding on $(1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}} \in H_0^1(\Omega)$, we finally get that

$$
\begin{aligned}
&\int_\Omega |\nabla \left((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}}\right)|^2 \\
&\leq \frac{p^2}{p+1}\Big(\int_{\{1-u\leq\varepsilon\}} (\frac{f(x)}{(1-u)^3})^{\frac{N}{2}}\Big)^{\frac{2}{N}}\Big(\int_\Omega |(1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}}|^{\frac{2N}{N-2}}\Big)^{\frac{N-2}{N}} + C \\
&\leq \frac{S_N^{-1}p^2}{p+1}\Big(\int_{\{1-u\leq\varepsilon\}} (\frac{f(x)}{(1-u)^3})^{\frac{N}{2}}\Big)^{\frac{2}{N}}\int_\Omega |\nabla \left((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}}\right)|^2 + C \\
&\leq \frac{1}{2}\int_\Omega |\nabla \left((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}}\right)|^2 + C\,,
\end{aligned}
$$

provided that $\varepsilon > 0$ is sufficiently small, where S_N is the Sobolev constant. Hence, by Sobolev embedding:

$$
\Big(\int_\Omega |(1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}}|^{\frac{2N}{N-2}}\Big)^{\frac{N-2}{N}} \leq S_N^{-1}\int_\Omega |\nabla \left((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}}\right)|^2 \leq C,
$$

and in turn

$$
\int_\Omega (1-T_k u)^{-\frac{pN}{N-2}} \leq C,
$$

where $C>0$ does not depend on k. Taking the limit as $k\to 0$, as before we get the validity of (2.3.10).

Now property (2.3.10) for $N\geq 2$ implies $u\leq C<1$, for a constant C depending on A, N, $\bar{u}$, Ω and f. Indeed, if $u(x_0)=1$ for some $x_0\in\Omega$, then $|1-u(x)| = |u(x_0)-u(x)| \leq C|x-x_0|^{\frac{2}{3}}$, as already remarked. This is in contradiction with (2.3.10) for p large. Since $\|u\|_\infty < 1$ implies that the right hand side of (2.3.5) is in $L^p(\Omega)$ for any $p>1$, by elliptic regularity theory $u \in C^1(\bar\Omega)$ and then, we can conclude from the maximum principle (in a weak form) that $0 < u \le \|u\|_\infty < 1$. ■

We have now:

Theorem 2.3.4. *For any dimension $1 \le N \le 7$ there exists a constant $C = C(N, \Omega, f) < 1$ independent of λ such that, for any $0 < \lambda < \lambda^*$ the minimal solution u_λ of $(S)_\lambda$ satisfies $\| u_\lambda \|_\infty \le K$.*
Consequently, $u^* = \lim\limits_{\lambda \uparrow \lambda^*} u_\lambda$ exists in the topology of $C^2(\bar\Omega)$ and is the unique classical solution of $(S)_{\lambda^*}$ among all the $H_0^1(\Omega)-$weak solutions. Moreover, it satisfies $\mu_{1,\lambda^*}(u^*) = 0$.

This result will follow from the following uniform energy estimate on semi-stable solutions for (2.3.5):

Proposition 2.3.5. *There exists a constant $C_p > 0$ such that, for any $H^1(\Omega)$-weak solution $u \le 1$ of (2.3.5) which is semi- stable:*

$$\int_\Omega \left(|\nabla \phi|^2 - \frac{2f(x)}{(1-u)^3}\phi^2 \right) dx \ge 0\,, \qquad \forall\, \phi \in H_0^1(\Omega), \tag{2.3.14}$$

there holds

$$\|\frac{f}{(1-u)^3}\|_{L^p(\Omega)} \le C_p$$

as long as $1 \le p < 1 + \frac{4}{3} + 2\sqrt{\frac{2}{3}}$.

Proof: First of all, let us remark that (2.3.5) on $u - \bar u \in H_0^1(\Omega)$ gives:

$$\begin{aligned}\int_\Omega \frac{f(x)}{(1-u)^2} &\le C\int_\Omega \frac{f(x)}{(1-u)^2}(1-\bar u)\\ &= C\Big(\int_\Omega \nabla u \nabla(u-\bar u) + \int_\Omega \frac{f(x)}{1-u}\Big)\\ &\le C\Big(\|u-\bar u\|_{H_0^1}^2 + \|u\|_{H_0^1}^2 + \epsilon\int_\Omega \frac{f(x)}{(1-u)^2} + \frac{1}{4\epsilon}\int_\Omega f(x)\Big)\end{aligned}$$

for any $\epsilon > 0$, because of the inequality $ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2$. Hence, for $\epsilon = \frac{1}{2C}$ we get:

$$\int_\Omega \frac{f(x)}{(1-u)^2} \le \frac{1}{2}\int_\Omega \frac{f(x)}{(1-u)^2} + C'$$

for some $C' > 0$, and then, $\int_\Omega \frac{f(x)}{(1-u)^2} < +\infty$. Now, (2.3.14) on $u - \bar u \in H_0^1(\Omega)$ gives that

$$\begin{aligned}\int_\Omega \frac{f(x)}{(1-u)^3} &\le 2C\int_\Omega \frac{f(x)}{(1-u)^3}(1-\bar u)^2\\ &\le C\Big(\int_\Omega |\nabla(u-\bar u)|^2 + \int_\Omega \frac{2f(x)}{1-u} + \int_\Omega \frac{4f(x)}{(1-u)^2}(u-\bar u)\Big)\\ &= C\Big(\int_\Omega |\nabla(u-\bar u)|^2 + \int_\Omega \frac{2f(x)}{1-u} + 4\int_\Omega \nabla u \nabla(u-\bar u)\Big) \le C.\end{aligned}$$

Fix $1 \leq p < 1+\frac{4}{3}+2\sqrt{\frac{2}{3}}$ in order to have $2-\frac{9(p-1)^2}{4(3p-2)} > 0$. Introduce, as in the previous proof, $T_k u = \min\{u, 1-k\}$, $0 < k < 1$. For k small, taking $(1-T_k u)^{-3p+2} - (1-\bar{u})^{-3p+2} \in H_0^1(\Omega)$ as a test function for (2.3.5) yields that

$$\begin{aligned}&(3p-2)\int_\Omega \Big(\frac{|\nabla T_k u|^2}{(1-T_k u)^{3p-1}} - \frac{\nabla u \nabla \bar{u}}{(1-\bar{u})^{3p-1}}\Big)\\ &= \int_\Omega \frac{f(x)}{(1-u)^2}\Big((1-T_k u)^{-3p+2} - (1-\bar{u})^{-3p+2}\Big).\end{aligned} \tag{2.3.15}$$

Moreover, by (2.3.14) and the simple inequality $(a+b)^2 \leq (1+\delta)a^2 + \frac{1+\delta}{\delta}b^2$ we get:

$$\begin{aligned}&2\int_\Omega \frac{f(x)}{(1-u)^3}\left((1-T_k u)^{-\frac{3(p-1)}{2}} - (1-\bar{u})^{-\frac{3(p-1)}{2}}\right)^2\\ &\leq \int_\Omega |\nabla\left((1-T_k u)^{-\frac{3(p-1)}{2}} - (1-\bar{u})^{-\frac{3(p-1)}{2}}\right)|^2\\ &\leq \frac{9(p-1)^2}{4}(1+\delta)\int_\Omega \frac{|\nabla T_k u|^2}{(1-T_k u)^{3p-1}} + C\\ &\leq \frac{9(p-1)^2}{4}(1+\delta)\int_\Omega \Big(\frac{|\nabla T_k u|^2}{(1-T_k u)^{3p-1}} - \frac{\nabla u \nabla \bar{u}}{(1-\bar{u})^{3p-1}}\Big) + C,\end{aligned} \tag{2.3.16}$$

for some $C > 0$ depending on p and $\delta > 0$. Inserting (2.3.15) into (2.3.16) and using $(1-T_k u)^{-1} \leq (1-u)^{-1}$ for $u \leq 1$, we get that

$$\begin{aligned}&2\int_\Omega \frac{f(x)}{(1-u)^3}\left((1-T_k u)^{-\frac{3(p-1)}{2}} - (1-\bar{u})^{-\frac{3(p-1)}{2}}\right)^2\\ &\leq \frac{9(p-1)^2(1+\delta)}{4(3p-2)}\int_\Omega \frac{f(x)}{(1-u)^2}\Big((1-T_k u)^{-3p+2} - (1-\bar{u})^{-3p+2}\Big) + C\\ &\leq \frac{9(p-1)^2(1+\delta)}{4(3p-2)}\int_\Omega \frac{f(x)}{(1-u)^3}(1-T_k u)^{-3(p-1)} + C\\ &\leq \frac{9(p-1)^2(1+\delta)^2}{4(3p-2)}\int_\Omega \frac{f(x)}{(1-u)^3}\left((1-T_k u)^{-\frac{3(p-1)}{2}} - (1-\bar{u})^{-\frac{3(p-1)}{2}}\right)^2 + C\end{aligned}$$

in view of (2.3.12), where $C > 0$ does not depend on k. Since $1 \leq p < 1+\frac{4}{3}+2\sqrt{\frac{2}{3}}$ and $2-\frac{9(p-1)^2(1+\delta)}{4(3p-2)} > 0$ for δ small, then we have

$$\int_\Omega \frac{f(x)}{(1-u)^3}(1-T_k u)^{-3(p-1)} \leq 2\int_\Omega \frac{f(x)}{(1-u)^3}\left((1-T_k u)^{-\frac{3(p-1)}{2}} - (1-\bar{u})^{-\frac{3(p-1)}{2}}\right)^2 + C \leq C$$

for some $C > 0$ in dependent of k. Taking the limit as $k \to 0$, we get that

$$\int_\Omega \frac{f(x)}{(1-u)^{3p}} \leq C.$$

Then, $\frac{f}{(1-u)^3} \in L^p(\Omega)$ for any $1 \le p < 1 + \frac{4}{3} + 2\sqrt{\frac{2}{3}}$ and proof is done. ■

Proof (of Theorem 2.3.4): The existence of u^* as a classical solution follows from Proposition 2.3.5 and Theorem 2.3.3, as long as $\frac{N}{2} < 1 + \frac{4}{3} + 2\sqrt{\frac{2}{3}}$ which happens when $N \le 7$.
Since $\mu_{1,\lambda} > 0$ on the minimal branch for any $\lambda < \lambda^*$, at the limit we have $\mu_{1,\lambda^*} \ge 0$. If now $\mu_{1,\lambda^*} > 0$ the Implicit Function Theorem could be applied to the operator $L_{u_{\lambda^*},\lambda^*}$, and would allow the continuation of the minimal branch $\lambda \mapsto u_\lambda$ beyond λ^*, which is a contradiction. Hence $\mu_{1,\lambda^*} = 0$. The uniqueness in the class of $H_0^1(\Omega)$-weak solutions then follows from Lemma 2.3.1. ■

2.3.3 Blow-up procedure: the power-law profiles

Our goal now is to study the effect of power-like permittivity profiles $f(x) \simeq |x|^\alpha$ on the problem $(S)_\lambda$ defined in the unit ball $B = B_1(0)$.

Since $|x|^\alpha$ is increasing in B, the moving plane method of Gidas, Ni and Nirenberg [53] does not guarantee radial symmetry of solutions to $(S)_\lambda$. However, the minimal branch is composed by radial solutions as the following result shows:

Proposition 2.3.6. *Let Ω and f be a radial domain and profile, respectively. Then, the minimal solutions of $(S)_\lambda$ are necessarily radially symmetric and consequently*

$$\lambda^*(\Omega, f) = \lambda_r^*(\Omega, f) = \sup\big\{\lambda;\ (S)_\lambda \text{ has a radial solution}\big\}.$$

Moreover, on a ball any radial solution of $(S)_\lambda$ attains its maximum at 0.

Proof: It is clear that $\lambda_r^*(\Omega, f) \le \lambda^*(\Omega, f)$, and the reverse will be proved if we establish that every minimal solution of $(S)_\lambda$ with $0 < \lambda < \lambda^*(\Omega, f)$ is radially symmetric. The recursive scheme defined in Theorem 2.1.2 gives that at each step u_n and therefore also the resulting limiting function –the minimal solution– is radially symmetric.
For a solution $u(r)$ on the ball of radius R, we have $u_r(0) = 0$ and

$$-u_{rr} - \frac{N-1}{r} u_r = \frac{\lambda f}{(1-u)^2} \quad \text{in} \quad (0, R)\,.$$

Hence, $-\frac{d(r^{N-1}u_r)}{dr} = \frac{\lambda f r^{N-1}}{(1-u)^2} \ge 0$, and therefore $u_r < 0$ in $(0, R)$ since $u_r(0) = 0$. This shows that $u(r)$ attains its maximum at 0. ■

The following result extends the compactness of the minimal branch in Theorem 2.3.4 to higher dimensions $N \ge 8$:

Theorem 2.3.7. *Assume $N \ge 8$ and $\alpha > \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. Let f be of the form:*

$$f(x) = |x|^\alpha g(x)\,, \quad g(x) \ge C > 0 \text{ in } B, \tag{2.3.17}$$

where $g \in C^1(\bar{B})$. Let $(\lambda_n)_n$ be such that $\lambda_n \to \lambda \in [0, \lambda^*]$ and u_n be a solution of $(S)_{\lambda_n}$ so that

$$\mu_{1,n} := \mu_{1,\lambda_n}(u_n) \geq 0 \,. \tag{2.3.18}$$

Then,

$$\sup_{n\in\mathbb{N}} \| u_n \|_\infty < 1.$$

In particular, the extremal solution $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ is a classical solution of $(S)_{\lambda^*}$ such that $\mu_{1,\lambda^*}(u^*) = 0$.

Proof: By Lemma 2.3.1 and (2.3.18) u_n coincides with the minimal solution u_{λ_n}, and by Proposition 2.3.6, u_n is radial and achieves its absolute maximum only at zero.
Given a permittivity profile $f(x)$ as in (2.3.17), in order to get Theorem 2.3.7, we want to show:

$$\sup_{n\in\mathbb{N}} \| u_n \|_\infty < 1, \tag{2.3.19}$$

provided $N \geq 8$ and $\alpha > \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. In particular, since the minimal branch u_λ is non decreasing in λ, by (2.3.19) along such a branch we would get that the extremal solution $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ is a solution of $(S)_{\lambda^*}$ so that $\mu_{1,\lambda^*}(u^*) \geq 0$. Property $\mu_{1,\lambda^*}(u^*) = 0$ must hold because otherwise, by Implicit Function Theorem, we could find solutions of $(S)_\lambda$ for $\lambda > \lambda^*$, which contradicts the definition of λ^*.

In order to prove (2.3.19), let us argue by contradiction. Up to a subsequence, assume that $u_n(0) = \max_B u_n \to 1$ as $n \to +\infty$. Since $\lambda = 0$ implies $u_n \to 0$ in $C^2(\bar{B})$, we can assume that $\lambda_n \to \lambda > 0$. Let $\varepsilon_n := 1 - u_n(0) \to 0$ as $n \to +\infty$ and introduce the following rescaled function:

$$U_n(y) = \frac{1 - u_n\big(\varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y\big)}{\varepsilon_n} \,, \quad y \in B_n := B_{\varepsilon_n^{-\frac{3}{2+\alpha}} \lambda_n^{\frac{1}{2+\alpha}}}(0). \tag{2.3.20}$$

The function U_n satisfies:

$$\begin{cases} \Delta U_n = \dfrac{|y|^\alpha g\big(\varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y\big)}{U_n^2} & \text{in } B_n, \\ U_n(y) \geq U_n(0) = 1, \end{cases} \tag{2.3.21}$$

and $B_n \to \mathbb{R}^N$ as $n \to +\infty$. This would reduce to a contradiction between (2.3.18) and the following Proposition 2.3.8. ■

Proposition 2.3.8. *There exists a subsequence $\{U_n\}_n$ defined in (2.3.20) such that $U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$, where U is a solution of the problem*

$$\begin{cases} \Delta U = g(0)\dfrac{|y|^\alpha}{U^2} & \text{in } R^N \,, \\ U(y) \geq U(0) = 1 & \text{in } \mathbb{R}^N \,. \end{cases} \tag{2.3.22}$$

If either $1 \le N \le 7$ *or* $N \ge 8$, $\alpha > \alpha_N$, *then there exists* $\phi_n \in C_0^\infty(B)$ *such that:*

$$\int_B \big(|\nabla \phi_n|^2 - \frac{2\lambda_n |x|^\alpha g(x)}{(1-u_n)^3}\phi_n^2\big) < 0.$$

The second part in Proposition 2.3.8 is based on Theorem 5.1.1, which characterizes the unstability of entire solutions of (2.3.22).

Proof: Let $R > 0$. For n large, decompose $U_n = U_n^1 + U_n^2$, where U_n^2 satisfies:

$$\begin{cases} \Delta U_n^2 = \Delta U_n & \text{in } B_R(0)\,, \\ U_n^2 = 0 & \text{on } \partial B_R(0)\,. \end{cases}$$

By (2.3.21) we get that on $B_R(0)$:

$$0 \le \Delta U_n \le R^\alpha \parallel g \parallel_\infty,$$

and standard elliptic regularity theory gives that U_n^2 is uniformly bounded in $C^{1,\beta}(B_R(0))$ for any $\beta \in (0,1)$. Up to a subsequence, we get that $U_n^2 \to U^2$ in $C^1(B_R(0))$. Since $U_n^1 = U_n \ge 1$ on $\partial B_R(0)$, by harmonicity $U_n^1 \ge 1$ in $B_R(0)$ and then the Harnack inequality gives

$$\sup_{B_{R/2}(0)} U_n^1 \le C_R \inf_{B_{R/2}(0)} U_n^1 \le C_R U_n^1(0) = C_R\big(1 - U_n^2(0)\big) \le C_R\big(1 + \sup_{n\in\mathbb{N}} |U_n^2(0)|\big) < \infty\,.$$

Hence, U_n^1 is uniformly bounded in $C^{1,\beta}(B_{R/4}(0))$ for any $\beta \in (0,1)$. Up to a further subsequence, we get that $U_n^1 \to U^1$ in $C^1(B_{R/4}(0))$ and then, $U_n \to U^1 + U^2$ in $C^1(B_{R/4}(0))$ for any $R > 0$. By a diagonal process and up to a subsequence, we find that $U_n \to U$ in $C^1_{\rm loc}(\mathbb{R}^N)$, where U is a solution of the equation (2.3.22).

If either $1 \le N \le 7$ or $N \ge 8$ and $\alpha > \alpha_N$, since $g(0) > 0$ Theorem 5.1.1 shows that $\mu_1(U) < 0$ and then, we find $\phi \in C_0^\infty(\mathbb{R}^N)$ so that:

$$\int \big(|\nabla \phi|^2 - 2g(0)\frac{|y|^\alpha}{U^3}\phi^2\big) < 0.$$

Defining now

$$\phi_n(x) = \big(\varepsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}\big)^{-\frac{N-2}{2}}\phi\big(\varepsilon_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}}x\big)\,,$$

then we have

$$\begin{aligned} \int_B \big(|\nabla \phi_n|^2 - \frac{2\lambda_n |x|^\alpha g(x)}{(1-u_n)^3}\phi_n^2\big) &= \int \big(|\nabla \phi|^2 - \frac{2|y|^\alpha}{U_n^3} g(\varepsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}y)\phi^2\big) \\ &\to \int \big(|\nabla \phi|^2 - 2g(0)\frac{|y|^\alpha}{U^3}\phi^2\big) < 0 \end{aligned}$$

as $n \to +\infty$, since ϕ has compact support and $U_n \to U$ in $C^1_{\rm loc}(\mathbb{R}^N)$. The proof of Proposition 2.3.8 is now complete. ■

2.4 Uniqueness of solutions

The purpose of this Section is to characterize the extremal solution when is singular and to discuss uniqueness of solutions to $(S)_\lambda$ when λ is small.

2.4.1 The extremal solution

We first note that, in view of the monotonicity in λ and the uniform boundedness of the first branch of solutions, the extremal function defined by $u^*(x) = \lim\limits_{\lambda \uparrow \lambda^*} u_\lambda(x)$ always exists, and can always be considered as a solution for $(S)_{\lambda^*}$ in a weak sense. Now, if there exists $C < 1$ such that $\| u_\lambda \|_\infty \leq C$ for any $\lambda < \lambda^*$ –just like in the case $1 \leq N \leq 7$– then we have seen in Theorem 2.3.4 that u^* is classical and is unique among $H_0^1(\Omega)-$weak solutions. In the sequel, we tackle the important case when u^* is a weak solution (i.e., in $H_0^1(\Omega)$) of $(S)_{\lambda^*}$ but singular: $\|u^*\|_\infty = 1$.

We shall borrow ideas from [13, 16], where the authors deal with the case of regular nonlinearities. However, unlike those papers where solutions are considered in a very weak sense, we consider here a more focussed and much simpler situation. We establish the following useful characterization of the extremal solution:

Theorem 2.4.1. *Assume f is a function satisfying (2.0.1) on a bounded domain Ω in $\mathbb{R}^N$. For $\lambda > 0$, consider u to be a $H_0^1(\Omega)-$weak solution of $(S)_\lambda$ such that $\| u \|_\infty = 1$. Then the following assertions are equivalent:*

1. *$\mu_{1,\lambda} \geq 0$, that is u satisfies*

$$\int_\Omega |\nabla \phi|^2 \geq \int_\Omega \frac{2\lambda f(x)}{(1-u)^3}\phi^2 \quad \forall \phi \in H_0^1(\Omega)\,,$$

2. *$\lambda = \lambda^*$ and $u = u^*$.*

By Lemma 2.3.1 u_λ is the unique semi-stable solution of $(S)_\lambda$. Since $\| u_\lambda \|_\infty < 1$ for any $\lambda \in (0, \lambda^*)$, we need – in order to prove Theorem 2.4.1– only to show that $(S)_\lambda$ does not have any $H_0^1(\Omega)$-weak solution for $\lambda > \lambda^*$. By the definition of λ^*, this is already true for classical solutions and we shall now extend such a property to the class of weak solutions:

Proposition 2.4.2. *If w is a $H_0^1(\Omega)$-weak supersolution of $(S)_\lambda$, then for any $\varepsilon \in (0, 1)$ there exists a classical solution w_ε of $(S)_{\lambda(1-\varepsilon)}$.*

Proof: For any $\psi \in C^2([0, 1])$ concave, increasing function so that $\psi(0) = 0$, we have that $\psi(w) \in H_0^1(\Omega)$ and we prove now that:

$$\int_\Omega \nabla \psi(w) \nabla \phi \geq \int_\Omega \frac{\lambda f}{(1-w)^2}\dot{\psi}(w)\phi \tag{2.4.1}$$

for any $0 \le \phi \in H_0^1(\Omega)$. Indeed, by concavity of ψ we get:

$$\int_\Omega \nabla\psi(w)\nabla\phi = \int_\Omega \dot\psi(w)\nabla w\nabla\phi = \int_\Omega \nabla w\nabla\Big(\dot\psi(w)\phi\Big) - \int_\Omega \ddot\psi(w)\phi|\nabla w|^2$$
$$\ge \int_\Omega \frac{\lambda f(x)}{(1-w)^2}\dot\psi(w)\phi$$

for any $0 \le \phi \in C_0^\infty(\Omega)$. By density and Fatou Theorem, we get the validity of (2.4.1). Now let $\varepsilon \in (0,1)$, and define

$$\psi_\varepsilon(w) := 1 - \Big(\varepsilon + (1-\varepsilon)(1-w)^3\Big)^{\frac{1}{3}}, \quad 0 \le w \le 1\,.$$

Since $\psi_\varepsilon \in C^2([0,1])$ is a concave, increasing function so that $\psi_\varepsilon(0) = 0$ and

$$\dot\psi_\varepsilon(w) = (1-\varepsilon)\frac{g\Big(\psi_\varepsilon(w)\Big)}{g(w)}, \quad g(s) := (1-s)^{-2},$$

by (2.4.1) we obtain that for any $0 \le \phi \in H_0^1(\Omega)$:

$$\int_\Omega \nabla\psi_\varepsilon(w)\nabla\phi \ge \int_\Omega \frac{\lambda f(x)}{(1-w)^2}\dot\psi_\varepsilon(w)\phi = \lambda(1-\varepsilon)\int_\Omega f(x)g\big(\psi_\varepsilon(w)\big)\phi$$
$$= \int_\Omega \frac{\lambda(1-\varepsilon)f(x)}{(1-\psi_\varepsilon(w))^2}\phi\,.$$

Hence, $\psi_\varepsilon(w)$ is a $H_0^1(\Omega)$-weak supersolution of $(S)_{\lambda(1-\varepsilon)}$ so that $0 \le \psi_\varepsilon(w) \le 1-\varepsilon^{\frac{1}{3}} < 1$. Since 0 is a subsolution for any $\lambda > 0$, we get the existence of a $H_0^1(\Omega)$-weak solution w_ε of $(S)_{\lambda(1-\varepsilon)}$ so that $0 \le w_\varepsilon \le 1-\varepsilon^{\frac{1}{3}}$. By standard elliptic regularity theory, w_ε is a classical solution of $(S)_{\lambda(1-\varepsilon)}$. ■

From above results of §2.3 and §2.4, the refined properties of steady states –such as regularity, stability, uniqueness, energy estimates and comparison results– are collected in the following Theorem and are shown to depend on the dimension of the ambient space and on the permittivity profile.

Theorem 2.4.3. *Assume f is a function satisfying (2.0.1) on a bounded domain Ω in $\mathbb{R}^N$, and consider $\lambda^* := \lambda^*(\Omega, f)$ as defined in Theorem 2.2.4. Then,*

1. *For any $0 \le \lambda < \lambda^*$, the minimal solution u_λ is the unique stable solution of $(S)_\lambda$: $\mu_{1,\lambda}(u_\lambda) > 0$. Moreover for each $x \in \Omega$, the function $\lambda \to u_\lambda(x)$ is strictly increasing and differentiable on $(0, \lambda^*)$.*

2. *If $1 \le N \le 7$ then –by means of energy estimates– one has $\sup_{\lambda\in(0,\lambda^*)} \| u_\lambda \|_\infty < 1$. If $N \ge 8$ and $\alpha > \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, such a property –by means of a blow-up procedure– is still true for $(S)_\lambda$ on the unit ball and power-like profiles as in (2.3.17).*

Consequently, $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ *exists in* $C^2(\bar{\Omega})$ *and is a solution for* $(S)_{\lambda^*}$ *such that* $\mu_{1,\lambda^*}(u^*) = 0$. *In particular,* u^* *–often referred to as the extremal solution of problem* $(S)_\lambda$*– is unique among all the* $H_0^1(\Omega)-$*weak solutions.*

3. *On the other hand, if* $N \geq 8$ *and* $0 \leq \alpha \leq \alpha_N$*, when* $f(x) = |x|^\alpha$ *and* Ω *is the unit ball, the extremal solution is* $u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ *and is therefore singular.*

We note that in general, the extremal function u^* exists in any dimension, does solve $(S)_{\lambda^*}$ in a weak sense and is the unique solution in an appropriate class. The above Theorem states that u^* is a classical solution for suitable dimensions N and parameters α. This will allow us to start another branch of non-minimal (unstable) solutions, which we will discuss in next two Chapters.

2.4.2 Uniqueness of low energy solutions for small voltage

In the following we focus on the uniqueness when λ is small enough. We first define non-minimal solutions of $(S)_\lambda$ as follows.

Definition 2.4.1. A solution $0 \leq u < 1$ is said to be a non-minimal positive solution of $(S)_\lambda$ if there exists another positive solution v of $(S)_\lambda$ and a point $x \in \Omega$ such that $u(x) > v(x)$.

Lemma 2.4.4. *Suppose* u *is a non-minimal solution of* $(S)_\lambda$ *with* $\lambda \in (0, \lambda^*)$*. Then* $\mu_1(\lambda, u) < 0$*, and the function* $w = u - u_\lambda$ *is in the negative space of* $L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3}$*.*

Proof: For a fixed $\lambda \in (0, \lambda^*)$, let u_λ be the minimal solution of $(S)_\lambda$. By Lemma 2.3.1, we have $w = u - u_\lambda \geq 0$ in Ω and $\mu_1(\lambda, u) < 0$. Since

$$-\Delta w - \frac{\lambda(2 - u - u_\lambda)f}{(1-u)^2(1-u_\lambda)^2} w = 0 \quad \text{in} \quad \Omega\,,$$

the strong maximum principle yields to $u_\lambda < u$ in Ω.
Let $\Omega_0 = \{x \in \Omega : f(x) = 0\}$. Direct calculations give that

$$\begin{aligned} -\Delta(u - u_\lambda) - \frac{2\lambda f}{(1-u)^3}(u - u_\lambda) &= \lambda f\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_\lambda)^2} - \frac{2}{(1-u)^3}(u - u_\lambda)\Big] \\ &= \begin{cases} 0\,, & x \in \Omega_0\,; \\ < 0\,, & x \in \Omega \setminus \Omega_0\,. \end{cases} \end{aligned}$$

From this we get

$$\langle L_{u,\lambda} w, w\rangle = \lambda \int_{\Omega \setminus \Omega_0} f\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_\lambda)^2} - \frac{2}{(1-u)^3}(u - u_\lambda)\Big](u - u_\lambda) < 0\,.$$

∎

Now we are able to prove the following uniqueness result:

Theorem 2.4.5. *For every $M > 0$ there exists $\lambda_1^*(M) \in (0, \lambda^*)$ such that for $\lambda \in (0, \lambda_1^*(M))$ equation $(S)_\lambda$ has a unique solution v satisfying*

1. $\|\frac{f}{(1-v)^3}\|_1 \le M$ *when* $N = 1$,
2. $\|\frac{f}{(1-v)^3}\|_{1+\epsilon} \le M$ *when* $N = 2$, *for some* $\epsilon > 0$,
3. $\|\frac{f}{(1-v)^3}\|_{N/2} \le M$ *when* $N \ge 3$.

Proof: For any fixed $\lambda \in (0, \lambda^*)$, let u_λ be the minimal solution of $(S)_\lambda$, and suppose $(S)_\lambda$ has a non-minimal solution u corresponding to cases 1), 2) or 3). The previous Lemma then gives

$$\int_\Omega |\nabla(u - u_\lambda)|^2 dx < \int_\Omega \frac{2\lambda(u-u_\lambda)^2 f(x)}{(1-u)^3} dx\,.$$

This implies in case $N \ge 3$ that

$$\begin{aligned} S_N\Big(\int_\Omega (u-u_\lambda)^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{N}} &< 2\lambda \int_\Omega \frac{f(x)}{(1-u)^3}(u-u_\lambda)^2 dx \\ &\le 2\lambda\Big(\int_\Omega \Big|\frac{f}{(1-u)^3}\Big|^{\frac{N}{2}}\Big)^{\frac{2}{N}}\Big(\int_\Omega (u-u_\lambda)^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{N}} \\ &\le 2\lambda M\Big(\int_\Omega (u-u_\lambda)^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{N}}, \end{aligned}$$

which is a contradiction if $\lambda < \frac{S_N}{2M}$ unless $u \equiv u_\lambda$. If $N = 1$, then we write

$$S\|u-u_\lambda\|_\infty^2 < 2\lambda \int_I \frac{f(x)}{(1-u)^3}(u-u_\lambda)^2 dx \le 2\lambda\|u-u_\lambda\|_\infty^2 \int_I \frac{f}{(1-u)^3} dx\,,$$

and the proof follows. A similar proof holds for dimension $N = 2$. ■

Remark 2.4.1. The above Theorem gives uniqueness for small λ among all solutions that either stay away from 1 or those that approach it slowly.

2.4.3 A monotonicity inequality

In this subsection we establish a monotonicity inequality for positive solutions of the following problem

$$\Delta u = \lambda |x|^\alpha u^{-2} \text{ in } \Omega, \tag{2.4.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N \ge 2$) with $0 \in \Omega$.

Since $0 < u \in C^2(\Omega)$, multiplying (2.4.2) by $\phi \cdot \nabla u$ and integrating by parts let us observe that

$$\int_\Omega \Big[\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial \phi^j}{\partial x_i} - \frac{1}{2}|\nabla u|^2 \frac{\partial \phi^i}{\partial x_i} + \lambda u^{-1}\frac{\partial |x|^\alpha}{\partial x_i}\phi^i + \lambda u^{-1}|x|^\alpha \frac{\partial \phi^i}{\partial x_i}\Big] dx = 0 \tag{2.4.3}$$

holds for all regular vector fields $\phi \in C_0^1(\Omega, \mathbb{R}^N)$ (summation over i and j is understood). We have the following result:

Theorem 2.4.6. *Let u be a positive solution of (2.4.2). Given $r_0 > 0$ so that $B(0, 2r_0) \subset \Omega$, then*

$$\begin{aligned}\mathcal{E}_u(r) := \quad & -\frac{3\lambda}{2} r^{-\mu} \int_{B(0,r)} |x|^\alpha u^{-1} dx + \frac{1}{4} \frac{d}{dr} \Big[r^{-\mu} \int_{\partial B(0,r)} u^2 dS \Big] \\ & -\frac{1}{4} r^{-\mu-1} \int_{\partial B(0,r)} u^2 dS \end{aligned} \tag{2.4.4}$$

is a nondecreasing function of r, for any $r \in (0, r_0)$, where $\mu = N - \frac{2}{3} + \frac{2\alpha}{3}$.

Remark 2.4.2. The monotonicity inequality (2.4.4) holds in a more general class, the non-negative solutions u of (2.4.2) of finite energy: $u \in H^1(\Omega)$ and $\int_\Omega u^{-1} < +\infty$, which are stationary: (2.4.3) holds. Also the nonlinearity u^{-2} can be replaced by u^{-p} for any $p \geq 1$. The following proof can be easily adapted.

Proof: : Fix $r_0 > 0$ so that $B(0, 2r_0) \subset \Omega$. Let r, $m > 0$ be such that $r + m < r_0$, and set $\phi(x) = \xi(|x|)x$, where

$$\xi(|x|) \equiv \begin{cases} 1 & \text{for } |x| \leq r, \\ 1 + \frac{r-|x|}{m} & \text{for } r \leq |x| \leq r+m, \\ 0 & \text{for } |x| \geq r+m. \end{cases}$$

We derive from (2.4.3), letting $m \to 0^+$, that the following identity holds

$$\begin{aligned} & \lambda N \int_{B(0,r)} |x|^\alpha u^{-1} dx - \frac{N-2}{2} \int_{B(0,r)} |\nabla u|^2 dx + \frac{r}{2} \int_{\partial B(0,r)} |\nabla u|^2 dS \\ & + \lambda \int_{B(0,r)} (x \cdot \nabla |x|^\alpha) u^{-1} dx - \lambda r \int_{\partial B(0,r)} |x|^\alpha u^{-1} dS = r \int_{\partial B(0,r)} (u_r)^2 dS. \end{aligned} \tag{2.4.5}$$

Since $x \cdot \nabla |x|^\alpha = \alpha |x|^\alpha$, (2.4.5) gives

$$\begin{aligned} & \lambda (N + \alpha) \int_{B(0,r)} |x|^\alpha u^{-1} dx - \frac{N-2}{2} \int_{B(0,r)} |\nabla u|^2 dx + \frac{r}{2} \int_{\partial B(0,r)} |\nabla u|^2 dS \\ & - \lambda r \int_{\partial B(0,r)} |x|^\alpha u^{-1} dS = r \int_{\partial B(0,r)} (u_r)^2 dS. \end{aligned} \tag{2.4.6}$$

On the other hand, multiplying (2.4.2) by u and integrating over $B(0, r)$ we find $\forall\ 0 < r < r_0$,

$$\int_{B(0,r)} |\nabla u|^2 dx = \int_{\partial B(0,r)} u u_r dS - \lambda \int_{B(0,r)} |x|^\alpha u^{-1} dx. \tag{2.4.7}$$

Taking the derivative of (2.4.7) with respect to r, we obtain

$$\int_{\partial B(0,r)} |\nabla u|^2 dS = \frac{d}{dr} \int_{\partial B(0,r)} u u_r dS - \lambda \int_{\partial B(0,r)} |x|^\alpha u^{-1} dS. \tag{2.4.8}$$

Substituting $\int_{B(0,r)} |\nabla u|^2 dx$ of (2.4.7) and $\int_{\partial B(0,r)} |\nabla u|^2 dS$ of (2.4.8) into (2.4.6), we obtain

$$\lambda \left(\frac{3N}{2} + \alpha - 1 \right) \int_{B(0,r)} |x|^\alpha u^{-1} dx - \frac{3}{2} \lambda r \int_{\partial B(0,r)} |x|^\alpha u^{-1} dS$$

$$+\frac{r}{2}\frac{d}{dr}\int_{\partial B(0,r)} uu_r dS - \frac{N-2}{2}\int_{\partial B(0,r)} uu_r dS = r\int_{\partial B(0,r)} (u_r)^2 dS.$$

This equality can be rewritten as

$$\begin{aligned} &-\frac{3\lambda}{2}\frac{d}{dr}\Big[r^{-\mu}\int_{B(0,r)} |x|^\alpha u^{-1}dx\Big] + \frac{1}{2}r^{-\mu}\frac{d}{dr}\Big[\int_{\partial B(0,r)} uu_r dS\Big] \\ &= r^{-\mu}\int_{\partial B(0,r)} \Big[(u_r)^2 + \frac{N-2}{2}r^{-1}uu_r\Big]dS, \end{aligned} \tag{2.4.9}$$

where $\mu = N - \frac{2}{3} + \frac{2\alpha}{3}$. Using the identity

$$\frac{d}{dr}\Big[\int_{\partial B(0,r)} u^2 dS\Big] = 2\int_{\partial B(0,r)} uu_r dS + (N-1)\int_{\partial B(0,r)} r^{-1}u^2 dS,$$

we have

$$\begin{aligned} &\frac{1}{2}\frac{d^2}{dr^2}\Big[r^{-\mu}\int_{\partial B(0,r)} u^2 dS\Big] - \frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(0,r)} uu_r dS\Big] \\ &= (N-\mu-1)r^{-\mu}\int_{\partial B(0,r)} \Big[\frac{(N-2-\mu)}{2}r^{-2}u^2 + r^{-1}uu_r\Big]dS. \end{aligned} \tag{2.4.10}$$

Note that

$$r^{-\mu}\frac{d}{dr}\Big[\int_{\partial B(0,r)} uu_r dS\Big] = \frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(0,r)} uu_r dS\Big] + \mu r^{-\mu-1}\int_{\partial B(0,r)} uu_r dS. \tag{2.4.11}$$

Substituting (2.4.10) and (2.4.11) into (2.4.9), we obtain that

$$\begin{aligned} &-\frac{3\lambda}{2}\frac{d}{dr}\Big[r^{-\mu}\int_{B(0,r)} |x|^\alpha u^{-1}dx\Big] + \frac{1}{4}\frac{d^2}{dr^2}\Big[r^{-\mu}\int_{\partial B(0,r)} u^2 dS\Big] \\ =\ & r^{-\mu}\int_{\partial B(0,r)} \Big[(u_r)^2 + \frac{2N-2\mu-3}{2}r^{-1}uu_r + \frac{1}{4}(N-\mu-1)(N-\mu-2)r^{-2}u^2\Big]dS, \end{aligned}$$

which yields to

$$\begin{aligned} &-\frac{3\lambda}{2}\frac{d}{dr}\Big[r^{-\mu}\int_{B(0,r)} |x|^\alpha u^{-1}dx\Big] + \frac{1}{4}\frac{d^2}{dr^2}\Big[r^{-\mu}\int_{\partial B(0,r)} u^2 dS\Big] \\ &-\frac{1}{4}\frac{d}{dr}\Big[r^{-\mu-1}\int_{\partial B(0,r)} u^2 dS\Big] = r^{-\mu}\int_{\partial B(0,r)} \Big(u_r + \frac{N-\mu-2}{2}r^{-1}u\Big)^2 dS \geq 0. \end{aligned}$$

This shows that

$$\begin{aligned} \mathcal{E}_u(r) = &-\frac{3\lambda}{2}r^{-\mu}\int_{B(0,r)} |x|^\alpha u^{-1}dx + \frac{1}{4}\frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(0,r)} u^2 dS\Big] \\ &-\tfrac{1}{4}r^{-\mu-1}\int_{\partial B(0,r)} u^2 dS \end{aligned}$$

is a nondecreasing function of r for $r \in (0, r_0)$, completing the proof of Theorem 2.4.6. ■

2.4.4 Uniqueness of solutions for small voltage

Theorem 2.4.5 shows that the minimal solution u_λ is unique among all the solutions that either stay away from 1 or approach 1 slowly. In order to remove such an integral condition, we establish here an intermediate result which will allows us to show in Chapter 4 uniqueness for small voltages on symmetric domains and with power-law permittivity profiles.

Theorem 2.4.7. *Let $f(x) = |x|^\alpha$ and Ω be a domain so that $0 \in \Omega$. Let $(\lambda_n)_n$ be a sequence such that $\lambda_n \to \lambda \in [0, \lambda^*]$ and let u_n be an associated non-minimal solution of $(S)_{\lambda_n}$. Assume that $u_n(0) = \max\limits_{\Omega} u_n$ and*

$$\max_{\Omega\setminus B_r(0)} u_n \le k_r < 1 \qquad \forall\, 0 < r \le r_0, \tag{2.4.12}$$

$$\nabla u_n(x) \cdot x \le 0 \qquad \forall\, x : \ |x| = r_0, \tag{2.4.13}$$

for some $r_0 < \frac{1}{2} dist\ (0, \partial\Omega)$.
Set $\varepsilon_n = 1 - u_n(0)$. If there exists a contant $C > 0$ so that

$$\lambda_n r^{\frac{4+2\alpha}{3}} \int_{S^{N-1}} \frac{1}{(1 - u_n(r\theta))^2} d\theta \le C \qquad \forall\, \epsilon^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} \le r \le r_0, \tag{2.4.14}$$

then $\lambda > 0$.

We argue by contradiction and assume $\lambda = 0$. Since u_n is a non-minimal solution, Theorem 2.4.5 shows that $\|u_n\|_\infty \to 1$ as $n \to +\infty$ (along a subsequence). Then, $\varepsilon_n \to 0$ as $n \to +\infty$ and, even more precisely:

$$\varepsilon_n^3 \lambda_n^{-1} \to 0 \quad \text{as} \quad n \to +\infty. \tag{2.4.15}$$

Indeed, otherwise we would have along some subsequence:

$$0 \le \frac{\lambda_n |x|^\alpha}{(1 - u_n)^2} \le C \frac{\lambda_n}{\varepsilon_n^2} \le C' \varepsilon_n \to 0 \text{ as } n \to +\infty.$$

But if the right hand side of $(S)_{\lambda_n}$ converges uniformly to 0, then elliptic regularity theory implies that $u_n \to u$ in $C^1(\bar{\Omega})$, where u is a harmonic function such that $u = 0$ on $\partial\Omega$, and hence $u \equiv 0$ on Ω. On the other hand, $\varepsilon_n \to 0$ implies that $\max\limits_{\Omega} u = 1$, a contradiction.

Define

$$U_n(y) = \frac{1 - u_n(\epsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y)}{\epsilon_n}, \ y \in \Omega_n := \{y : \ \epsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y \in \Omega\}. \tag{2.4.16}$$

The function U_n satisfies:

$$\Delta U_n = |y|^\alpha U_n^{-2} \ \text{ in } \Omega_n, \quad U_n(0) = 1, \quad U_n(y) \ge 1. \tag{2.4.17}$$

Let us observe that by (2.4.15) $\Omega_n \to \mathbb{R}^N$ as $n \to +\infty$. We shall show that such U_n does not exist for n large.

Proof (of Theorem 2.4.7): Set $R_n = \epsilon_n^{-\frac{3}{2+\alpha}} \lambda_n^{\frac{1}{2+\alpha}} r_0$, then we have:

$$U_n(R_n\theta) = R_n^{\frac{2+\alpha}{3}} \lambda_n^{-\frac{1}{3}} r_0^{-\frac{2+\alpha}{3}} (1-u_n)(r_0\theta)\ ,\ \ \forall\, \theta \in S^{N-1}.$$

Therefore, if we define

$$v_n(s,\theta) = |y|^{-\frac{2+\alpha}{3}} U_n(y), \quad |y| = e^s,$$

we get that for $s \in [0, T_n]$, where $T_n = \ln R_n$, v_n satisfies the equation:

$$v_{ss} + \left(N - \frac{2}{3} + \frac{2\alpha}{3}\right) v_s + \Delta_{S^{N-1}} v + \frac{2+\alpha}{3}\left(N - \frac{4}{3} + \frac{\alpha}{3}\right) v = v^{-2}. \tag{2.4.18}$$

Proposition 2.3.8 shows that (up to a subsequence) $U_n \to U$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$, where U is a solution of (2.3.22) with $g(0) = 1$. Since $U \geq 1$, by (2.4.12) we obtain that for n sufficiently large

$$\tfrac{1}{C} \leq v_n(s,\theta) = e^{-\frac{2+\alpha}{3}s} U_n(e^s\theta) \leq C, \qquad \forall\, 0 \leq s \leq 1,\ \theta \in S^{N-1} \tag{2.4.19}$$

$$\tfrac{1}{C} \leq \lambda_n^{\frac{1}{3}} s^{\frac{2+\alpha}{3}} v_n(T_n + \ln \tfrac{s}{r_0}, \theta) = (1-u_n)(s\theta) \leq 1 \quad \forall\, r \leq s \leq r_0,\ \theta \in S^{N-1}, \tag{2.4.20}$$

where $r > 0$ is any given number less than r_0. Since

$$(v_n)_s(s,\theta) + \frac{2+\alpha}{3} v_n(s,\theta) = e^{\frac{1-\alpha}{3}s} (U_n)_r(e^s\theta) \quad \forall\, (s,\theta) \in (-\infty, T_n] \times S^{N-1} \tag{2.4.21}$$

and $v_n(s,\theta) \geq U_n(e^s\theta) \geq U_n(0) = 1$ in $(-\infty, 0] \times S^{N-1}$, let us rewrite estimates (2.4.13), (2.4.14) in terms of v_n as follows:

$$(v_n)_s(T_n,\theta) + \tfrac{2+\alpha}{3} v_n(T_n,\theta) \geq 0 \qquad \forall\, \theta \in S^{N-1}, \tag{2.4.22}$$

$$\textstyle\int_{S^{N-1}} \frac{1}{v_n^2(s,\theta)} d\theta \leq C \lambda_n^{-\frac{1}{3}} \qquad \forall\, s \leq T_n. \tag{2.4.23}$$

We now use the monotonicity property in Theorem 2.4.6:

$$\mathcal{E}_{U_n}(r) := -\frac{3}{2} r^{-\mu} \int_{B(0,r)} \frac{|y|^\alpha}{U_n} dy + \frac{1}{4}\frac{d}{dr}\Big[r^{-\mu} \int_{\partial B(0,r)} U_n^2 dS\Big] - \frac{1}{4} r^{-\mu-1} \int_{\partial B(0,r)} U_n^2 dS$$

is a nondecreasing function of r, where $\mu = N - \frac{2}{3} + \frac{2\alpha}{3}$. A simple calculation implies that, under the change $U_n \to v_n(s,\theta)$, the function $\mathcal{E}_{U_n}(r)$ is just a positive multiple of

$$E_{v_n}(s) = w_n'(s) - 6h_n(s),$$

where

$$w_n(s) = \int_{S^{N-1}} v_n^2(s,\theta) d\theta, \quad h_n(s) = \int_{-\infty}^s d\tau e^{\mu(\tau - s)} \int_{S^{N-1}} \frac{d\theta}{v_n(\tau,\theta)}.$$

Hence $E_{v_n}(s)$ is a nondecreasing function of s for any $s \in [0, T_n]$.
By (2.4.23) we deduce that $\int_{S^{N-1}} \frac{d\theta}{v_n(s,\theta)} \le C\lambda_n^{-\frac{1}{6}}$ for any $s \le T_n$ and in turn,

$$0 \le h_n(s) \le C\lambda_n^{-\frac{1}{6}} \text{ for any } s \le T_n. \tag{2.4.24}$$

Introduce now $\overline{v}_n(s) = \int_{S^{N-1}} v_n(s,\theta)d\theta$. Observe that by (2.4.18) $\overline{v}_n$ satisfies the equation:

$$\overline{v}_n'' + \left(N - \frac{2}{3} + \frac{2\alpha}{3}\right)\overline{v}_n' + \frac{2+\alpha}{3}\left(N - \frac{4}{3} + \frac{\alpha}{3}\right)\overline{v} = \int_{S^{N-1}} \frac{d\theta}{v_n^2(s,\theta)}. \tag{2.4.25}$$

As far as the estimate of w_n', we claim that

$$w_n'(T_n) = -\frac{4+2\alpha}{3} w_n(T_n)\left(1 + o(1)\right) \quad \text{as } n \to +\infty. \tag{2.4.26}$$

Indeed, by (2.4.21) we get that

$$e^{(N-\frac{4}{3}+\frac{\alpha}{3})s}\left((v_n)_s(s,\theta) + \frac{2+\alpha}{3}v_n(s,\theta)\right) = e^{(N-1)s}(U_n)_r(e^s\theta) \to 0 \quad \text{as } s \to -\infty. \tag{2.4.27}$$

Hence, by (2.4.25) and (2.4.27) it follows that

$$\overline{v}_n'(T_n) + \frac{2+\alpha}{3}\overline{v}_n(T_n) = \int_{-\infty}^{T_n} e^{(N-\frac{4}{3}+\frac{\alpha}{3})(s-T_n)} \int_{S^{N-1}} \frac{d\theta}{v_n^2(s,\theta)}.$$

Then, by (2.4.20) and (2.4.23) we deduce that

$$\begin{aligned}
\overline{v}_n'(T_n) + \frac{2+\alpha}{3}\overline{v}_n(T_n) &\le C\lambda_n^{-\frac{1}{3}} \int_{-\infty}^{T_n + \ln\frac{r}{r_0}} e^{(N-\frac{4}{3}+\frac{\alpha}{3})(s-T_n)} \\
+C\lambda_n^{\frac{2}{3}} \int_{T_n+\ln\frac{r}{r_0}}^{T_n} e^{(N-\frac{4}{3}+\frac{\alpha}{3})(s-T_n)} &\le C\lambda_n^{-\frac{1}{3}}(\frac{r}{r_0})^{N-\frac{4}{3}+\frac{\alpha}{3}} + C\lambda_n^{\frac{2}{3}}
\end{aligned}$$

for any $0 < r \le r_0$. Therefore, it implies:

$$\overline{v}_n'(T_n) + \frac{2+\alpha}{3}\overline{v}_n(T_n) \le \lambda_n^{-\frac{1}{3}}o_r(1) + C\lambda_n^{\frac{2}{3}}, \tag{2.4.28}$$

where $o_r(1) \to 0$ as $r \to 0$ uniformly in n. By (2.4.20) observe that $v_n(T_n,\theta) = O(\lambda_n^{-\frac{1}{3}})$ uniformly in θ and $w_n(T_n) \ge C\lambda_n^{-\frac{2}{3}}$ for some $C > 0$. Hence, $(\lambda_n^{\frac{1}{3}} w_n(T_n))^{-1} = O(\lambda_n^{\frac{1}{3}}) = o(1)$ and by (2.4.28) now we deduce:

$$\begin{aligned}
w_n'(T_n) &= 2\int_{S^{N-1}} v_n(T_n,\theta)(v_n)_s(T_n,\theta)d\theta \\
&= 2\int_{S^{N-1}} v_n(T_n,\theta)[(v_n)_s(T_n,\theta) + \frac{2+\alpha}{3}v_n(T_n,\theta)]d\theta - \frac{4+2\alpha}{3}w_n(T_n) \\
&= \lambda_n^{-\frac{2}{3}}o_r(1) + O(\lambda_n^{\frac{1}{3}}) - \frac{4+2\alpha}{3}w_n(T_n) \\
&= -\frac{4+2\alpha}{3}w_n(T_n)\left(1 + o_r(1) + O(\lambda_n)\right),
\end{aligned}$$

in view of (2.4.22). Our claim on $w_n'(T_n)$ is established.

Thus, it now follows from (2.4.26) that

$$E_{v_n}(T_n) \le w_n'(T_n) \le -C\lambda_n^{-\frac{2}{3}}.$$

The monotonicity of $E_{v_n}(s)$ in s implies that

$$w_n'(s) - 6h_n(s) \le -C\lambda_n^{-\frac{2}{3}} \text{ for any } s \in [0, T_n],$$

and hence,

$$h_n(s) \ge C\left(\lambda_n^{-\frac{2}{3}} + w_n'(s)\right) \text{ for any } s \in [0, T_n]. \tag{2.4.29}$$

Integrating (2.4.29) from 0 to T_n and using (2.4.19) and (2.4.24), we conclude that

$$T_n\lambda_n^{-\frac{1}{6}} \ge CT_n\lambda_n^{-\frac{2}{3}} + C\lambda_n^{-\frac{2}{3}} - C > CT_n\lambda_n^{-\frac{2}{3}}$$

and then, $\lambda_n^{\frac{1}{2}} \ge C > 0$. A contradiction. This completes the proof of Theorem 2.4.7. ■

2.5 Bifurcation diagrams for power-law case

The bifurcation diagrams shown in Figure 2.2 actually reflect the radially symmetric situation, and our emphasis in this section is on whether there is a better chance to analyze mathematically the higher branches of solutions in this case. The classical work of Joseph-Lundgren [75] and many that followed can be adapted to this situation when the permittivity profile is constant. However, the case of a power-law permittivity profile $f(x) = |x|^\alpha$ defined in a unit ball already presents a much richer situation. We now present various analytical and numerical evidences for various conjectures relating to this case, some of which will be further discussed in next three Chapters.

Consider the domain Ω to be a unit ball $B_1(0) \subset \mathbb{R}^N$ $(N \ge 1)$, and let $f(x) = |x|^\alpha$ $(\alpha \ge 0)$. We analyze in this case the branches of radially symmetric solutions of $(S)_\lambda$ for $\lambda \in (0, \lambda^*]$. In this case, $(S)_\lambda$ reduces to

$$\begin{cases} -u_{rr} - \dfrac{N-1}{r}u_r = \dfrac{\lambda r^\alpha}{(1-u)^2}\,, & 0 < r \le 1, \\ u'(0) = 0\,, \quad u(1) = 0\,. \end{cases} \tag{2.5.1}$$

Here $r = |x|$ and $0 < u = u(r) < 1$ for $0 < r < 1$.
Consider first the following initial value problem:

$$\begin{cases} U'' + \dfrac{N-1}{r}U' = \dfrac{r^\alpha}{U^2}\,, & r > 0\,, \\ U'(0) = 0\,, \quad U(0) = 1\,. \end{cases} \tag{2.5.2}$$

Observe that $U > 1$ in $(0, +\infty)$. For any $\gamma > 0$, we can define a solution $u_\gamma(r)$ of (2.5.1) as

$$u_\gamma(r) = 1 - \lambda^{\frac{1}{3}} \gamma^{-\frac{2+\alpha}{3}} U(\gamma r).$$

The parameter λ and the maximum value of u_γ: $u_\gamma(0)$, depend on γ in the following way:

$$\begin{cases} u_\gamma(0) = 1 - \dfrac{1}{U(\gamma)}\,, \\ \lambda = \dfrac{\gamma^{2+\alpha}}{U^3(\gamma)}\,, \end{cases} \tag{2.5.3}$$

where the second relation guarantees the boundary condition $u_\gamma(1) = 0$.

As was done in §2.2.3, one can numerically integrate the initial value problem (2.5.2) and use the results to compute the complete bifurcation diagram for (2.5.1). We show such a computation of $u(0)$ versus λ defined in (2.5.3) for the slab domain ($N = 1$) in Figure 2.3. In this case, one observes from the numerical results that when $N = 1$, and $0 \le \alpha \le 1$, there exist exactly two solutions for $(S)_\lambda$ whenever $\lambda \in (0, \lambda^*)$. On the other hand, the situation becomes more complex for $\alpha > 1$ as $u(0) \to 1$. This leads to the question of determining the asymptotic behavior of $U(r)$ as $r \to \infty$. Towards this end, we proceed as follows.

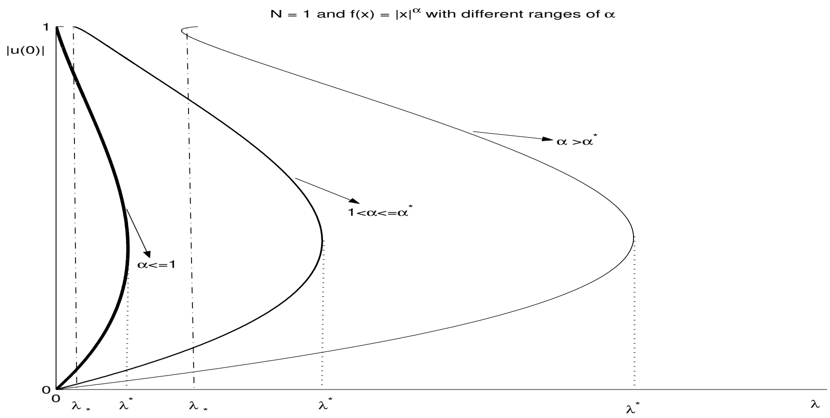

Figure 2.3: *Plots of $u(0)$ versus λ for profile $f(x) = |x|^\alpha$ ($\alpha \ge 0$) defined in the slab domain ($N = 1$). The numerical experiments point to a constant $\alpha_1 > 1$ (analytically given in (2.5.5)) such that the bifurcation diagrams are greatly different for the different ranges of α: $0 \le \alpha \le 1$, $1 < \alpha \le \alpha_1$ and $\alpha > \alpha_1$.*

Setting $v(s) = r^{-\frac{2+\alpha}{3}} U(r) > 0$, $r = e^s$, by (2.4.18) we have that

$$v'' + \left(N - \frac{2}{3} + \frac{2\alpha}{3}\right) v' + \frac{2+\alpha}{3} \left(N - \frac{4}{3} + \frac{\alpha}{3}\right) v = v^{-2}\,. \tag{2.5.4}$$

We can already identify from this equation the following regimes.

Case 1. Assume that

$$N = 1 \text{ and } 0 \le \alpha \le 1.$$

In this case, there is no positive equilibrium point for (2.5.4), which means that the bifurcation diagram vanishes at $\lambda = 0$. Then, one infers that in this case, there exist exactly two solutions for $\lambda \in (0, \lambda^*)$ and just one for $\lambda = \lambda^*$.

Case 2. N and α satisfy either one of the following conditions:

$$N = 1 \text{ and } \alpha > 1,$$
$$N \ge 2.$$

There exists then a positive equilibrium point V_e of (2.5.4):

$$v_e = \sqrt[3]{\frac{9}{(2+\alpha)(3N+\alpha-4)}} > 0\,.$$

Linearizing around this equilibrium point by writing

$$v = v_e + \delta e^{\sigma\eta}\,, \quad 0 < \delta << 1,$$

we obtain that

$$\sigma^2 + \frac{3N+2\alpha-2}{3}\sigma + \frac{(2+\alpha)(3N+\alpha-4)}{3} = 0\,.$$

Such an equation admits the following solutions:

$$\sigma_\pm = -\frac{3N+2\alpha-2}{6} \pm \frac{\sqrt{\Delta}}{6}\,,$$

with

$$\Delta = -8\alpha^2 - (24N-16)\alpha + (9N^2 - 84N + 100)\,.$$

We note that $\sigma_\pm < 0$ whenever $\Delta \ge 0$. Now define

$$\alpha_1 = -\frac{1}{2} + \frac{3}{4}\sqrt{6}\,, \quad \alpha_N = \frac{3N - 14 - 4\sqrt{6}}{4+2\sqrt{6}} \ \ (N \ge 8)\,. \tag{2.5.5}$$

Next, we discuss the ranges of N and α such that $\Delta \ge 0$ or $\Delta < 0$.

Case 2.A. N and α satisfy one of the following:

$$N = 1 \quad \text{with} \quad 1 < \alpha \le \alpha_1\,, \tag{2.5.6a}$$

$$N \ge 8 \quad \text{with} \quad 0 \le \alpha \le \alpha_N\,. \tag{2.5.6b}$$

In this case, we have $\triangle \ge 0$ and

$$v(s) \sim \Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + \delta_1 e^{-\frac{3N+2\alpha-2-\sqrt{\Delta}}{6}s} + \cdots\,, \quad \text{as} \quad s \to +\infty\,.$$

Further, we conclude that

$$U(r) \sim r^{\frac{2+\alpha}{3}}\Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + \delta_1 r^{-\frac{N-2}{2}+\frac{\sqrt{\Delta}}{6}} + \cdots , \quad \text{as} \quad r \to +\infty .$$

In both cases, the branch monotonically approaches the value 1 as $\gamma \to +\infty$ ($u_\gamma(0) \uparrow 1$ as $\gamma \to +\infty$). Moreover, since $\lambda = \gamma^{2+\alpha}/U^3(\gamma)$, we have

$$\lambda \uparrow \lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9} \quad \text{as} \quad \gamma \to \infty , \tag{2.5.7}$$

which is an important critical threshold for the voltage.

In the case (2.5.6*a*) illustrated by Figure 2.3, we have $\lambda_* < \lambda^*$, and the number of solutions increase but remains finite as λ approaches λ_*. On the other hand, in the case of (2.5.6*b*) illustrated by Figure 2.4, we have $\lambda_* = \lambda^*$, and there seems to be only one branch of solutions.

Case 2.B. N and α satisfy one of the following conditions:

$$N = 1 \quad \text{with} \quad \alpha > \alpha_1 , \tag{2.5.8a}$$

$$2 \le N \le 7 \quad \text{with} \quad \alpha \ge 0 , \tag{2.5.8b}$$

$$N \ge 8 \quad \text{with} \quad \alpha > \alpha_N . \tag{2.5.8c}$$

In this case, we have $\Delta < 0$ and

$$v(s) \sim \Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + \delta_1 e^{-\frac{3N+2\alpha-2}{6}s} cos\big(\frac{\sqrt{-\Delta}}{6}s + C_2\big) + \cdots \quad \text{as} \quad s \to +\infty .$$

We also have for $r \to +\infty$,

$$U(r) \sim r^{\frac{2+\alpha}{3}}\Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + \delta_1 r^{-\frac{N-2}{2}} cos\big(\frac{\sqrt{-\Delta}}{6}\ln r + C_2\big) + \cdots , \tag{2.5.9}$$

and from the fact that $\lambda = \gamma^{2+\alpha}/U^3(\gamma)$ we get again that

$$\lambda \sim \lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9} \quad \text{as} \quad \gamma \to \infty .$$

Note the oscillatory behavior of $U(r)$ in (2.5.9) for large r, which means that λ is expected to oscillate around the value $\lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9}$ as $\gamma \to \infty$, as well as $u_\gamma(0)$. The diagrams above point to the existence of a sequence $\{\lambda_i\}$ satisfying

$$\lambda_0 = 0 , \quad \lambda_{2k} \nearrow \lambda_* \quad \text{as} \quad k \to \infty ,$$

$$\lambda_1 = \lambda^* , \quad \lambda_{2k-1} \searrow \lambda_* \quad \text{as} \quad k \to \infty ,$$

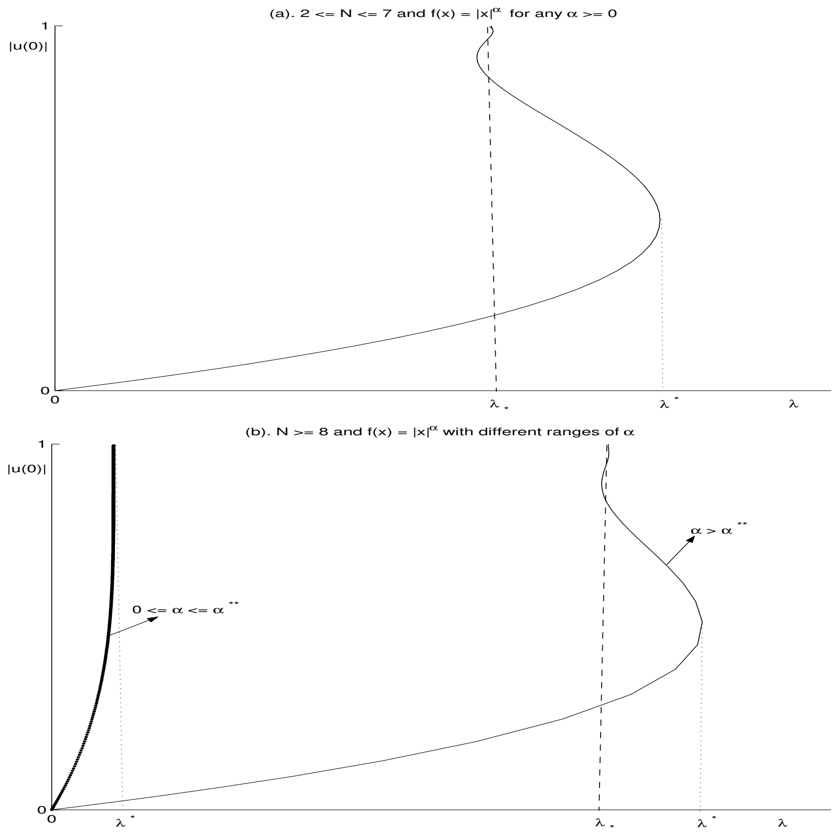

Figure 2.4: *Top figure: Plots of* $u(0)$ *versus* λ *for* $2 \leq N \leq 7$*, where* $u(0)$ *oscillates around the value* λ_* *defined in* (2.5.7) *and* u^* *is regular. Bottom figure: Plots of* $u(0)$ *versus* λ *for* $N \geq 8$*: when* $0 \leq \alpha \leq \alpha_N$*, there exists a unique solution for* $(S)_\lambda$ *with* $\lambda \in (0, \lambda^*)$ *and* u^* *is singular; when* $\alpha > \alpha_N$*,* $u(0)$ *oscillates around the value* λ_* *defined in* (2.5.7) *and* u^* *is regular.*

such that exactly $2k+1$ solutions for $(S)_\lambda$ exist when $\lambda \in (\lambda_{2k}, \lambda_{2k+2})$, while there are exactly $2k$ solutions when $\lambda \in (\lambda_{2k+1}, \lambda_{2k-1})$. Furthermore, $(S)_\lambda$ has infinitely many solutions at $\lambda = \lambda_*$.
The three cases (2.5.8a), (2.5.8b) and (2.5.8c) considered here for N and α are illustrated by the diagrams in Figure 2.3, Figure 2.4(a) and Figure 2.4(b), respectively.

We now conclude from above that the bifurcation diagrams show four possible regimes –at least if the domain is a ball:

A. There is exactly one branch of solution for $0 < \lambda < \lambda^*$. This regime occurs when $N \geq 8$ and $0 \leq \alpha \leq \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. The results of this section actually show that in this range, the first branch of solutions "disappears" at λ^* which happens to be equal to $\lambda_*(\alpha, N) = \frac{(2+\alpha)(3N+\alpha-4)}{9}$.

B. There exists an infinite number of branches of solutions. This regime occurs when

- $N = 1$ and $\alpha \geq \alpha_1 = -\frac{1}{2} + \frac{3}{4}\sqrt{6}$;
- $2 \leq N \leq 7$ and $\alpha \geq 0$;
- $N \geq 8$ and $\alpha > \alpha_N$.

In this case, $\lambda_*(\alpha, N) < \lambda^*$ and the multiplicity becomes arbitrarily large as λ approaches –from either side– $\lambda_*(\alpha, N)$ at which there is a touchdown solution u (i.e., $\| u \|_\infty = 1$).

C. There exists a finite number of branches of solutions. In this case, we have again that $\lambda_*(\alpha, N) < \lambda^*$, but now the branch approaches the value 1 monotonically, and the number of solutions increases but remains finite as λ approaches $\lambda_*(\alpha, N)$. This regime occurs when $N = 1$ and $1 < \alpha \leq \alpha_1$.

D. There exist exactly two branches of solutions for $0 < \lambda < \lambda^*$ and one solution for $\lambda = \lambda^*$. The bifurcation diagram disappears when it returns to $\lambda = 0$. This regime occurs when $N = 1$ and $0 \leq \alpha \leq 1$.

Some of these questions will be considered in next three Chapters. A detailed and involved analysis of compactness along unstable branches will be discussed there, as well as some information about the second bifurcation point.

2.6 Some comments

Main results of this Chapter can be found in [51, 64]. Since the first aim of studying MEMS modeling is to understand the pull-in voltage λ^*, in this Chapter we have applied the method of super- and sub- solutions to showing the existence of λ^*. In order to study analytic bounds of λ^*, we have also used some other methods, such as Pohozaev-type arguments, Bandle's Schwarz symmetrization (cfr. [9]) and etc. However, our analytic bounds of λ^* in Theorem

2.2.4 are not optimal generally, except the special case (2.2.17). This leads us to conjecture that there should be a better estimate for λ^*, involving the distribution of f in Ω.

The second aim of this Chapter has been to discuss semi-stable solutions of $(S)_\lambda$. From the strictly mathematical point of view, it turned out that –at least for $f \equiv 1$– there already exist in the literature many interesting results concerning properties of semi-stable solutions for Dirichlet boundary value problems of the form $-\Delta u = \lambda h(u)$ where h is a regular nonlinearity (e.g., $h(u) = e^u$ or $(1+u)^p$ for $p > 1$). See for example the seminal papers [31, 75, 76] and also [20] for a survey on the subject and an exhaustive list of related references. If the profile $f \equiv 1$, even the case of singular nonlinearities –involved in MEMS devices– had already been considered in [18] and in a more general context in [82]. Some usual analysis of the minimal branch (composed of semi-stable solutions) can be extended to cover the singular situation $(S)_\lambda$. However, some new arguments are still needed to be developed to cover our situation where $f(x)$ is assumed to vanish at somewhere. For example, generally energy estimates are necessary in §2.3.2 for the compactness of semi-stable solutions.

Uniqueness and multiplicity of solutions for $(S)_\lambda$ are discussed in §2.4-2.6 by applying various analytical and numerical techniques. A complete characterization of singular extremal solution u^* for $(S)_{\lambda^*}$ is well-understood in §2.4.1, while the regularity of u^* depends on a critical dimension $N^* = N^*(f)$. It seems from bifurcation diagrams of §2.5 that for middle dimension $2 \le N \le 7$, the uniqueness of solutions for $(S)_\lambda$ holds for small λ, while infinite multiplicity of solutions holds at some $0 < \lambda_* < \lambda^*$. We have also proved a result which will allows us to establish uniqueness and infinite multiplicity for middle dimension $2 \le N \le 7$ and for power-law profile $f(|x|) = |x|^\alpha$ ($\alpha \ge 0$) on the unit ball. In case $f = 1$ and $N = 2$, we shall be able to extend such uniqueness and infinite multiplicity result to symmetric domains in the coordinate variables.
For the case $f(x) = |x|^\alpha$ in a ball, the bifurcation diagrams of §2.5 are quite interesting: the "critical" dimension $N^* = N^*(\alpha)$ of $(S)_\lambda$ is exactly equal to 7 provided $\alpha = 0$, while the "critical" dimension N^* may be larger than 7 for $\alpha > 0$ large enough. This evidence gives hints that the zero property of profile $f(x)$ may push ahead the "critical" dimension of $(S)_\lambda$. We also note the following elliptic problem

$$\begin{cases} -\Delta u = \dfrac{\lambda |x|^\alpha}{(1-u)^2}\,, & x \in B_1, \\ u(x) = 0\,, & x \in \partial B_1 \end{cases} \tag{2.6.1}$$

with $0 \le u < 1$ on B_1, where $\alpha \ge 0$ and B_1 is a unit ball in $\mathbb{R}^N$. The radially symmetric solutions of (2.6.1) are studied in §2.5, by using bifurcation diagrams and asymptotic analysis. Notice that for $\alpha = 0$, any positive solution of (2.6.1) must be radially symmetric, due to the well-known result of Gidas, Ni and Nirenberg [53]. Here is a natural question: when $\alpha > 0$, do there exist non-symmetric positive solutions of (2.6.1)? It is our conjecture that the answer is positive for sufficiently large α.

Chapter 3

Compactness along Lower Branches

In this Chapter we continue the analysis of the problem:

$$\begin{cases} -\Delta u = \dfrac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$(S)_\lambda$}$$

where $\lambda > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and f satisfies (2.0.1). Following the notations and terminology of Chapter 2, the solutions of $(S)_\lambda$ are considered to be in the classical sense, and the *minimal solution* u_λ of $(S)_\lambda$ is the classical solution of $(S)_\lambda$ satisfying $u_\lambda(x) \leq u(x)$ in Ω for any solution u of $(S)_\lambda$.
For any solution u of $(S)_\lambda$, we introduce the linearized operator at u defined by:

$$L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3},$$

and its corresponding eigenvalues $\{\mu_{k,\lambda}(u); k = 1, 2, ...\}$. Note that the first eigenvalue is simple and is given by:

$$\mu_{1,\lambda}(u) = \inf \left\{ \langle L_{u,\lambda}\phi, \phi \rangle_{H_0^1(\Omega)} ; \phi \in C_0^\infty(\Omega), \int_\Omega \phi^2 dx = 1 \right\}$$

with the infimum being attained at a first, positive eigenfunction ϕ_1, while the second eigenvalue is given by the formula:

$$\mu_{2,\lambda}(u) = \inf \left\{ \langle L_{u,\lambda}\phi, \phi \rangle_{H_0^1(\Omega)} ; \phi \in C_0^\infty(\Omega), \int_\Omega \phi^2 dx = 1 \text{ and } \int_\Omega \phi\phi_1 dx = 0 \right\}.$$

This construction can be iterated to obtain the k-th eigenvalue $\mu_{k,\lambda}(u)$ with the convention that eigenvalues are repeated according to their multiplicities.

When the minimal branch is compact, in §3.1 we provide the existence of a second, unstable solution U_λ of $(S)_\lambda$, for λ in a small deleted left neighborhood of λ^*. We give also a Mountain Pass variational characterization of U_λ, see Theorem 3.1.3. §3.2 is concerned with the compactness of the second branch of $(S)_\lambda$ by applying a blow-up analysis, and the main result is given in Theorem 3.2.1. The multiplicity result is studied in Theorem 3.3.1 of §3.3.

3.1 A second solution of Mountain Pass type

The first result we state is quite standard:

Lemma 3.1.1. *Let u_λ be the minimal branch of $(S)_\lambda$ and $u^* = \lim\limits_{\lambda \uparrow \lambda^*} u_\lambda$ be the extremal solution. Suppose $\|u_\lambda\|_\infty \leq C$ for any $\lambda < \lambda^*$, for some $C < 1$. Then there exists $\delta > 0$ such that the solutions of $(S)_\lambda$ near (λ^*, u^*) form a curve $\rho(s) = (\bar{\lambda}(s), v(s))$, $|s| < \delta$, so that*

$$\bar{\lambda}(0) = \lambda^*,\ \bar{\lambda}'(0) = 0,\ \bar{\lambda}''(0) < 0\,, \quad \text{and} \quad v(0) = u^*,\ v' \mid_{\lambda=0} > 0 \text{ in } \Omega\,.$$

In particular, if $1 \leq N \leq 7$ (or $N \geq 8$, $\alpha > \alpha_N$ for power- like profiles on the unit ball), then for λ close from the left to λ^* there exists a unique second branch U_λ of solutions for $(S)_\lambda$, bifurcating from u^*, such that

$$\mu_{1,\lambda}(U_\lambda) < 0 \quad \text{while} \quad \mu_{2,\lambda}(U_\lambda) > 0.$$

Proof: The proof is similar to a related result of Crandall and Rabinowitz (cfr. [30, 31]), so we will be brief. First, the $L^\infty(\Omega)-$bound on u_λ and standard regularity theory show that u_λ is uniformly bounded in $C^1(\bar{\Omega})$. Since $f \in C^{0,\beta}(\bar{\Omega})$ for some $\beta \in (0, 1]$, in turn we get that $\|u_\lambda\|_{C^{2,\beta}(\bar{\Omega})} \leq C < +\infty$. It follows that $u^* = \lim\limits_{\lambda \uparrow \lambda^*} u_\lambda$ exists in $C^2(\bar{\Omega})$ and is a classical solution of $(S)_{\lambda^*}$. Since $\frac{\lambda^* f(x)}{(1-u^*)^2}$ is nonnegative, Theorem 3.2 of [30] characterizes the solution set of $(S)_\lambda$ near (λ^*, u^*): it is a curve parametrized as $(\bar{\lambda}(s), v(s))$, $|s| < \delta$, so that $\bar{\lambda}(0) = \lambda^*$, $\bar{\lambda}'(0) = 0$, $v(0) = u^*$ and $v' \mid_{\lambda=0} > 0$ in Ω. The same computation as in Theorem 4.8 in [30] gives that $\bar{\lambda}''(0) < 0$. In particular, if $1 \leq N \leq 7$ (or $N \geq 8$, $\alpha > \alpha_N$ for power-like profiles on the unit ball) then our Theorem 2.4.3 gives the compactness of u_λ, and the theory of Crandall and Rabinowitz in [31] implies, for λ close to λ^*, the existence of a unique second branch U_λ of solutions for $(S)_\lambda$, bifurcating from u^*. By Lemma 2.4.4, we get

$$\mu_{1,\lambda}(U_\lambda) < 0 \quad \text{while} \quad \mu_{2,\lambda}(U_\lambda) > 0, \tag{3.1.1}$$

as it follows from $\mu_{2,\lambda}(U_\lambda) \to \mu_{2,\lambda^*}(u^*) > 0$ as $\lambda \to \lambda^*$. ∎

We shall give now a variational characterization for both the stable and unstable solutions u_λ, U_λ in the following sense: when $1 \leq N \leq 7$, there exists $\delta > 0$ such that for any $\lambda \in (\lambda^* - \delta, \lambda^*)$, the minimal solution u_λ is a local minimum for some regularized energy functional $J_{\varepsilon,\lambda}$ on $H_0^1(\Omega)$, while the second solution U_λ is a Mountain Pass of $J_{\varepsilon,\lambda}$.

Since the nonlinearity $g(u) = \frac{1}{(1-u)^2}$ is singular at $u = 1$, we need to consider a regularized C^1 nonlinearity $g_\varepsilon(u)$, $0 < \varepsilon < 1$, of the following form:

$$g_\varepsilon(u) = \begin{cases} \dfrac{1}{(1-u)^2} & u \le 1-\varepsilon\,, \\ \dfrac{1}{\varepsilon^2} - \dfrac{2(1-\varepsilon)}{p\varepsilon^3} + \dfrac{2}{p\varepsilon^3(1-\varepsilon)^{p-1}}u^p & u \ge 1-\varepsilon\,, \end{cases} \tag{3.1.2}$$

where $p > 1$ if $N = 1, 2$ and $1 < p < \frac{N+2}{N-2}$ if $3 \le N \le 7$. For $\lambda \in (0, \lambda^*)$, we study the regularized semilinear elliptic problem:

$$\begin{cases} -\Delta u = \lambda f(x) g_\varepsilon(u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.3}$$

From a variational viewpoint, the action functional associated to (3.1.3) is

$$J_{\varepsilon,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega f(x) G_\varepsilon(u) dx\,, \quad u \in H_0^1(\Omega)\,,$$

where $G_\varepsilon(u) = \displaystyle\int_{-\infty}^{u} g_\varepsilon(s) ds$.

In view of Theorem 2.3.4, we now fix $0 < \varepsilon < \frac{1 - \|u^*\|_\infty}{2}$. For $\lambda \uparrow \lambda^*$, the minimal solution u_λ of $(S)_\lambda$ is still a solution of (3.1.3) so that $\mu_1\big(-\Delta - \lambda f(x) g'_\varepsilon(u_\lambda)\big) > 0$. The following holds:

Lemma 3.1.2. *Let $1 \le N \le 7$. For $\lambda \uparrow \lambda^*$, the minimal solution u_λ of $(S)_\lambda$ is a local minimum of $J_{\varepsilon,\lambda}$ on $H_0^1(\Omega)$.*

Proof: First, we show that u_λ is a local minimum of $J_{\varepsilon,\lambda}$ in $C^1(\bar\Omega)$. Indeed, since

$$\mu_{1,\lambda} := \mu_1\big(-\Delta - \lambda f(x) g'_\varepsilon(u_\lambda)\big) > 0$$

and $u_\lambda < 1 - \varepsilon$, we have the following inequality:

$$\int_\Omega |\nabla\phi|^2 dx - 2\lambda \int_\Omega \frac{f(x)}{(1-u_\lambda)^3}\phi^2 dx \ge \mu_{1,\lambda}\int_\Omega \phi^2 \tag{3.1.4}$$

for any $\phi \in H_0^1(\Omega)$. Now, take any $\phi \in H_0^1(\Omega) \cap C^1(\bar\Omega)$ such that $\|\phi\|_{C^1} \le \delta_\lambda$. Since $u_\lambda \le 1 - \frac{3}{2}\varepsilon$, if $\delta_\lambda \le \frac{\varepsilon}{2}$, then $u_\lambda + \phi \le 1 - \varepsilon$ and we have that:

$$\begin{aligned} &J_{\varepsilon,\lambda}(u_\lambda + \phi) - J_{\varepsilon,\lambda}(u_\lambda) \\ &= \frac{1}{2}\int_\Omega |\nabla\phi|^2 dx + \int_\Omega \nabla u_\lambda \cdot \nabla\phi dx - \lambda \int_\Omega f(x)\Big(\frac{1}{1-u_\lambda-\phi} - \frac{1}{1-u_\lambda}\Big) \\ &\ge \frac{\mu_{1,\lambda}}{2}\int_\Omega \phi^2 - \lambda\int_\Omega f(x)\Big(\frac{1}{1-u_\lambda-\phi} - \frac{1}{1-u_\lambda} - \frac{\phi}{(1-u_\lambda)^2} - \frac{\phi^2}{(1-u_\lambda)^3}\Big)\,, \end{aligned} \tag{3.1.5}$$

where we have applied (3.1.4). Since now

$$\Big|\frac{1}{1-u_\lambda-\phi}-\frac{1}{1-u_\lambda}-\frac{\phi}{(1-u_\lambda)^2}-\frac{\phi^2}{(1-u_\lambda)^3}\Big| \le C|\phi|^3$$

for some $C>0$, (3.1.5) gives that

$$J_{\varepsilon,\lambda}(u_\lambda+\phi)-J_{\varepsilon,\lambda}(u_\lambda) \ge \big(\frac{\mu_{1,\lambda}}{2}-C\lambda \parallel f \parallel_\infty \delta_\lambda\big)\int_\Omega \phi^2>0$$

provided δ_λ is small enough. This proves that u_λ is a local minimum of $J_{\varepsilon,\lambda}$ in the $C^1(\bar\Omega)$ topology. Since $g_\varepsilon(u)$ has a subcritical growth

$$0 \le g_\varepsilon(u) \le C_\varepsilon(1+|u|^p), \tag{3.1.6}$$

we can directly apply Theorem 1 in [14] to get that u_λ is a local minimum of $J_{\varepsilon,\lambda}$ in $H_0^1(\Omega)$. ■

The proof for the existence of a second solution for (3.1.3) relies on the standard Mountain Pass Theorem [3]. For self-containedness, we include the statement of this Theorem:

Mountain Pass Theorem: Suppose $J_{\varepsilon,\lambda}$ is a C^1-functional defined on a Banach space E satisfying (PS)-condition and

1. there exists a neighborhood U of u_λ in E and a constant $\sigma>0$ such that $J_{\varepsilon,\lambda}(v_1) \ge J_{\varepsilon,\lambda}(u_\lambda)+\sigma$ for all $v_1 \in \partial U$;
2. $\exists v_2 \notin U$ such that $J_{\varepsilon,\lambda}(v_2) \le J_{\varepsilon,\lambda}(u_\lambda)$.

Defining

$$\Gamma = \Big\{\gamma \in C([0,1],E) : \gamma(0)=u_\lambda\,, \quad \gamma(1)=v_2\Big\},$$

then

$$c_{\varepsilon,\lambda}=\inf_{\gamma\in\Gamma}\max_{0\le t\le 1}\Big\{J_{\varepsilon,\lambda}(\gamma(t)) : t\in(0,1)\Big\}$$

is a critical value of $J_{\varepsilon,\lambda}$.

The following result holds:

Theorem 3.1.3. *Let $1 \le N \le 7$. There exists $\delta>0$ such that for any $\lambda\in(\lambda^*-\delta,\lambda^*)$, the second solution U_λ given by Lemma 3.1.1 is a Mountain Pass solution for $J_{\varepsilon,\lambda}$ on $H_0^1(\Omega)$.*

We briefly sketch the proof of Theorem 3.1.3 as follows. Since u_λ is a local minimum for $J_{\varepsilon,\lambda}$ for $\lambda\uparrow\lambda^*$, by Mountain Pass Theorem we can show the existence of a second solution $U_{\varepsilon,\lambda}$ for (3.1.3). Using the subcritical growth (3.1.6) of $g_\varepsilon(u)$ and the inequality:

$$\theta G_\varepsilon(u) \le u g_\varepsilon(u) \quad \text{for } u \ge M_\varepsilon, \tag{3.1.7}$$

for some C_ε, $M_\varepsilon > 0$ large and $\theta = \frac{p+3}{2} > 2$, we will obtain that $J_{\varepsilon,\lambda}$ satisfies the Palais-Smale condition and, by elliptic regularity theory, we get the uniform convergence of $U_{\varepsilon,\lambda}$ as $\lambda \uparrow \lambda^*$. On the other hand, a similar proof as in Lemma 2.3.1 shows that the convexity of $g_\varepsilon(u)$ ensures that problem (3.1.3) has a unique solution u^* –the extremal solution of $(S)_\lambda$– at $\lambda = \lambda^*$. It allows us to deduce that $U_{\varepsilon,\lambda} \to u^*$ in $C(\bar\Omega)$ as $\lambda \uparrow \lambda^*$, and implies $U_{\varepsilon,\lambda} \le 1-\varepsilon$ for λ close to λ^*. Therefore, $U_{\varepsilon,\lambda}$ is a second solution for $(S)_\lambda$ bifurcating from u^*. Since $U_{\varepsilon,\lambda}$ is a MP solution and $(S)_\lambda$ has exactly two solutions u_λ, U_λ for $\lambda \uparrow \lambda^*$ (cfr. Lemma 3.1.1), it finally yields to $U_{\varepsilon,\lambda} = U_\lambda$.

In order to complete the details for the proof of Theorem 3.1.3, we first need to show that $J_{\varepsilon,\lambda}$ has a Mountain-Pass geometry in $H_0^1(\Omega)$. Since $f \not\equiv 0$, fix some small ball $B_{2r} \subset \Omega$ of radius $2r$, $r > 0$, so that $\int_{B_r} f(x)dx > 0$. Take a cut-off function χ so that $\chi = 1$ on B_r and $\chi = 0$ outside B_{2r}. Let $w_\varepsilon = (1-\varepsilon)\chi \in H_0^1(\Omega)$. We have that:

$$J_{\varepsilon,\lambda}(w_\varepsilon) \le \frac{(1-\varepsilon)^2}{2}\int_\Omega |\nabla\chi|^2 dx - \frac{\lambda}{\varepsilon^2}\int_{B_r} f(x) \to -\infty$$

as $\varepsilon \to 0$, and uniformly for λ far away from zero. Since

$$J_{\varepsilon,\lambda}(u_\lambda) = \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2 dx - \lambda\int_\Omega \frac{f(x)}{1-u_\lambda}dx \to \frac{1}{2}\int_\Omega |\nabla u^*|^2 dx - \lambda^*\int_\Omega \frac{f(x)}{1-u^*}dx$$

as $\lambda \to \lambda^*$, we can find that for $\varepsilon > 0$ small, the inequality

$$J_{\varepsilon,\lambda}(w_\varepsilon) < J_{\varepsilon,\lambda}(u_\lambda) \tag{3.1.8}$$

holds for any λ close to λ^*.

Fix now $\varepsilon > 0$ small enough so that (3.1.8) holds for λ close to λ^*, and define

$$c_{\varepsilon,\lambda} = \inf_{\gamma\in\Gamma}\max_{u\in\gamma} J_{\varepsilon,\lambda}(u),$$

where $\Gamma = \{\gamma : [0,1] \to H_0^1(\Omega); \gamma \text{ continuous and } \gamma(0) = u_\lambda,\ \gamma(1) = w_\varepsilon\}$. We can then use the Mountain Pass Theorem to get a solution $U_{\varepsilon,\lambda}$ of (3.1.3) for λ close to λ^*, provided the Palais-Smale condition holds at level c. We next prove this (PS)-condition in the following form:

Lemma 3.1.4. *Assume that $\{w_n\} \subset H_0^1(\Omega)$ satisfies*

$$J_{\varepsilon,\lambda_n}(w_n) \le C\,, \qquad J'_{\varepsilon,\lambda_n}(w_n) \to 0 \quad \text{in } H^{-1} \tag{3.1.9}$$

for $\lambda_n \to \lambda > 0$. Then the sequence $(w_n)_n$ is uniformly bounded in $H_0^1(\Omega)$ and therefore admits a convergent subsequence in $H_0^1(\Omega)$.

Proof: By (3.1.9) we have that:

$$\int_\Omega |\nabla w_n|^2 dx - \lambda_n \int_\Omega f(x) g_\varepsilon(w_n) w_n dx = o(\| w_n \|_{H_0^1})$$

as $n \to +\infty$ and then,

$$\begin{aligned} C &\geq \frac{1}{2}\int_\Omega |\nabla w_n|^2 dx - \lambda_n \int_\Omega f(x) G_\varepsilon(w_n) dx \\ &= \big(\frac{1}{2} - \frac{1}{\theta}\big)\int_\Omega |\nabla w_n|^2 dx + \lambda_n \int_\Omega f(x)\big(\frac{1}{\theta} w_n g_\varepsilon(w_n) - G_\varepsilon(w_n)\big) dx + o(\| w_n \|_{H_0^1}) \\ &\geq \big(\frac{1}{2} - \frac{1}{\theta}\big)\int_\Omega |\nabla w_n|^2 dx + o(\| w_n \|_{H_0^1}) - C_\varepsilon \\ &\quad + \lambda_n \int_{\{w_n \geq M_\varepsilon\}} f(x)\big(\frac{1}{\theta} w_n g_\varepsilon(w_n) - G_\varepsilon(w_n)\big) dx \\ &\geq \big(\frac{1}{2} - \frac{1}{\theta}\big)\int_\Omega |\nabla w_n|^2 dx + o(\| w_n \|_{H_0^1}) - C_\varepsilon \end{aligned}$$

in view of (3.1.7). Hence, $\sup_{n\in\mathbb{N}} \| w_n \|_{H_0^1} < +\infty$.

Since p is subcritical, the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ provides that, up to a subsequence, $w_n \to w$ weakly in $H_0^1(\Omega)$ and strongly in $L^{p+1}(\Omega)$, for some $w \in H_0^1(\Omega)$. By (3.1.9) we get that $\int_\Omega |\nabla w|^2 = \lambda \int_\Omega f(x) g_\varepsilon(w) w$, and then, by (3.1.6), we deduce that

$$\begin{aligned} \int_\Omega |\nabla (w_n - w)|^2 &= \int_\Omega |\nabla w_n|^2 - \int_\Omega |\nabla w|^2 + o(1) \\ &= \lambda_n \int_\Omega f(x) g_\varepsilon(w_n) w_n - \lambda \int_\Omega f(x) g_\varepsilon(w) w + o(1) \to 0 \end{aligned}$$

as $n \to +\infty$. ■

Proof (of Theorem 3.1.3): Fix $\varepsilon > 0$ small. Consider for any $\lambda \in (\lambda^* - \delta, \lambda^*)$ the Mountain Pass solution $U_{\varepsilon,\lambda}$ of (3.1.3) at energy level $c_{\varepsilon,\lambda}$, where $\delta > 0$ is small enough. Since $c_{\varepsilon,\lambda} \leq c_{\varepsilon,\lambda^*-\delta}$ for any $\lambda \in (\lambda^* - \delta, \lambda^*)$, by Lemma 3.1.4 we get that $U_{\varepsilon,\lambda_n} \to U^*$ in $H_0^1(\Omega)$ along a subsequence $\lambda_n \to \lambda^*$, for some $U^* \in H_0^1(\Omega)$. Then, by (3.1.6) and elliptic regularity theory, we get that $U_{\varepsilon,\lambda_n}$ is uniformly bounded in $C^{2,\beta}(\bar{\Omega})$ for some $\beta \in (0, 1]$. Hence, up to a further subsequence, $U_{\varepsilon,\lambda_n} \to U^*$ in $C^2(\bar{\Omega})$, where U^* is a solution for problem (3.1.3) at $\lambda = \lambda^*$. Also u^* is a solution for (3.1.3) at $\lambda = \lambda^*$ so that $\mu_1\big(-\Delta - \lambda^* f(x) g_\varepsilon'(u^*)\big) = 0$. As already remarked, by convexity of $g_\varepsilon(u)$ it follows that u^* is the unique solution of this equation and therefore $U^* = u^*$. Since along any convergent sequence of $U_{\varepsilon,\lambda}$ as $\lambda \uparrow \lambda^*$ the limit is always u^*, we get that $\lim_{\lambda\uparrow\lambda^*} U_{\varepsilon,\lambda} = u^*$ in $C^2(\bar{\Omega})$. Therefore, since $u^* \leq 1 - 2\varepsilon$, there exists $\delta > 0$ so that for any $\lambda \in (\lambda^* - \delta, \lambda^*)$ $U_{\varepsilon,\lambda} \leq u^* + \varepsilon \leq 1 - \varepsilon$ and hence, $U_{\varepsilon,\lambda}$ is a solution of $(S)_\lambda$. Since the Mountain Pass energy level $c_{\varepsilon,\lambda}$ satisfies $c_{\varepsilon,\lambda} > J_{\varepsilon,\lambda}(u_\lambda)$, we have that $U_{\varepsilon,\lambda} \neq u_\lambda$ and then $U_{\varepsilon,\lambda} = U_\lambda$ for any $\lambda \in (\lambda^* - \delta, \lambda^*)$. Note that by Lemma 3.1.1, we know that u_λ, U_λ are the only solutions of $(S)_\lambda$ as $\lambda \uparrow \lambda^*$. ■

Let us stress that the argument of Theorem 3.1.3 works also for problem $(S)_\lambda$ on the unit ball with $f(x)$ in the form (2.3.17) provided $N \geq 8$, $\alpha > \alpha_N$. This leads to the following Proposition for the higher dimensional case:

Proposition 3.1.5. *Theorem 3.1.3 is still true when Ω is a ball, and $f(x)$ is as in (2.3.17) provided $N \geq 8$ and $\alpha > \alpha_N$.*

3.2 Compactness along the second branch

In this section, we are interested in continuing the second branch till the second bifurcation point, by means of the Implicit Function Theorem. Our main result of this section is the following compactness for $2 \leq N \leq 7$:

Theorem 3.2.1. *Assume $2 \leq N \leq 7$, and suppose f in the form*

$$f(x) = \Big(\prod_{i=1}^{k} |x - p_i|^{\alpha_i}\Big) g(x)\,, \quad g(x) \geq C > 0 \;\; in \;\; \Omega, \tag{3.2.1}$$

where $g \in C^1(\bar{\Omega})$, for some points $p_i \in \Omega$ and exponents $\alpha_i \geq 0$. Let $(\lambda_n)_n$ be a sequence such that $\lambda_n \to \lambda \in [0, \lambda^*]$ and let u_n be an associated solution such that

$$\mu_{2,n} := \mu_{2,\lambda_n}(u_n) \geq 0. \tag{3.2.2}$$

Then, $\sup\limits_{n\in\mathbb{N}} \| u_n \|_\infty < 1$. Moreover, if in addition $\mu_{1,n} := \mu_{1,\lambda_n}(u_n) < 0$, then necessarily $\lambda > 0$.

Remark 3.2.1. We remark that Theorem 3.2.1 yields to an another proof – based on a blow-up argument– of the compactness result for minimal solutions established in Chapter 2 by means of some energy estimates, though under the more stringent assumption (3.2.1) on $f(x)$. We expect that the same result should be true for radial solutions on the unit ball for $N \geq 8$, $\alpha > \alpha_N$, and f as in (2.3.17).

In order to prove Theorem 3.2.1, we now assume that f is in the form (3.2.1), and let $(u_n)_n$ be a solution sequence for $(S)_{\lambda_n}$ where $\lambda_n \to \lambda \in [0, \lambda^*]$.

3.2.1 Blow-up analysis

Assume that the sequence $(u_n)_n$ is not compact, which means that up to passing to a subsequence, we may assume that $\max\limits_{\Omega} u_n \to 1$ as $n \to \infty$. Let x_n be a maximum point of u_n in Ω (i.e., $u_n(x_n) = \max\limits_{\Omega} u_n$). We would like to identify the limiting profile of u_n around x_n.

More generally, let $y_n \in \Omega$ be a sequence of points so that $u_n(y_n) \to 1^-$ as $n \to +\infty$, and set $\mu_n = 1 - u_n(y_n)$. As we will see later, for our purposes it is not restrictive to assume that $\mu_n^3 \lambda_n^{-1} \to 0$ and $y_n \to p \in \bar{\Omega}$ as $n \to +\infty$. Depending on the location of p and the rate of $|y_n - p|$, the length scale to see around y_n some non trivial limiting profile is the following:

$$r_n = \begin{cases} \mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} & \text{if } p \notin Z\,, \\ \mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |y_n - p_i|^{-\frac{\alpha_i}{2}} & \text{if } p = p_i\,,\ \mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha_i + 2} \to +\infty \text{ as } n \to +\infty\,, \\ \mu_n^{\frac{3}{2+\alpha_i}} \lambda_n^{-\frac{1}{2+\alpha_i}} & \text{if } \limsup\limits_{n \to +\infty} \mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha_i+2} < +\infty, \end{cases} \tag{3.2.3}$$

where $Z = \{p_1, \ldots, p_k\}$ is the zero set of the potential $f(x)$, and $\alpha_1, \ldots, \alpha_k$ are the related multiplicities given by (3.2.1). Let us remark that $\mu_n^3 \lambda_n^{-1} \to 0$ implies $r_n \to 0$ as $n \to +\infty$. Only to give an idea, let us establish the following rough correspondence: the first situation in the definition of r_n corresponds to a blow up at some point outside Z, the second one to a "slow" blow up at some $p_i \in Z$, while the third one is a "fast" blow at some $p_i \in Z$. We now introduce the following rescaled function around y_n:

$$U_n(y) = \frac{1 - u_n(r_n y + y_n)}{\mu_n}\,, \qquad y \in \Omega_n = \frac{\Omega - y_n}{r_n}\,. \tag{3.2.4}$$

Since $U_n(0) = 1$ by construction, in order to get a limiting profile equation we should add a condition avoiding vanishing on compact sets of Ω_n. Let us remark that, for the maximum point x_n of u_n and $\varepsilon_n = 1 - u_n(x_n)$, the associated rescaled function U_n satisfies: $U_n \geq U_n(0) = 1$ in Ω_n.

Proposition 3.2.2. *Assume that*

$$\mu_n^3 \lambda_n^{-1} \to 0 \quad \text{as } n \to +\infty \tag{3.2.5}$$

and

$$U_n \geq C > 0 \quad \text{in } \Omega_n \cap B_{R_n}(0), \tag{3.2.6}$$

for some $R_n \to +\infty$ as $n \to +\infty$. Then, up to a subsequence, $U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$, where U is a solution of the problem

$$\begin{cases} \Delta U = s \dfrac{|y + y_0|^\gamma}{U^2} & \text{in } R^N\,, \\ U(y) \geq C > 0 & \text{in } \mathbb{R}^N \end{cases} \tag{3.2.7}$$

for some $s > 0$, $\gamma \in \{0, \alpha_1, \ldots, \alpha_k\}$ and $y_0 \in \mathbb{R}^N$ (depending on the type of blow up). Moreover, there exists a function $\phi_n \in C_0^\infty(\Omega)$ such that:

$$\int_\Omega \left(|\nabla \phi_n|^2 - \frac{2\lambda_n f(x)}{(1 - u_n)^3} \phi_n^2 \right) < 0 \tag{3.2.8}$$

and Supp $\phi_n \subset B_{Mr_n}(y_n)$ for some $M > 0$.

Proof: As already remarked, observe that (3.2.5) implies $r_n \to 0$ as $n \to +\infty$. Based on Lemma 3.2.3 below, in case $p \in \partial\Omega$ (3.2.5) provides us with a stronger estimate:

$$r_n(\text{dist }(y_n, \partial\Omega))^{-1} \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.2.9}$$

Indeed, by contradiction and up to a subsequence, assume that $r_n d_n^{-1} \to \delta > 0$ as $n \to +\infty$, where $d_n := \text{dist }(y_n, \partial\Omega)$, and then, $d_n \to 0$ as $n \to +\infty$. We introduce the following rescaling W_n:

$$W_n(y) = \frac{1 - u_n(d_n y + y_n)}{\mu_n}, \quad y \in A_n = \frac{\Omega - y_n}{d_n} \cap B_{r_n d_n^{-1} R_n}(0)\,.$$

Since $R_n \to +\infty$ and $r_n d_n^{-1} \to \delta > 0$, we get that $r_n d_n^{-1} R_n \to +\infty$ and, in view of $d_n \to 0$, $A_n \to T_1$ as $n \to +\infty$, where T_1 is a hyperspace containing 0 so that dist $(0, \partial T_1) = 1$. The function W_n solves problem (3.2.13) with $h_n(y) = \frac{\lambda_n d_n^2}{\mu_n^3} f(d_n y + y_n)$. The lower bound (3.2.6) rewrites as

$$W_n \geq C > 0 \quad \text{in } A_n.$$

We have that:

$$\| h_n \|_\infty \leq \frac{\lambda_n d_n^2}{\mu_n^3} \| f \|_\infty \leq \frac{2}{\delta} \| f \|_\infty$$

and $W_n = \frac{1}{\mu_n} \to +\infty$ on $\partial A_n \cap B_2(0)$ as $n \to \infty$. By Lemma 3.2.3 we get that (3.2.15) must hold, a contradiction to Hopf Lemma applied to u_n on $\partial A_n \cap B_2(0)$. This gives the validity of (3.2.9).

Now, in case $p \in \partial\Omega$ (3.2.9) gives that

$$\text{dist }(0, \partial\Omega_n) = \frac{\text{dist }(y_n, \partial\Omega)}{r_n} \to +\infty$$

as $n \to +\infty$, which implies that $\Omega_n \to \mathbb{R}^N$ as $n \to +\infty$. For $p \in \Omega$, since $r_n \to 0$ as $n \to +\infty$, it is clearly true that $\Omega_n \to \mathbb{R}^N$ as $n \to +\infty$. Introduce the following notation

$$f_i(x) = \Big(\prod_{j=1,\, j \neq i}^{k} |x - p_j|^{\alpha_j} \Big) g(x)\,, \tag{3.2.10}$$

then U_n satisfies $\Delta U_n = \frac{f_n(y)}{U_n^2}$ in Ω_n, where $f_n(y)$ is given by:

$$f_n = \begin{cases} f(r_n y + y_n) & \text{if } p \notin Z\,, \\ |\frac{r_n}{|y_n - p_i|} y + \frac{y_n - p_i}{|y_n - p_i|}|^{\alpha_i} f_i(r_n y + y_n) & \text{if } p = p_i\,,\ \mu_n^{-3}\lambda_n |y_n - p_i|^{\alpha_i + 2} \to \infty \text{ as } n \to \infty\,, \\ |y + \frac{y_n - p_i}{r_n}|^{\alpha_i} f_i(r_n y + y_n) & \text{if } \limsup\limits_{n \to +\infty} \mu_n^{-3}\lambda_n |y_n - p_i|^{\alpha_i + 2} < +\infty. \end{cases}$$

Only in the latter situation $\limsup_{n\to+\infty} \mu_n^{-3}\lambda_n|y_n - p_i|^{\alpha_i+2} < +\infty$, up to a subsequence, we assume that

$$\frac{y_n - p_i}{r_n} \to y_0 \quad \text{as } n \to +\infty. \tag{3.2.11}$$

Arguing as in Proposition 2.3.8, up to a subsequence, we get that $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$. According to the three situations described in the definition (3.2.3) of r_n, the function $U \geq C > 0$ is a solution of (3.2.7) with: $s = f(p)$, $\gamma = 0$ in the first case; $s = f_i(p)$, $\gamma = 0$ in the second case; $s = f_i(p)$, $\gamma = \alpha_i$ and y_0 as in (3.2.11) in the third case. Set $f_\infty(y) := \lim_{n\to+\infty} f_n(y) = s|y + y_0|^\gamma$.

Since $2 \leq N \leq 7$ and $s > 0$, Theorem 5.1.1 gives that $\mu_1(U) < 0$ and then, we find $\phi \in C_0^\infty(\mathbb{R}^N)$ so that

$$\int \Big(|\nabla\phi|^2 - \frac{2f_\infty(y)}{U^3}\phi^2\Big) < 0.$$

Defining now $\phi_n(x) = r_n^{-\frac{N-2}{2}}\phi(\frac{x-y_n}{r_n})$, we then have

$$\int_\Omega \Big(|\nabla\phi_n|^2 - \frac{2\lambda_n f(x)}{(1-u_n)^3}\phi_n^2\Big) = \int \Big(|\nabla\phi|^2 - \frac{2f_n(y)}{U_n^3}\phi^2\Big) \to \int \Big(|\nabla\phi|^2 - \frac{2f_\infty(y)}{U^3}\phi^2\Big) < 0$$

as $n \to +\infty$, since ϕ has compact support and $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R})$. ■

Remark 3.2.2. In the case of fast blow up at p_i: $\limsup_{n\to+\infty} \mu_n^{-3}\lambda_n|y_n - p_i|^{\alpha_i+2} < +\infty$, Proposition 3.2.2 is still true if, instead of condition (3.2.6), we assume:

$$U_n \geq C|y + \frac{y_n - p_i}{r_n}|^{\frac{\alpha_i}{3}} \quad \text{in } \Omega_n \cap B_{R_n}(0), \tag{3.2.12}$$

for some $R_n \to +\infty$ as $n \to +\infty$ and $C > 0$. Recall that in this situation $r_n = \mu_n^{\frac{3}{2+\alpha_i}}\lambda_n^{-\frac{1}{2+\alpha_i}}$. By (3.2.12), we get easily that on $\Omega_n \cap B_{R_n}(0)$:

$$0 \leq \Delta U_n \leq C\big|y + \frac{y_n - p_i}{r_n}\big|^{\frac{\alpha_i}{3}}.$$

Up to a subsequence, we get that $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, where $U \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{-y_0\})$ is a solution of

$$\begin{cases} \Delta U = |y + y_0|^{\alpha_i}\dfrac{f_i(p_i)}{U^2} & \text{in } \mathbb{R}^N \setminus \{-y_0\}\,, \\ U(y) \geq C|y + y_0|^{\frac{\alpha_i}{3}} & \text{in } \mathbb{R}^N\,. \end{cases}$$

By Hopf Lemma, we have that $U(-y_0) > 0$. Indeed, let B some ball so that $-y_0 \in \partial B$ and assume by contradiction that $U(-y_0) = 0$. Since

$$-\Delta U + c(y)U = 0 \text{ in } B\,, \ \ U \in C^2(B) \cap C(\bar{B})\,, \ \ U(y) > U(-y_0) \text{ in } B,$$

and $c(y) = f_i(p_i)\frac{|y+y_0|^{\alpha_i}}{U^3} \geq 0$ is a bounded function, by Hopf Lemma we get that $\partial_\nu U(-y_0) < 0$, where ν is the unit outward normal of B. Hence, along the outward normal direction of B at $-y_0$ U becomes negative in contradiction with the positivity of U. Hence, $U(-y_0) > 0$ and $U(y) \geq C := \inf_{R^N} U(y) > 0$ in $\mathbb{R}^N$. The argument now goes as in the proof of Proposition 3.2.2.

Let us remark that, in general, we are not able to prove that a blow up point p is always far away from $\partial\Omega$, even though we suspect it to be true. However, the weaker estimate (3.2.5), which holds in general, yields to (3.2.9), based on the following Lemma:

Lemma 3.2.3. *Let h_n be a function on a smooth bounded domain A_n in $\mathbb{R}^N$. Let W_n be a solution of the problem*

$$\begin{cases} \Delta W_n = \dfrac{h_n(x)}{W_n^2} & \text{in } A_n, \\ W_n(y) \geq C > 0 & \text{in } A_n, \\ W_n(0) = 1, \end{cases} \tag{3.2.13}$$

for some $C > 0$. Assume that $\sup_{n\in\mathbb{N}} \| h_n \|_\infty < +\infty$ and $A_n \to T_\mu$ as $n \to +\infty$ for some $\mu \in (0, +\infty)$, where T_μ is a hyperspace so that $0 \in T_\mu$ and dist $(0, \partial T_\mu) = \mu$. Then for sufficiently large n, either

$$\min_{\partial A_n \cap \overline{B_{2\mu}(0)}} W_n \leq C \tag{3.2.14}$$

or

$$\min_{\partial A_n \cap \overline{B_{2\mu}(0)}} \partial_\nu W_n \leq 0, \tag{3.2.15}$$

where ν is the unit outward normal of A_n.

Proof: Assume that $\partial_\nu W_n > 0$ on $\partial A_n \cap \overline{B_{2\mu}(0)}$. Let

$$G(x) = \begin{cases} -\dfrac{1}{2\pi} \log \dfrac{|x|}{2\mu} & \text{if } N = 2 \\ c_N\big(\dfrac{1}{|x|^{N-2}} - \dfrac{1}{(2\mu)^{N-2}}\big) & \text{if } N \geq 3 \end{cases}$$

be the Green function of the operator $-\Delta$ in $B_{2\mu}(0)$ with homogeneous Dirichlet boundary condition, where $c_N = \frac{1}{(N-2)|\partial B_1(0)|}$ and $|\cdot|$ stands for the Lebesgue measure.
Here and in the sequel, when there is no ambiguity on the domain, ν and dS will denote the unit outward normal and the boundary integration element of the corresponding domain. By the representation formula we have that:

$$\begin{aligned} W_n(0) = &- \int_{A_n \cap B_{2\mu}(0)} \Delta W_n(x) G(x) dx - \int_{\partial A_n \cap B_{2\mu}(0)} W_n(x) \partial_\nu G(x) dS \\ &+ \int_{\partial A_n \cap B_{2\mu}(0)} \partial_\nu W_n(x) G(x) dS - \int_{\partial B_{2\mu}(0) \cap A_n} W_n(x) \partial_\nu G(x) dS. \end{aligned} \tag{3.2.16}$$

Note that on ∂T_μ we have

$$-\partial_\nu G(x) = \begin{cases} \frac{1}{2\pi}\frac{x}{|x|^2}\cdot\nu > 0 & \text{if } N=2\,; \\ (N-2)c_N\frac{x}{|x|^N}\cdot\nu > 0 & \text{if } N\geq 3\,. \end{cases} \tag{3.2.17}$$

Since $\partial A_n \to \partial T_\mu$ as $n\to\infty$, it yields that for sufficiently large n,

$$\partial_\nu G(x) \leq 0 \quad \text{on } \partial A_n \cap B_{2\mu}(0). \tag{3.2.18}$$

Hence, by (3.2.16), (3.2.18) and the assumptions on W_n, we then get for sufficiently large n,

$$1 \geq -\int_{A_n\cap B_{2\mu}(0)} \frac{h_n(x)}{W_n^2(x)}G(x)dx - \big(\min_{\partial A_n\cap B_{2\mu}(0)} W_n\big)\int_{\partial A_n\cap B_{2\mu}(0)} \partial_\nu G(x)dS\,,$$

since $G(x)\geq 0$ in $B_{2\mu}(0)$ and $\partial_\nu G(x)\leq 0$ on $\partial B_{2\mu}(0)$. On the other hand, we have

$$\Big|\int_{A_n\cap B_{2\mu}(0)} \frac{h_n(x)}{W_n^2(x)}G(x)\Big| \leq C\,,$$

and (3.2.17) also implies that for sufficiently large n,

$$-\int_{\partial A_n\cap B_{2\mu}(0)} \partial_\nu G(x)d\sigma(x) \to -\int_{\partial T_\mu\cap B_{2\mu}(0)} \partial_\nu G(x)dS > 0.$$

Then for sufficiently large n, $1\geq -C + C^{-1}\big(\min_{\partial A_n\cap B_{2\mu}(0)} W_n\big)$ for some $C>0$ large enough. Therefore, we conclude that for sufficiently large n, $\min_{\partial A_n\cap B_{2\mu}(0)} W_n$ is bounded and the proof is complete. ■

3.2.2 Spectral confinement

Let us now assume the validity of (3.2.2), namely $\mu_{2,n} := \mu_{2,\lambda_n}(u_n) \geq 0$ for any $n\in\mathbb{N}$. This information will play a crucial role in controlling the number k of "blow up points" (for $(1-u_n)^{-1}$) in terms of the spectral information on u_n. Indeed, roughly speaking, we can estimate k with the number of negative eigenvalues of L_{u_n,λ_n} (with multiplicities). In particular, assumption (3.2.2) implies that "blow up" can occur only along the sequence x_n of maximum points of u_n in Ω.

Proposition 3.2.4. *Assume $2\leq N\leq 7$, and suppose f as in (3.2.1). Let $\lambda_n\to\lambda\in[0,\lambda^*]$ and u_n be an associated solution. Assume that $u_n(x_n)=\max_\Omega u_n \to 1$ as $n\to+\infty$. Then, there exist constants $C>0$ and $N_0\in\mathbb{N}$ such that*

$$\big(1-u_n(x)\big) \geq C\lambda_n^{\frac{1}{3}} d(x)^{\frac{\alpha}{3}}|x-x_n|^{\frac{2}{3}}\,, \quad \forall\, x\in\Omega\,,\ n\geq N_0\,, \tag{3.2.19}$$

where $d(x)^{\frac{\alpha}{3}} = \min\{|x-p_i|^{\frac{\alpha_i}{3}} : i=1,\dots,k\}$ is a "distance function" from the zero set $\{p_1,\dots,p_k\}$ of $f(x)$.

Proof: Let $\varepsilon_n = 1 - u_n(x_n)$. Then, $\varepsilon_n \to 0$ as $n \to +\infty$ and, even more precisely:

$$\varepsilon_n^3 \lambda_n^{-1} \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.2.20}$$

Indeed, otherwise we would have along some subsequence:

$$0 \le \frac{\lambda_n f(x)}{(1-u_n)^2} \le \frac{\lambda_n}{\varepsilon_n^2} \parallel f \parallel_\infty \le C\varepsilon_n \to 0 \text{ as } n \to +\infty.$$

But if the right hand side of $(S)_{\lambda_n}$ tends uniformly to zero, then elliptic regularity theory implies that, up to a subsequence, $u_n \to u$ in $C^1(\bar\Omega)$, where u is a harmonic function such that $u = 0$ on $\partial\Omega$. Hence, $u \equiv 0$ in Ω. On the other hand, $\varepsilon_n \to 0$ implies that $\max\limits_\Omega u = 1$, a contradiction.
By (3.2.20), (3.2.5) holds and we can apply Proposition 3.2.2 to obtain a function $\phi_n \in C_0^\infty(\Omega)$ such that (3.2.8) holds, together with a specific control on Supp ϕ_n.

By contradiction, assume now that (3.2.19) is false: up to a subsequence, then there exists a sequence $y_n \in \Omega$ such that

$$\begin{aligned} &\lambda_n^{-\frac13} d(y_n)^{-\frac{\alpha}{3}} |y_n - x_n|^{-\frac23} \big(1 - u_n(y_n)\big) \\ &= \lambda_n^{-\frac13} \min_{x\in\Omega} \Big(d(x)^{-\frac{\alpha}{3}} |x - x_n|^{-\frac23} \big(1 - u_n(x)\big)\Big) \to 0 \quad as \quad n \to +\infty\,. \end{aligned} \tag{3.2.21}$$

Then, $\mu_n := 1 - u_n(y_n) \to 0$ as $n \to \infty$ and (3.2.21) can be rewritten as:

$$\frac{\mu_n^{\frac32} \lambda_n^{-\frac12}}{|x_n - y_n|\, d(y_n)^{\frac{\alpha}{2}}} \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.2.22}$$

We now want to explain the meaning of the crucial choice (3.2.21). Let β_n be a sequence of positive numbers so that

$$R_n := \beta_n^{-\frac12} \min\{d(y_n)^{\frac12}, |x_n - y_n|^{\frac12}\} \to +\infty \quad \text{as} \quad n \to +\infty. \tag{3.2.23}$$

Let us introduce the following rescaled function:

$$\hat U_n(y) = \frac{1 - u_n(\beta_n y + y_n)}{\mu_n}\,, \quad y \in \hat\Omega_n = \frac{\Omega - y_n}{\beta_n}.$$

Formula (3.2.21) implies:

$$\begin{aligned} \mu_n &= d(y_n)^{\frac{\alpha}{3}} |y_n - x_n|^{\frac23} \min_{x\in\Omega} \Big(d(x)^{-\frac{\alpha}{3}} |x - x_n|^{-\frac23} \big(1 - u_n(x)\big)\Big) \\ &\le \mu_n d(y_n)^{\frac{\alpha}{3}} |y_n - x_n|^{\frac23} d(\beta_n y + y_n)^{-\frac{\alpha}{3}} |\beta_n y + y_n - x_n|^{-\frac23} \hat U_n(y). \end{aligned}$$

Since

$$\frac{d(\beta_n y + y_n)}{d(y_n)} = \min\{\Big|\frac{y_n - p_i}{d(y_n)} + \frac{\beta_n}{d(y_n)} y\Big| : i = 1, \ldots, k\} \ge 1 - \frac{\beta_n}{d(y_n)}|y|$$

in view of $|y_n - p_i| \geq d(y_n)$, by (3.2.23) we get that:

$$\hat{U}_n(y) \geq \big(1 - \frac{\beta_n R_n}{d(y_n)}\big)^{\frac{\alpha}{3}} \big(1 - \frac{\beta_n R_n}{|x_n - y_n|}\big)^{\frac{2}{3}} \geq \big(\frac{1}{2}\big)^{\frac{2+\alpha}{3}}$$

for any $y \in \hat{\Omega}_n \cap B_{R_n}(0)$. Hence, whenever (3.2.23) holds, we get the validity of (3.2.6) for the rescaled function $\hat{U}_n$ at y_n with respect to β_n.

We need to discuss all the possible types of blow up at y_n.

$1^{\textbf{st}}$ **Case** Assume that $y_n \to q \notin \{p_1, \ldots, p_k\}$. By (3.2.22) we get that $\mu_n^3 \lambda_n^{-1} \to 0$ as $n \to +\infty$. Since $d(y_n) \geq C > 0$, let $\beta_n = \mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}$ and, by (3.2.22) we get that (3.2.23) holds. Associated to y_n, μ_n, define $\hat{U}_n$, $\hat{\Omega}_n$ as in (3.2.4). We have from above that (3.2.6) holds by the validity of (3.2.23) for our choice of β_n. Hence, Proposition 3.2.2 applied to $\hat{U}_n$ gives the existence of $\psi_n \in C_0^\infty(\Omega)$ such that (3.2.8) holds and Supp $\psi_n \subset B_{M\mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}(y_n)$ for some $M > 0$.

In the worst case $x_n \to q$, given U_n be as in (3.2.4) associated to x_n, ε_n, we get by scaling that, for $x = \varepsilon_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} y + x_n$,

$$\begin{aligned} &\lambda_n^{-\frac{1}{3}} \big(d(x)^{-\frac{\alpha}{3}} |x - x_n|^{-\frac{2}{3}} \big(1 - u_n(x)\big)\big) \\ &\geq C \lambda_n^{-\frac{1}{3}} \big(|x - x_n|^{-\frac{2}{3}} \big(1 - u_n(x)\big)\big) = C|y|^{-\frac{2}{3}} U_n(y) \geq C_R > 0 \end{aligned}$$

uniformly in n and $y \in B_R(0)$ for any $R > 0$. Then,

$$\frac{\varepsilon_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n - y_n|} \to 0 \quad \text{as} \quad n \to +\infty.$$

Hence, in this situation ϕ_n and ψ_n have disjoint compact supports and obviously, it remains true when $x_n \to p \neq q$. Hence, $\mu_{2,n} < 0$ in contradiction with (3.2.2).

$2^{\textbf{nd}}$ **Case** Assume that $y_n \to p_i$ in a “slow” way:

$$\mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha+2} \to +\infty \quad \text{as} \quad n \to +\infty.$$

Let now $\beta_n = \mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |y_n - p_i|^{-\frac{\alpha}{2}}$. Since $d(y_n) = |y_n - p_i|$ in this situation, we get that:

$$\frac{d(y_n)}{\beta_n} = \mu_n^{-\frac{3}{2}} \lambda_n^{\frac{1}{2}} |y_n - p_i|^{\frac{\alpha+2}{2}} \to +\infty,$$

and (3.2.22) exactly gives

$$\frac{|x_n - y_n|}{\beta_n} = \frac{|x_n - y_n|}{\mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |y_n - p_i|^{-\frac{\alpha}{2}}} \to +\infty \tag{3.2.24}$$

as $n \to +\infty$. Hence, (3.2.23) holds. Associated to μ_n, y_n, define now $\hat{U}_n$, $\hat{\Omega}_n$ according to (3.2.4). Since (3.2.6) follows by (3.2.23), Proposition 3.2.2 for $\hat{U}_n$ gives some $\psi_n \in C_0^\infty(\Omega)$

such that (3.2.8) holds and Supp $\psi_n \subset B_{M\mu_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|y_n-p_i|^{-\frac{\alpha}{2}}}(y_n)$ for some $M > 0$. If $x_n \to p \neq p_i$, then clearly ϕ_n, ψ_n have disjoint compact supports leading to $\mu_{2,n} < 0$ in contradiction with (3.2.2). If also $x_n \to p_i$, we can easily show by scaling that:

1) if $\varepsilon_n^{-3}\lambda_n|x_n - p_i|^{\alpha+2} \to +\infty$ as $n \to +\infty$, given U_n be as in (3.2.4) associated to x_n, ε_n, we get that for $x = \varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n - p_i|^{-\frac{\alpha}{2}}y + x_n$,

$$\lambda_n^{-\frac{1}{3}}\big(d(x)^{-\frac{\alpha}{3}}|x - x_n|^{-\frac{2}{3}}\big(1 - u_n(x)\big)\big)$$
$$= |y|^{-\frac{2}{3}}U_n(y)\big|\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n - p_i|^{-\frac{\alpha+2}{2}}y + \frac{x_n - p_i}{|x_n - p_i|}\big|^{-\frac{\alpha}{3}} \geq C_R > 0$$

uniformly in n and $y \in B_R(0)$ for any $R > 0$. Then,

$$\frac{\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n - p_i|^{-\frac{\alpha}{2}}}{|x_n - y_n|} \to 0 \quad \text{as} \quad n \to +\infty,$$

and hence, by (3.2.24) ϕ_n and ψ_n have disjoint compact supports leading to $\mu_{2,n} < 0$, which contradicts (3.2.2).

2) if $\varepsilon_n^{-3}\lambda_n|x_n - p_i|^{\alpha+2} \leq C$ as $n \to +\infty$, given U_n be as in (3.2.4) associated to x_n, ε_n, we get that for $x = \varepsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}y + x_n$,

$$\begin{aligned}\lambda_n^{-\frac{1}{3}}\big(d(x)^{-\frac{\alpha}{3}}|x - x_n|^{-\frac{2}{3}}\big(1 - u_n(x)\big)\big) &= |y|^{-\frac{2}{3}}U_n(y)\big|y + \varepsilon_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}}(x_n - p_i)\big|^{-\frac{\alpha}{3}}\\ &\geq D_R|y|^{-\frac{2}{3}}U_n(y) \geq C_R > 0\end{aligned}$$

uniformly in n and $y \in B_R(0)$ for any $R > 0$. Then,

$$\frac{\varepsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}}{|x_n - y_n|} \to 0 \quad \text{as} \quad n \to +\infty,$$

and hence, by (3.2.24) ϕ_n and ψ_n have disjoint compact supports leading to a contradiction.

3^{rd} **Case** Assume that $y_n \to p_i$ in a "fast" way:

$$\mu_n^{-3}\lambda_n|y_n - p_i|^{\alpha+2} \leq C.$$

Since $d(y_n) = |y_n - p_i|$, by (3.2.22) we get that

$$\frac{|y_n - p_i|}{|x_n - y_n|} = \frac{\mu_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}}{|x_n - y_n|\,|y_n - p_i|^{\frac{\alpha}{2}}}\big(\mu_n^{-3}\lambda_n|y_n - p_i|^{\alpha+2}\big)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to +\infty, \tag{3.2.25}$$

and then for n large

$$\frac{|x_n - p_i|}{|y_n - p_i|} \geq \frac{|x_n - y_n|}{|y_n - p_i|} - 1 \geq 1 \ , \quad \frac{|x_n - p_i|}{|x_n - y_n|} \geq 1 - \frac{|y_n - p_i|}{|x_n - y_n|} \geq \frac{1}{2}. \tag{3.2.26}$$

Since $\varepsilon_n \le \mu_n$, (3.2.22) and (3.2.26) give that

$$\begin{aligned} &\varepsilon_n^{-3}\lambda_n|x_n-p_i|^{\alpha+2} \\ &\ge \Big(\frac{\mu_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}}{|x_n-y_n|\,|y_n-p_i|^{\frac{\alpha}{2}}}\Big)^{-2}\Big(\frac{|x_n-p_i|}{|x_n-y_n|^{\frac{2}{\alpha+2}}\,|y_n-p_i|^{\frac{\alpha}{\alpha+2}}}\Big)^{\alpha+2} \\ &\ge C\Big(\frac{\mu_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}}{|x_n-y_n|\,d(y_n)^{\frac{\alpha}{2}}}\Big)^{-2} \to +\infty \quad \text{as } n\to+\infty. \end{aligned} \tag{3.2.27}$$

The meaning of (3.2.27) is the following: once y_n provides a fast blowing up sequence at p_i, then no other fast blow up at p_i can occur as (3.2.27) states for x_n.

Let $\beta_n = \mu_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}$. By (3.2.22) and (3.2.25) we get that

$$\begin{aligned} \frac{\beta_n}{|x_n-y_n|} &= \mu_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}|x_n-y_n|^{-1} \\ &= \Big(\frac{\mu_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}}{|x_n-y_n|\,d(y_n)^{\frac{\alpha}{2}}}\Big)^{\frac{2}{2+\alpha}}\Big(\frac{|y_n-p_i|}{|x_n-y_n|}\Big)^{\frac{\alpha}{2+\alpha}} \to 0 \quad \text{as } n\to+\infty. \end{aligned} \tag{3.2.28}$$

However, since u_n blows up fast at p_i along y_n, we have $\beta_n^{-1}d(y_n) \le C$ and then, (3.2.23) does not hold. Letting as before

$$\hat{U}_n(y) = \frac{1-u_n(\beta_n y + y_n)}{\mu_n}\,,\ \ y\in\hat{\Omega}_n = \frac{\Omega - y_n}{\beta_n},$$

we need to refine the analysis before in order to get some estimate for $\hat{U}_n$ even when only (3.2.28) does hold. Formula (3.2.21) gives that:

$$\begin{aligned} \hat{U}_n(y) &\ge |y_n-p_i|^{-\frac{\alpha}{3}}|y_n-x_n|^{-\frac{2}{3}}|\beta_n y+y_n-p_i|^{\frac{\alpha}{3}}|\beta_n y+y_n-x_n|^{\frac{2}{3}} \\ &= \big(\mu_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}}|y_n-p_i|\big)^{-\frac{\alpha}{3}}\Big|\frac{\beta_n}{|x_n-y_n|}y+\frac{y_n-x_n}{|x_n-y_n|}\Big|^{\frac{2}{3}} \\ &\quad\cdot\big|y+\mu_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}}(y_n-p_i)\big|^{\frac{\alpha}{3}} \\ &\ge C\big(\mu_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}}|y_n-p_i|\big)^{-\frac{\alpha}{3}}\big|y+\mu_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}}(y_n-p_i)\big|^{\frac{\alpha}{3}} \end{aligned} \tag{3.2.29}$$

for $|y| \le R_n = (\frac{|x_n-y_n|}{\beta_n})^{\frac{1}{2}}$, and $R_n \to +\infty$ as $n\to+\infty$ by (3.2.28). Since (3.2.29) implies that (3.2.12) holds for μ_n, y_n, $\hat{U}_n$, Proposition 3.2.2 provides some $\psi_n \in C_0^\infty(\Omega)$ such that (3.2.8) holds and Supp $\psi_n \subset B_{M\mu_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}}(y_n)$ for some $M>0$.

Since y_n cannot lie in any ball centered at x_n and radius of order of scale parameters ($\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}$ or $\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n-p_i|^{-\frac{\alpha}{2}}$), we get from (3.2.28) that ϕ_n and ψ_n have disjoint compact supports leading to $\mu_{2,n} < 0$, a contradiction to (3.2.2). This completes the proof of Proposition 3.2.4. ∎

3.2.3 Compactness issues

We are now in position to give the proof of Theorem 3.2.1. Assume $2 \leq N \leq 7$, and let f be as in (3.2.1). Let $(\lambda_n)_n$ be a sequence such that $\lambda_n \to \lambda \in [0, \lambda^*]$ and let u_n be an associated solution such that (3.2.2) holds, namely

$$\mu_{2,n} := \mu_{2,\lambda_n}(u_n) \geq 0.$$

The essential ingredient will be the estimate of Proposition 3.2.4 combined with the uniqueness result of Lemma 2.3.1.

Proof (of Theorem 3.2.1): Let x_n be the maximum point of u_n in Ω and, up to a subsequence, assume by contradiction that $u_n(x_n) = \max_{\Omega} u_n(x) \to 1$ as $n \to \infty$. Then Proposition 3.2.4 gives that for some $C > 0$ and $N_0 \in \mathbb{N}$ large,

$$u_n(x) \leq 1 - C\lambda_n^{\frac{1}{3}} d(x)^{\frac{\alpha}{3}} |x - x_n|^{\frac{2}{3}}$$

for any $x \in \Omega$ and $n \geq N_0$, where $d(x)^{\frac{\alpha}{3}} = \min\{|x - p_i|^{\frac{\alpha_i}{3}} : i = 1, \dots, k\}$ stands for a "distance function" from the zero set of $f(x)$. Thus, we have that:

$$0 \leq \frac{\lambda_n f(x)}{(1 - u_n)^2} \leq C \frac{f(x)}{d(x)^{\frac{2\alpha}{3}}} \frac{\lambda_n^{\frac{1}{3}}}{|x - x_n|^{\frac{4}{3}}} \tag{3.2.30}$$

for any $x \in \Omega$ and $n \geq N_0$. Since by (3.2.1)

$$\left|\frac{f(x)}{d(x)^{\frac{2\alpha}{3}}}\right| \leq |x - p_i|^{\frac{\alpha_i}{3}} \|f_i\|_\infty \leq C$$

for x close to p_i, f_i as in (3.2.10), we get that $\frac{f(x)}{d(x)^{\frac{2\alpha}{3}}}$ is a bounded function on Ω and then, (3.2.30) gives that $\lambda_n f(x)/(1 - u_n)^2$ is uniformly bounded in $L^s(\Omega)$, for any $1 < s < \frac{3N}{4}$. Standard elliptic regularity theory now implies that u_n is uniformly bounded in $W^{2,s}(\Omega)$. By Sobolev's imbedding Theorem, u_n is uniformly bounded in $C^{0,\beta}(\bar{\Omega})$ for any $0 < \beta < 2/3$. Up to a subsequence, we get that $u_n \to u_0$ weakly in $H_0^1(\Omega)$ and strongly in $C^{0,\beta}(\bar{\Omega})$, $0 < \beta < 2/3$, where u_0 is a Hölderian function solving weakly in $H_0^1(\Omega)$ the equation:

$$\begin{cases} -\Delta u_0 = \dfrac{\lambda f(x)}{(1 - u_0)^2} & \text{in } \Omega\,, \\ 0 \leq u_0 \leq 1 & \text{in } \Omega\,, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, by uniform convergence

$$\max_{\Omega} u_0 = \lim_{n \to +\infty} \max_{\Omega} u_n = 1$$

and, in particular $u_0 > 0$ in Ω. Clearly, $\lambda > 0$ since any weak harmonic function in $H_0^1(\Omega)$ is identically zero. To reach a contradiction, we shall first show that $\mu_{1,\lambda}(u_0) \geq 0$ and then deduce from the uniqueness, stated in Lemma 2.3.1, of the semi-stable solution u_λ that $u_0 = u_\lambda$. But $\max_\Omega u_\lambda < 1$ for any $\lambda \in [0, \lambda^*]$, contradicting $\max_\Omega u_0 = 1$. Hence, the claimed compactness must hold.

If in addition to (3.2.2) we assume that $\mu_{1,n} < 0$, then $\lambda > 0$. Indeed, if $\lambda_n \to 0$, then by compactness and standard regularity theory, we get that $u_n \to u_0$ in $C^2(\bar\Omega)$, where u_0 is a harmonic function so that $u_0 = 0$ on $\partial\Omega$. Then, $u_0 = 0$ and $u_n \to 0$ in $C^2(\bar\Omega)$. But the only branch of solutions for $(S)_\lambda$ bifurcating from 0 for λ small is the branch of minimal solutions u_λ and then, $u_n = u_{\lambda_n}$ for n large contradicting $\mu_{1,n} < 0$.

In order to complete the proof, we need only to show that

$$\mu_{1,\lambda}(u_0) = \inf\left\{\int_\Omega \big(|\nabla\phi|^2 - \frac{2\lambda f(x)}{(1-u_0)^3}\phi^2\big)dx;\ \phi \in C_0^\infty(\Omega) \text{ and } \int_\Omega \phi^2 dx = 1\right\} \geq 0. \quad (3.2.31)$$

Indeed, first Proposition 3.2.2 implies the existence of a function $\phi_n \in C_0^\infty(\Omega)$ so that

$$\int_\Omega \big(|\nabla\phi_n|^2 - \frac{2\lambda_n f(x)}{(1-u_n)^3}\phi_n^2\big) < 0. \quad (3.2.32)$$

Moreover, Supp $\phi_n \subset B_{r_n}(x_n)$ and $r_n \to 0$ as $n \to +\infty$. Recall that $p = \lim_{n\to+\infty} x_n$.

By contradiction, if (3.2.31) were false, then there should exist $\phi_0 \in C_0^\infty(\Omega)$ such that

$$\int_\Omega \big(|\nabla\phi_0|^2 - \frac{2\lambda f(x)}{(1-u_0)^3}\phi_0^2\big) < 0. \quad (3.2.33)$$

We will replace ϕ_0 with a truncated function ϕ_δ with $\delta > 0$ small enough, so that (3.2.33) is still true while $\phi_\delta = 0$ in $B_{\delta^2}(p) \cap \Omega$. In this way, ϕ_n and ϕ_δ would have disjoint compact supports in contradiction to $\mu_{2,n} \geq 0$.

Let $\delta > 0$ and set $\phi_\delta = \chi_\delta\phi_0$, where χ_δ is a cut-off function defined as:

$$\chi_\delta(x) = \begin{cases} 0 & |x-p| \leq \delta^2\,, \\ 2 - \dfrac{\log|x-p|}{\log\delta} & \delta^2 \leq |x-p| \leq \delta\,, \\ 1 & |x-p| \geq \delta\,. \end{cases}$$

By Lebesgue's Theorem, we have:

$$\int_\Omega \frac{2\lambda f(x)}{(1-u_0)^3}\phi_\delta^2 \to \int_\Omega \frac{2\lambda f(x)}{(1-u_0)^3}\phi_0^2 \quad \text{as } \delta \to 0\,. \quad (3.2.34)$$

For the gradient term, we have the expansion:

$$\int_\Omega |\nabla\phi_\delta|^2 = \int_\Omega \phi_0^2|\nabla\chi_\delta|^2 + \int_\Omega \chi_\delta^2|\nabla\phi_0|^2 + 2\int_\Omega \chi_\delta\phi_0\nabla\chi_\delta\nabla\phi_0\,.$$

The following estimates hold:

$$0 \leq \int_\Omega \phi_0^2|\nabla\chi_\delta|^2 \leq \|\phi_0\|_\infty^2 \int_{\delta^2\leq|x-p|\leq\delta} \frac{1}{|x-p|^2\log^2\delta} \leq \frac{C}{\log\frac{1}{\delta}}$$

and

$$\left|2\int_\Omega \chi_\delta\phi_0\nabla\chi_\delta\nabla\phi_0\right| \le \frac{2\|\phi_0\|_\infty\|\nabla\phi_0\|_\infty}{\log\frac{1}{\delta}}\int_{B_1(0)}\frac{1}{|x|},$$

which provide

$$\int_\Omega |\nabla\phi_\delta|^2 \to \int_\Omega |\nabla\phi_0|^2 \quad \text{as} \quad \delta \to 0. \tag{3.2.35}$$

Combining (3.2.33)-(3.2.35), we get that

$$\int_\Omega \left(|\nabla\phi_\delta|^2 - \frac{2\lambda f(x)}{(1-u_0)^3}\phi_\delta^2\right) < 0$$

for $\delta > 0$ sufficiently small. This completes the proof of (3.2.31) and therefore, Theorem 3.2.1 is completely established. ■

We finally mention that Theorem 5.1.1 is the main tool to control the blow up behavior of a possible non compact sequence of solutions. The usual asymptotic analysis for equations with Sobolev critical nonlinearity, based on some energy bounds (usually $L^{\frac{2N}{N-2}}(\Omega)$-bounds), does not work in our context. In view of Chapter 2, a possible loss of compactness can be related to a blow up in $L^{\frac{3N}{2}}(\Omega)$-norm along the sequence. Essentially, the blow up associated to a sequence u_n (in the sense of the blowing up of $(1-u_n)^{-1}$) corresponds exactly to the blow up of the $L^{\frac{3N}{2}}(\Omega)$-norm. We replace these energy bounds by some spectral information and, based on Theorem 5.1.1, we then provide an estimate of the number of blow up points (counted with their "multiplicities") in terms of the Morse index along the sequence.

3.3 The second bifurcation point

In this section, we discuss the second bifurcation point for $(S)_\lambda$. First of all, we will say that a curve $(\lambda(t), V(t)) \in C([0,1];\mathbb{R}\times C^2(\bar{\Omega}))$ with a finite number of self- intersections is a "second branch" of $(S)_\lambda$ if $V(t)$ solves $(S)_{\lambda(t)}$ with $\mu_{2,\lambda(t)}(V(t)) \ge 0$ and $V(t) = U_{\lambda(t)}$ for any t in a small deleted left neighborhood of 1, where $\lambda(1) = \lambda^*$.
Let us remark that $\lambda(t) < \lambda^*$ for any $0 \le t < 1$. Indeed, if $\lambda(\bar{t}) = \lambda^*$, we would have by Lemma 2.3.1 $V(\bar{t}) = u^*$ and by Lemma 3.1.1 $V(t) = U_{\lambda(t)}$ for t close to $\bar{t}$, having in such a way an infinite number of self-intersections.
We define now the second bifurcation point to be defined as:

$$\lambda_2^* = \inf\{\lambda(0) : \ (\lambda(t), V(t)) \text{ is a "second branch" of } (S)_\lambda\}\,.$$

Theorem 3.3.1. *Assume f satisfies (3.2.1). Then, for $2 \le N \le 7$ we have $\lambda_2^* \in (0,\lambda^*)$ and for any $\lambda \in (\lambda_2^*, \lambda^*)$ there exist at least two solutions u_λ and V_λ of $(S)_\lambda$ so that*

$$\mu_{1,\lambda}(V_\lambda) < 0 \quad \text{while} \quad \mu_{2,\lambda}(V_\lambda) \ge 0.$$

In particular, for $\lambda = \lambda_2^*$ there exists a second solution, namely $V^* := \lim_{\lambda\downarrow\lambda_2^*} V_\lambda$ so that

$$\mu_{1,\lambda_2^*}(V^*) < 0 \quad \text{and} \quad \mu_{2,\lambda_2^*}(V^*) = 0.$$

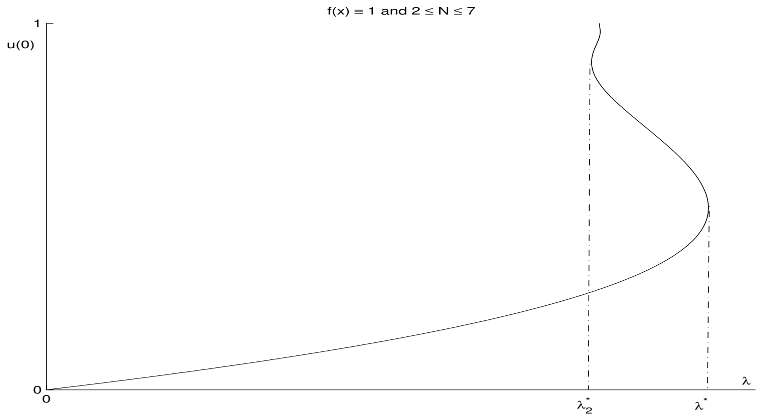

Figure 3.1: *Plots of* $u(0)$ *versus* λ *for the case where* $f(x) \equiv 1$ *is defined in the unit ball* $B_1(0) \subset \mathbb{R}^N$ *with dimension* $2 \le N \le 7$*, where* λ^* *(resp.* λ_2^**) is the first (resp. second) turning point.*

One can compare Theorem 3.3.1 with the multiplicity result of [2] for nonlinearities of the form $\lambda u^q + u^p$ $(0 < q < 1 < p)$, where the authors show that for p subcritical, there exists a second –Mountain Pass– solution for any $\lambda \in [0, \lambda^*)$. On the other hand, when p is critical, the second branch blows up as $\lambda \to 0$ (see also [4] for a related problem). We note that in our situation, the second branch cannot approach the value $\lambda = 0$ as illustrated by the bifurcation diagram Figure 3.1.

Proof: We claim that: for any $\lambda \in (\lambda_2^*, \lambda^*)$, there exists a solution V_λ such that

$$\mu_{1,\lambda} := \mu_{1,\lambda}(V_\lambda) < 0 \quad \forall\, \lambda \in (\lambda_2^*, \lambda^*). \tag{3.3.1}$$

In particular, $V_\lambda \neq u_\lambda$ provides a second solution different from the minimal one.

By definition of λ_2^*, for any $\lambda \in (\lambda_2^*, \lambda^*)$, let us consider a "second branch" $(\lambda(t), V(t))$ so that $\lambda(0) < \lambda$. Given $\bar{t}$ so that $\lambda = \lambda(\bar{t})$, the solution V_λ is found as $V_\lambda := V(\bar{t})$. Clearly, (3.3.1) is true. Indeed, since $\lambda(1) = \lambda^*$ and $V(t) = U_{\lambda(t)}$ for $t \uparrow 1$, by Lemma 3.1.1 $\mu_{1,\lambda(t)}(V(t)) = \mu_{1,\lambda(t)}(U_{\lambda(t)}) < 0$ for t close from the left to 1. Now $\mu_{1,\lambda(t)}(V(t)) = 0$ is not allowed because it would imply by Lemma 2.3.1 that $V(t) = u_{\lambda(t)}$. By the stability of minimal solutions of $(S)_\lambda$ for $\lambda < \lambda^*$ (see Theorem 2.4.3), it follows that $V(t) = u^*$ and $\lambda(t) = \lambda^*$, which is not possible as already observed. Hence, $\mu_{1,\lambda(t)}(V(t)) < 0$ for any $t \in [0, 1)$ and the claim is established.

Taking a sequence $\lambda_n \downarrow \lambda_2^*$, apply Theorem 3.2.1 to get $\lambda_2^* = \lim\limits_{n\to+\infty} \lambda_n > 0$, $\sup\limits_{n\in\mathbb{N}} \| V_{\lambda_n} \|_\infty < 1$. By elliptic regularity theory, up to a subsequence $V_{\lambda_n} \to V^*$ in $C^2(\bar{\Omega})$, where V^* is a solution for $(S)_{\lambda_2^*}$. As before, $\mu_{1,\lambda_2^*}(V^*) < 0$ and by continuity $\mu_{2,\lambda_2^*}(V^*) \ge 0$.

Suppose $\mu_{2,\lambda_2^*}(V^*) > 0$, let us fix some $\varepsilon > 0$ so small that $0 \le V^* \le 1 - 2\varepsilon$ and consider the truncated nonlinearity $g_\varepsilon(u)$ as in (3.1.2). Clearly, V^* is a solution of (3.1.3) at $\lambda = \lambda_2^*$

so that $-\Delta - \lambda_2^* f(x) g_\varepsilon'(V^*)$ has no zero eigenvalues, since $\mu_{1,\lambda_2^*}(V^*) < 0$ and $\mu_{2,\lambda_2^*}(V^*) > 0$. Namely, V^* solves $N(\lambda_2^*, V^*) = 0$, where N is a map from $\mathbb{R} \times C^{2,\alpha}(\bar\Omega)$ into $C^{2,\alpha}(\bar\Omega)$, $\alpha = \min\{\alpha_1, \dots, \alpha_k\}$, defined as:

$$N : (\lambda, V) \longrightarrow V + \Delta^{-1}\big(\lambda f(x) g_\varepsilon(V)\big).$$

Moreover,

$$\partial_V N(\lambda_2^*, V^*) = \text{Id } + \Delta^{-1}\big(\frac{2\lambda_2^* f(x)}{(1-V^*)^3}\big)$$

is an invertible map since $-\Delta - \lambda_2^* f(x) g_\varepsilon'(V^*)$ has no zero eigenvalues. The Implicit Function Theorem gives the existence of a curve W_λ, $\lambda \in (\lambda_2^* - \delta, \lambda_2^* + \delta)$, of solutions for (3.1.3) so that $\lim\limits_{\lambda \to \lambda_2^*} W_\lambda = V^*$ in $C^{2,\alpha}(\bar\Omega)$. Up to take δ smaller, this convergence implies that $\mu_{2,\lambda}(W_\lambda) > 0$ and $W_\lambda \leq 1 - \varepsilon$ for any $\lambda \in (\lambda_2^* - \delta, \lambda_2^* + \delta)$. Hence, for λ close to λ_2^* W_λ is a solution of $(S)_\lambda$ so that $\mu_{2,\lambda}(W_\lambda) > 0$ which allows us to extend some "second branch" for values $\lambda < \lambda_2^*$, contradicting the definition of λ_2^*. Therefore, $\mu_{2,\lambda_2^*}(V^*) = 0$, which completes the proof of Theorem 3.3.1. ■

Let $(\lambda(t), V(t))$ be a "second branch" of $(S)_\lambda$. By (3.1.1), for $\delta >$ small, we have that $L_{V(t),\lambda(t)}$ is invertible for $t \in (1-\delta, 1)$, and, as long as it remains invertible, we can use the Implicit Function Theorem to find $(\lambda(t), V(t))$ as the unique smooth extension of the curve U_λ (in principle U_λ exists only for λ close to λ^*). Let now λ^{**} be defined in the following way:

$$\lambda^{**} = \inf\{\lambda(0) : (\lambda(t), V(t)) \text{ is a "second branch" of } (S)_\lambda \text{ s.t. } \mu_{2,\lambda(t)}(V(t)) > 0 \, \forall t \in [0,1]\}.$$

As already observed, $\mu_{1,\lambda(t)}(V(t)) < 0$ for any $t \in [0,1)$ along any "second branch" $(\lambda(t), V(t))$. Then, $\lambda_2^* \leq \lambda^{**}$ and there exists a smooth curve $(\lambda(t), V(t))$ so that $\lambda(t) \downarrow \lambda^{**}$ as $t \downarrow 0$ which is the unique maximal extension of the curve U_λ. This is what the second branch is supposed to be. If now $\lambda_2^* < \lambda^{**}$, then there is no longer uniqueness for the extension and the "second branch" is defined only as one of potentially many continuous extensions of U_λ.
It remains open whether λ_2^* is the second turning point for the solution diagram of $(S)_\lambda$ or if the "second branch" simply disappears at $\lambda = \lambda_2^*$. Note that if the "second branch" does not disappear, then it can continue for λ less than λ_2^* but only along solutions whose first two eigenvalues are negative.

3.4 Some comments

Main results of this Chapter are published in [38]. The standard bifurcation theory of Crandall and Rabinowitz [30, 31] gives that once the compactness of minimal branch solutions of $(S)_\lambda$ is true, then there exists a second solution U_λ of $(S)_\lambda$ on the deleted left neighborhood of λ^*. In §3.1 we provide the Mountain Pass variational characterization of such a solution U_λ. Different from the regular nonlinearity case (e.g. $\lambda(1+u)^p$ in [50]), since the solution u

of $(S)_\lambda$ is restricted between 0 and 1, we need to smoothly truncate the singular nonlinear term $\frac{\lambda f}{(1-u)^2}$ into a subcritical case (3.1.2), and then we can apply the standard Mountain Pass Theorem, together with the uniqueness of extremal solutions.

Blow-up techniques of Elliptic PDEs were well-developed and applied in the past few years, and an important and original contribution in this direction can be found in the monograph [35] by O. Druet, E. Hebey and F. Robert. In §3.2 we have applied the blow-up analysis to our singular nonlinear case, where the compactness of solutions with Morse index-1 (lying along the second branch) has been established by using spectral information. Let us quote the recent paper [25] concerning the compactness of unstable branches is considered for singular nonlinearities in a larger class than $(1-u)^{-2}$.
As far as we know, there are no such compactness results in the case of regular nonlinearities, marking a substantial difference with the singular situation. As a byproduct, we have followed the second branch of bifurcation diagrams and proved the existence of a second solution for λ in a natural range, by means of the Implicit Function Theorem. An unsolved problem has also been given in §3.3.

Chapter 4

Description of Higher Branches

In this Chapter we further study the Dirichlet boundary value problem:

$$\begin{cases} -\Delta u = \dfrac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (S)_\lambda$$

where $\lambda \geq 0$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and f is as in (3.2.1). Recall from Chapter 2 that the compactness of semi-stable solutions of $(S)_\lambda$ is well-known so far by energy estimates. However, for a sequence of unstable solutions it is in general very difficult to show energy estimates, as for example it happens along the second branch U_λ. In Chapter 3, we exploit that the Morse index is 1 along the second branch, by developing a different approach to face non compactness phenomena based on such a spectral information. Since in general it is relatively much easier to construct solutions satisfying good spectral information (for example, by variational methods), the boundedness of Morse index seems to be a very natural assumption in the study of compactness issues for $(S)_\lambda$. One of the main purposes of the Chapter is to characterize the compactness of solutions sequence for $(S)_\lambda$ in terms of spectral informations. As a byproduct, we also give a uniqueness result for λ close to 0 and close to λ^* in the class of all solutions with uniformly bounded Morse index. The second main purpose is to establish an infinite multiplicity result in symmetric situations, namely the existence of a curve composed by solutions of $(S)_\lambda$ having many infinitely turning points. The essential tool is a general uniqueness result for $(S)_\lambda$ when λ is a small voltage, without bounds on the energy or on the Morse index.

This Chapter is a continuation and a strong improvement of former two Chapters. In §4.1 we improve the approach of §3.2 for the second branch and discuss the compactness in the class of solutions for $(S)_\lambda$ with uniformly bounded Morse index, as stated in Theorem 4.1.1. The main tool here is a non existence result for solutions of $(S)_\lambda$ with finite Morse index and finite singular set (where the solutions reach the value 1). In §4.2 we apply Theorem 4.1.1 to deriving some consequences, see Theorems 4.2.1∼4.2.3.

4.1 Compactness issues

In this section we extend the approach developed in Chapter 3 to deal with the compactness of higher branches along which blow up could occur at many finitely points (not only the maximum point as for the second branch). Here is the main result of this section.

Theorem 4.1.1. *Assume* $2 \leq N \leq 7$*, and let* f *be such that (3.2.1) holds. Let* $\{\lambda_n\}_{n\in\mathbb{N}}$ *be a sequence such that* $\lambda_n \to \lambda \in [0, \lambda^*]$ *and let* u_n *be an associated solution such that:*

$$\sup_{n\in\mathbb{N}} m(u_n, \lambda_n) < +\infty. \tag{4.1.1}$$

Then,

$$\sup_{n\in\mathbb{N}} \| u_n \|_\infty < 1. \tag{4.1.2}$$

Moreover, if in addition $\mu_{1,n} := \mu_{1,\lambda_n}(u_n) < 0$*, then necessarily* $\lambda > 0$.
(Here and in the sequel, $\mu_{k,\lambda}(u)$ *denotes the k-th eigenvalue of* $L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3}$ *with the convention that eigenvalues are repeated according to their multiplicities, and the Morse index* $m(u,\lambda)$ *is the number of negative eigenvalues of* $L_{u,\lambda}$ *).*

Remark 4.1.1. Estimate (4.1.2) will be sometimes referred to as a "compactness property" of the solutions set of $(S)_\lambda$. Indeed, by elliptic regularity theory, for any $k \in \mathbb{N}$ the set $\{u : u \text{ is a solution of } (S)_\lambda\,,\ m(u,\lambda) \leq k\}$ is a compact set in $C^m(\bar\Omega)$-norm, where $m \geq 1$ depends on the regularity of $f(x)$.

In next subsections we will develop some useful tools in deriving the compactness result.

4.1.1 Regularity properties

As an application of §2.3.2, we provide the following non-existence result which in particular, excludes solutions of (2.3.5) with finite Morse index and finite singular set.

Theorem 4.1.2. *Suppose* $2 \leq N \leq 7$*, and let* $u \in C(\bar\Omega)$ *be a* $H^1(\Omega)$*-weak solution of (2.3.5) so that* $\|u\|_\infty \leq 1$ *and the singular set* $S = \{x \in \Omega : u(x) = 1\}$ *is a non empty set. Assume that* u *has finite Morse index: i.e, there exists a finite dimensional subspace* $T \subset H_0^1(\Omega)$ *so that*

$$\int_\Omega \Big(|\nabla\phi|^2 - \frac{2f(x)}{(1-u)^3}\phi^2\Big) \geq 0\,, \quad \text{for any } \phi \in T^\perp = \big\{\phi \in H_0^1(\Omega) : \int_\Omega \nabla\phi\nabla\psi = 0 \,\forall\, \psi \in T\big\}. \tag{4.1.3}$$

Then, the singular set S *has no isolated points.*

Proof: Assume by contradiction that $x_0 \in S$ is an isolated point of S. Let δ_0 be such that $B_{2\delta_0}(x_0) \cap S = \{x_0\}$. We want to show that:

$$\int_{B_\delta} \Big(|\nabla\phi|^2 - \frac{2f(x)}{(1-u)^3}\phi^2\Big) \geq 0\,, \quad \text{for any } \phi \in H_0^1(B_\delta), \tag{4.1.4}$$

for some small $0 < \delta \le \delta_0$, where $B_\delta := B_\delta(x_0)$.

By contradiction, assume that (4.1.4) were false for any $0 < \delta \le \delta_0$. Then, there exist $\phi_0 \in C_0^\infty(B_{\delta_0})$ such that

$$\int_{B_{\delta_0}} \Big(|\nabla \phi_0|^2 - \frac{2f(x)}{(1-u)^3}\phi_0^2 \Big) < 0. \tag{4.1.5}$$

We claim that one can assume $\phi_0 = 0$ in B_δ for some small $0 < \delta < \delta_0$. Indeed, otherwise, we can replace ϕ_0 with a truncated function ϕ_δ, $\delta > 0$ small, so that (4.1.5) is still true while $\phi_\delta = 0$ in B_δ. For proving this claim, we set $\phi_\delta = \chi_\delta \phi_0$, where χ_δ is a cut-off function defined as:

$$\chi_\delta(x) = \begin{cases} 0 & |x - x_0| \le \delta\,, \\ 2\Big(1 - \dfrac{\log|x-x_0|}{\log \delta}\Big) & \delta \le |x-x_0| \le \sqrt{\delta}\,, \\ 1 & |x-x_0| \ge \sqrt{\delta}\,. \end{cases}$$

By Lebesgue's Theorem, we have:

$$\int_{B_{\delta_0}} \frac{2f(x)}{(1-u)^3}\phi_\delta^2 \to \int_{B_{\delta_0}} \frac{2f(x)}{(1-u)^3}\phi_0^2 \quad \text{as } \delta \to 0. \tag{4.1.6}$$

For the gradient term, we have the expansion:

$$\int_{B_{\delta_0}} |\nabla \phi_\delta|^2 = \int_{B_{\delta_0}} \phi_0^2 |\nabla \chi_\delta|^2 + \int_{B_{\delta_0}} \chi_\delta^2 |\nabla \phi_0|^2 + 2\int_{B_{\delta_0}} \chi_\delta \phi_0 \nabla \chi_\delta \nabla \phi_0\,.$$

The following estimates hold:

$$0 \le \int_{B_{\delta_0}} \phi_0^2 |\nabla \chi_\delta|^2 \le 4\|\phi_0\|_\infty^2 \int_{\delta \le |x-x_0| \le \sqrt{\delta}} \frac{1}{|x-x_0|^2 \log^2 \delta} \le \frac{C}{\log \frac{1}{\delta}}$$

and

$$\Big|2\int_{B_{\delta_0}} \chi_\delta \phi_0 \nabla \chi_\delta \nabla \phi_0\Big| \le \frac{4\|\phi_0\|_\infty \|\nabla \phi_0\|_\infty}{\log \frac{1}{\delta}} \int_{B_1(0)} \frac{1}{|x|},$$

which provide

$$\int_{B_{\delta_0}} |\nabla \phi_\delta|^2 \to \int_{B_{\delta_0}} |\nabla \phi_0|^2 \quad \text{as } \delta \to 0. \tag{4.1.7}$$

Combining (4.1.6) and (4.1.7), we get that $\phi_\delta = 0$ in B_δ and

$$\int_{B_{\delta_0}} \Big(|\nabla \phi_\delta|^2 - \frac{2f(x)}{(1-u)^3}\phi_\delta^2 \Big) < 0$$

for $\delta > 0$ sufficiently small, and the claim is proved.
In this way, we find $0 < \delta_1 < \delta_0$ small and $\phi_0 \in C_0(B_{\delta_0} \backslash B_{\delta_1}) \cap H_0^1(\Omega)$ such that (4.1.5) holds.

Since by contradiction we are assuming that (4.1.4) is false for any $\delta > 0$, we can now iterate the argument to find a strictly decreasing sequence δ_n and $\phi_n \in C_0(B_{\delta_n} \setminus B_{\delta_{n+1}}) \cap H_0^1(\Omega)$ such that:

$$\int_{B_{\delta_n}} \left(|\nabla \phi_n|^2 - \frac{2f(x)}{(1-u)^3} \phi_n^2 \right) < 0.$$

Since $\{\phi_n\}_{n\in\mathbb{N}}$ are mutually hortogonal having disjoint supports, we get an infinite dimensional set $M = \text{Span}\ \{\phi_n : n \in \mathbb{N}\} \subset H_0^1(\Omega)$ so that

$$\int_{\Omega} \left(|\nabla \phi|^2 - \frac{2f(x)}{(1-u)^3} \phi^2 \right) < 0 \quad \forall\, \phi \in M.$$

Since M is an infinite dimensional subspace of $H_0^1(\Omega)$, we have that $M \cap T^\perp \neq \emptyset$, in contradiction with (4.1.3). Hence, (4.1.4) holds for some $\delta = \delta(x_0) \le \delta_0$.

Using elliptic regularity theory, we get that $u \in C^1_{\text{loc}}(B_{2\delta_0} \setminus \{x_0\})$. Since $u \in C^1(\partial B_\delta)$ and $\max\limits_{\partial B_\delta} u < 1$ in view of $0 < \delta \le \delta_0$, we extend it on B_δ as a function $\bar u \in C^1(\bar B_\delta)$ satisfying $0 \le \bar u \le \|\bar u\|_{\infty, B_\delta} < 1$. Since (4.1.4) holds on B_δ, we can apply Theorem 2.3.3 and Proposition 2.3.5 to obtaining that $\|u\|_{\infty,B_\delta} < 1$, which contradicts $u(x_0) = 1$. Therefore, S has no isolated points. ■

4.1.2 A pointwise estimate

Let $2 \le N \le 7$. Assume f in the form (3.2.1), and let $(u_n)_n$ be a solutions sequence of $(S)_\lambda$ associated to $\lambda_n \to \lambda \in [0, \lambda^*]$. Since we want to show that $\sup\limits_{n\in\mathbb{N}} \|u_n\|_\infty < 1$, by contradiction and up to a subsequence, in the sequel we will assume $u_n(x_n) = \max\limits_\Omega u_n \to 1^-$ as $n \to +\infty$, where x_n is a maximum point of u_n.

The crucial assumption is the validity of (4.1.1), namely $m(u_n, \lambda_n) \le k$ for any $n \in \mathbb{N}$ and some $k \in \mathbb{N}$. This information, combined with Proposition 3.2.2, will allows us to control the blow up behavior of u_n. Indeed, the following pointwise estimate on u_n is available:

Theorem 4.1.3. *Assume $2 \le N \le 7$ and f as in (3.2.1). Let u_n be a solution of $(S)_\lambda$ associated to $\lambda_n \in [0, \lambda^*]$, and assume that $\lambda_n \to \lambda$ and $u_n(x_n) = \max\limits_\Omega u_n \to 1$ as $n \to +\infty$. Then, up to a subsequence, there exist constants $C > 0$, $N_0 \in \mathbb{N}$ and m-sequences $x_n^1, \dots, x_n^m$, $m \le k$, such that*

$$1 - u_n(x) \ge C \lambda_n^{\frac{1}{3}}\ (d(x)^\alpha)^{\frac{1}{3}}\ d_n(x)^{\frac{2}{3}}\,, \quad \forall\, x \in \Omega\,,\ \forall\, n \ge N_0\,, \tag{4.1.8}$$

where $d(x)^\alpha := \min\{|x - p_i|^{\alpha_i} : i = 1, \dots, k\}$ is a "distance function" from Z and $d_n(x) = \min\{|x - x_n^i| : i = 1, \dots, m\}$ is the distance function from $\{x_n^1, \dots, x_n^m\}$.
More precisely, letting r_n^i be associated to x_n^i and $\varepsilon_n^i := 1 - u_n(x_n^i)$ by means of (3.2.3), for any $i, j = 1, \dots, m$, $i \neq j$, there holds:

$$(\varepsilon_n^i)^3 \lambda_n^{-1} \to 0\,,\ U_n^i(y) = \frac{1 - u_n(r_n^i y + x_n^i)}{\varepsilon_n^i} \to U^i(y) \text{ in } C^1_{loc}(\mathbb{R}^N)\,,\ \frac{r_n^i + r_n^j}{|x_n^i - x_n^j|} \to 0 \tag{4.1.9}$$

as $n \to +\infty$, where $\varepsilon_n^i := 1 - u_n(x_n^i)$ and U^i satisfies an equation of type (3.2.7). In addition, there exist m-sequences of test functions $\phi_n^1, \dots, \phi_n^m \in C_0^\infty(\Omega)$ so that

$$\int_\Omega \Big(|\nabla \phi_n^i|^2 - \frac{2\lambda_n f(x)}{(1-u_n)^3}(\phi_n^i)^2\Big) < 0\,, \quad \textit{Supp } \phi_n^i \subset B_{Mr_n^i}(x_n^i)\,, \quad \forall\, i = 1, \dots, m, \tag{4.1.10}$$

for some $M > 0$ large.

Proof: Let $\varepsilon_n^1 = 1 - u_n(x_n^1)$, where x_n^1 is a maximum point of u_n. Let r_n^1 be associated to x_n^1 according to (3.2.3). Recalling the validity of (3.2.20) for the maximum point x_n^1 of u_n, Proposition 3.2.2 gives that, up to a subsequence:

$$\frac{1 - u_n(r_n^1 y + x_n^1)}{\varepsilon_n^1} \to U^1(y) \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N) \text{ as } n \to +\infty,$$

where U^1 satisfies an equation of type (3.2.7), together with the existence of $\phi_n^1 \in C_0^\infty(\Omega)$ such that (3.2.8) holds with Supp $\phi_n^1 \subset B_{Mr_n^1}(x_n^1)$ for some $M > 0$.
If (4.1.8) is true for some subsequence of u_n with x_n^1, we take $m = 1$ and the proof is done. Otherwise, we proceed by an inductive method. Indeed, assume that, up to a subsequence, we have already found l-sequences $x_n^1, \dots, x_n^l$, associated $r_n^1, \dots, r_n^l$ (defined by (3.2.3)) and test functions $\phi_n^1, \dots, \phi_n^l \in C_0^\infty(\Omega)$ so that (4.1.9) and (4.1.10) hold at l-th step. If (4.1.8) holds for some subsequence of u_n with $x_n^1, \dots, x_n^l$, we take $m = l$ and the proof is done. Otherwise, up to a subsequence, we will show the existence of x_n^{l+1}, r_n^{l+1} and ϕ_n^{l+1} so that (4.1.9) and (4.1.10) are still true at $(l+1)$-th step. Since (4.1.9) and (4.1.10) at l-th step imply that $\phi_n^1, \dots, \phi_n^l$ have mutually disjoint compact supports, we get that $m(u_n, \lambda_n) \geq l$. Then, by (4.1.1) the inductive process must stop after a finite number of steps, say m steps, with $m \leq k$, and (4.1.8) holds with $x_n^1, \dots, x_n^m$.

In order to complete the proof, we need to show how the induction process works. Assume that (4.1.9) and (4.1.10) hold at l-th step and (4.1.8) is not true for any subsequence of u_n with $x_n^1, \dots, x_n^l$. Let $x_n^{l+1} \in \Omega$ be such that

$$\begin{aligned} &\lambda_n^{-\frac{1}{3}} \left(d(x_n^{l+1})^\alpha\right)^{-\frac{1}{3}} d_n(x_n^{l+1})^{-\frac{2}{3}} \Big(1 - u_n(x_n^{l+1})\Big) \\ &= \lambda_n^{-\frac{1}{3}} \min_{x \in \Omega} \Big((d(x)^\alpha)^{-\frac{1}{3}} d_n(x)^{-\frac{2}{3}} \big(1 - u_n(x)\big)\Big) \to 0 \end{aligned} \tag{4.1.11}$$

as $n \to +\infty$, where $d_n(x)$ is the distance function from $\{x_n^1, \dots, x_n^l\}$. Let $\varepsilon_n^{l+1} := 1 - u_n(x_n^{l+1})$.
Formula (4.1.11) gives a lot of information about the blow up around x_n^{l+1}. First of all, it can be rewritten in the more convenient form:

$$\frac{(\varepsilon_n^{l+1})^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n^{l+1} - x_n^i|\, |x_n^{l+1} - p_j|^{\frac{\alpha_j}{2}}} \to 0 \quad \text{as } n \to +\infty\,,\ \forall\, i = 1, \dots, l,\ j = 1, \dots, k. \tag{4.1.12}$$

The inductive assumption gives $\frac{r_n^i + r_n^j}{|x_n^i - x_n^j|} \to 0$ as $n \to +\infty$ for any $i, j = 1, \dots, l$, $i \neq j$. Then, by definition of r_n^j we get for $|y| \le R$ and $n \ge n_R$:

$$\lambda_n^{-\frac{1}{3}} \left(d(r_n^j y + x_n^j)^\alpha\right)^{-\frac{1}{3}} d_n(r_n^j y + x_n^j)^{-\frac{2}{3}} \Big(1 - u_n(r_n^j y + x_n^j)\Big)$$
$$= \begin{cases} (d(r_n^j y + x_n^j)^\alpha)^{-\frac{1}{3}} |y|^{-\frac{2}{3}} U_n^j(y) & \text{if } x_n^j \to p \notin Z \\ |\frac{r_n^j}{|x_n^j - p_i|} y + \frac{x_n^j - p_i}{|x_n^j - p_i|}|^{-\frac{\alpha_i}{3}} |y|^{-\frac{2}{3}} U_n^j(y) & \text{if } x_n^j \to p_i \in Z,\ (\varepsilon_n^j)^{-3} \lambda_n |x_n^j - p_i|^{\alpha_i + 2} \to +\infty \\ |y + (r_n^j)^{-1}(x_n^j - p_i)|^{-\frac{\alpha_i}{3}} |y|^{-\frac{2}{3}} U_n^j(y) & \text{if } (\varepsilon_n^j)^{-3} \lambda_n |x_n^j - p_i|^{\alpha_i + 2} \le C \end{cases}$$

for any $j = 1, \dots, l$. By inductive assumption, we have that $U_n^j(y) = \frac{1 - u_n(r_n^j y + x_n^j)}{\varepsilon_n^j} \to U^j(y)$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ as $n \to +\infty$ for any $j = 1, \dots, l$. Associating (eventually) to x_n^j the limit point y_0 as in (3.2.11), we get that:

$$\lambda_n^{-\frac{1}{3}} \left(d(r_n^j y + x_n^j)^\alpha\right)^{-\frac{1}{3}} d_n(r_n^j y + x_n^j)^{-\frac{2}{3}} \Big(1 - u_n(r_n^j y + x_n^j)\Big)$$
$$\to \begin{cases} (d(p)^\alpha)^{-\frac{1}{3}} |y|^{-\frac{2}{3}} U^j(y) & \text{if } x_n^j \to p \notin Z\,, \\ |y|^{-\frac{2}{3}} U^j(y) & \text{if } x_n^j \to p_i \in Z,\ (\varepsilon_n^j)^{-3} \lambda_n |x_n^j - p_i|^{\alpha_i + 2} \to \infty\,, \\ |y + y_0|^{-\frac{\alpha_i}{3}} |y|^{-\frac{2}{3}} U^j(y) & \text{if } (\varepsilon_n^j)^{-3} \lambda_n |x_n^j - p_i|^{\alpha_i + 2} \le C \end{cases}$$

uniformly for $|y| \le R$ as $n \to +\infty$. Since U^j is bounded away from zero, then (4.1.11) gives also that x_n^{l+1} cannot asymptotically lie in the balls centered at x_n^i of radius $\approx r_n^i$, $i = 1, \dots, l$, namely:

$$\frac{r_n^i}{|x_n^{l+1} - x_n^i|} \to 0 \quad \text{as } n \to +\infty\,,\ \ \forall\, i = 1, \dots, l. \tag{4.1.13}$$

Finally, the choice of x_n^{l+1} as a minimum point in (4.1.11) gives that:

$$\frac{1 - u_n(\beta_n y + x_n^{l+1})}{\varepsilon_n^{l+1}} \ge \Big(\frac{d(\beta_n y + x_n^{l+1})^\alpha}{d(x_n^{l+1})^\alpha}\Big)^{\frac{1}{3}} \Big(\frac{d_n(\beta_n y + x_n^{l+1})}{d_n(x_n^{l+1})}\Big)^{\frac{2}{3}}, \tag{4.1.14}$$

for any sequence β_n. Indeed, by the following chain of estimates:

$$\begin{aligned} \varepsilon_n^{l+1} &\le (d(x_n^{l+1})^\alpha)^{\frac{1}{3}} d_n(x_n^{l+1})^{\frac{2}{3}} \min_{x \in \Omega} \Big((d(x)^\alpha)^{-\frac{1}{3}} d_n(x)^{-\frac{2}{3}} \big(1 - u_n(x)\big)\Big) \\ &\le (d(x_n^{l+1})^\alpha)^{\frac{1}{3}} d_n(x_n^{l+1})^{\frac{2}{3}} (d(\beta_n y + x_n^{l+1})^\alpha)^{-\frac{1}{3}} d_n(\beta_n y + x_n^{l+1})^{-\frac{2}{3}} \Big(1 - u_n(\beta_n y + x_n^{l+1})\Big), \end{aligned}$$

the validity of (4.1.14) follows. Here and in the sequel of the proof, the crucial point to establish the validity of (3.2.6) (or (3.2.12)) for suitable rescaled functions around x_n^{l+1} is exactly given by the validity of (4.1.14). By (4.1.12), we get that in particular $(\varepsilon_n^{l+1})^3 \lambda_n^{-1} \to 0$ as $n \to +\infty$. We need now to discuss all the possible types of blow up at x_n^{l+1}.

1st **Case** Assume that $x_n^{l+1} \to q \notin Z$. Associated to x_n^{l+1}, let $r_n^{l+1} = (\varepsilon_n^{l+1})^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}$ be defined according to (3.2.3). Then, $|x_n^{l+1} - p_j| \geq C > 0$ for any $j = 1, \dots, k$ which reduces (4.1.12) to:

$$\frac{r_n^{l+1}}{|x_n^{l+1} - x_n^i|} \to 0 \quad \text{as } n \to +\infty\,, \ \forall\, i = 1, \dots, l, \tag{4.1.15}$$

and then, $R_n = \left(\frac{d_n(x_n^{l+1})}{r_n^{l+1}}\right)^{\frac{1}{2}} \to +\infty$ as $n \to +\infty$. Since $\{r_n^{l+1} y + x_n^{l+1} \,:\, |y| \leq R_n\}$ is uniformly far away from Z, (4.1.14) shows that:

$$U_n^{l+1}(y) := \frac{1 - u_n(r_n^{l+1} y + x_n^{l+1})}{\varepsilon_n^{l+1}} \geq C_0 \big(1 - \frac{r_n^{l+1} R_n}{d_n(x_n^{l+1})}\big)^{\frac{2}{3}} \geq \frac{C_0}{2}$$

for n large and $y \in \frac{\Omega - x_n^{l+1}}{r_n^{l+1}} \cap B_{R_n}(0)$. We have used here the following estimate:

$$\begin{aligned}\frac{d_n(\beta_n y + x_n^{l+1})}{d_n(x_n^{l+1})} &= \min \Big\{ |\frac{x_n^{l+1} - x_n^i}{d_n(x_n^{l+1})} + \frac{\beta_n}{d_n(x_n^{l+1})} y| \,:\, i = 1, \dots, l \Big\} \\ &\geq 1 - \frac{\beta_n}{d_n(x_n^{l+1})} |y|.\end{aligned} \tag{4.1.16}$$

Up to a subsequence, Proposition 3.2.2 provides $U_n^{l+1} \to U^{l+1}$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ as $n \to +\infty$, U^{l+1} being a solution of an equation of type (3.2.7), and some $\phi_n^{l+1} \in C_0^\infty(\Omega)$ such that (3.2.8) holds with Supp $\phi_n^{l+1} \subset B_{M r_n^{l+1}}(x_n^{l+1})$ for some $M > 0$. By (4.1.15), combined with (4.1.13), we get that (4.1.9) and (4.1.10) are still true at $(l+1)$-th step, as needed.

2nd **Case** Assume that $x_n^{l+1} \to p_j$ with the following rate:

$$(\varepsilon_n^{l+1})^{-3} \lambda_n |x_n^{l+1} - p_j|^{\alpha_j + 2} \to +\infty \quad \text{as } n \to +\infty.$$

Let $r_n^{l+1} = (\varepsilon_n^{l+1})^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |x_n^{l+1} - p_j|^{-\frac{\alpha_j}{2}}$ according to (3.2.3). By (4.1.12) we get that (4.1.15) still holds and then, $R_n = \left(\min\{\frac{|x_n^{l+1} - p_j|}{r_n^{l+1}}, \frac{d_n(x_n^{l+1})}{r_n^{l+1}}\}\right)^{\frac{1}{2}} \to +\infty$ as $n \to +\infty$. Since $\{r_n^{l+1} y + x_n^{l+1} \,:\, |y| \leq R_n\}$ is uniformly close to $p_j \in Z$, estimates (4.1.14) and (4.1.16) imply:

$$\begin{aligned} U_n^{l+1}(y) := \frac{1 - u_n(r_n^{l+1} y + x_n^{l+1})}{\varepsilon_n^{l+1}} &\geq \Big(\frac{|r_n^{l+1} y + x_n^{l+1} - p_j|}{|x_n^{l+1} - p_j|}\Big)^{\frac{\alpha_j}{3}} \Big(\frac{d_n(r_n^{l+1} y + x_n^{l+1})}{d_n(x_n^{l+1})}\Big)^{\frac{2}{3}} \\ &\geq \Big(1 - \frac{r_n^{l+1} R_n}{|x_n^{l+1} - p_j|}\Big)^{\frac{\alpha_j}{3}} \Big(1 - \frac{r_n^{l+1} R_n}{d_n(x_n^{l+1})}\Big)^{\frac{2}{3}} \geq \frac{1}{2} \end{aligned}$$

for n large and $|y| \leq R_n$. Up to a subsequence, Proposition 3.2.2 provides $U_n^{l+1} \to U^{l+1}$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ as $n \to +\infty$, where U^{l+1} solves an equation of type (3.2.7), and the existence of $\phi_n^{l+1} \in C_0^\infty(\Omega)$ such that (3.2.8) holds, Supp $\phi_n^{l+1} \subset B_{M r_n^{l+1}}(x_n^{l+1})$ for some $M > 0$. Finally, (4.1.13) with (4.1.15) gives that (4.1.9) and (4.1.10) are still true at $(l+1)$-th step, also in this second case.

3rd Case Assume that $x_n^{l+1} \to p_j$ and

$$(\varepsilon_n^{l+1})^{-3}\lambda_n |x_n^{l+1} - p_j|^{\alpha_j+2} \le C.$$

By (4.1.12) $x_n^{l+1} \neq p_j$ and for any $i = 1, \dots, l$ there holds:

$$\frac{|x_n^{l+1} - p_j|}{|x_n^{l+1} - x_n^i|} = \frac{(\varepsilon_n^{l+1})^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n^{l+1} - x_n^i|\, |x_n^{l+1} - p_j|^{\frac{\alpha_j}{2}}} \cdot \Big((\varepsilon_n^{l+1})^{-3}\lambda_n |x_n^{l+1} - p_j|^{\alpha_j+2} \Big)^{\frac{1}{2}} \to 0 \quad \text{as } n \to \infty. \tag{4.1.17}$$

Let $r_n^{l+1} = (\varepsilon_n^{l+1})^{\frac{3}{2+\alpha_j}} \lambda_n^{-\frac{1}{2+\alpha_j}}$ according to (3.2.3). By (4.1.12) and (4.1.17) we get that for any $i = 1, \dots, l$:

$$\frac{r_n^{l+1}}{|x_n^{l+1} - x_n^i|} = \Big(\frac{(\varepsilon_n^{l+1})^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n^{l+1} - x_n^i|\, |x_n^{l+1} - p_j|^{\frac{\alpha_j}{2}}} \Big)^{\frac{2}{2+\alpha_j}} \cdot \Big(\frac{|x_n^{l+1} - p_j|}{|x_n^{l+1} - x_n^i|} \Big)^{\frac{\alpha_j}{2+\alpha_j}} \to 0 \quad \text{as } n \to \infty, \tag{4.1.18}$$

providing the validity of (4.1.15). Let $R_n = (\frac{d_n(x_n^{l+1})}{r_n^{l+1}})^{\frac{1}{2}} \to +\infty$ as $n \to +\infty$. Since $\{r_n^{l+1} y + x_n^{l+1} : |y| \le R_n\}$ is uniformly close to $p_j \in Z$, by (4.1.14) and (4.1.16) we get:

$$\begin{aligned} U_n^{l+1}(y) := \frac{1 - u_n(r_n^{l+1} y + x_n^{l+1})}{\varepsilon_n^{l+1}} &\ge \Big(\frac{|r_n^{l+1} y + x_n^{l+1} - p_j|}{|x_n^{l+1} - p_j|} \Big)^{\frac{\alpha_j}{3}} \Big(1 - \frac{r_n^{l+1} R_n}{d_n(x_n^{l+1})} \Big)^{\frac{2}{3}} \\ &\ge \frac{1}{2} \Big(\frac{|x_n^{l+1} - p_j|}{r_n^{l+1}} \Big)^{-\frac{\alpha_j}{3}} \Big| y + \frac{x_n^{l+1} - p_j}{r_n^{l+1}} \Big|^{\frac{\alpha_j}{3}} \\ &\ge C \Big| y + \frac{x_n^{l+1} - p_j}{r_n^{l+1}} \Big|^{\frac{\alpha_j}{3}} \end{aligned}$$

for n large and $|y| \le R_n$, where $C > 0$ is a constant. We have used that $\frac{|x_n^{l+1} - p_j|}{r_n^{l+1}} \le C$, which is true for assumption in this case. We use now Proposition 3.2.2 in combination with Remark 3.2.2 to get that, up to a subsequence, $U_n^{l+1} \to U^{l+1}$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ as $n \to +\infty$ and U^{l+1} is a solution of an equation of type (3.2.7). Moreover, we find $\phi_n^{l+1} \in C_0^\infty(\Omega)$ such that (3.2.8) holds and Supp $\phi_n^{l+1} \subset B_{M r_n^{l+1}}(x_n^{l+1})$, $M > 0$. Since (4.1.13) together with (4.1.18) gives the validity of (4.1.9) and (4.1.10) at $(l+1)$-th step, the induction scheme also works in this last case and the proof of Theorem 4.1.3 is complete. ■

4.1.3 Compactness of unstable branches

We are now in position to give the proof of Theorem 4.1.1. The essential ingredient will be the pointwise estimate of Theorem 4.1.3. The contradiction will come out from the non existence result of Theorem 4.1.2.

Proof (of Theorem 4.1.1): By contradiction, up to a subsequence, let us assume that $\max_{\Omega} u_n \to 1$ as $n \to +\infty$. Up to a further subsequence, Theorem 4.1.3 gives the existence of m-sequences $x_n^1, \dots, x_n^m$ so that $x_n^i \to x^i \in \bar{\Omega}$ as $n \to +\infty$ and the following pointwise estimate holds:

$$1 - u_n(x) \geq C\lambda_n^{\frac{1}{3}} \left(d(x)^\alpha\right)^{\frac{1}{3}} d_n(x)^{\frac{2}{3}} \tag{4.1.19}$$

for any $x \in \Omega$ and $n \geq N_0$, for some $C > 0$ and $N_0 \in \mathbb{N}$ large, where $d(x)^\alpha = \min\{|x - p_i|^{\alpha_i} : i = 1, \dots, k\}$ and $d_n(x) = \min\{|x - x_n^i| : i = 1, \dots, m\}$. Therefore, we get the following bounds in Ω:

$$0 \leq \frac{\lambda_n f(x)}{(1 - u_n)^2} \leq C \frac{f(x)}{(d(x)^\alpha)^{\frac{2}{3}}} \frac{\lambda_n^{\frac{1}{3}}}{d_n(x)^{\frac{4}{3}}}, \tag{4.1.20}$$

for some $C > 0$. Since by (3.2.1)

$$\left| \frac{f(x)}{(d(x)^\alpha)^{\frac{2}{3}}} \right| \leq |x - p_i|^{\frac{\alpha_i}{3}} \|f_i\|_\infty \leq C$$

for $|x - p_i| \leq \delta$ and f_i as in (3.2.10), we get that $\frac{f(x)}{(d(x)^\alpha)^{\frac{2}{3}}}$ is a bounded function on Ω. Hence, by (4.1.20) $\frac{\lambda_n f(x)}{(1-u_n)^2}$ is uniformly bounded in $L^s(\Omega)$, for any $1 < s < \frac{3N}{4}$. By elliptic regularity theory and Sobolev embeddings, up to a subsequence, we get that u_n converges weakly in $H_0^1(\Omega)$ and strongly in $C(\bar{\Omega})$ to a limit function $u_0 \in C(\bar{\Omega}) \cap H_0^1(\Omega)$ as $n \to +\infty$. In particular, it holds that $\max_{\Omega} u_0 = 1$, by means of the uniform convergence of u_n to u_0, and then, $S = \{x \in \Omega : u_0(x) = 1\}$ is a non empty set.

If $\lambda = \lim_{n \to +\infty} \lambda_n = 0$, then (4.1.20) gives $\frac{\lambda_n f(x)}{(1-u_n)^2} \to 0$ in $L^s(\Omega)$ as $n \to +\infty$, for any $1 < s < \frac{3N}{4}$. So, $u_0 \in H_0^1(\Omega)$ is a weak harmonic function and then, it should vanish identically, in contradiction to $\max_{\Omega} u_0 = 1$.

Hence, we have $\lambda = \lim_{n \to +\infty} \lambda_n > 0$, and by (4.1.19) we get that $u_0 < 1$ in $\Omega \setminus \{x^1, \dots, x^m, p_1, \dots, p_k\}$. In particular, the set S is finite because $S \subset \{x^1, \dots, x^m, p_1, \dots, p_k\}$.

Since $\frac{\lambda_n f(x)}{(1-u_n)^2}$ is uniformly bounded in $L^s(\Omega)$ for any $1 < s < \frac{3N}{4}$ and $\frac{\lambda_n f(x)}{(1-u_n)^2} \to \frac{\lambda f(x)}{(1-u_0)^2}$ uniformly on compact sets in $\bar{\Omega} \setminus \{x^1, \dots, x^m, p_1, \dots, p_k\}$, we get that

$$\frac{\lambda_n f(x)}{(1 - u_n)^2} \to \frac{\lambda f(x)}{(1 - u_0)^2} \qquad \text{weakly in } L^s(\Omega)\,,\ 1 < s < \frac{3N}{4}. \tag{4.1.21}$$

Taking now the limit of the equation satisfied by u_n, by (4.1.21) we get that $u_0 \in C(\bar{\Omega})$ is a $H^1(\Omega)$-weak solution of:

$$\begin{cases} -\Delta u_0 = \dfrac{\lambda f(x)}{(1 - u_0)^2} & \text{in } \Omega\,, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1.22}$$

Since

$$\int_\Omega \left(|\nabla\phi|^2 - \frac{2\lambda_n f(x)}{(1-u_n)^3}\phi^2\right) \to \int_\Omega \left(|\nabla\phi|^2 - \frac{2\lambda f(x)}{(1-u_0)^3}\phi^2\right)$$

for any $\phi \in C_0^\infty(\Omega)$ in view of (4.1.21), by (4.1.1) we get that u_0 has a finite Morse index according to definition (4.1.3). Since the set $S = \{x \in \Omega : u_0(x) = 1\}$ is a non empty finite set, by Theorem 4.1.2 such a solution u_0 cannot exist and we reach a contradiction. Hence, (4.1.2) holds.

If we also assume that $\mu_{1,n} < 0$, then $\lambda > 0$. Indeed, if $\lambda_n \to 0$, then by compactness and elliptic regularity theory, we would get $u_n \to u_0$ in $C^1(\bar\Omega)$, where u_0 is a harmonic function so that $u_0 = 0$ on $\partial\Omega$. Then, $u_0 \equiv 0$ and $u_n \to 0$ in $C^1(\bar\Omega)$. Hence, $\mu_{1,n} = \mu_{1,\lambda_n}(u_n) \to \mu_{1,0}(0) > 0$ as $n \to +\infty$, a contradiction. ■

4.2 Some consequences

In this section, we derive some consequences of Theorem 4.1.1. In view of Theorem 4.1.1, we first show a posteriori the equivalence among energy bounds and Morse index bounds. Indeed, we provide the following characterization of blow up sequences u_n (in the sense of blow up of $(1-u_n)^{-1}$), to be compared with [7, 8] in the context of polinomial subcritical nonlinearities:

Theorem 4.2.1. *Assume $2 \le N \le 7$ and suppose f as in (3.2.1). Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lambda_n \to \lambda \in [0, \lambda^*]$ and let u_n be an associated solution of $(S)_{\lambda_n}$. Then, the following are equivalent:*

1. $\max\limits_\Omega u_n \to 1$ *as* $n \to +\infty$ *;*
2. $\displaystyle\int_\Omega \left(\frac{f(x)}{(1-u_n)^3}\right)^{\frac{N}{2}} \to +\infty$ *as* $n \to +\infty$ *;*
3. $m(u_n, \lambda_n) \to +\infty$ *as* $n \to +\infty$ *.*

Proof: (1) ⇒ (2) Assume that $\max\limits_\Omega u_n \to 1$ as $n \to +\infty$. If $\int_\Omega \left(\frac{f(x)}{(1-u_n)^3}\right)^{\frac{N}{2}} \le C < \infty$ along a subsequence, the right hand side of $(S)_\lambda$ would be uniformly bounded in $L^{\frac{3N}{4}}$. By elliptic regularity theory and Sobolev embeddings, $u_n \to u_0$ weakly in $H_0^1(\Omega)$ and strongly in $C(\bar\Omega)$, where u_0 is a $H^1(\Omega)$-weak solution of $(S)_\lambda$ with $\lambda = \lim\limits_{n\to+\infty} \lambda_n$ so that $\int_\Omega \left(\frac{f(x)}{(1-u_0)^3}\right)^{\frac{N}{2}} < \infty$ and $0 \le u_0 \le 1$. By Theorem 2.3.3 we get $\|u_0\|_\infty < 1$ and, by uniform convergence, $\|u_n\|_\infty \to \|u_0\|_\infty < 1$ as $n \to +\infty$, a contradiction. Hence, necesssarily $\int_\Omega \left(\frac{f(x)}{(1-u_n)^3}\right)^{\frac{N}{2}} \to +\infty$ as $n \to +\infty$, which gives (2).

(2) ⇒ (1) The viceversa is trivial as it follows by the following inequality:

$$\int_\Omega \left(\frac{f(x)}{(1-u_n)^3}\right)^{\frac{N}{2}} \le \frac{\|f\|_\infty^{\frac{N}{2}}}{(1-\|u_n\|_\infty)^{\frac{3N}{2}}}|\Omega|,$$

where $|\cdot|$ stands for the Lebesgue measure.

(1) ⇒ (3) Assume $\max_\Omega u_n \to 1$ as $n \to +\infty$, then Theorem 4.1.1 directly implies $m(u_n, \lambda_n) \to +\infty$ as $n \to +\infty$.

(3) ⇒ (1) Since as before $\frac{f(x)}{(1-u_n)^3} \le \frac{\|f\|_\infty}{(1-\|u_n\|_\infty)^3}$, by the variational characterization of the eigenvalues we get that

$$\mu_{k,\lambda_n}(u_n) \ge \mu_k(L_n)\,, \qquad L_n := -\Delta - \frac{2\lambda_n\|f\|_\infty}{(1-\|u_n\|_\infty)^3},$$

where $\mu_k(L_n)$ stands for the k-th eigenvalue of the operator L_n. Indeed, for operator L in the form $L = -\Delta - c(x)$, $c(x) \in L^s(\Omega)$ for some $s > \frac{N}{2}$, let us recall that:

$$\mu_1(L) = \inf_{\phi\in H_0^1(\Omega)\,,\,\phi\ne 0} \frac{< L\phi, \phi >}{\int_\Omega \phi^2},$$

$$\mu_k(L) = \sup\Big\{\inf_{\phi\in M^\perp\,,\,\phi\ne 0} \frac{< L\phi, \phi >}{\int_\Omega \phi^2} \;:\; M \subset H_0^1(\Omega) \text{ linear, } \dim(M) = k-1\Big\} \quad \forall\, k \ge 2,$$

where $< \cdot, \cdot >$ is the standard inner product in $H_0^1(\Omega)$ and $M^\perp$ is the hortogonal space of M in $H_0^1(\Omega)$ with respect to this inner product.
Therefore, point (3) implies that the Morse index of L_n, the number of negative eigenvalues of L_n, blows up as $n \to +\infty$. Hence, the constant function $\frac{2\lambda_n\|f\|_\infty}{(1-\|u_n\|_\infty)^3} \to +\infty$ as $n \to +\infty$ and then, the validity of point (1) is established. ■

As a direct consequence of Theorem 4.1.1, Theorems 4.2.2 and 4.2.3 below show that some features of bifurcation diagrams on the ball hold for general domains. Theorem 4.2.2 is concerned with the following uniqueness result.

Theorem 4.2.2. *Assume $2 \le N \le 7$. Let f be as in (3.2.1). For any fixed $k \in \mathbb{N}$ there exists $\delta > 0$ small so that*

1. *for $\lambda \in (0, \delta)$ the minimal solution u_λ is the unique solution u of $(S)_\lambda$ with $m(u, \lambda) \le k$;*

2. *for $\lambda \in (\lambda^* - \delta, \lambda^*)$ u_λ and U_λ are the unique solutions u of $(S)_\lambda$ with $m(u, \lambda) \le k$.*

As far as point (1) in Theorem 4.2.2 is concerned, in [42] the authors show that problem $(S)_\lambda$ on a two-dimensional annulus with $f(x) = 1$ has exactly two radial solutions for any $\lambda \in (0, \lambda^*)$. The second solution -the non minimal one- has Morse index unbounded in a neighborhood of $\lambda = 0$.

Proof (of Theorem 4.2.2): (1) Let $\lambda_n \to 0$ as $n \to +\infty$ and associated solutions u_n of $(S)_\lambda$ so that $m(u_n, \lambda_n) \le k$, $k \in \mathbb{N}$. Theorem 4.1.1 implies that $\mu_{1,n} \ge 0$ for n large. By the characterization of the minimal solution u_λ as the only semi-stable solution, we get that $u_n = u_{\lambda_n}$ for n large. Hence, necessarily there exists $\delta = \delta_k > 0$ so that u_λ is the unique solution u of $(S)_\lambda$ with $m(u, \lambda) \le k$ for any $\lambda \in (0, \delta)$.

(2) Let $\lambda_n \to \lambda^*$ as $n \to +\infty$ and associated solutions u_n with $m(u_n, \lambda_n) \le k$, for some $k \in \mathbb{N}$. By Theorem 4.1.1 we get that $\sup\limits_{n\in\mathbb{N}} \|u_n\|_\infty < 1$. By elliptic regularity theory, u_n is uniformly bounded in $C^{1,\beta}(\bar\Omega)$, for some $\beta \in (0,1)$. Up to a subsequence, $u_n \to u_0$ in $C^1(\bar\Omega)$ as $n \to +\infty$, where u_0 is a $C^1(\bar\Omega)$-solution of $(S)_\lambda$ with $\lambda = \lambda^*$ so that $\max\limits_\Omega u_0 < 1$. Recall from Chapter 2 that equation $(S)_{\lambda^*}$ admits a unique $H_0^1(\Omega)-$weak solution, the extremal one u^*. Then, $u_n \to u^*$ in $C^1(\bar\Omega)$ as $n \to +\infty$. By [31], in a C^1-small neighborhood of u^* problem $(S)_\lambda$ has only the two solutions u_λ, U_λ for λ close to λ^*. Hence, either $u_n = u_{\lambda_n}$ or $u_n = U_{\lambda_n}$ and the uniqueness result follows. ■

Finally, based on a degree argument, we conclude this section by showing the following existence of a sequence of solutions whose Morse index blows up.

Theorem 4.2.3. *Assume $2 \le N \le 7$ and suppose f as in (3.2.1). Then there exist a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ and associated solution u_n of $(S)_\lambda$ so that*

$$m(u_n, \lambda_n) \to +\infty \qquad \text{as } n \to +\infty.$$

Proof: Let us define the solution set $\mathcal{V}$ as

$$\mathcal{V} = \{(\lambda, u) \in [0, +\infty) \times E : \ u \text{ is a solution of } (S)_\lambda\},$$

where $E = \{u \in C^1(\bar\Omega) : \ u = 0 \text{ on } \partial\Omega\}$ is endowed with the standard norm. By contradiction and in view of the equivalence of Theorem 4.2.1, let us assume that

$$\sup_{(\lambda,u)\in\mathcal{V}} \max_\Omega u \le 1 - 2\delta, \tag{4.2.1}$$

for some $\delta \in (0, \frac{1}{2})$. Hence, $\mathcal{V}$ is a compact set in $[0, +\infty) \times E$. By Theorem 4.2.2 we can fix $\lambda_1, \lambda_2 \in (0, \lambda^*)$, $\lambda_1 < \lambda_2$, so that $(S)_\lambda$ possesses:

– for λ_1, only the (non degenerate) minimal solution u_{λ_1} which satisfies $m(u_{\lambda_1}, \lambda_1) = 0$;

– for λ_2, only the two (non degenerate) solutions u_{λ_2}, U_{λ_2} satisfying $m(u_{\lambda_2}, \lambda_2) = 0$, $m(U_{\lambda_2}, \lambda_2) = 1$.

Define the projection of $\mathcal{V}$ onto E:

$$\mathcal{U} = \{u \in E : \ \exists\, \lambda \text{ so that } (\lambda, u) \in \mathcal{V}\},$$

and we consider a δ-neighborhood of $\mathcal{U}$ in E:

$$\mathcal{U}_\delta = \{u \in E : \ \text{dist}_E(u, \mathcal{U}) \le \delta\}.$$

Note that (4.2.1) gives

$$\sup_{u\in\mathcal{U}_\delta}\max_\Omega u \le 1-\delta.$$

Regularize the nonlinearity $(1-u)^{-2}$ in the following way:

$$g_\delta(u) = \begin{cases} (1-u)^{-2} & \text{if } u \le 1-\delta\,, \\ \delta^{-2} & \text{if } u \ge 1-\delta, \end{cases}$$

such that, for any fixed λ, proving the existence of solutions for $(S)_\lambda$ in $\mathcal{U}_\delta$ is equivalent to finding zeroes of the map $T_\lambda = Id - K_\lambda : E \to E$, where $K_\lambda(u) = -\Delta^{-1}(\lambda f(x) g_\delta(u))$ is a compact operator and Δ^{-1} is the laplacian resolvent with homogeneous Dirichlet boundary condition. We can define the Leray-Schauder degree d_λ of T_λ on $\mathcal{U}_\delta$ with respect to zero, since by definition of $\mathcal{U}$ (the set of all solutions) $\partial\mathcal{U}_\delta$ does not contain any solution of $(S)_\lambda$ for any value of λ. Since d_λ is well defined for any $\lambda \in [0,\lambda^*]$, by omotopy $d_{\lambda_1} = d_{\lambda_2}$.
To get a contradiction, let us now compute d_{λ_1} and d_{λ_2}. Since the only zero of T_{λ_1} in $\mathcal{U}_\delta$ is u_{λ_1} with Morse index zero, we have $d_{\lambda_1} = 1$. Since T_{λ_2} has in $\mathcal{U}_\delta$ exactly two zeroes u_{λ_2} and U_{λ_2} with Morse index zero and one, respectively, we have $d_{\lambda_2} = 1-1 = 0$. This contradicts $d_{\lambda_1} = d_{\lambda_2}$, and the proof is complete. ■

Let us finally point out that the equivalence among points (1) and (2) in Theorem 4.2.1 is already proved in Chapter 2, even if it is not stated. Moreover, a weaker form of Theorem 4.2.2(1) is already shown in Chapter 2 as the uniqueness, for small voltages λ, in the class of solutions of bounded energy (see Theorem 2.4.5).

4.3 Power-law problems on symmetric domains

In this section we consider uniqueness for small voltages and infinite multiplicity of positive solutions for the following problem

$$-\Delta u = \frac{\lambda |x|^\alpha}{(1-u)^2} \text{ in } \Omega,\ 0<u<1 \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega, \qquad (P)_\lambda$$

where the parameter $\lambda > 0$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain.
The study of problem $(P)_\lambda$ will be focussed on the two following symmetric situations:

$$u(x) = u(|x|)\,,\ \ \Omega = B\,,\ \ \alpha \ge 0\,,\ \ N \ge 2 \qquad (4.3.1a)$$

$$\Omega \text{ well-behaved}\,,\ \ \alpha = 0\,,\ \ N = 2, \qquad (4.3.1b)$$

where B is the unit ball. Here, Ω denotes a well-behaved domain in $\mathbb{R}^2$ if $0 \in \Omega$, Ω is invariant under the 2 reflections in the coordinate planes and, for any $0 < t < s < \max_\Omega x_i$, $(I-P_i)D_{i,s} \subseteq (I-P_i)D_{i,t}$, where P_i is the orthogonal projection onto Span $\{e_i\}$, $D_{i,s} = \{x \in \Omega :\ x_i = s\}$ and $\{e_1, e_2\}$ is the usual basis in $\mathbb{R}^2$. Examples of such domains include balls, ellipses, rectangles, etc.

Let us define the solution set:

$$\mathcal{V} = \{(\lambda, u) \in [0, +\infty) \times E : \ u \text{ is a solution of } (P)_\lambda\},$$

where $E = \{u = u(|x|) \in C^1(\bar{B}) : \ u = 0 \text{ on } \partial B\}$ in case (4.3.1a) and $E = \{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ in case (4.3.1b). Motivated by the bifurcation diagrams of Chapter 2, the following result was established by Guo and Wei in [67, 68]:

Theorem 4.3.1. *Assume either (4.3.1a) or (4.3.1b). Then we have*

1. $\lambda_* = \inf\{\lambda > 0 : \ (u, \lambda) \in \mathcal{V}$ *for some non-minimal* $u\} > 0$.
2. *When (4.3.1b) holds, then there exists a curve* $(\lambda(t), u(t))$ *in* $\mathcal{V}$, $t \geq 0$, *bifurcating from* $(0, 0)$ *and* $\|u(t)\|_\infty \to 1$ *as* $t \to +\infty$, *having infinitely many bifurcation points in* $\mathcal{V}$. *When (4.3.1a) holds, the same happens for either* $2 \leq N \leq 7$ *or* $N \geq 8$, $\alpha > \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, *and the bifurcation points are turning points, where the branch* $(\lambda(t), u(t))$ *locally "bends back".*

Remark 4.3.1. For $N \geq 2$, Theorem 4.3.1 gives in the radial setting a rigorous proof for the Conjecture B stated in section §2.5.

4.3.1 Some preliminaries

We introduce some tools which will be crucial to deal with the situation (4.3.1b). The following one-dimensional Sobolev inequality (cfr. Propositions 1.3&1.4 of [1]) plays a key role in the estimates of this section:

Proposition 4.3.2. *It holds*

$$\Big(\int_0^\pi u^{-2}dx\Big)\Big(\int_0^\pi (u^2 - u_x^2)dx\Big) \leq \pi^2, \tag{4.3.2}$$

for all $u \in H^+ = \big\{u \in H^1(\mathbb{R}) : \ u(x) > 0, \ u(\pi + x) = u(x) \ \forall\, x \in \mathbb{R}\big\}$.

Following [1], this subsection is mainly devoted to the proof of Proposition 4.3.2. First, we introduce the following notations:

$$X = \big\{u \in H^+ : \int_0^\pi \frac{e^{2ix}}{u^2(x)}dx = 0\big\}\,, \quad X_0 = \big\{u \in X : \oint \frac{dx}{u^2} = 1\big\}\,,$$

where $\oint g(x)dx$ is the average of the function $g(x)$ over $[0, \pi]$. For each $(\lambda, \theta) \in (0, \infty) \times [0, \pi)$, consider the change of variables on $[0, \pi]$ given by

$$x = \psi_{\lambda,\theta}(y) = \begin{cases} \theta + \frac{\pi}{2} + \arctan\big(\lambda^2 \tan(y - \frac{\pi}{2})\big) & \text{if } 0 \leq y \leq \frac{\pi}{2} - \arctan\big(\frac{1}{\lambda^2}\tan(\theta - \frac{\pi}{2})\big) \\ \theta - \frac{\pi}{2} + \arctan\big(\lambda^2 \tan(y - \frac{\pi}{2})\big) & \text{if } \frac{\pi}{2} - \arctan\big(\frac{1}{\lambda^2}\tan(\theta - \frac{\pi}{2})\big) < y \leq \pi, \end{cases}$$

where $\tan(\theta - \frac{\pi}{2}) \ |_{\theta=0} = -\infty$. Observe that $\psi_{\lambda,\theta}$ has a jump discontinuity of π at $\frac{\pi}{2} - \arctan\big(\frac{1}{\lambda^2}\tan(\theta - \frac{\pi}{2})\big)$ when $0 < \theta < \pi$. By the periodicity of functions in H^+, $\psi_{\lambda,\theta}$ induces an action on H^+ as follows: for any $u \in H^+$, let

$$u_{\lambda,\theta}(y) = U_{\lambda^{-1}}(y)u(\psi_{\lambda,\theta}(y)) = U_\lambda^{-1}(\psi_{\lambda,\theta}(y) - \theta)u(\psi_{\lambda,\theta}(y)),$$

where $U_\lambda(x) = (\lambda^2 \sin^2 x + \lambda^{-2} \cos^2 x)^{\frac{1}{2}}$. Using

$$\frac{d\psi_{\lambda,\theta}}{dy} = U_\lambda^2(\psi_{\lambda,\theta}(y) - \theta),$$

one can verify that

$$u^3_{\lambda,\theta}(y)\left((u_{\lambda,\theta})_{yy}(y) + u_{\lambda,\theta}(y)\right) = u^3(x)\left(u_{xx}(x) + u(x)\right)\big|_{x=\psi_{\lambda,\theta}(y)}$$

for all $(\lambda, \theta) \in (0, \infty) \times [0, \pi)$. Starting from $u \equiv 1$, we see that all the positive solutions of

$$u_{xx} + u = \frac{1}{u^3} \tag{4.3.3}$$

are given by $U_\lambda(x - \theta)$, where $\lambda > 0$ and $\theta \in [0, \pi)$. Also, we have

Lemma 4.3.3. *The functionals* $\oint \frac{dx}{u^2}$ *and* $\oint (u^2 - u_x^2)dx$ *are invariants under the action* $u \to u_{\lambda,\theta}$*, for all* $(\lambda, \theta) \in (0, \infty) \times [0, \pi)$*.*

Lemma 4.3.4. *For any* $u \in H^+$*, there exists* $\lambda \in (0, 1]$ *such that* $u_{\lambda,\theta} \in X$*, for some* $\theta \in [0, \pi)$*.*

Proof: Given $u \in H^+$, let us define

$$F(u, \lambda, \theta) = \int_0^\pi \frac{e^{2iy}}{u^2_{\lambda,\theta}(y)} dy = \int_0^\pi \frac{dx}{u^2(x)} \frac{\left(\lambda^{-2}\cos^2(x-\theta) - \lambda^2 \sin^2(x-\theta)\right) + i \sin(2x - 2\theta)}{\lambda^{-2}\cos^2(x-\theta) + \lambda^2 \sin^2(x-\theta)}.$$

Since for $u \in X$ we can simply take $\lambda = 1$, we need only to consider the case $u \in H^+ \setminus X$. We argue in a variational way and we define

$$\Phi(\lambda, \theta) = \int_0^\pi \frac{dx}{u^2(x)} \ln\left(\lambda^{-2}\cos^2(x-\theta) + \lambda^2 \sin^2(x-\theta)\right), \quad \lambda \in (0, 1],\ \theta \in \mathbb{R}.$$

Since $\Phi(\lambda, \theta) \to +\infty$ as $\lambda \to 0^+$ uniformly on $\theta \in \mathbb{R}$, by $\pi-$periodicity $\Phi(\lambda, \theta)$ admits a point of minimum $(\lambda_u, \theta_u) \in (0, 1] \times [0, \pi)$: $\Phi(\lambda_u, \theta_u) = \min_{(0,1]\times\mathbb{R}} \Phi$.
Assumption $u \notin X$ provides the existence of some $\theta_0 \in (0, \pi)$ such that

$$\int_0^\pi \frac{dx}{u^2(x)} \cos(2x - 2\theta_0) = \cos(2\theta_0)\left(\int_0^\pi \frac{\cos(2x)}{u^2(x)} dx\right) + \sin(2\theta_0)\left(\int_0^\pi \frac{\sin(2x)}{u^2(x)} dx\right) < 0.$$

Since $\Phi(1, \theta) \equiv 0$ and

$$\frac{\partial \Phi}{\partial \lambda}(\lambda, \theta_0) \to -2\int_0^\pi \frac{dx}{u^2(x)} \cos(2x - 2\theta_0) > 0$$

as $\lambda \to 1^-$, we have that $\lambda_u < 1$. The extremality property $\nabla\Phi(\lambda_u, \theta_u) = 0$ for $\lambda_u < 1$ rewrites as $F(u, \lambda_u, \theta_u) = 0$. The proof is complete. ■

Proof (of Proposition 4.3.2): Let $\{u_j\}$ be a maximizing sequence of (4.3.2) satisfying $\oint u_j^{-2}dx = 1$. By Lemmata 4.3.3 and 4.3.4, we may also assume that $u_j \in X_0$ and, up to a translation, $u_j(0) = u_j(\pi) = 1$.
Set $F(u) = \int_0^\pi (u^2 - u_x^2)dx$. Since $\{u_j\}$ is a maximizing sequence of (4.3.2), we have that $F(u_j) \geq F(1) - \frac{\pi}{2} = \frac{\pi}{2} > 0$ for j large. Therefore,

$$\int_0^\pi (u_j)_x^2 dx \leq \int_0^\pi u_j^2 dx \quad \text{for } j \text{ large}, \tag{4.3.4}$$

and the sequence $\bar{u}_j = \|u_j\|_\infty^{-1} u_j$ is uniformly bounded in $H^1_{\text{loc}}(\mathbb{R})$.
We claim that $\{u_j\}$ is uniformly bounded. If not, up to a subsequence, $\bar{u}_j$ converges uniformly to some non-negative function $\bar{u} \in H^1(\mathbb{R})$ satisfying

$$\bar{u} \in H_0^1(0,\pi)\,, \qquad \int_0^\pi \bar{u}_x^2 dx \leq \int_0^\pi \bar{u}^2 dx\,.$$

Then, by Poincaré inequality $\int_0^\pi \bar{u}_x^2 dx = \int_0^\pi \bar{u}^2 dx$ and $\bar{u}(x) = \sin x$ in $(0,\pi)$ is the first eigenfunction of $-\Delta$ in $H_0^1(0,\pi)$. Property $u_j \in X_0$ can be rewritten as

$$\oint \frac{\sin^2 x}{u_j^2} dx = \frac{1}{2}.$$

Since by Lebesgue's Theorem $\int_\delta^{\pi-\delta} \bar{u}_j^{-2} \sin^2 x dx \to \pi - 2\delta$ and $\oint \bar{u}_j^{-2} dx = \|u_j\|_\infty^2 \to +\infty$ as $j \to +\infty$, we have that

$$\begin{aligned} \frac{\int_0^\pi \bar{u}_j^{-2} \sin^2 x dx}{\int_0^\pi \bar{u}_j^{-2} dx} &= \frac{\int_0^\delta \bar{u}_j^{-2} \sin^2 x dx + \int_\delta^{\pi-\delta} \bar{u}_j^{-2} \sin^2 x dx + \int_{\pi-\delta}^\pi \bar{u}_j^{-2} \sin^2 x dx}{\int_0^\pi \bar{u}_j^{-2} dx} \\ &\leq 2\sin^2\delta + \frac{\int_\delta^{\pi-\delta} \bar{u}_j^{-2} \sin^2 x dx}{\int_0^\pi \bar{u}_j^{-2} dx} \to 2\sin^2\delta \text{ as } j \to +\infty, \end{aligned}$$

for any small δ. Hence,

$$\frac{\int_0^\pi \bar{u}_j^{-2} \sin^2 x dx}{\int_0^\pi \bar{u}_j^{-2} dx} \to 0 \text{ as } j \to +\infty,$$

in contradiction with

$$\frac{\oint \bar{u}_j^{-2} \sin^2 x dx}{\oint \bar{u}_j^{-2} dx} = \oint \frac{\sin^2 x}{u_j^2} dx = \frac{1}{2}.$$

Hence, $\{u_j\}$ is uniformly bounded. By (4.3.4) we get that

$$|\log u_j(x) - \log u_j(y)| = |\int_y^x \frac{(u_j)_x}{u_j} dx| \leq \Big(\int_0^\pi \frac{dx}{u_j^2}\Big)^{\frac{1}{2}} \Big(\int_0^\pi (u_j)_x^2 dx\Big)^{\frac{1}{2}} \leq \Big(\int_0^\pi u_j^2 dx\Big)^{\frac{1}{2}} \leq C,$$

for any $x, y \in [0,\pi]$. Since $u_j(0) = 1$, u_j has also a uniform positive lower bound.

As before, we can show that $u_j \to u \in H^+$ uniformly and weakly in $H^1_{\text{loc}}(\mathbb{R})$. Since $u > 0$, by Lebesgue's Theorem $u \in X_0$ and $F(u) \geq \limsup_{j\to+\infty} F(u_j) \geq \frac{\pi}{2}$. Hence, $F(u)$ achieves at u the supremum on $H^+ \cap \{\oint \frac{dx}{u^2} = 1\}$, where u solves the equation

$$u_{xx} + u = \frac{\beta}{u^3}\ , \ \ \beta = F(u) > 0.$$

Since $v = \beta^{-\frac{1}{4}} u$ solves (4.3.3), v has to coincide with $U_\lambda(x-\theta)$, for some $0 < \lambda \leq 1$ and $0 \leq \theta < \pi$. Since

$$\int_0^\pi \frac{\cos(2x)}{U_\lambda^2(x)} dx = \int_0^\pi \frac{\cos^2 x - \sin^2 x}{\lambda^2 \sin^2 x + \lambda^{-2}\cos^2 x} dx = \pi \frac{\lambda^2 - 1}{\lambda^2 + 1},$$

let us observe that $U_\lambda \in X$ only for $\lambda = 1$. Since $u, v \in X$, then $\lambda = 1$ and $v = 1$. Since $\oint u^{-2} = 1$, we get that $F(u) = \beta = \pi^2$ and the proof of Proposition 4.3.2 is complete. ■

To conclude, we introduce the following useful Lemma (see Lemma 2.2 in [71]):

Lemma 4.3.5. *Assume that $\phi \geq 0$ is a smooth function on $\bar{B}_r \subset \mathbb{R}^2$ such that $\Delta\phi + \phi^2 \geq 0$. Then, there exist universal constants c, $\eta_0 > 0$ such that $\int_{B_r} \phi dx \leq \eta_0$ implies $\phi(x) \leq \frac{c}{r^2}\int_{B_r} \phi dx$ for $x \in B_{\frac{r}{2}}$.*

Proof: By the scaling $\phi(x) \to r^2\phi(rx)$, we may assume $r = 1$. We claim that

$$K := \max_{|x|\leq 1} (1 - |x|)^2 \phi(x) \leq 1.$$

Otherwise, if $K > 1$ we choose $\xi \in B_1$ such that $(1 - |\xi|)^2\phi(\xi) = K$. Setting $\sigma = 1 - |\xi|$, then for $x \in B_{\frac{\sigma}{2}}(\xi)$ we have $\phi(x) \leq \frac{4K}{\sigma^2}$. Hence $\psi(x) = \frac{\sigma^2}{4K}\phi(\xi + \frac{\sigma}{2\sqrt{K}}x)$ is well-defined in B_1, and satisfies

$$-\Delta\psi \leq \psi^2, \quad \psi \leq 1 \text{ on } B_1; \quad \psi(0) = \frac{1}{4}, \quad \int_{B_1} \psi dx = \int_{B_{\frac{\sigma}{2\sqrt{K}}}(\xi)} \phi(x) dx \leq \eta_0\,.$$

Therefore, we have $-\Delta\psi \leq \psi$. Using elliptic estimates (cfr. page 67 in [69]), we know that $\psi(0) \leq c\int_{B_1} \psi dx \leq c\eta_0$, where c is a suitable constant. Choosing η_0 small enough such that $c\eta_0 < \frac{1}{4}$, a contradiction arises.

Since $K \leq 1$, then we have $\phi(x) \leq 16$ and $-\Delta\phi \leq 16\phi$ on $B_{3/4}$. So again by elliptic estimates we get that $\phi(x) \leq c\int_{B_{\frac{3}{4}}} \phi dx \leq c\int_{B_1} \phi dx \leq c\eta_0$ on $B_{1/2}$. This completes the proof of Lemma 4.3.5. ■

4.3.2 Uniqueness for small voltages λ

This subsection is devoted to the proof of Theorem 4.3.1(1). In the symmetric situations (4.3.1a) and (4.3.1b), we are able to improve Theorems 2.4.5 & 4.2.3 where uniqueness for λ small was established in the class of solutions with uniformly finite energy, bounded Morse index respectively. We argue by contradiction and suppose there are sequences $\{\lambda_n\}$, $\{u_n\}$ with $\lambda_n \to 0$ as $n \to \infty$ and u_n a non-minimal solution of $(P)_{\lambda_n}$. By the above quoted Theorems, $\|u_n\|_\infty \to 1$ as $n \to \infty$ and the argument now relies on Theorem 2.4.7. Our next task is to show the validity of (2.4.12), (2.4.13) and (2.4.14) along the sequence u_n.

In the sequel of this subsection, let $\lambda_n > 0$ be a sequence and let u_n be a solution of $(P)_{\lambda_n}$ so that

$$\lambda_n \to \lambda \geq 0\,, \quad \varepsilon_n := 1 - \|u_n\|_\infty \to 0^+ \quad \text{as } n \to \infty. \tag{4.3.5}$$

The validity of (2.4.12) is based on Pohozaev's identity as the following Lemma shows:

Lemma 4.3.6. *Assume either (4.3.1a) or (4.3.1b). Let $\lambda_n > 0$ be a sequence and let u_n be a solution of $(P)_{\lambda_n}$ so that (4.3.5) holds. Let $r_0 < \frac{1}{2} dist\,(0, \partial\Omega)$. For any $0 < r \leq r_0$ there exists $k_r < 1$ such that*

$$\max_{\Omega \setminus B_r(0)} u_n \leq k_r < 1.$$

Proof: Consider first the situation (4.3.1a). We apply Pohozaev's identity (2.2.13) on $B_1(0)$ to get

$$\frac{\dot{u}_n^2(1)}{2} = \lambda_n (N+\alpha) \int_0^1 \frac{s^{N-1+\alpha} u_n(s)}{1-u_n(s)} ds - \frac{N-2}{2} \lambda_n \int_0^1 \frac{s^{N-1+\alpha} u_n(s)}{(1-u_n(s))^2} ds.$$

Then, for some $C > 0$ we have that

$$\dot{u}_n^2(1) \leq C \lambda_n \int_0^1 \frac{s^{N-1+\alpha}}{1-u_n(s)} ds.$$

The equation $(P)_{\lambda_n}$ satisfied by u_n rewrites as: $-(s^{N-1}\dot{u}_n(s))' = \lambda_n s^{N-1+\alpha}(1-u_n(s))^{-2}$. We then easily get that $\dot{u}_n \leq 0$ in $(0,1]$ and

$$\dot{u}_n(1) = -\lambda_n \int_0^1 \frac{s^{N-1+\alpha}}{(1-u_n(s))^2} ds.$$

Thus, we obtain that

$$\begin{aligned} \Big(\int_0^1 \frac{s^{N-1+\alpha}}{(1-u_n(s))^2} ds \Big)^2 &= \lambda_n^{-2} \dot{u}_n^2(1) \leq C \lambda_n^{-1} \int_0^1 \frac{s^{N-1+\alpha}}{1-u_n(s)} ds \\ &\leq C \lambda_n^{-1} \Big(\int_0^1 \frac{s^{N-1+\alpha}}{(1-u_n(s))^2} ds \Big)^{\frac{1}{2}} \Big(\int_0^1 s^{N-1+\alpha} ds \Big)^{\frac{1}{2}}, \end{aligned}$$

which implies

$$\int_0^1 \frac{s^{N-1+\alpha}}{(1-u_n(s))^2} ds \leq C \lambda_n^{-\frac{2}{3}}. \tag{4.3.6}$$

Since

$$\dot{u}_n \leq 0 \quad \text{in } (0,1], \tag{4.3.7}$$

we see that

$$\frac{1}{(1-u_n(s))^2} \leq C \int_{\frac{r}{4}}^{s} \frac{t^{N-1+\alpha}}{(1-u_n(t))^2} dt \tag{4.3.8}$$

for any $s \in [\frac{r}{2}, 1]$. Thus, it follows from (4.3.6) and (4.3.8) that

$$\lambda_n \frac{|x|^\alpha}{(1-u_n(x))^2} \leq \lambda_n C \int_0^1 t^{N-1+\alpha}(1-u_n(t))^{-2} dt \leq C\lambda_n^{\frac{1}{3}} \tag{4.3.9}$$

for any $|x| \in [\frac{r}{2}, 1]$.

Let us consider now the second situation (4.3.1b). In such a case, Pohozaev's identity (2.2.13) implies that

$$\int_{\partial\Omega} \Big(\frac{\partial u_n}{\partial \nu}\Big)^2 x \cdot \nu dS = 4\lambda_n \int_\Omega \frac{dx}{1-u_n} - 4\lambda_n |\Omega|,$$

where ν is the unit outward normal vector of $\partial\Omega$. Since Ω is well-behaved, we have that

$$x \cdot \nu \geq C > 0 \quad \forall\, x \in \partial\Omega.$$

Therefore, it follows that

$$\int_{\partial\Omega} \Big(\frac{\partial u_n}{\partial \nu}\Big)^2 dS \leq C\lambda_n \int_\Omega \frac{dx}{1-u_n}. \tag{4.3.10}$$

On the other hand, it follows from $(P)_{\lambda_n}$ that

$$\int_{\partial\Omega} \frac{\partial u_n}{\partial \nu} dS = -\lambda_n \int_\Omega \frac{dx}{(1-u_n)^2}.$$

Hence, we have

$$\begin{aligned} \Big(\int_\Omega \frac{dx}{(1-u_n)^2}\Big)^2 \leq C\lambda_n^{-2} \int_{\partial\Omega} \Big(\frac{\partial u_n}{\partial \nu}\Big)^2 dS &\leq C\lambda_n^{-1} \int_\Omega \frac{dx}{1-u_n} \\ &\leq C\lambda_n^{-1} \Big(\int_\Omega \frac{dx}{(1-u_n)^2}\Big)^{\frac{1}{2}}, \end{aligned} \tag{4.3.11}$$

which implies

$$\int_\Omega \frac{dx}{(1-u_n)^2} \leq C\lambda_n^{-\frac{2}{3}}.$$

By the standard moving plane argument in [70], it follows that for any $x \in \Omega \setminus B_{\frac{r}{2}}(0)$, there exists a piece of cone with vertex at x: Γ_x with (i) $\text{meas}(\Gamma_x) \geq \gamma > 0$ uniformly on x, (ii) $\Gamma_x \subset \Omega$, (iii) $(1-u_n(y))^{-2} \geq (1-u_n(x))^{-2}$ for any $y \in \Gamma_x$. Thus, this and (4.3.11) imply that

$$\frac{1}{(1-u_n(x))^2} \leq \frac{1}{\text{meas}(\Gamma_x)} \int_{\Gamma_x} \frac{dy}{(1-u_n)^2} \leq \gamma^{-1} C\lambda_n^{-\frac{2}{3}},$$

and hence,

$$\frac{\lambda_n}{(1-u_n(x))^2} \le C\lambda_n^{\frac{1}{3}} \tag{4.3.12}$$

for any $x \in \Omega \setminus B_{\frac{r}{2}}(0)$.

Let k_n be the solution of the problem

$$-\Delta k_n = C\lambda_n^{\frac{1}{3}} \text{ in } \Omega \setminus B_{\frac{r}{2}}(0), \quad k_n = 0 \text{ on } \partial\Omega, \quad k_n = 1 \text{ on } \partial B_{\frac{r}{2}}(0).$$

Then, by (4.3.9) and (4.3.12) the comparison principle implies that

$$u_n \le k_n \text{ in } \Omega \setminus B_{\frac{r}{2}}(0).$$

Since $k_n = k_0 + C\lambda_n^{\frac{1}{3}} k_1$, where

$$\Delta k_0 = 0 \text{ in } \Omega \setminus B_{\frac{r}{2}}(0), \quad k_0 = 0 \text{ on } \partial\Omega, \quad k_0 = 1 \text{ on } \partial B_{\frac{r}{2}}(0),$$

and

$$\Delta k_1 = 1 \text{ in } \Omega \setminus B_{\frac{r}{2}}(0), \quad k_1 = 0 \text{ on } \partial\Omega \cup \partial B_{\frac{r}{2}}(0),$$

we have

$$u_n \le k_0 + C\lambda_n^{\frac{1}{3}} k_1.$$

Note that the maximum principle implies $\max_{\Omega \setminus B_r(0)} k_0 < 1$ and $|k_1(x)| \le C$ for $x \in \Omega \setminus B_r(0)$. Therefore, for n sufficiently large $u_n(x) \le k_r < 1$ for any $x \in \Omega \setminus B_r(0)$. ■

When (4.3.1b) holds, the shape of Ω and the moving planes method of [53] imply that u_n is even in x_i, $i = 1, 2$, and

$$\nabla u_n \cdot x \le 0 \quad \text{in } \Omega. \tag{4.3.13}$$

In the situation (4.3.1a), (4.3.7) gives exactly (4.3.13) also in this case. In particular, (2.4.13) holds in general.

To conclude the proof of Theorem 4.3.1(1), finally we need to establish (2.4.14). We have the following general result:

Lemma 4.3.7. *Assume either (4.3.1a) or (4.3.1b). Let $\lambda_n > 0$ be a sequence and let u_n be a solution of $(P)_{\lambda_n}$ so that (4.3.5) holds. Then, for any $0 < r_0 < \frac{1}{2}dist\,(0, \partial\Omega)$*

$$r^{-\frac{2+\alpha}{3}} \int_{S^{N-1}} (1-u_n)(r\theta)d\theta \le C \qquad \forall\, \epsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} \le r \le r_0, \tag{4.3.14}$$

and

$$\lambda_n r^{\frac{4+2\alpha}{3}} \int_{S^{N-1}} \frac{1}{(1-u_n(r\theta))^2} d\theta \le C \qquad \forall\, \epsilon^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} \le r \le r_0, \tag{4.3.15}$$

for some $C > 0$.

Proof: Let us recall some facts and notations in §4.2.2. By (2.4.15) $\varepsilon_n^3\lambda_n^{-1} \to 0$ as $n \to +\infty$. According to (2.4.16), the function $U_n(y) = \frac{1-u_n(\epsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}y)}{\epsilon_n}$ solves:

$$\Delta U_n = |y|^\alpha U_n^{-2} \text{ in } \Omega_n, \quad U_n(0) = 1, \quad U_n(y) \ge 1,$$

where $\Omega_n := \{y : \epsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}} y \in \Omega\} \to \mathbb{R}^N$ as $n \to +\infty$. Given $0 < r_0 < \frac{1}{2}\text{dist}\,(0, \partial\Omega)$, set $R_n = e^{T_n} = \epsilon_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}} r_0$ and let $v_n(s,\theta) = |y|^{-\frac{2+\alpha}{3}} U_n(y)$, $|y| = e^s$, be the Emden-Fowler transformation of U_n. According to (2.4.18), the function v_n solves in $(-\infty, T_n) \times S^{N-1}$:

$$v_{ss} + \left(N - \frac{2}{3} + \frac{2\alpha}{3}\right) v_s + \Delta_{S^{N-1}} v + \frac{2+\alpha}{3}\left(N - \frac{4}{3} + \frac{\alpha}{3}\right) v = v^{-2}.$$

Since up to a subsequence $U_n \to U \ge 1$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, according to (2.4.19) we have that

$$\frac{1}{C} \le v_n(s,\theta) \le C\,, \quad |(v_n)_s(s,\theta)| + |\nabla_\theta v_n(s,\theta)| \le C \qquad \forall\, 0 \le s \le 1,\ \theta \in S^{N-1} \tag{4.3.16}$$

for n large. Lemma 4.3.6, combined with elliptic estimates, yields to

$$|\nabla u_n(x)| \le C \ \text{ for } x \in \Omega \setminus B_r(0)$$

for some $C = C_r > 0$, and then it readily implies:

$$\frac{1}{C} \le \lambda_n^{\frac{1}{3}} v_n(s,\theta) \le C\,, \quad |(v_n)_s(s,\theta)| + |\nabla_\theta v_n(s,\theta)| = O(\lambda_n^{-\frac{1}{3}}) \ \forall\, T_n - 1 \le s \le T_n \tag{4.3.17}$$

uniformly in $\theta \in S^{N-1}$, in accordance with (2.4.20). By (2.4.21) estimate (4.3.13) gives

$$(v_n)_s(s,\theta) + \frac{2+\alpha}{3} v_n(s,\theta) \ge 0 \qquad \forall\, (s,\theta) \in (-\infty, T_n] \times S^{N-1}. \tag{4.3.18}$$

In terms of v_n, the desired estimates (4.3.14), (4.3.15) rewrite as

$$\int_{S^{N-1}} v_n(s,\theta)d\theta \le C\lambda_n^{-\frac{1}{3}}\,, \quad \int_{S^{N-1}} \frac{1}{v_n^2(s,\theta)} d\theta \le C\lambda_n^{-\frac{1}{3}} \quad \forall\, 0 \le s \le T_n. \tag{4.3.19}$$

Denote

$$w_n(s) = \int_{S^{N-1}} v_n^2(s,\theta)d\theta\,, \quad \overline{v}_n(s) = \int_{S^{N-1}} v_n(s,\theta)d\theta\,.$$

For convenience, in the following we omit the subscript n of w_n, $\overline{v}_n$ and v_n. Note that

$$\overline{v}_{ss} + \left(N - \frac{2}{3} + \frac{2\alpha}{3}\right)\overline{v}_s + \frac{2+\alpha}{3}\left(N - \frac{4}{3} + \frac{\alpha}{3}\right)\overline{v} = \int_{S^{N-1}} \frac{d\theta}{v^2} \tag{4.3.20}$$

$$w_{ss} + \left(N - \frac{2}{3} + \frac{2\alpha}{3}\right) w_s + \frac{4+2\alpha}{3}\left(N - \frac{4}{3} + \frac{\alpha}{3}\right) w = 2\int_{S^{N-1}} \frac{d\theta}{v} \tag{4.3.21}$$

$$+2\int_{S^{N-1}} (|\nabla_\theta v|^2 + v_s^2)d\theta.$$

Multiplying (2.4.18) by v_s and integrating on $[0,T_n]\times S^{N-1}$, we get that

$$\int_0^{T_n}\int_{S^{N-1}} v_s^2(s,\theta)dsd\theta \le C\lambda_n^{-\frac{2}{3}}$$

for some $C>0$, by means of (4.3.16) and (4.3.17). Therefore, it holds

$$\int_0^{t}\int_{S^{N-1}} v_s^2(s,\theta)d\theta ds \le C\lambda_n^{-\frac{2}{3}} \text{ for } t\in[0,T_n] \tag{4.3.22}$$

for some $C>0$. The proof proceeds now with the following three steps.

Step 1: We claim that

$$\int_{S^{N-1}}\frac{d\theta}{v^2(s,\theta)} \le (2+\alpha)\overline{v}_s+(2+\alpha)(N-\frac{4}{3}+\frac{\alpha}{3})\overline{v} \text{ for } s\in(-\infty,T_n]. \tag{4.3.23}$$

In fact, (4.3.18) and (4.3.20) imply that

$$\begin{aligned}&\overline{v}_{sss}+\left(N-\frac{2}{3}+\frac{2\alpha}{3}\right)\overline{v}_{ss}+\frac{2+\alpha}{3}\left(N-\frac{4}{3}+\frac{\alpha}{3}\right)\overline{v}_s=-2\int_{S^{N-1}}\frac{v_s}{v^3}d\theta\\ &\le\frac{4+2\alpha}{3}\int_{S^{N-1}}\frac{d\theta}{v^2}=\frac{4+2\alpha}{3}\left[\overline{v}_{ss}+\left(N-\frac{2}{3}+\frac{2\alpha}{3}\right)\overline{v}_s+\frac{2+\alpha}{3}\left(N-\frac{4}{3}+\frac{\alpha}{3}\right)\overline{v}\right].\end{aligned}$$

Setting $q(s)=\overline{v}_s-\frac{4+2\alpha}{3}\overline{v}$, above estimate yields to

$$q_{ss}+\left(N-\frac{2}{3}+\frac{2\alpha}{3}\right)q_s+\frac{2+\alpha}{3}\left(N-\frac{4}{3}+\frac{\alpha}{3}\right)q\le 0$$

for any $s\le T_n$ or equivalently,

$$\left[e^{\frac{2+\alpha}{3}s}\left(q_s+(N-\frac{4}{3}+\frac{\alpha}{3})q\right)\right]_s\le 0 \text{ for } s\in(-\infty,T_n]. \tag{4.3.24}$$

Note that

$$e^{\frac{2+\alpha}{3}s}\overline{v}(s)\to|S^{N-1}|\,,\qquad e^{\frac{2+\alpha}{3}s}\Big(\overline{v}_s(s)+\frac{2+\alpha}{3}\overline{v}(s)\Big)\to 0\quad\text{as } s\to-\infty,$$

and the equation (4.3.20) satisfied by $\overline{v}$ now implies

$$e^{\frac{2+\alpha}{3}s}\overline{v}_{ss}(s)\to-(\frac{2+\alpha}{3})^2|S^{N-1}|\quad\text{as } s\to-\infty.$$

Then, we have

$$e^{\frac{2+\alpha}{3}s}q(s)\to-(2+\alpha)|S^{N-1}|\,,\quad e^{\frac{2+\alpha}{3}s}q_s(s)\to\frac{(2+\alpha)^2}{9}|S^{N-1}| \text{ as } s\to-\infty. \tag{4.3.25}$$

Therefore, (4.3.25) implies that

$$e^{\frac{2+\alpha}{3}s}\Big(q_s(s)+(N-\frac{4}{3}+\frac{\alpha}{3})q(s)\Big)\to -(N-\frac{14}{9}+\frac{2\alpha}{9})(2+\alpha)|S^{N-1}| \quad \text{as} \quad s\to-\infty\,.$$

It now deduces from (4.3.24) that, for any $s\in(-\infty,T_n]$,

$$e^{\frac{2+\alpha}{3}s}\Big(q_s(s)+(N-\frac{4}{3}+\frac{\alpha}{3})q(s)\Big)\le -(N-\frac{14}{9}+\frac{2\alpha}{9})(2+\alpha)|S^{N-1}|\le 0, \tag{4.3.26}$$

which by (4.3.20) reduces to

$$\begin{aligned}
0 &\ge q_s(s)+\Big(N-\frac{4}{3}+\frac{\alpha}{3}\Big)q(s)\\
&= \overline{v}_{ss}+\Big(N-\frac{8}{3}-\frac{\alpha}{3}\Big)\overline{v}_s-\frac{4+2\alpha}{3}\Big(N-\frac{4}{3}+\frac{\alpha}{3}\Big)\overline{v}\\
&= -(2+\alpha)\overline{v}_s-(2+\alpha)(N-\frac{4}{3}+\frac{\alpha}{3})\overline{v}+\int_{S^{N-1}}\frac{d\theta}{v^2}.
\end{aligned}$$

Therefore, we conclude that (4.3.23) holds:

$$\int_{S^{N-1}}\frac{d\theta}{v^2}\le(2+\alpha)\overline{v}_s+(2+\alpha)(N-\frac{4}{3}+\frac{\alpha}{3})\overline{v}\ \text{ for } s\in(-\infty,T_n].$$

Step 2: We claim that

$$\int_{S^{N-1}}\frac{d\theta}{v^2(s,\theta)}\le C\Big(\lambda_n^{-\frac{1}{3}}+\int_{S^{N-1}}v_s^2d\theta\Big). \tag{4.3.27}$$

To prove (4.3.27), we consider the function

$$J(s)=\int_{S^{N-1}}\Big[|\nabla_\theta v|^2(s,\theta)-v_s^2(s,\theta)-\frac{2+\alpha}{3}\left(N-\frac{4}{3}+\frac{\alpha}{3}\right)v^2(s,\theta)-\frac{2}{v(s,\theta)}\Big]d\theta.$$

Multiplying (2.4.18) by v_s and integrating on S^{N-1}, we obtain that

$$J_s(s)=2\left(N-\frac{2}{3}+\frac{2\alpha}{3}\right)\int_{S^{N-1}}v_s^2(s,\theta)d\theta>0,$$

which gives

$$J(0)\le J(s)\le J(T_n)\ \text{ for } s\in[0,T_n].$$

It follows from (4.3.16), (4.3.17) that

$$|J(s)|\le C\lambda_n^{-\frac{2}{3}}\ \text{ for } s\in[0,T_n],$$

and hence

$$\Big|\int_{S^{N-1}}|\nabla_\theta v|^2(s,\theta)d\theta-\frac{2+\alpha}{3}\left(N-\frac{4}{3}+\frac{\alpha}{3}\right)\int_{S^{N-1}}v^2(s,\theta)d\theta\Big| \le C\Big(\lambda_n^{-\frac{2}{3}}+\int_{S^{N-1}}v_s^2(s,\theta)d\theta+\int_{S^{N-1}}\frac{d\theta}{v(s,\theta)}\Big). \tag{4.3.28}$$

On the other hand, it follows from the Young's inequality and (4.3.23) that

$$\int_{S^{N-1}}\frac{d\theta}{v(s,\theta)}\le C\Big(\int_{S^{N-1}}\frac{d\theta}{v^2(s,\theta)}+1\Big)\le C\big(\overline{v}_s+\overline{v}+1\big)\le C\big(1+\overline{v}+\int_{S^{N-1}}v_s^2(s,\theta)d\theta\big).$$

Then we deduce from (4.3.28) and the Young's inequality that for any $\epsilon>0$,

$$\begin{aligned}&|\int_{S^{N-1}}|\nabla_\theta v|^2(s,\theta)d\theta-\frac{2+\alpha}{3}\left(N-\frac{4}{3}+\frac{\alpha}{3}\right)\int_{S^{N-1}}v^2(s,\theta)d\theta|\\ &\le C\Big(\lambda_n^{-\frac{2}{3}}+\int_{S^{N-1}}v_s^2(s,\theta)d\theta+\overline{v}(s)\Big)\\ &\le C\Big(\lambda_n^{-\frac{2}{3}}+\int_{S^{N-1}}v_s^2(s,\theta)d\theta\Big)+\epsilon\int_{S^{N-1}}v^2(s,\theta)d\theta+C_\epsilon.\end{aligned}$$

When (4.3.1a) holds, the function v is radial and for $\epsilon<\frac{2+\alpha}{3}\left(N-\frac{4}{3}+\frac{\alpha}{3}\right)$ we get that

$$v^2(s)\le C\Big(\lambda_n^{-\frac{2}{3}}+v_s^2(s)\Big). \tag{4.3.29}$$

In the situation (4.3.1b), the function v is symmetric with respect to reflections in the coordinate planes. In particular, $v(s,\cdot)$ is a π-periodic function. Choosing ϵ such that $\frac{4}{9}+\epsilon<1$ and using the Sobolev's inequality (4.3.2), we obtain that

$$\int_{S^1}v^2(s,\theta)d\theta\le C\Big(\lambda_n^{-\frac{2}{3}}+\int_{S^1}v_s^2(s,\theta)d\theta+\frac{1}{\int_{S^1}\frac{d\theta}{v^2(s,\theta)}}\Big). \tag{4.3.30}$$

We collect (4.3.29) and (4.3.30) in the following estimate:

$$\int_{S^{N-1}}v^2(s,\theta)d\theta\le C\Big(\lambda_n^{-\frac{2}{3}}+\int_{S^{N-1}}v_s^2(s,\theta)d\theta+\frac{1}{\int_{S^{N-1}}\frac{d\theta}{v^2(s,\theta)}}\Big), \tag{4.3.31}$$

which holds in both situations (4.3.1a), (4.3.1b). By Jensen's inequality (s^2 is a convex function) and (4.3.23) we get that

$$\int_{S^{N-1}}v^2(s,\theta)d\theta\ge C\overline{v}^2\ge C\Big(\int_{S^{N-1}}\frac{d\theta}{v^2(s,\theta)}\Big)^2-C\int_{S^{N-1}}v_s^2(s,\theta)d\theta.$$

Multiplying (4.3.31) by $\int_{S^{N-1}}\frac{d\theta}{v^2(s,\theta)}$, the previous estimate yields to

$$\Big(\int_{S^{N-1}}\frac{d\theta}{v^2(s,\theta)}\Big)^3\le C\Big(\lambda_n^{-\frac{2}{3}}+\int_{S^{N-1}}v_s^2(s,\theta)d\theta\Big)\Big(\int_{S^{N-1}}\frac{d\theta}{v^2(s,\theta)}\Big)+C.$$

Therefore, the validity of (4.3.27) follows:

$$\int_{S^{N-1}} \frac{d\theta}{v^2(s,\theta)} \le C\Big[\lambda_n^{-\frac{1}{3}} + \Big(\int_{S^{N-1}} v_s^2(s,\theta)d\theta\Big)^{\frac{1}{2}}\Big].$$

Step 3: Estimate (4.3.19) holds.

By (4.3.20) and Step 2, we have

$$\overline{v}_{ss} + \left(N - \frac{2}{3} + \frac{2\alpha}{3}\right)\overline{v}_s + \frac{2+\alpha}{3}\left(N - \frac{4}{3} + \frac{\alpha}{3}\right)\overline{v} \le C\Big(\lambda_n^{-\frac{1}{3}} + (\int_{S^{N-1}} v_s^2(s,\theta)d\theta)^{\frac{1}{2}}\Big),$$

which rewrites as:

$$\left[e^{\frac{2+\alpha}{3}s}\left(\overline{v}_s + (N - \frac{4}{3} + \frac{\alpha}{3})\overline{v}\right)\right]_s \le Ce^{\frac{2+\alpha}{3}s}\Big(\lambda_n^{-\frac{1}{3}} + (\int_{S^{N-1}} v_s^2(s,\theta)d\theta)^{\frac{1}{2}}\Big).$$

Hence, by (4.3.16) and (4.3.22) we obtain that:

$$\begin{aligned}
&\overline{v}_s(t) + (N - \frac{4}{3} + \frac{\alpha}{3})\overline{v}(t) \le C\Big(\lambda_n^{-\frac{1}{3}} + \int_0^t e^{\frac{2+\alpha}{3}(s-t)}ds(\int_{S^{N-1}} v_s^2(s,\theta)d\theta)^{\frac{1}{2}}\Big) \\
&\le C\Big(\lambda_n^{-\frac{1}{3}} + (\int_0^t e^{\frac{2+\alpha}{3}(s-t)} \int_{S^{N-1}} v_s^2(s,\theta)dsd\theta)^{\frac{1}{2}}\Big) \\
&\le C\lambda_n^{-\frac{1}{3}} \quad \text{for } t \in [0, T_n],
\end{aligned} \tag{4.3.32}$$

in view of Jensen's inequality applied to the concave function $\sqrt{s}$. Let s_n be so that $\overline{v}(s_n) = \max\limits_{s\in[0,T_n]} \overline{v}(s)$. If $s_n = 0, T_n$, by (4.3.16), (4.3.17) we get that

$$\overline{v}(s) \le C\lambda_n^{-\frac{1}{3}}, \quad \int_{S^{N-1}} \frac{d\theta}{v^2(s,\theta)} \le C\lambda_n^{-\frac{1}{3}} \quad \text{for } s \in (-\infty, T_n]$$

by means of (4.3.23). If $s_n \in (0, T_n)$, $\overline{v}_s(s_n) = 0$ and (4.3.32) implies that $\overline{v}(s_n) \le C\lambda_n^{-\frac{1}{3}}$. Finally, (4.3.23) yields to

$$\max_{s\in[0,T_n]} \int_{S^{N-1}} \frac{d\theta}{v^2(s,\theta)} \le \max_{s\in[0,T_n]} \int_{S^{N-1}} v(s,\theta)d\theta \le C\lambda_n^{-\frac{1}{3}}$$

also in this case. So, (4.3.19) holds and the proof is complete. ■

4.3.3 Infinite multiplicity of solutions

In this subsection, we establish Theorem 4.3.1(2) by borrowing some ideas of Dancer's paper [32]. Assume that either (4.3.1a) or (4.3.1b) holds.

Proof (of Theorem 4.3.1(2)): Set $E = \{u = u(|x|) \in C^1(\bar{B}) :\ u = 0 \text{ on } \partial B\}$ in case (4.3.1a) and $E = \{u \in C^1(\bar{\Omega}) :\ u = 0 \text{ on } \partial\Omega\}$ in case (4.3.1b). Let G be the resolvant of $-\Delta$ on E. By Theorem 2.4.3, the minimal solution u_λ is stable: $\mu_{1,\lambda}(u_\lambda) > 0$, which implies the invertibility on E of the linearized operator $\mathrm{Id} - \lambda A(u_\lambda)$ for any $0 \le \lambda < \lambda^*$, where $A(u)(\phi) = G\left(\frac{|x|^\alpha}{(1-u)^3}\phi\right)$. By the Implicit Function Theorem, the minimal branch (λ, u_λ) is a simple curve in $\mathcal{V}$, where

$$\mathcal{V} = \{(\lambda, u) \in [0, +\infty) \times E :\ u \text{ is a solution of } (P)_\lambda\}.$$

Buffoni, Dancer and Toland in [19] developed a fine bifurcation theory in the spirit of [30, 31]. We follow section 2.1 in [19] to find an analytic curve $(\hat{\lambda}(t), \hat{u}(t))$, $t \ge 0$, in $\mathcal{V}$ such that $(\hat{\lambda}(0), \hat{u}(0)) = (0, 0)$, $\|\hat{u}(t)\|_\infty \to 1$ as $t \to \infty$ and $\mathrm{Id} - \hat{\lambda}(t)A(\hat{u}(t))$ is invertible on E except at isolated points. By the Implicit Function Theorem, the curve $(\hat{\lambda}(t), \hat{u}(t))$ can only have isolated intersections. If we now use the usual trick of finding a minimal continuum in $\{(\hat{\lambda}(t), \hat{u}(t)) :\ t \ge 0\}$ joining $(0, 0)$ to "infinity", we obtain a continuous curve $(\lambda(t), u(t))$ in $\mathcal{V}$ with no self-intersections which is only piecewise analytic. Clearly, $\mathrm{Id} - \lambda(t)A(u(t))$ is still invertible on E except at isolated points. To obtain a minimal irreducible continuum, we can use the arguments of [32] as follows. Let $\mu_{k,\lambda(t)}(u(t))$ be the $k-$th eigenvalue counting multiplicity of the operator

$$L_{u(t),\lambda(t)} = -\Delta - \frac{2\lambda(t)|x|^\alpha}{(1-u(t))^3} \tag{4.3.33}$$

on E. According to the above comments, $\mu_{k,\lambda(t)}(u(t))$ is continuous, piecewise analytic and have only isolated zeroes. We will show below that $\mu_{k,\lambda(t)}(u(t)) < 0$ for large t. Namely, for any $M \in \mathbb{N}$ (4.3.33) has at least M negative eigenvalues for t large. Hence, there exists a sequence $t_n \to \infty$ as $n \to \infty$ such that the number of negative eigenvalues of (4.3.33) changes across t_n (recall that $\mu_{k,\lambda(0)}(u(0)) = \mu_k(-\Delta) > 0$ for any $k \in \mathbb{N}$). We claim that $(\lambda(t_n), u(t_n))$ is a bifurcation point. Otherwise, near $(\lambda(t_n), u(t_n))$ the solution set is a curve parameterized by λ and the critical groups- as defined in Chang [27]- of these solutions must be locally independent of λ by homotopy invariance. The formula for the critical groups at a non-degenerate point (see [27], page 33) implies that the Morse index has to be constant in a deleted neighborhood of $(\lambda(t_n), u(t_n))$, in contradiction with the choice of t_n. There is a minor technical point here: we need to work in the completion $\bar{E}$ of E in $\|\cdot\|_{H_0^1}$. Since $\|u(t_n)\|_\infty < \tau < 1$, we can smoothly truncate the nonlinearity $\frac{1}{(1-u)^2}$ to be $\frac{1}{(1-\tau)^2}$ for $\tau < u < 1$. For the truncation, the argument above makes sense on $\bar{E}$. Note that the truncation does not affect the solution set close to $(\lambda(t_n), u(t_n))$ in $\mathbb{R} \times \bar{E}$.
The bifurcation point $(\lambda(t_n), u(t_n))$ is either a turning point, i.e. the point where $(\lambda(t), u(t))$ changes direction (the branch locally "bends back") or a point of secondary bifurcation. In case (4.3.1a), the following Lemma

Lemma 4.3.8. *Assume (4.3.1a). For any $\kappa \in (0, 1)$, there is at most one $(\lambda, u) \in \mathcal{V}$ such that $u(0) = \kappa$.*

implies that $(\lambda(t_n), u(t_n))$ is not a secondary bifurcation point and then, is necessarily a turning point.

Since $\|u(t)\|_\infty \to 1$ as $t \to +\infty$, by Theorem 4.2.1 we get that the Morse index of $m(u(t), \lambda(t)) \to +\infty$ as $t \to +\infty$. In case (4.3.1b), our claim on $\mu_{k,\lambda(t)}(u(t)) < 0$ for large t easily follows. As far as (4.3.1a), we would like to show that, the radial Morse index $m_r(u(t), \lambda(t)) \to +\infty$ as $t \to +\infty$.
We argue by contradiction. Suppose there is a sequence $t_n \to +\infty$ so that $m_r(u_n, \lambda_n) \le C$, where $\lambda_n = \lambda(t_n)$ and $u_n = u(t_n)$. Up to a subsequence, assume that $\lambda_n \to \lambda \ge 0$ as $n \to +\infty$. Since $\|u_n\|_\infty \to 1$ as $n \to +\infty$, u_n is a non-minimal radial solutions of $(P)_{\lambda_n}$ and by Theorem 4.3.1(1), we have that $\lambda > 0$. Set $\epsilon_n = 1 - \|u_n\|_\infty \to 0$ as $n \to +\infty$ and introduce the rescaled function $U_n(y)$ according to (2.3.20):

$$U_n(y) = \frac{1 - u_n\big(\varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y\big)}{\varepsilon_n}\ ,\ \ y \in B_n := \{y :\ \epsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y \in B\}.$$

A limiting argument shows that, up to a subsequence, $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ as $n \to \infty$, where U is a radial solution of $(r^{N-1}\dot{U})' = \frac{r^{N-1+\alpha}}{U^2}$ on $\mathbb{R}^N$ such that $U(r) \ge U(0) = 1$. By Lemma 4.3.7, we have the validity of (4.3.14), (4.3.15) which rewrite in terms of U_n as

$$\frac{1}{C} r^{\frac{2+\alpha}{3}} \le U_n(r) \le C r^{\frac{2+\alpha}{3}}\ , \quad 1 \le r \le R_n,$$

for some $R_n \to +\infty$ as $n \to +\infty$. In the limit we get

$$\frac{1}{C} r^{\frac{2+\alpha}{3}} \le U(r) \le C r^{\frac{2+\alpha}{3}}\ , \quad r \ge 1, \tag{4.3.34}$$

and hence

$$\lim_{r \to \infty} r^{-\frac{2+\alpha}{3}} U(r) = \Big(\frac{2+\alpha}{3}(N - 2 + \frac{2+\alpha}{3})\Big)^{-\frac{1}{3}}. \tag{4.3.35}$$

The proof of (4.3.35) is a little variant of the proof of Theorem 1.1 of [60]. See also Theorem 1.2 of [66]. First, by (4.3.35) we see that

$$2r^\alpha U^{-3}(r) \sim \frac{2(2+\alpha)}{3}\Big(N - 2 + \frac{2+\alpha}{3}\Big) r^{-2} \ \text{ as } r \to \infty.$$

By explicitly solving the Euler equation

$$-\ddot{k} - \frac{N-1}{r}\dot{k} - \frac{\mu}{r^2} k = 0,$$

one finds that any non-trivial solution is in the form

$$k(r) = r^{-\frac{N-2}{2}} \Big[A_1 \cos\left(\sqrt{\mu - \frac{(N-2)^2}{4}} \ln y\right) + A_2 \sin\left(\sqrt{\mu - \frac{(N-2)^2}{4}} \ln y\right)\Big],\ A_1, A_2 \in \mathbb{R},$$

and has infinitely many (and unbounded) positive zeroes whenever $\mu > \frac{(N-2)^2}{4}$. A simple calculation implies that

$$\frac{2(2+\alpha)}{3}\Big(N-2+\frac{2+\alpha}{3}\Big) > \frac{(N-2)^2}{4}$$

provided either $2 \le N \le 7$ or $N \ge 8$, $\alpha > \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. We now have that the unique solution h of

$$-\ddot{h}(r) - \frac{N-1}{r}\dot{h}(r) = \frac{2r^\alpha}{U^3(r)}h(r),\ h(0)=1,\ \dot{h}(0)=0 \tag{4.3.36}$$

has infinitely many positive zeroes. Given $M > 0$, by continuous dependence we get that the unique solution q of

$$-\ddot{q}(r) - \frac{N-1}{r}\dot{q}(r) = \frac{2r^\alpha}{U^3(r)}q(r) + \sigma q(r),\ \ h(0)=1,\ \dot{h}(0)=0, \tag{4.3.37}$$

has at least M positive, large zeroes, for a sufficiently small negative number σ. Let $q_i(y)$ be the function defined to be $q(|y|)$ for $|y|$ between the $i-$th and $(i+1)-$th zero of q and to be zero otherwise, $i = 1,\dots,M-1$. Then, $q_i \in H^1(\mathbb{R}^N)$ are orthogonal both in $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$. Multiplying (4.3.37) by q_i and integrating between these zeroes it yields to

$$Q(q_i) = \int_{\mathbb{R}^N}\Big[|\nabla q_i|^2 - \frac{2|y|^\alpha}{U^3}q_i^2\Big]dy < 0.$$

Since q_i has compact support, it easily follows that

$$q_i \in H_0^1(B_n)\,,\ \int_{B_n}|\nabla q_i|^2(y)dy - \frac{2|y|^\alpha\epsilon_n^3}{(1-u_n)^3(\tau_n y)}q_i^2(y)dy < 0$$

for any $i = 1,\dots,M-1$, where $\tau_n = \epsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}$. Hence, returning to the original scaling, we see that, for any $i = 1,\dots,M-1$,

$$\int_B |\nabla\tilde{q}_i|^2 - \frac{2\lambda_n|x|^\alpha}{(1-u_n)^3}\tilde{q}_i^2 < 0$$

for n large, where $\tilde{q}_i(x) = q_i(\tau_n^{-1}x) \in H_0^1(B)$. Since $\tilde{q}_i$ are radial functions which are orthogonal both in $L^2(B)$ and $H_0^1(B)$, $i = 1,\dots,M-1$, by the variational characterization of the eigenvalues $m_r(u_n,\lambda_n) \ge M-1$ for n large. Since M is arbitrary, we get that $m_r(u_n,\lambda_n) \to +\infty$ as $n \to +\infty$. A contradiction. The proof of Theorem 4.3.1(2) is complete. ■

Finally, we give the proof of Lemma 4.3.8.

Proof (of Lemma 4.3.8): Let (λ_1,u_1), $(\lambda_2,u_2) \in \mathcal{V}$ such that $u_1(0)=u_2(0)=\kappa$. Recall that u_1, u_2 are radial solutions under the assumtpion (4.3.1a). The function

$$U_i(r) = \frac{1-u_i\left((1-\kappa)^{\frac{3}{2+\alpha}}\lambda_i^{-\frac{1}{2+\alpha}}r\right)}{1-\kappa}\,,\quad r \in I_i = \Big[0,(1-\kappa)^{-\frac{3}{2+\alpha}}\lambda_i^{\frac{1}{2+\alpha}}\Big],$$

for $i = 1, 2$, satisfies

$$\ddot{U}(r) + \frac{N-1}{r}\dot{U}(r) = r^{\alpha}U^{-2}(r), \quad U(0) = 1, \quad \dot{U}(0) = 0 \tag{4.3.38}$$

on the associated interval I_i. Standard ODE theory implies that (4.3.38) has a unique solution $U(r)$ and then, U_i coincides with U on I_i, $i = 1, 2$. On the other hand, since

$$U_1\left((1-\kappa)^{-\frac{3}{2+\alpha}}\lambda_1^{\frac{1}{2+\alpha}}\right) = U_2\left((1-\kappa)^{-\frac{3}{2+\alpha}}\lambda_2^{\frac{1}{2+\alpha}}\right) = \frac{1}{1-\kappa},$$

we get that

$$U\left((1-\kappa)^{-\frac{3}{2+\alpha}}\lambda_1^{\frac{1}{2+\alpha}}\right) = U\left((1-\kappa)^{-\frac{3}{2+\alpha}}\lambda_2^{\frac{1}{2+\alpha}}\right). \tag{4.3.39}$$

The solution U of (4.3.38) is easily seen to be increasing: $\dot{U}(r) > 0$ for $r > 0$. Then (4.3.39) implies that $\lambda_1 = \lambda_2$. Then, $U_1 = U_2 = U$ on $I_1 = I_2$ and then, $u_1 = u_2$. The proof of Lemma 4.3.8 is complete. ■

4.4 The one dimensional problem

In this section, we discuss the compactness of solutions for $(S)_\lambda$ in one dimensional case.

Theorem 4.4.1. *Let I be a bounded interval in $\mathbb{R}$ and $f \in C^1(\bar{I})$ be such that $f \geq C > 0$ in I. Let $(u_n)_n$ be a solution sequence for $(S)_{\lambda_n}$ on I, where $\lambda_n \to \lambda \in (0, \lambda^*]$. Assume for any $n \in \mathbb{N}$ and k large enough,*

$$\mu_{k,n} := \mu_{k,\lambda_n}(u_n) \geq 0. \tag{4.4.1}$$

Then, $\sup\limits_{n\in\mathbb{N}} \| u_n \|_\infty < 1$.

Proof: Let $I = (a, b)$ be a bounded interval in $\mathbb{R}$. Assume $f \in C^1(\bar{I})$ so that $f \geq C > 0$ in I. We study solutions u_n of $(S)_{\lambda_n}$ in the form

$$\begin{cases} -\ddot{u}_n = \dfrac{\lambda_n f(x)}{(1-u_n)^2} & \text{in } I\,, \\ 0 < u_n < 1 & \text{in } I\,, \\ u_n(a) = u_n(b) = 0. \end{cases} \tag{4.4.2}$$

Assume that u_n satisfy (4.4.1) and $\lambda_n \to \lambda \in (0, \lambda^*]$. Let $x_n \in I$ be a maximum point: $u_n(x_n) = \max\limits_I u_n$. If $(u_n)_n$ is not compact, then up to a subsequence, we may assume that $u_n(x_n) \to 1$ with $x_n \to x_0 \in \bar{I}$ as $n \to +\infty$. Away from x_0, u_n is uniformly far away from 1. Otherwise, by the maximum principle we would have $u_n \to 1$ on an interval of positive measure, and then $\mu_{k,\lambda_n}(u_n) < 0$ for any k and n large, a contradiction.

Assume, for example, that $a \leq x_0 < b$. By elliptic regularity theory, $\dot{u}_n(x)$ is uniformly bounded for x far away from x_0. Letting $\varepsilon > 0$, we multiply (4.4.2) by $\dot{u}_n$ and integrate on $(x_n, x_0+\varepsilon)$:

$$\begin{aligned}\dot{u}_n^2(x_n) - \dot{u}_n^2(x_0+\varepsilon) &= \int_{x_n}^{x_0+\varepsilon} \frac{2\lambda_n f(s)\dot{u}_n(s)}{(1-u_n(s))^2} ds \\ &= \frac{2\lambda_n f(x_0+\varepsilon)}{1-u_n(x_0+\varepsilon)} - \frac{2\lambda_n f(x_n)}{1-u_n(x_n)} - \int_{x_n}^{x_0+\varepsilon} \frac{2\lambda_n \dot{f}(s)}{1-u_n(s)} ds.\end{aligned}$$

Then, for n large:

$$\begin{aligned}\dot{u}_n^2(x_n) + \frac{C\lambda}{1-u_n(x_n)} &\leq \dot{u}_n^2(x_0+\varepsilon) + 2\lambda_n \frac{f(x_0+\varepsilon)}{1-u_n(x_0+\varepsilon)} - 2\lambda_n \int_{x_n}^{x_0+\varepsilon} \frac{\dot{f}(s)}{1-u_n(s)} ds \\ &\leq C_\varepsilon + 4\lambda \parallel \dot{f} \parallel_\infty \frac{x_0+\varepsilon-x_n}{1-u_n(x_n)}\end{aligned}$$

since $u_n(x_n)$ is the maximum value of u_n in I. Choosing $\varepsilon > 0$ sufficiently small, we get that for any n large: $\frac{1}{1-u_n(x_n)} \leq C_\varepsilon$, contradicting $u_n(x_n) \to 1$ as $n \to +\infty$. ■

Even in one dimensional case, we can still define the second turning point λ_2^* as in §3.3. We don't know whether $\lambda_2^* = 0$ (this is indeed the case when $f(x) = 1$, see [93]) or $\lambda_2^* > 0$. In the latter situation, there would exist a solution V^* for $(S)_{\lambda_2^*}$ which could be –in some cases– the second turning point. Let us remark that for $N = 1$, the multiplicity result of Theorem 3.3.1 holds also for any $\lambda \in (\lambda_2^*, \lambda^*)$.

4.5 Some comments

Main results of this Chapter are from [36, 67, 68]. The equivalence between compactness and finite Morse index of solutions for elliptic problems can be traced back to A. Bahri and P.-L. Lions' work [7, 8], where the authors dealt with superlinear elliptic equations with regular nonlinearities (see also [33, 40]). It is interesting to look insights into spectral and related properties of solutions for $(S)_\lambda$, such as whether the Morse index does not change within each branch, and whether the Morse index must be increasing one by one once the solution passes through each turning point along the bifurcation diagram of $(S)_\lambda$. Note that in this text such an analysis already exists for the first two branches on general domains and for any radial branch on the unit ball. Essentially, we have proved in Chapters 3 & 4 that, the Morse index of solutions for $(S)_\lambda$ along the first two branches is equal to 0 and 1, respectively.

One of the main results of this Chapter is Theorem 4.2.1, where we have introduced the equivalence between compactness, energy bounds and Morse index of solutions for $(S)_\lambda$. Theorem 4.2.3 has proved the existence of singular solutions for $(S)_\lambda$ with $2 \leq N \leq 7$. Note that a typical singular solution of $(S)_\lambda$ with $2 \leq N \leq 7$ is $u(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ provided that $f(x) = |x|^\alpha$, $\lambda = \frac{(2+\alpha)(3N+\alpha-4)}{9}$ and Ω is a unit ball in $\mathbb{R}^N$.

The uniqueness and infinite multiplicity of Theorem 4.3.1 hold for the following more general elliptic problem

$$-\Delta u = \frac{\lambda}{(1-u)^p} \ \text{ in } \Omega, \ \ 0 < u < 1 \ \text{ in } \Omega, \ \ u = 0 \ \text{ on } \partial\Omega, \qquad (S_{\lambda,p})$$

where $1 \le p \le 2$, and $\Omega \subset \mathbb{R}^2$ is a "rather" symmetric domain: balls, ellipses, rectangles, and etc. The critical observation for studying $(S_{\lambda,p})$ is that we need to use the Sobolev inequality (4.3.2), and we leave the details to interested readers. Note that $(S_{\lambda,1})$ arises in the study of singular minimal hypersurfaces with symmetry, see [81] and the references therein. However, $(S_{\lambda,p})$ for general $p > 0$ also arises from chemical catalyst kinetics, see [17, 34]. It is interesting to address whether such uniqueness and infinite multiplicity of $(S_{\lambda,p})$ hold for any $p \ge 1$.

It is also interesting to look insights into whether such uniqueness and infinite multiplicity of $(S_{\lambda,2})$ hold for any middle dimension $2 \le N \le 7$. For that, it seems quite necessary to discuss the higher dimensional version of the Sobolev inequality (4.3.2), and to classify the solutions of the following supercritical problem with $N \ge 2$:

$$\Delta U = \frac{1}{U^2} \quad \text{in } \mathbb{R}^2, \ \ U(y) \ge U(0) = 1.$$

Chapter 5

A limiting equation

In this section, we will focus on qualitative properties of solutions for the limiting equation

$$\begin{cases} \Delta U = \dfrac{|y|^\alpha}{U^2} & \text{in } \mathbb{R}^N, \\ U(y) \geq C > 0 & \text{in } \mathbb{R}^N. \end{cases}$$

In literature, an extensive analysis of such limiting problems with smooth polinomial and exponential nonlinearities, where $-U^{-2}$ is replaced by U^p, $p > 1$, and e^U respectively. Liouville-type and classification results are available. For $\alpha = 0$, Gidas and Spruck in [54] consider the subcritical case ($p > 1$ when $N = 2$ and $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$) and the celebrated papers [24], [53] (see also [29]) gives a full description in the critical case $p = \frac{N+2}{N-2}$, $N \geq 3$. For $N = 2$, a similar classification is available for the exponential nonlinearity e^U in [29] for $\alpha = 0$ and in [89] for $\alpha > 0$.
A study of stability properties can be also pursued. For $\alpha = 0$ and U^p, $p > 1$, we refer to [8] for changing-sign solutions with finite Morse index in the subcritical case and to [40] for possibly unbounded solutions semi-stable outside a compact set. For e^U, let us quote [37, 41] for stable solutions when $2 \leq N \leq 9$ and [33] for finite Morse index solutions when $N = 3$. Cabré and Capella in [21] deals with general smooth nonlinearities (convex and increasing) in the radial setting when $\alpha = 0$.

In §5.1 we will focus on stability properties. §5.2 will be devoted to a symmetry result for a solution U arising from a limiting procedure in case (4.3.1b).

5.1 Linear instability in low dimensions

The following Theorem characterizes the stability for solutions of our limiting problem:

Theorem 5.1.1. *Assume either* $1 \leq N \leq 7$ *or* $N \geq 8$ *and* $\alpha > \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. *Let* U *be* *a solution of*

$$\begin{cases} \Delta U = \dfrac{|y|^\alpha}{U^2} & \textit{in } \mathbb{R}^N, \\ U(y) \geq C > 0 & \textit{in } \mathbb{R}^N. \end{cases} \tag{5.1.1}$$

Then, there holds

$$\mu_1(U) = \inf \left\{ \int_{\mathbb{R}^N} \big(|\nabla \phi|^2 - \frac{2|y|^\alpha}{U^3} \phi^2 \big);\ \phi \in C_0^\infty(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \phi^2 = 1 \right\} < 0\,. \tag{5.1.2}$$

Moreover, if $N \geq 8$ and $0 \leq \alpha \leq \alpha_N$, then there exists at least one solution U of (5.1.1) satisfying $\mu_1(U) \geq 0$.

Let us quote the extension of Theorem 5.1.1 given by the first author in [37]: the nonlinearity $-U^{-2}$ can be replaced by e^U, U^p with $p > 1$ and $-U^p$ for $p < 0$ and corresponding sharp critical dimension & critical α_N are found.

Proof: By contradiction, assume that

$$\mu_1(U) = \inf \left\{ \int_{\mathbb{R}^N} \big(|\nabla \phi|^2 - \frac{2|y|^\alpha}{U^3} \phi^2 \big)\,;\ \phi \in C_0^\infty(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \phi^2\, dx = 1 \right\} \geq 0.$$

By the density of $C_0^\infty(\mathbb{R}^N)$ in $D^{1,2}(\mathbb{R}^N)$, we have

$$\int |\nabla \phi|^2 \geq 2 \int \frac{|y|^\alpha}{U^3} \phi^2\,, \quad \forall\, \phi \in D^{1,2}(\mathbb{R}^N)\,. \tag{5.1.3}$$

In particular, the test function $\phi = \frac{1}{(1+|y|^2)^{\frac{N-2}{4}+\frac{\delta}{2}}} \in D^{1,2}(\mathbb{R}^N)$ applied in (5.1.3) gives that

$$\int \frac{|y|^\alpha}{(1+|y|^2)^{\frac{N-2}{2}+\delta} U^3} \leq C \int \frac{1}{(1+|y|^2)^{\frac{N}{2}+\delta}} < +\infty$$

for any $\delta > 0$. Therefore, we have

$$\begin{aligned} \int \frac{1}{(1+|y|^2)^{\frac{N-2-\alpha}{2}+\delta} U^3} &= \int_{B_1} \frac{(1+|y|^2)^{\frac{\alpha}{2}}}{(1+|y|^2)^{\frac{N-2}{2}+\delta} U^3} + \int_{B_1^c} \frac{(1+|y|^2)^{\frac{\alpha}{2}}}{(1+|y|^2)^{\frac{N-2}{2}+\delta} U^3} \\ &\leq C \int_{B_1} \frac{1}{U^3} + C \int_{B_1^c} \frac{|y|^\alpha}{(1+|y|^2)^{\frac{N-2}{2}+\delta} U^3} \\ &\leq C + C \int \frac{|y|^\alpha}{(1+|y|^2)^{\frac{N-2}{2}+\delta} U^3}\,, \end{aligned} \tag{5.1.4}$$

which gives

$$\int \frac{1}{(1+|y|^2)^{\frac{N-2-\alpha}{2}+\delta} U^3} \leq C + C \int \frac{1}{(1+|y|^2)^{\frac{N}{2}+\delta}} < +\infty\,. \tag{5.1.5}$$

Step 1. We want to show that (5.1.3) allows us to perform the following Moser-type iteration scheme: for any $0 < q < 4 + 2\sqrt{6}$ and β there holds

$$\int \frac{1}{(1+|y|^2)^{\beta-1-\frac{\alpha}{2}} U^{q+3}} \leq C_q\big(1 + \int \frac{1}{(1+|y|^2)^{\beta} U^q}\big) \tag{5.1.6}$$

(provided the second integral is finite).

Indeed, let $R > 0$ and consider a smooth radial cut-off function η so that: $0 \leq \eta \leq 1$, $\eta = 1$ in $B_R(0)$, $\eta = 0$ in $\mathbb{R}^N \setminus B_{2R}(0)$. Multiplying (5.1.1) by $\frac{\eta^2}{(1+|y|^2)^{\beta-1} U^{q+1}}$, $q > 0$, and integrating by parts we get:

$$\begin{aligned}
&\int \frac{|y|^\alpha \eta^2}{(1+|y|^2)^{\beta-1} U^{q+3}} \\
&= \frac{4(q+1)}{q^2} \int \Big|\nabla\Big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}} U^{\frac{q}{2}}}\Big)\Big|^2 - \frac{4(q+1)}{q^2} \int \frac{1}{U^q}\Big|\nabla\Big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\Big)\Big|^2 \\
&\quad - \frac{q+2}{q^2} \int \nabla(\frac{1}{U^q}) \nabla\Big(\frac{\eta^2}{(1+|y|^2)^{\beta-1}}\Big) \\
&= \frac{4(q+1)}{q^2} \int \Big|\nabla\Big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}} U^{\frac{q}{2}}}\Big)\Big|^2 - \frac{2}{q} \int \frac{1}{U^q}\Big|\nabla\Big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\Big)\Big|^2 \\
&\quad + \frac{2(q+2)}{q^2} \int \frac{1}{U^q} \frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}} \Delta\Big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\Big),
\end{aligned}$$

where the relation $\Delta(\psi)^2 = 2|\nabla\psi|^2 + 2\psi\Delta\psi$ is used in the second equality. Then, by (5.1.3) we deduce that

$$\begin{aligned}
&(8q + 8 - q^2) \int \frac{|y|^\alpha \eta^2}{(1+|y|^2)^{\beta-1} U^{q+3}} \\
&\leq C_q' \int \frac{1}{U^q}\Big(\big|\nabla\big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\big)\big|^2 + \frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\big|\Delta\big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\big)\big|\Big).
\end{aligned}$$

Assuming $|\nabla\eta| \leq \frac{C}{R}$ and $|\Delta\eta| \leq \frac{C}{R^2}$, it is straightforward to see that:

$$\begin{aligned}
&\big|\nabla\big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\big)\big|^2 + \frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\big|\Delta\big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\big)\big| \\
&\leq C\Big(\frac{1}{(1+|y|^2)^\beta} + \frac{1}{R^2(1+|y|^2)^{\beta-1}} \chi_{B_{2R}(0)\setminus B_R(0)}\Big]
\end{aligned}$$

for some constant C independent of R. Then,

$$(8q + 8 - q^2) \int \frac{|y|^\alpha \eta^2}{(1+|y|^2)^{\beta-1} U^{q+3}} \leq C_q'' \int \frac{1}{(1+|y|^2)^\beta U^q}.$$

Let $q_+ = 4 + 2\sqrt{6}$. For any $0 < q < q_+$, we have $8q + 8 - q^2 > 0$ and therefore:

$$\int \frac{|y|^\alpha \eta^2}{(1+|y|^2)^{\beta-1} U^{q+3}} \leq C_q \int \frac{1}{(1+|y|^2)^\beta U^q},$$

where C_q does not depend on $R > 0$. Taking the limit as $R \to +\infty$, we get that:

$$\int \frac{|y|^\alpha}{(1+|y|^2)^{\beta-1} U^{q+3}} \le C_q \int \frac{1}{(1+|y|^2)^\beta U^q},$$

and then, the validity of (5.1.6) easily follows from the same argument of (5.1.4).

Step 2. Let either $1 \le N \le 7$ or $N \ge 8$ and $\alpha > \alpha_N$. We want to show that

$$\int \frac{1}{(1+|y|^2) U^q} < +\infty \tag{5.1.7}$$

for some $0 < q < q_+ = 4 + 2\sqrt{6}$.

Indeed, set $\beta_0 = \frac{N-2-\alpha}{2} + \delta$, $\delta > 0$, and $q_0 = 3$. By (5.1.5) we get that

$$\int \frac{1}{(1+|y|^2)^{\beta_0} U^{q_0}} < +\infty.$$

Let $\beta_i = \beta_0 - i(1 + \frac{\alpha}{2})$ and $q_i = q_0 + 3i$, $i \in \mathbb{N}$. Since $q_0 < q_1 < q_+ = 4 + 2\sqrt{6} < q_2$, we can iterate (5.1.6) exactly two times to get that:

$$\int \frac{1}{(1+|y|^2)^{\beta_2} U^{q_2}} < +\infty \tag{5.1.8}$$

where $\beta_2 = \frac{N-6-3\alpha}{2} + \delta$, $q_2 = 9$.
Let $0 < q < q_+ = 4 + 2\sqrt{6} < 9$. By (5.1.8) and Hölder inequality we get that:

$$\begin{aligned}
&\int \frac{1}{(1+|y|^2) U^q} \\
&= \int \frac{(1+|y|^2)^{\frac{q}{9}(\frac{6-N}{2} - \delta + \frac{3}{2}\alpha)}}{U^q} \cdot \frac{1}{(1+|y|^2)^{\frac{q}{9}(\frac{6-N}{2} - \delta + \frac{3}{2}\alpha) + 1}} \\
&\le \Big(\int \frac{1}{(1+|y|^2)^{\beta_2} U^{q_2}} \Big)^{\frac{q}{9}} \Big(\int \frac{1}{(1+|y|^2)^{\frac{q}{9-q}(\frac{6-N}{2} - \delta + \frac{3}{2}\alpha) + \frac{9}{9-q}}} \Big)^{\frac{9-q}{9}} < +\infty
\end{aligned}$$

provided $-\frac{2q}{9-q}\beta_2 + \frac{18}{9-q} > N$ or equivalently

$$q > \frac{9N - 18}{6 - 2\delta + 3\alpha}. \tag{5.1.9}$$

To ensure (5.1.9) for some $\delta > 0$ small and $q < q_+$ at the same time, it requires $\frac{3N-6}{2+\alpha} < q_+$ or equivalently

$$1 \le N \le 7 \quad \text{or} \quad N \ge 8\,, \ \alpha > \alpha_N = \frac{3N - 14 - 4\sqrt{6}}{4 + 2\sqrt{6}}.$$

Our assumptions then provide the existence of some $0 < q < q_+ = 4+2\sqrt{6}$ such that (5.1.7) holds.

Step 3. We are ready to obtain a contradiction. Let $0 < q < 4+2\sqrt{6}$ be such that (5.1.7) holds, and suppose η is the cut-off function of Step 1. Using equation (5.1.1) we compute:

$$\begin{aligned}
&\int |\nabla(\frac{\eta}{U^{\frac{q}{2}}})|^2 - \int \frac{2|y|^\alpha}{U^3}(\frac{\eta}{U^{\frac{q}{2}}}) \\
&= \frac{q^2}{4}\int \frac{\eta^2|\nabla U|^2}{U^{q+2}} + \int \frac{|\nabla\eta|^2}{U^q} + \frac{1}{2}\int \nabla(\eta^2)\nabla(\frac{1}{U^q}) - \int \frac{2|y|^\alpha\eta^2}{U^{q+3}} \\
&= -\frac{q^2}{4(q+1)}\int \nabla U\cdot\nabla(\frac{\eta^2}{U^{q+1}}) + \int \frac{|\nabla\eta|^2}{U^q} \\
&\quad + \tfrac{q+2}{4(q+1)}\int \nabla(\eta^2)\nabla(\frac{1}{U^q}) - \int \frac{2|y|^\alpha\eta^2}{U^{q+3}} \\
&= -\frac{8q+8-q^2}{4(q+1)}\int \frac{|y|^\alpha\eta^2}{U^{q+3}} + \int \frac{|\nabla\eta|^2}{U^q} - \frac{q+2}{4(q+1)}\int \frac{\Delta\eta^2}{U^q}\,.
\end{aligned}$$

Since $0 < q < 4+2\sqrt{6}$, we have $8q+8-q^2 > 0$ and

$$\begin{aligned}
&\int |\nabla(\frac{\eta}{U^{\frac{q}{2}}})|^2 - \int \frac{2|y|^\alpha}{U^3}(\frac{\eta}{U^{\frac{q}{2}}})^2 \\
&\le -\frac{8q+8-q^2}{4(q+1)}\int_{B_1(0)} \frac{|y|^\alpha\eta^2}{U^{q+3}} + O\Big(\frac{1}{R^2}\int_{B_{2R}(0)\setminus B_R(0)} \frac{1}{U^q}\Big) \\
&\le -\frac{8q+8-q^2}{4(q+1)}\int_{B_1(0)} \frac{|y|^\alpha\eta^2}{U^{q+3}} + O\Big(\int_{|y|\ge R} \frac{1}{(1+|y|^2)U^q}\Big).
\end{aligned}$$

Since (5.1.7) implies: $\displaystyle\lim_{R\to+\infty}\int_{|y|\ge R}\frac{1}{(1+|y|^2)U^q} = 0$, we get that for R large

$$\begin{aligned}
&\int |\nabla(\frac{\eta}{U^{q/2}})|^2 - \int \frac{2|y|^\alpha}{U^3}(\frac{\eta}{U^{q/2}})^2 \\
&\le -\frac{8q+8-q^2}{4(q+1)}\int_{B_1(0)} \frac{|y|^\alpha}{U^{q+3}} + O\Big(\int_{|y|\ge R} \frac{1}{(1+|y|^2)U^q}\Big) < 0.
\end{aligned}$$

A contradiction to (5.1.3). Hence, (5.1.2) holds and the proof of the first part of Theorem 5.1.1 is complete.

We now deal with the second part of Theorem 5.1.1. Consider a sequence $\lambda_n \to \lambda^*$ and corresponding minimal solution u_n of $(S)_{\lambda_n}$ on the unit ball B with $f(x) = |x|^\alpha$. For $N \ge 8$ and $0 \le \alpha \le \alpha_N$, Theorem 2.4.3(3) provides that the extremal solution $u^* = \lim\limits_{n\to+\infty} u_n$ is singular and then, $\|u_n\|_\infty \to +\infty$ as $n \to +\infty$. Theorem 2.4.3(1) ensures that

$$\int_B \big(|\nabla\phi_n|^2 - \frac{2\lambda_n|x|^\alpha}{(1-u_n)^3}\phi_n^2\big)dx \ge 0 \qquad \forall\phi_n \in C_0^\infty(B)\,.$$

Let $\varepsilon_n = 1 - \|u_n\|_\infty \to 0$ as $n \to +\infty$ and $U_n(y)$ be the rescaled function given in (2.3.20), defined on $B_n = B_{\varepsilon_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}}}(0) \to \mathbb{R}^N$ as $n \to +\infty$. Then, by Proposition 2.3.8 $U_n \to U$ in $C^1_{\rm loc}(\mathbb{R}^N)$ (up to a subsequence), where U solves (5.1.1). Define

$$\phi_n(x) = \left(\varepsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}\right)^{-\frac{N-2}{2}}\phi\left(\varepsilon_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}}x\right),$$

for any given $\phi \in C_0^\infty(\mathbb{R}^N)$. Then we have

$$\begin{aligned}\int \Big(|\nabla\phi|^2 - \frac{2|y|^\alpha}{U^3}\phi^2\Big)dy &= \lim_{n\to\infty}\int \Big(|\nabla\phi|^2 - \frac{2|y|^\alpha}{U_n^3}\phi^2\Big)dy \\ &= \lim_{n\to\infty}\int_B \Big(|\nabla\phi_n|^2 - \frac{2\lambda_n|x|^\alpha}{(1-u_n)^3}\phi_n^2\Big)dx \geq 0\,,\end{aligned}$$

since ϕ has compact support and $U_n \to U$ in $C^1_{\rm loc}(\mathbb{R}^N)$. Then, the function U is a semi-stable solution of (5.1.1) and the proof of Theorem 5.1.1 is complete. ■

5.2 A radial symmetry result in $\mathbb{R}^2$

We continue the analysis of power-law equations on symmetric domains of §4.3. We deal with problem $(P)_\lambda$ in case (4.3.1b):

$$-\Delta u = \frac{\lambda}{(1-u)^2} \text{ in } \Omega,\ 0 < u < 1 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \qquad (P)_\lambda$$

where the parameter $\lambda > 0$, and $\Omega \subset \mathbb{R}^2$ is a well-behaved domain. Let us recall that Ω denotes a well-behaved domain in $\mathbb{R}^2$ if $0 \in \Omega$, Ω is invariant under the 2 reflections in the coordinate planes and, for any $0 < t < s < \max\limits_\Omega x_i$, $(I - P_i)D_{i,s} \subseteq (I - P_i)D_{i,t}$, where P_i is the orthogonal projection onto Span $\{e_i\}$, $D_{i,s} = \{x \in \Omega :\ x_i = s\}$ and $\{e_1, e_2\}$ is the usual basis in $\mathbb{R}^2$.

Let now $\lambda_n \to \lambda \geq 0$ as $n \to +\infty$. Let u_n be an associated non-compact sequence of solutions: $\|u_n\|_\infty \to 1$ as $n \to +\infty$. Since the minimal branch of $(P)_\lambda$ is compact for $\lambda \to 0$, Theorem 4.3.1 provides that $\lambda > 0$. In the situation (4.3.1b), let us recall that, by the moving planes method of [53], u_n is even in x_i, $i = 1, 2$, and $\nabla u_n \cdot x \leq 0$ in Ω.
Set $\epsilon_n = 1 - \|u_n\|_\infty \to 0$ as $n \to +\infty$ and introduce the rescaled function $U_n(y)$ according to (2.3.20):

$$U_n(y) = \frac{1 - u_n\left(\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}y\right)}{\varepsilon_n}\,,\ y \in \Omega_n := \{y :\ \epsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}y \in \Omega\}. \qquad (5.2.1)$$

Then, U_n satisfies $\Delta U_n = \frac{1}{U_n^2}$ in Ω_n and by Proposition 2.3.8, $U_n \to U$ in $C^1_{\rm loc}(\mathbb{R}^2)$ as $n \to \infty$ (up to a subsequence). Clearly, $U(y)$ is an even function and satisfies

$$\Delta U = \frac{1}{U^2} \quad \text{in } \mathbb{R}^2\,, \quad U(y) \geq U(0) = 1. \qquad (5.2.2)$$

Based on the following classification result in [66]:

Theorem 5.2.1. *A solution U of (5.2.2) is radially symmetric if and only if*

$$\lim_{|y|\to+\infty}\Big(U(y)-(\frac{4}{9})^{-\frac{1}{3}}|y|^{\frac{2}{3}}\Big)=0. \tag{5.2.3}$$

we provide the following radial symmetry result in $\mathbb{R}^2$:

Theorem 5.2.2. *Let U be a solution of (5.2.2) obtained as the limit of U_n in (5.2.1) as $n\to+\infty$. Then, U is radially symmetric.*

We need first two preliminary results.

Lemma 5.2.3. *Let u_n be a solution of $(P)_{\lambda_n}$, where $\lambda_n\to\lambda>0$ as $n\to+\infty$. Assume the validity of (4.3.14) and (4.3.15) in the situation (4.3.1b). Then, there exists $C>0$ such that*

$$1-u_n(r\theta)\ge Cr^{\frac{2}{3}} \qquad \forall\,\epsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}\le r\le r_0,\ \theta\in S^1. \tag{5.2.4}$$

Proof: We adopt the same notations in the proof of Lemma 4.3.7. Let us have in mind that we are considering situation (4.3.1b) for which in particular $N=2$ and $\alpha=0$. Let us recall that (4.3.14) and (4.3.15) can be re-formulated in terms of v_n as (4.3.19). We want to prove the estimate

$$v_n(s,\theta)\ge C \qquad \forall\,0\le s\le T_n,\ \theta\in S^1, \tag{5.2.5}$$

which is simply equivalent to (5.2.4). We omit the subscript n of v_n.
Let (s_n,θ_n) be a point of minimum of $v(s,\theta)$ in $[0,T_n]\times S^1$. Set $v_{min}:=\min_{[0,T_n]\times S^1} v = v(s_n,\theta_n)$. If either $0\le s_n\le 1$ or $T_n-1\le s_n\le T_n$, by (4.3.16) and (4.3.17) we see that $v_{min}\ge C>0$. We consider now the case $1<s_n<T_n-1$. Set $m(s,\theta)=\frac{1}{v^3(s,\theta)}$. Let $\hat\nabla=(\partial_s,\partial_\theta)$, $\hat\Delta=\partial_{ss}+\partial_{\theta\theta}$ be the gradient, laplacian in $(s,\theta)-$coordinates, respectively. By (2.4.18), the function m satisfies the equation

$$\hat\Delta m=\frac{12}{v^5}|\hat\nabla v|^2+4\frac{v_s}{v^4}+\frac{4}{3}\frac{1}{v^3}-\frac{3}{v^6}\ge-\frac{4}{3}m_s-3m^2+\frac{4}{3}m.$$

Therefore, m satisfies

$$\hat\Delta m+\frac{4}{3}m_s+3m^2\ge 0 \quad \text{in } [s_n-1,s_n+1]\times S^1,$$

and

$$m_{max}:=\max_{[0,T_n]\times S^1} m=m(s_n,\theta_n).$$

Define $\hat m(s,\theta)=e^{\frac{2}{3}(s-s_n)}m(s,\theta)$. An easy calculation implies that $\hat m$ satisfies

$$\hat\Delta\hat m+3e^{-\frac{2}{3}(s-s_n)}\hat m^2\ge 0.$$

Given $C_0=3e^{\frac{2}{3}}$, we have that

$$\Delta\hat m+C_0\hat m^2\ge 0 \quad \text{in } [s_n-1,s_n+1]\times S^1, \tag{5.2.6}$$

where $[s_n-1,s_n+1]\subset[0,T_n]$. The function $\tilde m(s,\theta)=C_0\hat m(s,\theta)$ satisfies now

$$\Delta\tilde m+\tilde m^2\ge 0\quad\text{in }[s_n-1,s_n+1]\times S^1.$$

Given $\hat B_r=\{(s,\theta):\ (s-s_n)^2+(\theta-\theta_n)^2<r^2\}$ for $r<1$, Lemma 4.3.5 now implies that there exist C, $\eta_0>0$, independent of n, such that

$$\tilde m(s,\theta)\le\frac{C}{r^2}\int_{\hat B_r}\tilde m(t,\theta)dtd\theta\ \text{ for }(s,\theta)\in\hat B_{\frac r2}$$

provided $\int_{\hat B_r}\tilde m(t,\theta)dtd\theta\le\eta_0$. Given $D>0$ so that $\int_{S^1}v^{-2}(s,\theta)d\theta\le D$ in view of (4.3.19), we can choose $0<r=(2C_0D)^{-1}e^{-\frac23}\eta_0v_{min}$. Since $r\ge1$ implies $v_{min}\ge C>0$, let us assume $r<1$ and then, $\hat B_r\subset(s_n-1,s_n+1)\times S^1$. We obtain that

$$\begin{aligned}\int_{\hat B_r}\tilde m(t,\theta)dtd\theta=\int_{\hat B_r}\frac{C_0e^{\frac23(t-s_n)}}{v^3(t,\theta)}dtd\theta&\le\int_{s_n-r}^{s_n+r}\int_{S^1}\frac{C_0e^{\frac23(t-s_n)}}{v^3(t,\theta)}dtd\theta\\&\le v_{min}^{-1}\int_{s_n-r}^{s_n+r}C_0e^{\frac23(t-s_n)}dt\int_{S^1}\frac{d\theta}{v^2(t,\theta)}\\&\le 2C_0De^{\frac23}v_{min}^{-1}r=\eta_0,\end{aligned}$$

which concludes that

$$C_0v_{min}^{-3}=\tilde m(s_n,\theta_n)\le\frac{C}{r^2}\int_{\hat B_r}\tilde m(s,\theta)dsd\theta=Cv_{min}^{-2}.$$

This implies that $v_{min}\ge C>0$, and we are done. ∎

Lemma 5.2.4. *Let u_n be a solution of $(P)_{\lambda_n}$, where $\lambda_n\to\lambda>0$ as $n\to+\infty$. Assume the validity of (4.3.14) and (4.3.15) in the situation (4.3.1b). Then, there exists $C>0$ such that*

$$1-u_n(r\theta)\le Cr^{\frac23}\qquad\forall\,\epsilon_n^{\frac32}\lambda_n^{-\frac12}\le r\le r_0,\ \theta\in S^1\tag{5.2.7}$$

and

$$|\partial_ru_n(r\theta)|\le Cr^{-\frac13}\qquad\forall\,\epsilon_n^{\frac32}\lambda_n^{-\frac12}\le r\le r_0,\ \theta\in S^1.\tag{5.2.8}$$

Proof: We adopt the same notations in the proof of Lemma 4.3.7 and observe that $N=2$, $\alpha=0$ in the situation (4.3.1b). Let us recall that (4.3.14) and (4.3.15) can be re-formulated in terms of v_n as (4.3.19). Estimate (5.2.7) rewrites as

$$v_n(s,\theta)\le C\qquad\forall\,0\le s\le T_n,\ \theta\in S^1.\tag{5.2.9}$$

In view of (2.4.21): $(v_n)_s(s,\theta)+\frac23v_n(s,\theta)=|y|^{\frac13}(U_n)_r(y)$ for $|y|=e^s$, (5.2.8) is equivalent to

$$|(v_n)_s(s,\theta)|\le C\qquad\forall\,0\le s\le T_n,\ \theta\in S^1\tag{5.2.10}$$

provided $|v_n| \le C$ for $0 \le s \le T_n$. In view of (5.2.5) given by Lemma 5.2.3, estimates (5.2.7), (5.2.8) are equivalent to establish the validity of (5.2.9), (5.2.10). We omit the subscript n of v_n, $\overline{v}_n$ and w_n, where $\overline{v}_n(s) = \int_{S^1} v_n(s,\theta)d\theta$ and $w_n(s) = \int_{S^1} v_n^2(s,\theta)d\theta$.

We first claim that

$$\max_{[0,T_n]} w(s) \le C. \tag{5.2.11}$$

Indeed, from (4.3.21) we deduce:

$$\int_{S^1} v_\theta^2(s,\theta)d\theta - \frac{4}{9}\int_{S^1} v^2(s,\theta)d\theta = \frac{1}{2}\Big[w_{ss}(s) + \frac{4}{3}w_s(s)\Big] - \int_{S^1} v_s^2(s,\theta)d\theta - \int_{S^1} \frac{d\theta}{v(s,\theta)},$$

which yields to

$$\begin{aligned}&\int_{S^1} v_\theta^2(s,\theta)d\theta - \frac{4}{9}\int_{S^1}\Big(v(s,\theta) - \frac{1}{2\pi}\overline{v}(s)\Big)^2 d\theta = \frac{1}{2}\Big[w_{ss}(s) + \frac{4}{3}w_s(s)\Big]\\ &-\int_{S^1} v_s^2(s,\theta)d\theta - \int_{S^1}\frac{d\theta}{v(s,\theta)} + \frac{4}{9\pi}\int_{S^1}\Big(v(s,\theta) - \frac{1}{2\pi}\overline{v}(s)\Big)\overline{v}d\theta + \frac{2}{9\pi}\overline{v}^2(s).\end{aligned} \tag{5.2.12}$$

Multiplying (2.4.18) by v_s and integrating on $[0,s]\times S^1$, we obtain

$$\int_{S^1} v_\theta^2(s,\theta)d\theta = \int_{S^1} v_s^2(s,\theta)d\theta + \frac{4}{9}\int_{S^1} v^2(s,\theta)d\theta + 2\int_{S^1}\frac{d\theta}{v(s,\theta)} + f(s), \tag{5.2.13}$$

where

$$\begin{aligned}f(s) = &-2\int_{S^1}\frac{d\theta}{v(0,\theta)} + \int_{S^1} v_\theta^2(0,\theta)d\theta - \frac{4}{9}\int_{S^1} v^2(0,\theta)d\theta\\ &-\int_{S^1} v_s^2(0,\theta)d\theta + \frac{8}{3}\int_0^s\int_{S^1} v_s^2(t,\theta)dtd\theta.\end{aligned}$$

Thus, we deduce from (4.3.21) and (5.2.13) that

$$w_{ss} + \frac{4}{3}w_s = 4\int_{S^1} v_s^2(s,\theta)d\theta + 6\int_{S^1}\frac{d\theta}{v(s,\theta)} + 2f(s). \tag{5.2.14}$$

For any $\epsilon > 0$, by (5.2.12) and (5.2.14) we get that

$$\int_{S^1} v_\theta^2(s,\theta)d\theta - (\frac{4}{9}+\epsilon)\int_{S^1}\Big(v(s,\theta) - \frac{1}{2\pi}\overline{v}(s)\Big)^2 d\theta = \int_{S^1} v_s^2(s,\theta)d\theta + 2\int_{S^1}\frac{d\theta}{v(s,\theta)} + f(s) + C_\epsilon\overline{v}^2(s),$$

in view of $ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2$. Using the Poincaré inequality:

$$\int_{S^1}\Big(v(s,\theta) - \frac{1}{2\pi}\overline{v}(s)\Big)^2 d\theta \le \int_{S^1} v_\theta^2(s,\theta)d\theta,$$

for $\epsilon < \frac{5}{9}$ it leads to

$$\int_{S^1} v_\theta^2(s,\theta)d\theta \le C\left(1 + \int_{S^1} v_s^2(s,\theta)d\theta + \int_{S^1}\frac{d\theta}{v(s,\theta)} + \overline{v}^2(s)\right),$$

in view of $|f(s)| \leq C$ on $[0, T_n]$ by (4.3.16), (4.3.22). By the Sobolev embedding theorem and the Poincaré inequality, we have that

$$\begin{aligned} v^2(s,\theta) &\leq C\left(\int_{S^1} v_\theta^2(s,\theta)d\theta + \int_{S^1} v^2(s,\theta)d\theta\right) \\ &\leq C\Big(1 + \overline{v}^2(s) + \int_{S^1} \frac{d\theta}{v(s,\theta)} + \int_{S^1} v_s^2(s,\theta)d\theta\Big). \end{aligned} \tag{5.2.15}$$

Integrating (5.2.15) on S^1 and using (4.3.19) in Lemma 4.3.7, we get that

$$w(s) \leq C\Big(1 + \int_{S^1} v_s^2(s,\theta)d\theta\Big).$$

Since $|f(s)| \leq C$, by (4.3.19) and (5.2.14) we deduce that

$$w(s) \leq C\Big(1 + w_{ss}(s) + \frac{4}{3}w_s(s)\Big). \tag{5.2.16}$$

Let s_n be the maximum point of w on $[0, T_n]$: $w(s_n) = \max\limits_{[0,T_n]} w(s)$. If $s_n = 0$, T_n, by (4.3.16) and (4.3.17) we see that $w(s_n) \leq C$. If $s_n \in (0, T_n)$, we get that $w_s(s_n) = 0$ and $w_{ss}(s_n) \leq 0$ and then, (5.2.16) leads to $w(s_n) \leq C$ also in this case. Hence, the claim (5.2.11) is established.

We are now ready to prove (5.2.9). Given $\hat{\Delta} = \partial_{ss} + \partial_{\theta\theta}$, recall that

$$\hat{\Delta}v + \frac{4}{3}v_s + \Big(\frac{4}{9} - \frac{1}{v^3}\Big)v = 0$$

by means of (2.4.18) and the function v^{-3} is uniformly bounded in $[0, T_n] \times S^1$ by (5.2.5) in Lemma 5.2.3. Let now (t_n, θ_n) be the maximum point of v on $[0, T_n] \times S^1$: $v(t_n, \theta_n) = \max\limits_{[0,T_n]\times S^1} v(s,\theta)$. When either $0 \leq t_n \leq 1$ or $T_n - 1 \leq t_n \leq T_n$, (4.3.16) and (4.3.17) yield to $v(t_n, \theta_n) \leq C$. For $1 < t_n < T_n - 1$, we have that

$$\hat{B}_1(t_n, \theta_n) := \{(s,\theta) :\ (s - t_n)^2 + (\theta - \theta_n)^2 < 1\} \subset (0, T_n) \times S^1.$$

Theorem 8.17 of [59] implies that

$$v(s_n, \theta_n) \leq C\Big(\int_{\hat{B}_1(t_n,\theta_n)} v^2(s,\theta)dsd\theta + 1\Big) \leq C \tag{5.2.17}$$

in view of (5.2.11). This completes the proof of (5.2.9).

By similar arguments, we can argue to obtain

$$\max_{[0,T_n]\times S^1} |v_s(s,\theta)| \leq C.$$

Indeed, we note that v_s satisfies the equation

$$\hat{\Delta} v_s + \frac{4}{3} v_{ss} + \Big(\frac{4}{9} + \frac{2}{v^3}\Big) v_s = 0. \tag{5.2.18}$$

Assume first that $\max_{[0,T_n]\times S^1} |v_s(s,\theta)| = \max_{[0,T_n]\times S^1} v_s(s,\theta)$ is attained at the point $(\hat{t}_n, \hat{\theta}_n)$. If either $0 \le \hat{t}_n \le 1$ or $T_n - 1 \le \hat{t}_n \le T_n$, (4.3.16) and (4.3.17) provide that $v_s(\hat{t}_n, \hat{\theta}_n) \le C$. Arguing as for (5.2.7), Theorem 8.17 of [59] and (4.3.22) imply that

$$v_s(\hat{t}_n, \hat{\theta}_n) \le C.$$

When $\max_{[0,T_n]\times S^1} |v_s(s,\theta)| = - \max_{[0,T_n]\times S^1} v_s(s,\theta)$, we can use the same arguments for the solution $-v_s$ of (5.2.18) and obtain

$$\max_{[0,T_n]\times S^1} |v_s(s,\theta)| \le C$$

also in this case. The proof of Lemma 5.2.4 is complete. ■

We are now in position to establish Theorem 5.2.2.

Proof (of Theorem 5.2.2): In view of Theorem 5.2.1, we just need to verify (5.2.3) on $U(y)$. Following the same notations as in the proof of Lemma 4.3.7, for $0 < r_0 < \frac{1}{2}\text{dist}\,(0, \partial\Omega)$ set $R_n = e^{T_n} = \epsilon_n^{-\frac{3}{2}} \lambda_n^{\frac{1}{2}} r_0$ and let $v_n(s,\theta) = |y|^{-\frac{2}{3}} U_n(y)$, $|y| = e^s$, be the Emden-Fowler transformation of U_n. According to (2.4.18), the function v_n solves in $(-\infty, T_n) \times S^1$:

$$v_{ss} + \frac{4}{3} v_s + v_{\theta\theta} + \frac{4}{9} v = v^{-2}.$$

Lemma 4.3.7 implies that (4.3.14), (4.3.15) hold true in the case (4.3.1b). We are in position to apply Lemmata 5.2.3 and 5.2.4: by (5.2.5) and (5.2.9), (5.2.10) respectively we get that

$$\frac{1}{C} \le v_n(s,\theta) \le C\,, \quad |(v_n)_s(s,\theta)| \le C \quad \forall\, 0 \le s \le T_n,\ \theta \in S^1 \tag{5.2.19}$$

for n large. Since (5.2.19) implies that $U_n(y) \ge C|y|^{\frac{2}{3}}$ in $1 \le |y| \le R_n$, we get that U satisfies $U(y) \ge C|y|^{\frac{2}{3}}$ in $\mathbb{R}^2$ in view of $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R}^2)$ as $n \to \infty$. Theorem 1.3 of [66] now gives

$$\lim_{|y|\to\infty} |y|^{-\frac{2}{3}} U(y) = \Big(\frac{4}{9}\Big)^{-\frac{1}{3}}. \tag{5.2.20}$$

Setting $V(s,\theta) = |y|^{-\frac{2}{3}} U(y)$, $|y| = e^s$, clearly we have that $v_n \to V$ in $C^1_{loc}([0,\infty) \times S^1)$ as $n \to \infty$, where V satisfies the equation

$$V_{ss} + \frac{4}{3} V_s + V_{\theta\theta} + \frac{4}{9} V = \frac{1}{V^2} \quad \text{in } [0,\infty) \times S^1.$$

Property (5.2.3) is equivalent in terms of $V(s,\theta)$ to establish

$$\lim_{s\to\infty} e^{\frac{2}{3}s} \Big(V(s,\theta) - (\frac{4}{9})^{-\frac{1}{3}} \Big) = 0. \tag{5.2.21}$$

Since $U(y)$ is an even function, for any s $V(s,\cdot)$ is a π-periodic function. Setting $W(s,\theta) = V(s,\theta) - V_\infty$ and $V_\infty = (\frac{4}{9})^{-\frac{1}{3}}$, then $W(s,\theta)$ satisfies

$$W_{ss} + \frac{4}{3}W_s + W_{\theta\theta} + \frac{4}{3}W = \ell(W), \tag{5.2.22}$$

where $\ell(t) = (t+V_\infty)^{-2} - V_\infty^{-2} + \frac{8}{9}t$. Note that ℓ is real analytic at $t=0$ and satisfies $\ell(0) = \ell'(0) = 0$, $\ell''(0) = 6V_\infty^{-4}$. We now introduce the function $Z(s,\theta) = W(s,\theta) - \overline{W}(s)$, where $\overline{f}(s) = \frac{1}{2\pi}\int_{S^1} f(s,\theta)d\theta$. Then Z satisfies the equation

$$Z_{ss} + \frac{4}{3}Z_s + Z_{\theta\theta} + \frac{4}{3}Z = j(W), \tag{5.2.23}$$

where

$$j(W) = \ell(W) - \overline{\ell(W)} = \ell(W) - \ell(\overline{W}) + \overline{\ell(\overline{W}) - \ell(W)} = o(Z) \text{ for } s \to \infty$$

in view of $\lim_{s\to\infty} W(s,\theta) = \lim_{s\to\infty}\overline{W}(s) = 0$ uniformly in θ. In particular, it holds $\lim_{s\to\infty} Z(s,\theta) = 0$ uniformly in θ.

Define $Y(s) = \int_{S^1} Z^2(s,\theta)d\theta$. Multiplying (5.2.23) by $2Z$ and integrating on S^1, we see that

$$Y_{ss}(s) + \frac{4}{3}Y_s(s) + \frac{8}{3}Y(s) - 2\int_{S^1} Z_s^2(s,\theta)d\theta - 2\int_{S^1} Z_\theta^2(s,\theta)d\theta = o(Y(s)).$$

On the other hand, for each fixed s $Z(s,\cdot)$ is a π-periodic function with $\overline{Z}(s) = 0$. By the Poincaré inequality on $\pi-$periodic functions, we get that

$$4\int_{S^1} Z^2(s,\theta)d\theta \le \int_{S^1} Z_\theta^2(s,\theta)d\theta.$$

Hence, we get that $Y_{ss}(s) + \frac{4}{3}Y_s(s) - \frac{16}{3}Y(s) + o(1)Y(s) \ge 0$, which implies

$$Y_{ss}(s) + \frac{4}{3}Y_s(s) - 4Y(s) \ge 0.$$

We claim that there is $C_0 > 1$ such that

$$Y(s) \le C_0 e^{-\tau s} \text{ for } s \text{ large}, \tag{5.2.24}$$

where $\tau = \frac{2}{3}(1+\sqrt{10})$. Since $X(s) = C_0 e^{-\tau s}$ is a solution of $X_{ss}(s) + \frac{4}{3}X_s(s) - 4X(s) = 0$, then

$$\begin{aligned} &(X-Y)_{ss} + \frac{4}{3}(X-Y)_s - 4(X-Y) \le 0, \\ &X(0) - Y(0) > 0, \quad \lim_{s\to\infty}(X(s) - Y(s)) = 0 \end{aligned} \tag{5.2.25}$$

provided $C_0 > Y(0)$. If $\min_{(0,\infty)}(X(s) - Y(s)) = X(s_0) - Y(s_0) < 0$ for some $s_0 \in (0,\infty)$, then $(X-Y)_s(s_0) = 0$ and $(X-Y)_{ss}(s_0) \ge 0$, in contradiction with (5.2.25). Estimate (5.2.24) now easily follows.

Setting $\tilde{Y}(t) = \sqrt{Y(s)}$, $t = e^{-s}$, there exists $t_0 > 0$ such that

$$\tilde{Y}(t) \leq \tilde{C}_0 t^{\frac{1+\sqrt{10}}{3}} \quad \text{for } t \geq t_0.$$

Decompose $W(s,\theta)$ as $W(s,\theta) = \overline{W}(s) + e^{-\frac{1+\sqrt{10}}{3}s}\tilde{W}(s,\theta)$. Arguing as in the proof of Theorem 5.4 in [66], similar arguments now yield to $\overline{W}(s) = O(e^{-\frac{1+\sqrt{10}}{3}s})$ and, for any integer $n \geq 0$,

$$\tilde{W}(s,\theta) \to \Phi(\theta) \text{ as } s \to \infty$$

uniformly in $C^n(S^1)$, where Φ is in the first eigenspace of the operator $-\frac{d^2}{d\theta^2}$ acting on the $\pi-$periodic functions on S^1. Namely

$$\Phi_{\theta\theta} + 4\Phi = 0, \quad \int_{S^1} \Phi(\theta)d\theta = 0.$$

It now implies that

$$\lim_{s\to\infty} e^{\frac{2}{3}s} W(s,\theta) = 0,$$

since $\frac{1}{3}(1+\sqrt{10}) > \frac{2}{3}$. This completes the proof of Theorem 5.2.2. ∎

Part II

Semilinear Parabolic Problems with Singular Nonlinearities

Chapter 6

Dynamic Deflection

The second part of this text is devoted to the dynamic deflection of the elastic membrane satisfying (1.2.17). Throughout this Chapter and unless mentioned otherwise, for convenience we study dynamic solutions of (1.2.17) in the form

$$\frac{\partial u}{\partial t} - \Delta u = \frac{\lambda f(x)}{(1-u)^2} \qquad \text{for} \quad x \in \Omega\,, \tag{6.0.1a}$$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega; \quad u(x,0) = 0 \quad \text{for} \quad x \in \Omega\,, \tag{6.0.1b}$$

where nonnegative $f \in C^\alpha(\bar\Omega)$ for some $\alpha \in (0,1]$ describes the permittivity profile of the elastic membrane shown in Figure 1.1, while $\lambda > 0$ characterizes the applied voltage, see §1.3.2. In this Chapter we deal with issues of global convergence, finite and infinite time "touchdown", and touchdown profiles as well as pull-in distance. Recall that a point $x_0 \in \bar\Omega$ is said to be a *touchdown point* for a solution $u(x,t)$ of (1.1), if for some $T \in (0,+\infty]$, we have $\lim\limits_{t_n \to T} u(x_0,t_n) = 1$. T is then said to be a –finite or infinite– touchdown time. For each such solution, we define its corresponding –possibly infinite– "first touchdown time":

$$T_\lambda(\Omega,f,u) = \inf\Big\{t \in (0,+\infty];\ \sup_{x\in\Omega} u(x,t) = 1\Big\}.$$

In §6.1, we analyze the relationship between the applied voltage λ, the permittivity profile f, and the dynamic deflection u of the elastic membrane. More precisely, for λ^* defined as in Theorem 2.2.4, we show in §§6.1.1 & 6.1.3 that if $\lambda \le \lambda^*$, then the unique dynamic solution of (6.0.1) must globally converge to its unique minimal steady-state, while if $\lambda > \lambda^*$, then the unique dynamic solution of (6.0.1) must touchdown at finite time, see §6.1.2. The latter occurrence referred to sometimes as *quenching*. Note that in the case where the unique minimal steady-state of (6.0.1) at $\lambda = \lambda^*$ is non-regular – which can happen if $N \ge 8$ – Theorem 2.2.4 and §6.1.2 surprisingly show that the corresponding dynamic solution may touchdown at infinite time.

In §6.2 we first compute some global convergence or touchdown behavior of (6.2.1) for different applied voltage λ, and we then prove rigorously the following surprising fact exhibited by the numerical simulations: for $\lambda > \lambda^*$, finite-time touchdown cannot occur at the zero points of the permittivity profile f.

§6.3 is focussed on the analysis and estimate of finite touchdown time, which often translates into useful information concerning the speed of the operation for many MEMS devices such as RF switches or micro-valves. In §6.4 we discuss touchdown profiles by the method of asymptotic analysis, and our purpose is to provide some information on the refined touchdown rate discussed in next Chapter.

§6.5 is devoted to the pull-in distance of MEMS devices, referred to as the maximum stable deflection of the elastic membrane before touchdown occurs. We provide numerical results for pull-in distance with some explicit examples, from which one can observe that both larger pull-in distance and pull-in voltage can be achieved by properly tailoring the permittivity profile. Some other unproved phenomena are also observed there, which may be interesting to some readers.

6.1 Global convergence or touchdown

In this section, we analyze the relationship between the applied voltage λ, the permittivity profile f, and the dynamic deflection u of (6.0.1). The main result of this section can be stated in the following theorem.

Theorem 6.1.1. *Suppose $\lambda^* := \lambda^*(\Omega, f)$ is the pull-in voltage defined in Theorem 2.2.4, then the following hold:*

1. *If $\lambda \le \lambda^*$, then there exists a unique solution $u(x,t)$ for (6.0.1) which globally converges as $t \to +\infty$, monotonically and pointwise to its unique minimal steady-state.*
2. *If $\lambda > \lambda^*$, then the unique solution $u(x,t)$ of (6.0.1) must touchdown at a finite time.*

First, we note the following uniqueness result.

Lemma 6.1.2. *Suppose u_1 and u_2 are solutions of (6.0.1) on the interval $[0,T]$ such that $\|u_i\|_{L^\infty(\bar\Omega\times[0,T])} < 1$ for $i = 1, 2$, then $u_1 \equiv u_2$.*

Proof: Indeed, the difference $U = u_1 - u_2$ then satisfies

$$U_t - \Delta U = \alpha U \quad \text{in} \quad \Omega \tag{6.1.1}$$

with initial data $U(x,0) = 0$ and zero boundary condition. Here

$$\alpha(x,t) = \frac{\lambda(2 - u_1 - u_2)f(x)}{(1-u_1)^2(1-u_2)^2}\,.$$

The assumption on u_1, u_2 implies that $\alpha(x,t) \in L^\infty(\bar\Omega \times [0,T])$. We now fix $T_1 \in [0,T]$ and consider the solution ϕ of the problem

$$\begin{cases} \phi_t + \Delta\phi + \alpha\phi = 0 & x \in \Omega,\ 0 < t < T_1, \\ \phi(x,T_1) = \theta(x) \in C_0(\Omega), & \\ \phi(x,t) = 0 & x \in \partial\Omega, \end{cases} \tag{6.1.2}$$

The standard linear theory (cf. Theorem 8.1 of [80]) gives that the solution of (6.1.2) is unique and bounded. Now multiplying (6.1.1) by ϕ, and integrating it on $\Omega \times [0, T_1]$, together with (6.1.2), yield that

$$\int_\Omega U(x, T_1)\theta(x)dx = 0$$

for arbitrary T_1 and $\theta(x)$, which implies that $U \equiv 0$, and we are done. ■

6.1.1 Global convergence when $\lambda < \lambda^*$

Theorem 6.1.3. *Suppose $\lambda^* := \lambda^*(\Omega, f)$ is the pull-in voltage defined in Theorem 2.2.4, then for $\lambda < \lambda^*$ there exists a unique global solution $u(x,t)$ for (6.0.1) which monotonically converges as $t \to +\infty$ to the unique minimal solution $u_\lambda(x)$ of $(S)_\lambda$.*

Proof:: This is standard and follows from the maximum principle combined with the existence of regular minimal steady-state solutions at this range of λ. Indeed, fix $0 < \lambda < \lambda^*$, and use Theorem 2.4.3 to obtain the existence of a unique minimal solution $u_\lambda(x)$ of $(S)_\lambda$. It is clear that the pair $\widetilde{u} \equiv 0$ and $\widehat{u} = u_\lambda(x)$ are sub- and super-solutions of (6.0.1) for all $t > 0$. This implies that the unique global solution $u(x,t)$ of (6.0.1) satisfies $1 > u_\lambda(x) \geq u(x,t) \geq 0$ in $\Omega \times (0, \infty)$.

By differentiating in time and setting $v = u_t$, we get for any fixed $t_0 > 0$

$$v_t = \Delta v + \frac{2\lambda f(x)}{(1-u)^3} v \qquad x \in \Omega\,,\ 0 < t < t_0\,, \tag{6.1.3a}$$

$$v(x,t) = 0 \quad x \in \partial\Omega\,, \quad v(x,0) \geq 0 \quad x \in \Omega\,. \tag{6.1.3b}$$

Here $\frac{2\lambda f(x)}{(1-u)^3}$ is a locally bounded non-negative function, and by the strong maximum principle, we get that $u_t = v > 0$ for $(x,t) \in \Omega \times (0, t_0)$ or $u_t \equiv 0$. The second case is impossible because otherwise $u(x,t) \equiv u_\lambda(x)$ for any $t > 0$. It follows that $u_t > 0$ holds for all $(x,t) \in \Omega \times (0,\infty)$, and since $u(x,t)$ is bounded, this monotonicity property implies that the unique global solution $u(x,t)$ converges to some function $u_s(x)$ as $t \to \infty$. Hence, $1 > u_\lambda(x) \geq u_s(x) > 0$ in Ω.

Next we claim that the limit $u_s(x)$ is a solution of $(S)_\lambda$. Indeed, consider a solution u_1 of the linear stationary boundary problem

$$-\Delta u_1 = \frac{\lambda f(x)}{(1-u_s)^2} \qquad x \in \Omega\,, \quad u_1 = 0 \qquad x \in \partial\Omega\,. \tag{6.1.4}$$

Let $w(x,t) = u(x,t) - u_1(x)$, then w satisfies

$$w_t - \Delta w = \lambda f(x)\big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_s)^2}\big]\,, \quad (x,t) \in \Omega \times (0,T)\,; \tag{6.1.5a}$$

$$w(x,t) = 0 \quad x \in \partial\Omega \times (0,T)\,; \quad w(x,0) = -u_1(x) \qquad x \in \Omega\,. \tag{6.1.5b}$$

Since the right side of (6.1.5a) converges to zero in $L^2(\Omega)$ as $t \to \infty$, a standard eigenfunction expansion implies that the solution w of (6.1.5) also converges to zero in $L^2(\Omega)$ as $t \to \infty$. This shows that $u(x,t) \to u_1(x)$ in $L^2(\Omega)$ as $t \to \infty$. But since $u(x,t) \to u_s(x)$ pointwise in Ω as $t \to \infty$, we deduce that $u_1(x) \equiv u_s(x)$ in $L^2(\Omega)$, which implies that $u_s(x)$ is also a solution for $(S)_\lambda$. The minimal property of $u_\lambda(x)$ then yields that $u_\lambda(x) \equiv u_s(x)$ on Ω, which follows that for every $x \in \Omega$, we have $u(x,t) \uparrow u_\lambda(x)$ as $t \to \infty$. ■

6.1.2 Touchdown at finite time when $\lambda > \lambda^*$

Recall from Theorem 2.2.4 that there is no solution for $(S)_\lambda$ as soon as $\lambda > \lambda^*$. Since the solution $u(x,t)$ of (6.0.1) –whenever it exists– is strictly increasing in time t ((see preceding theorem)), then there must be $T \leq \infty$ such that $u(x,t)$ reaches 1 at some point of $\bar\Omega$ as $t \to T^-$. Otherwise, a proof similar to Theorem 6.1.3 would imply that $u(x,t)$ would converge to its steady-state which is then the unique minimal solution u_λ of $(S)_\lambda$, contrary to the hypothesis that $\lambda > \lambda^*$. Therefore for this case, it only remains to know whether the touchdown time is finite or infinite. This is exactly what we prove in the following.

Theorem 6.1.4. *Suppose $\lambda^* := \lambda^*(\Omega, f)$ is the pull-in voltage defined in Theorem 2.2.4, then for $\lambda > \lambda^*$, there exists a finite time $T_\lambda(\Omega, f)$ at which the unique solution $u(x,t)$ of* (6.0.1) *must touchdown. Moreover, if $\inf_{x\in\Omega} f(x) > 0$, then we have the bound*

$$T_\lambda(\Omega, f) \leq T_{0,\lambda} := \frac{8(\lambda+\lambda^*)^2}{3\inf_{x\in\Omega} f(x)(\lambda-\lambda^*)^2(\lambda+3\lambda^*)}\Big[1+\Big(\frac{\lambda+3\lambda^*}{2\lambda+2\lambda^*}\Big)^{1/2}\Big]. \qquad (6.1.6)$$

We start by transforming the problem from a touchdown situation (i.e. quenching) into a blow-up problem where a concavity method can be used. For that, we set $V = 1/(1-u)$ which reduces (6.0.1) to the following parabolic problem

$$V_t - \Delta V = -\frac{2|\nabla V|^2}{V} + \lambda f(x) V^4 \qquad \text{for} \quad x \in \Omega\,, \qquad (6.1.7\text{a})$$

$$V(x,t) = 1 \qquad \text{for} \quad x \in \partial\Omega\,, \qquad (6.1.7\text{b})$$

$$V(x,0) = 1 \qquad \text{for} \quad x \in \Omega\,. \qquad (6.1.7\text{c})$$

This transformation implies that when $\lambda > \lambda^*$, the solution of (6.1.7) must blow up (in finite or infinite time) and that there is no solution for the corresponding stationary equation:

$$\Delta V - \frac{2|\nabla V|^2}{V} + \lambda f(x) V^4 = 0\,, \quad x \in \Omega\,; \qquad V = 1\,, \quad x \in \partial\Omega\,. \qquad (6.1.8)$$

Therefore, proving finite touchdown time of u for (6.0.1) is equivalent to showing finite blow-up time of the solution V for (6.1.7).

In the case where $\inf_{x\in\Omega} f(x) = 0$, we will also need to consider the stationary problem on a subset $\Omega_\epsilon := \{x \in \Omega : f(x) > \epsilon\}$ of Ω, where $\epsilon > 0$ is small enough. We recall from §2.1.2 the following properties for the corresponding pull-in voltage $\lambda^*(\Omega_\epsilon, f)$:

$$\lambda^*(\Omega_\epsilon, f) \geq \lambda^* = \lambda^*(\Omega, f) \text{ and } \lim_{\epsilon\to 0} \lambda^*(\Omega_\epsilon, f) = \lambda^*.$$

For the proof, we shall first analyze the following auxiliary parabolic equation

$$v_t - \Delta v = -\frac{2|\nabla v|^2}{v} + \lambda a^2 t^2 f(x) v^4 \quad \text{for} \quad x \in \Omega\,, \tag{6.1.9a}$$

$$v = 1 \quad \text{for} \quad x \in \partial\Omega\,, \tag{6.1.9b}$$

$$v(x,0) = 1 \quad \text{for} \quad x \in \Omega\,, \tag{6.1.9c}$$

where $a > 0$ is a given constant.

Lemma 6.1.5. *Suppose v is a solution of* (6.1.9) *up to a finite time $\bar{T}$, then $\big(\frac{v_t}{v^4}\big)_t \geq 0$ for all $t < \bar{T}$.*

Proof: Dividing (6.1.9a) by v^4, we obtain

$$\frac{v_t}{v^4} = \frac{\Delta v}{v^4} - \frac{2|\nabla v|^2}{v^5} + \lambda a^2 t^2 f(x)\,.$$

Setting $w = v^{-3}$, then direct calculations show that

$$w_t - \Delta w + \frac{2|\nabla w|^2}{3w} + 3\lambda a^2 t^2 f(x) = 0\,. \tag{6.1.10}$$

Differentiate (6.1.10) twice with respect to t, we obtain

$$\begin{aligned}\Big(\frac{|\nabla w|^2}{w}\Big)_{tt} &= \Big(\frac{2\nabla w \nabla w_t}{w} - \frac{|\nabla w|^2 w_t}{w^2}\Big)_t \\ &= \frac{2\nabla w \nabla w_{tt}}{w} + \frac{2|\nabla w_t|^2}{w} - \frac{4\nabla w \nabla w_t w_t}{w^2} - \frac{|\nabla w|^2 w_{tt}}{w^2} + \frac{2|\nabla w|^2 w_t^2}{w^3}\,,\end{aligned}$$

which means that the function

$$z = w_{tt} = -3\Big(\frac{v_t}{v^4}\Big)_t \tag{6.1.11}$$

satisfies

$$\begin{aligned}L(z) :&= z_t - \Delta z + \frac{4\nabla w}{3w}\nabla z - \frac{2|\nabla w|^2}{3w^2} z \\ &= -6\lambda a^2 f(x) - \frac{2}{3}\Big[\frac{2|\nabla w_t|^2}{w} + \frac{2|\nabla w|^2 w_t^2}{w^3} - \frac{4\nabla w \nabla w_t w_t}{w^2}\Big] \\ &\leq -6\lambda a^2 f(x)\,,\end{aligned}$$

after an application of Cauchy-Schwarz inequality. Hence we have

$$L(z) \leq -6\lambda a^2 f(x) \leq 0\,. \tag{6.1.12}$$

Now from (6.1.9) and the definition of z, we have $z(x,0) = 0$ and $z = 0$ on $\partial\Omega$. Since the coefficients of L remain bounded as long as v is bounded, we conclude from the maximum principle that $z(x,t) \leq 0$ holds for all $t < \bar{T}$. This completes the proof of Lemma 6.1.5. ■

Proof of Theorem 6.1.4: Let $\lambda > \lambda^*$ and let $\epsilon > 0$ be small enough so that $\lambda > \lambda^*(\Omega_\epsilon, f) \geq \lambda^*$. Let $\lambda' = \lambda - \lambda^* > 0$, and set

$$a_\epsilon = \frac{3\epsilon\lambda'(4\lambda^* + \lambda')}{4(2\lambda^* + \lambda')}\Big[1 - \Big(\frac{4\lambda^* + \lambda'}{2(2\lambda^* + \lambda')}\Big)^{1/2}\Big], \tag{6.1.13a}$$

and

$$T^\epsilon_{0,\lambda} = \frac{1}{a_\epsilon} = \frac{8(\lambda + \lambda^*)^2}{3\epsilon(\lambda - \lambda^*)^2(\lambda + 3\lambda^*)}\Big[1 + \Big(\frac{\lambda + 3\lambda^*}{2\lambda + 2\lambda^*}\Big)^{1/2}\Big] < +\infty\,. \tag{6.1.13b}$$

Consider now a solution v of (6.1.9) corresponding to $\lambda = \lambda^* + \lambda'$ and a_ϵ as defined in (6.1.13a). We first establish the following

Claim: There exists $x_\epsilon \in \Omega$ with $f(x_\epsilon) > \epsilon$ such that $v(x_\epsilon, t) \to \infty$ as $t \nearrow T^\epsilon_{0,\lambda}$.
Indeed, let $t_\epsilon = \frac{1}{a_\epsilon}\Big[\frac{4\lambda^*+\lambda'}{2(2\lambda^*+\lambda')}\Big]^{1/2}$ in such a way that

$$t_\epsilon < T^\epsilon_{0,\lambda} \quad \text{and} \quad a_\epsilon^2 t_\epsilon^2(\lambda^* + \frac{\lambda'}{2}) = \lambda^* + \frac{\lambda'}{4}.$$

We claim that there exists $x_\epsilon \in \Omega_\epsilon$ such that

$$\Delta v(x_\epsilon, t_\epsilon) - \frac{2|\nabla v(x_\epsilon, t_\epsilon)|^2}{v(x_\epsilon, t_\epsilon)} + (\lambda^* + \frac{\lambda'}{4})f(x_\epsilon)|v(x_\epsilon, t_\epsilon)|^4 > 0\,. \tag{6.1.14}$$

Indeed, otherwise we get that for all $x \in \Omega_\epsilon$

$$\Delta v(x, t_\epsilon) - \frac{2|\nabla v(x, t_\epsilon)|^2}{v(x, t_\epsilon)} + (\lambda^* + \frac{\lambda'}{4})f(x_\epsilon)|v(x, t_\epsilon)|^4 \leq 0\,. \tag{6.1.15}$$

Since $v(x, t_\epsilon) \geq 1$ on Ω and hence on Ω_ϵ, this means that the function $\bar{v}(x) = v(x, t_\epsilon)$ is a supersolution for the equation

$$\Delta V - \frac{2|\nabla V|^2}{V} + \lambda f(x)V^4 = 0\,, \quad x \in \Omega_\epsilon\,; \qquad V = 1\,, \quad x \in \partial\Omega_\epsilon\,. \tag{6.1.16}$$

Since $\underline{v} \equiv 1$ is obviously a subsolution of (6.1.16), it follows that the latter has a solution which contradicts the fact that $\lambda = \lambda^* + \frac{\lambda'}{4} > \lambda^*(f, \Omega_\epsilon) \geq \lambda^*$. Hence assertion (6.1.14) is verified.

On the other hand, we do get from (6.1.9) that for $t = t_\epsilon$ and every $x \in \Omega$,

$$v_t = \Delta v - \frac{2|\nabla v|^2}{v} + (\lambda^* + \frac{\lambda'}{4})f(x)v^4 + \frac{\lambda'}{2}a_\epsilon^2 t_\epsilon^2 f(x)v^4\,. \tag{6.1.17}$$

We then deduce from (6.1.17) and (6.1.14) that at the point (x_ϵ, t_ϵ), we have

$$\frac{v_t}{v^4} \geq \frac{\lambda'}{2}a_\epsilon^2 t_\epsilon^2 f(x_\epsilon) > 0.$$

Applying Lemma 6.1.5, we then get for all (x_ϵ, t), $t_\epsilon \le t < T^\epsilon_{0,\lambda}$ that:

$$\frac{v_t}{v^4} \ge \frac{\lambda'}{2} a_\epsilon^2 t_\epsilon^2 f(x_\epsilon) > 0. \tag{6.1.18}$$

Integrating (6.1.18) with respect to t in $(t_\epsilon, T^\epsilon_{0,\lambda})$, we obtain since $f(x_\epsilon) \ge \epsilon$ that:

$$\frac{1}{3}\big(1 - v^{-3}(x_\epsilon, T^\epsilon_{0,\lambda})\big) \ge \frac{\lambda'}{2} a_\epsilon^2 t_\epsilon^2 f(x_\epsilon)(T^\epsilon_{0,\lambda} - t_\epsilon) \ge \frac{\lambda'}{2} a_\epsilon^2 t_\epsilon^2 \epsilon (T^\epsilon_{0,\lambda} - t_0) = \frac{1}{3}.$$

It follows that $v(x_\epsilon, t) \to \infty$ as $t \nearrow T^\epsilon_{0,\lambda}$, and the claim is proved.

To complete the proof of Theorem 6.1.4, we note that since $a_\epsilon^2 t^2 \le 1$ for all $t \le T^\epsilon_{0,\lambda}$, we obtain from (6.1.9) that

$$v_t \le \Delta v - \frac{2|\nabla v|^2}{v} + \lambda f(x) v^4\,, \quad (x,t) \in \Omega \times (0, T^\epsilon_{0,\lambda}).$$

Setting $w = V - v$, where V is the solution of (6.1.7), then w satisfies

$$w_t - \Delta w - \frac{2\nabla(V+v)}{V}\nabla w + \Big[\lambda(V^2+v^2)(V+v)f(x) + \frac{2|\nabla v|^2}{Vv}\Big] w \ge 0\,, \quad (x,t) \in \Omega \times (0, T^\epsilon_{0,\lambda}).$$

Here the coefficients of ∇w and w are bounded functions as long as V and v are both bounded. It is also clear that $w = 0$ on $\partial\Omega$ and $w(x,0) = 0$. Applying the maximum principle, we reduce that $w \ge 0$ and thus $V \ge v$. Consequently, V must also blow up at some finite time $T \le T^\epsilon_{0,\lambda}$, which means that u must touchdown at some finite time prior to $T^\epsilon_{0,\lambda}$.

Note that we have really proved that for any $\epsilon > 0$, there exists $\lambda^*_\epsilon \ge \lambda^*$ such that for any $\lambda > \lambda^*_\epsilon$, the solution of (6.0.1) touches down at a time prior to

$$T^\epsilon_{0,\lambda} = \frac{1}{3\max\{\epsilon, \inf_\Omega f\}} \frac{8(\lambda+\lambda^*)^2}{(\lambda-\lambda^*)^2(\lambda+3\lambda^*)}\Big[1 + \Big(\frac{\lambda+3\lambda^*}{2\lambda+2\lambda^*}\Big)^{1/2}\Big] < +\infty\,. \tag{6.1.19}$$

Moreover $\lambda^*_\epsilon \to \lambda^*$ as $\epsilon \to 0$. In the case where $\inf_{x\in\Omega} f(x) > 0$, formula (6.1.19) reduces to our second claim in Theorem 6.1.4. ■

6.1.3 Global convergence or touchdown in infinite time for $\lambda = \lambda^*$

In order to complete the proof of Theorem 6.1.1, the rest is to discuss the dynamic behavior of (6.0.1) at $\lambda = \lambda^*$. For this critical case, there exists a unique steady-state w^* of (6.0.1) obtained as a pointwise limit of the minimal solution u_λ as $\lambda \uparrow \lambda^*$. If w^* is regular (i.e, if it is a classical solution such as in the case when $N \le 7$) a similar proof as in the case where $\lambda < \lambda^*$, yields the existence of a unique solution $u^*(x,t)$ which globally converges to the unique steady-state w^* as $t \to \infty$. On the other hand, if w^* is a non-regular steady-state, *i.e.* if $\|w^*\|_\infty = 1$, the situation is complicated as we shall still prove global convergence to the extremal solution, which then amounts to a touchdown in infinite time.

Throughout this subsection, we shall consider the unique solution $0 \le u^* = u^*(x,t) < 1$ for the problem

$$u_t^* - \Delta u^* = \frac{\lambda^* f(x)}{(1-u^*)^2} \qquad \text{for} \quad (x,t) \in \Omega \times [0,t^*)\,, \tag{6.1.20a}$$

$$u^*(x,t) = 0 \qquad \text{for} \quad x \in \partial\Omega \times [0,t^*)\,, \tag{6.1.20b}$$

$$u^*(x,0) = 0 \qquad \text{for} \quad x \in \Omega\,, \tag{6.1.20c}$$

where t^* is the maximal time for existence. We shall use techniques developed in [13] to establish the following

Theorem 6.1.6. *If w^* is a non-regular minimal steady-state of* (6.1.20)*, then there exists a unique global solution u^* of* (6.1.20) *such that $u^*(x,t) \le w^*(x)$ for all $t < \infty$, while $u^*(x,t) \to w^*(x)$ as $t \to \infty$. In particular, $\lim_{t\to+\infty} \|u^*(x,t)\|_\infty = 1$.*

The proof of Theorem 6.1.6 needs to use the following lemma.

Lemma 6.1.7. *Consider the function $\delta(x) := \mathrm{dist}(x, \partial\Omega)$, then for any $0 < T < \infty$, there exists $\varepsilon_1 = \varepsilon_1(T)$ such that for $0 < \varepsilon \le \varepsilon_1$ the solution Z^ε of the problem*

$$\begin{aligned} Z_t - \Delta Z &= -\varepsilon f(x) && \text{in} \quad \Omega \times (0,\infty)\,, \\ Z(x,t) &= 0 && \text{on} \quad \partial\Omega \times (0,\infty)\,, \\ Z(x,0) &= \delta(x) && \text{in} \quad \Omega \end{aligned}$$

satisfies $Z^\varepsilon \ge 0$ on $[0,T] \times \bar{\Omega}$.

Proof: Let $\big(T(t)\big)_{t\ge0}$ be the heat semigroup with Dirichlet boundary condition, and consider the solution ξ_0 of

$$-\Delta \xi_0 = 1 \quad \text{in} \quad \Omega\,; \quad \xi_0 = 0 \quad \text{on} \quad \partial\Omega\,.$$

then we have

$$\xi_0 = T(t)\xi_0 + \int_0^t T(s) 1_\Omega ds$$

for all $t \ge 0$. Since $T(t)\xi_0 \ge 0$, it follows that

$$\int_0^t T(s) 1_\Omega ds \le \xi_0 \le C\delta \quad \text{for all} \quad t \ge 0\,. \tag{6.1.21}$$

On the other hand, we have

$$Z^\varepsilon(t) = T(t)\delta - \varepsilon f \int_0^t T(s) 1_\Omega ds\,,$$

and so we have $Z^\varepsilon(t) \geq T(t)\delta - \varepsilon C\delta$. Consider now c_0, $c_1 > 0$ such that $c_0\phi_1 \leq \delta \leq c_1\phi_1$, where ϕ_1 is the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$, associated to the eigenvalue μ_1. We have

$$T(t)\delta \geq c_0 T(t)\phi_1 = c_0 e^{-\mu_1 t}\phi_1 \geq \frac{c_0}{c_1} e^{-\mu_1 t}\delta\,.$$

Therefore, we have $Z^\varepsilon(t) \geq \big(\frac{c_0}{c_1}e^{-\mu_1 t} - C\varepsilon\big)\delta$. And hence it follows that $Z^\varepsilon(t) \geq 0$ on $[0,T]$ provided $\varepsilon \leq \frac{c_0}{c_1 C}e^{-\mu_1 T}$. ■

Proof of Theorem 6.1.6: We proceed in four steps.

Claim 1. We have that $u^*(x,t) \leq w^*(x)$ for all $(x,t) \in \Omega \times [0,t^*)$. Indeed, fix any $T < t^*$ and let ξ be the solution of the backward heat equation:

$$\begin{aligned} \xi_t - \Delta\xi = h(x,t) \qquad &\text{in} \quad \Omega\times(0,T)\,,\\ \xi|_{\partial\Omega} = 0\,, \qquad &\xi(T) = 0\,, \end{aligned}$$

where $h(x,t) \geq 0$ is in $\Omega \times (0,T)$. Multiplying (6.1.20) by ξ and integrating on $\Omega\times(0,T)$ we find that

$$\int_0^T\int_\Omega u^* h\,dxdt = \int_0^T\int_\Omega \frac{\lambda^*\xi f(x)}{(1-u^*)^2}\,dxdt\,.$$

On the other hand,

$$-\int_0^T\int_\Omega w^*\xi_t\,dxdt = \int_\Omega w^*\xi(0)dx \quad \text{and} \quad -\int_0^T\int_\Omega w^*\Delta\xi\,dxdt = \int_0^T\int_\Omega \frac{\lambda^*\xi f(x)}{(1-w^*)^2}\,dxdt.$$

Therefore, we have

$$\begin{aligned} \int_0^T\int_\Omega (u^*-w^*)h\,dxdt &\leq \int_\Omega w^*\xi(0)dx + \int_0^T\int_\Omega (u^*-w^*)h\,dxdt\\ &= \int_0^T\int_\Omega \Big(\frac{1}{(1-u^*)^2} - \frac{1}{(1-w^*)^2}\Big)\lambda^*\xi f(x)\,dxdt\\ &\leq C\int_0^T\int_{\{u^*\geq w^*\}} \Big(\frac{1}{(1-u^*)^2} - \frac{1}{(1-w^*)^2}\Big)\xi\,dxdt\\ &\leq C\int_0^T\int_\Omega (u^*-w^*)^+\xi\,dxdt, \end{aligned}$$

since $\|u^*\|_\infty < 1$ for $t\in[0,T)$. Therefore, we have

$$\int_0^T\int_\Omega (u^*-w^*)h\,dxdt \leq C\Big(\int_0^T\int_\Omega [(u^*-w^*)^+]^2\,dxdt\Big)^{1/2}\Big(\int_0^T\int_\Omega \xi^2\,dxdt\Big)^{1/2}.$$

On the other hand, $\xi(x,t) = \int_t^T T(s-t)h(x,s)ds$, where $T(t)$ is the heat semigroup with Dirichlet boundary condition, and hence

$$\|\xi(x,t)\|_{L^2}^2 \leq \Big(\int_t^T \|h(x,s)\|_{L^2}ds\Big)^2 \leq (T-t)\int_0^T\int_\Omega h^2\,dxdt\,.$$

Therefore,

$$\int_0^T \int_\Omega \xi^2 \, dxdt \le \frac{T^2}{2} \int_0^T \int_\Omega h^2 \, dxdt,$$

and so,

$$\int_0^T \int_\Omega (u^* - w^*) h \, dxdt \le \frac{CT}{\sqrt{2}} \Big(\int_0^T \int_\Omega [(u^* - w^*)^+]^2 \, dxdt \Big)^{1/2} \Big(\int_0^T \int_\Omega h^2 \, dxdt \Big)^{1/2}.$$

Letting h converge to $(u^* - w^*)^+$ in L^2, and since $u^* - w^* \in L^1(\Omega)$ we have

$$\int_0^T \int_\Omega [(u^* - w^*)^+]^2 \, dxdt \le \frac{CT}{\sqrt{2}} \int_0^T \int_\Omega [(u^* - w^*)^+]^2 \, dxdt \,,$$

which gives that $u^* \le w^*$ provided $C^2 T^2 < 2$, and our first claim follows.

Claim 2. There exist $0 < \tau_1 < t^*$, and C_0, $c_0 > 0$ such that for all $x \in \Omega$

$$u^*(x, \tau_1) \le \min\{C_0 \delta(x); \, w^*(x) - c_0 \delta(x)\} \,. \tag{6.1.22}$$

Fix $0 < \tau < t^*$ sufficiently small, and let v be the solution of

$$v_t - \Delta v = \frac{\lambda^* f(x)}{(1 - v)^2} \qquad \text{for} \quad (x, t) \in \Omega \times [0, \bar{T}) \,, \tag{6.1.23a}$$

$$v(x, t) = 0 \qquad \text{for} \quad x \in \partial\Omega \times [0, \bar{T}) \,, \tag{6.1.23b}$$

$$v(x, 0) = v_0 = u^*(x, \tau) \qquad \text{for} \quad x \in \Omega \,, \tag{6.1.23c}$$

where $[0, \bar{T})$ is the maximal interval of existence for v. Similarly to Claim 1, we can show that $0 \le v \le w^*$. Choose now $K > 1$ sufficiently large such that the path $z(x, t) := u^*(x, t) + \frac{1}{K} T(t) v_0$ satisfies $\|z(x, t)\|_\infty \le 1$ for $0 \le t < \bar{T}$. We then have

$$\begin{aligned} z_t - \Delta z &= \frac{\lambda^* f(x)}{(1 - u^*)^2} \le \frac{\lambda^* f(x)}{(1 - z)^2} && \text{in} \quad \Omega \times (0, \bar{T}) \,, \\ z(x, t) &= 0 && \text{on} \quad \partial\Omega \times (0, \bar{T}) \,, \\ z(x, 0) &= \frac{v_0(x)}{K} && \text{in} \quad \Omega \,, \end{aligned}$$

and the maximum principle gives that $z \le v$. Consider now a function $\gamma : [0, \infty) \to R$ such that $\gamma(t) > 0$ and

$$T(t) v_0 \ge K \gamma(t) \delta \text{ on } \Omega. \tag{6.1.24}$$

We then get

$$u^* \le v - \frac{1}{K} T(t) v_0 \le w^* - \frac{1}{K} T(t) v_0 \le w^* - \gamma(t) \delta \quad \text{for} \quad 0 \le t < \bar{T} \,. \tag{6.1.25}$$

On the other hand, for any $0 \le t \le T < t^*$, u^* is bounded by some constant $M < 1$ on $\bar{\Omega} \times [0, T]$ such that

$$u^* \le MT(t)1_\Omega + \frac{C}{(1-M)^2} \int_0^t T(s)1_\Omega ds \,.$$

Consider now a function $C : [0, \infty) \to R$ such that $T(t)1_\Omega \le C(t)\delta$ for $t \ge 0$, which means that

$$u^* \le MC(t)\delta + C(M)C\delta$$

for any $0 \le t \le T$, where (6.1.21) is applied. This combined with (6.1.25) conclude the proof of Claim (6.1.22).

Claim 3. For $0 < \varepsilon < 1$ there exists w_ε satisfying $\|w_\varepsilon\|_\infty < 1$ and

$$\int_\Omega \nabla w_\varepsilon \nabla \varphi \ge \int_\Omega \big(\frac{1}{(1-w_\varepsilon)^2} - \varepsilon\big)\lambda^* \varphi f(x) \tag{6.1.26}$$

for all $\varphi \in H_0^1(\Omega)$ with $\varphi \ge 0$ on Ω. Moreover, there exists $0 < \varepsilon_1 \le 1$ such that for $0 < \varepsilon < \varepsilon_1$, we also have

$$0 \le w_\varepsilon(x) - \frac{c_0}{2}\delta(x) \quad \text{for} \quad x \in \Omega \tag{6.1.27}$$

c_0 being as in (6.1.22).

To prove (6.1.26), we set

$$g(w^*) = \frac{1}{(1-w^*)^2}\,, \quad h(w^*) = \int_0^{w^*} \frac{ds}{g(s)}\,, \quad 0 \le w^* < 1\,. \tag{6.1.28}$$

For any $\varepsilon \in (0, 1)$ we also set

$$\widetilde{g}(w^*) = \frac{1}{(1-w^*)^2} - \varepsilon\,, \quad \widetilde{h}(w^*) = \int_0^{w^*} \frac{ds}{\widetilde{g}(s)}\,, \quad 0 \le w^* < 1\,, \tag{6.1.29}$$

and $\phi_\varepsilon(w^*) := \widetilde{h}^{-1}\big(h(w^*)\big)$. It is easy to check that $\phi_\varepsilon(0) = 0$ and $0 \le \phi_\varepsilon(s) < s$ for $s \ge 0$, and ϕ_ε is increasing and concave with

$$\phi'_\varepsilon(s) = \frac{g(\phi_\varepsilon(s)) - \varepsilon}{g(s)} > 0\,.$$

Setting $w_\varepsilon = \phi_\varepsilon(w^*)$, we have for any $\varphi \in H_0^1(\Omega)$ with $\varphi \ge 0$ on Ω,

$$\begin{aligned} \int_\Omega \nabla w_\varepsilon \nabla \varphi = \int_\Omega \phi'_\varepsilon(w^*) \nabla w^* \nabla \varphi &= \int_\Omega \nabla w^* \nabla \left(\phi'_\varepsilon(w^*)\varphi\right) - \int_\Omega \phi''_\varepsilon(w^*)\varphi |\nabla w^*|^2 \\ \ge \int_\Omega \frac{\lambda^* f(x)}{(1-w^*)^2}\phi'_\varepsilon(w^*)\varphi &= \int_\Omega \Big(\frac{1}{(1-w_\varepsilon)^2} - \varepsilon\Big)\lambda^* \varphi f(x)\,, \end{aligned}$$

which gives (6.1.26) for any $\varepsilon \in (0, \varepsilon_0)$.

In order to prove (6.1.27), we set

$$\eta(x) = \min\{w^*(x), (C_0 + c_0)\delta(x)\} \quad \text{and} \quad \eta_\varepsilon = \phi_\varepsilon \circ \eta\,,$$

where $\phi_\varepsilon(\cdot)$ is defined above, and C_0 and c_0 are as in (6.1.22). Since $\eta \le w^*$ and ϕ_ε is increasing, we have $\eta_\varepsilon \le \phi_\varepsilon(w^*) = w_\varepsilon$. Applying (6.1.22) we get that

$$0 \le \eta(x) - c_0\delta(x) \quad \text{on} \quad \Omega\,. \tag{6.1.30}$$

We also note that $\eta_\varepsilon = \phi_\varepsilon(\eta) \le \eta \le M$ with $M = (C_0 + c_0)\delta(x)$, and $\phi'_\varepsilon(s) \to 1$ as $\varepsilon \to 0$ uniformly in $[0, 1]$. Therefore, for some $\theta \in (0, 1)$ we have

$$\begin{aligned}\eta - \eta_\varepsilon = \eta - \big(\phi_\varepsilon(\eta) - \phi_\varepsilon(0)\big) = \eta\big(1 - \phi'_\varepsilon(\theta\eta)\big) &\le \eta \sup_{\{0\le s\le 1\}} (1 - \phi'_\varepsilon(s))\\ \le (C_0 + c_0)\delta \sup_{\{0\le s\le 1\}} (1 - \phi'_\varepsilon(s)) &\le \frac{c_0}{2}\delta\end{aligned}$$

provided ε small enough, which gives

$$\eta \le \eta_\varepsilon + \frac{c_0}{2}\delta\,. \tag{6.1.31}$$

We now conclude from (6.1.30) and (6.1.31) that

$$0 \le \eta - c_0\delta \le \eta_\varepsilon - \frac{c_0}{2}\delta \le w_\varepsilon - \frac{c_0}{2}\delta$$

for small $\varepsilon > 0$, and (6.1.27) is therefore proved.

To complete the proof of Theorem 6.1.6, we assume that $t^* < \infty$ and we shall work towards a contradiction. In view of Claim 3), we let $\varepsilon > 0$ be small enough so that $0 \le w_\varepsilon - \frac{c_0}{2}\delta$. Use Lemma 6.1.7 and choose $K > 2$ large enough such that the solution Z of the problem

$$\begin{aligned} Z_t - \Delta Z &= -\varepsilon\lambda^* f(x) && \text{in} \quad \Omega \times (0, t^*)\,,\\ Z(x,t) &= 0 && \text{on} \quad \partial\Omega \times (0, t^*)\,,\\ Z(x,0) &= \frac{c_0}{K}\delta && \text{in} \quad \Omega \end{aligned}$$

satisfies $0 \le Z < 1 - u^*$ on $\bar\Omega \times (0, t^*)$. Let v be the solution of

$$\begin{aligned} v_t - \Delta v &= \big(\frac{1}{(1-|v|)^2} - \varepsilon\big)\lambda^* f(x) && \text{in} \quad \Omega \times (0, s^*)\,,\\ v(x,t) &= 0 && \text{on} \quad \partial\Omega \times (0, s^*)\,,\\ v(x,0) &= w_\varepsilon && \text{in} \quad \Omega\,, \end{aligned}$$

where $[0, s^*)$ is the maximal interval of existence for v. Setting $z(x,t) = Z(x,t) + u^*(x,t)$ for $0 \le t < t^*$, we then have $0 \le u^* \le z < 1$ and

$$\begin{aligned} z_t - \Delta z = \Big(\frac{1}{(1-u^*)^2} - \varepsilon\Big)\lambda^* f(x) &\le \Big(\frac{1}{(1-z)^2} - \varepsilon\Big)\lambda^* f(x) && \text{in} \quad \Omega \times (0, t^*)\,,\\ z(x,t) &= 0 && \text{on} \quad \partial\Omega \times (0, t^*)\,,\\ z(x,0) &= \frac{c_0}{K}\delta(x) \le w_\varepsilon(x) && \text{in} \quad \Omega\,. \end{aligned}$$

Now the maximum principle gives that $z \le v$ on $\Omega \times (0, \min\{s^*, t^*\})$, and in particular we have $0 \le v$ on $\Omega \times (0, \min\{s^*, t^*\})$. Furthermore, the maximum principle and (6.1.26) also yield that $v \le w_\varepsilon$. Since $\|w_\varepsilon\|_\infty < 1$ we necessarily have $t^* < s^* = \infty$. Therefore, $u^* \le z \le v \le w_\varepsilon$ on $[0, t^*)$, which implies that $\|u^*\|_\infty < 1$ at $t = t^*$, which contradicts to our initial assumption that u^* is not a regular solution. ■

6.2 Location of touchdown points

In this section, we first present a couple of numerical simulations for different domains, different permittivity profiles, and various values of λ, by applying an implicit Crank-Nicholson scheme (see [64] for details), on the problem

$$\frac{\partial u}{\partial t} - \Delta u = -\frac{\lambda f(x)}{(1+u)^2} \quad \text{for} \quad x \in \Omega\,, \tag{6.2.1a}$$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega\,; \quad u(x,0) = 0 \quad \text{for} \quad x \in \Omega\,, \tag{6.2.1b}$$

in the following two choices for the domain Ω

$$\Omega : [-1/2, 1/2] \quad (\text{slab})\,; \qquad \Omega : x^2 + y^2 \le 1 \quad (\text{unit disk})\,. \tag{6.2.2}$$

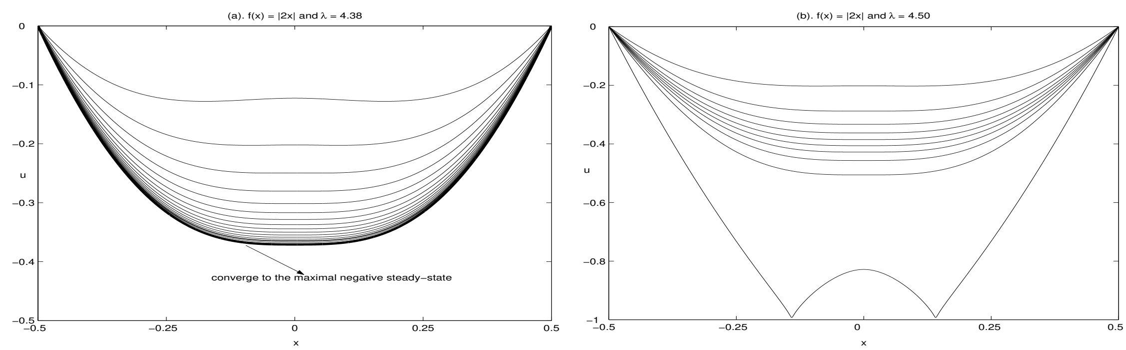

Figure 6.1: *Left Figure: u versus x for $\lambda = 4.38$. Right Figure: u versus x for $\lambda = 4.50$. Here we consider* (6.2.1) *with $f(x) = |2x|$ in the slab domain.*

Simulation 1: We consider $f(x) = |2x|$ for a permittivity profile in the slab domain $-1/2 \le x \le 1/2$. Here the number of the meshpoints is chosen as $N = 2000$ for the plots u versus x at different times. Figure 6.1(a) shows, for $\lambda = 4.38$, a typical sequence of solutions u for (6.2.1) approaching to the maximal negative steady-state. In Figure 6.1(b) we take $\lambda = 4.50$ and plot u versus x at different times $t = 0$, 0.1880, 0.3760, 0.5639, 0.7519, 0.9399, 1.1279, 1.3159, 1.5039, 1.6918, 1.879818, and a touchdown behavior is observed at

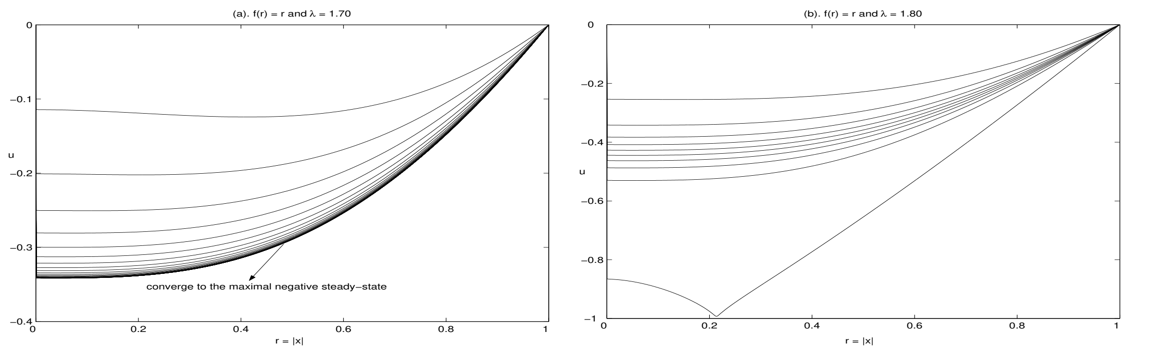

Figure 6.2: *Left Figure:* u *versus* r *for* $\lambda = 1.70$. *Right Figure:* u *versus* r *for* $\lambda = 1.80$. *Here we consider* (6.2.1) *with* $f(r) = r$ *in the unit disk domain.*

two different nonzero points $x = \pm 0.14132$. These numerical results and Theorem 6.1.1 point to a pull-in voltage $4.38 \leq \lambda^* < 4.50$.

Simulation 2: Here we consider $f(r) = r$ for a permittivity profile in the unit disk domain. The number of meshpoints is again chosen to be $N = 2000$ for the plots u versus r at different times. Figure 6.2(a) shows how for $\lambda = 1.70$, a typical sequence of solutions u for (6.2.1) approach to the maximal negative steady-state. In Figure 6.2(b) we take $\lambda = 1.80$ and plot u versus r at different times $t = 0$, 0.4475, 0.8950, 1.3426, 1.7901, 2.2376, 2.6851, 3.1326, 3.5802, 4.0277, 4.4751942, and a touchdown behavior is observed at the nonzero points $r = 0.21361$. Again these numerical results point to a pull-in voltage $1.70 \leq \lambda^* < 1.80$.

One can observe from above that touchdown points at finite time are not the zero points of the varying permittivity profile f, a fact firstly observed and conjectured in [64]. The main purpose of this section is to analyze this phenomena. Here is the main result without any additional assumption on the domain.

Theorem 6.2.1. *Suppose* $u(x,t)$ *is a touchdown solution of* (6.0.1) *at a finite time* T, *then we have*

1. *The permittivity profile* f *cannot vanish on an isolated set of touchdown points in* Ω.
2. *On the other hand, zeroes of the permittivity profile can be locations of touchdown in infinite time.*

Note that Theorem 6.2.1 holds for any bounded domain. The proof of Theorem 6.2.1 is based on the following Harnack-type estimate.

Lemma 6.2.2. *For any compact subset* K *of* Ω *and any* $m > 0$, *there exists a constant* $C = C(K,m) > 0$ *such that* $\sup_{x\in K}|u(x)| \leq C < 1$ *whenever* u *satisfies*

$$\Delta u \geq \frac{m}{(1-u)^2} \quad x \in \Omega\,; \quad 0 \leq u < 1 \quad x \in \Omega\,. \tag{6.2.3}$$

Proof: Setting $v = 1/(1-u)$, then (6.2.3) gives that v satisfies

$$\frac{\Delta v}{v^2} - \frac{2|\nabla v|^2}{v^3} \geq m v^2 \quad \text{in } \Omega\,,$$

which means that v is a subsolution of the "linear" equation $\Delta v = 0$ in Ω. In order to apply the Harnack inequality on v, we need to show that for balls $B_r \subset \Omega$, we have that $v \in L^3(B_r)$ with an L^3-norm that only depends on m and the radius r.

Without loss of generality, we may assume $0 \in K \subset \overline{\Omega}$. Let $B_r = B_r(0) \subset K$ be the ball centered at $x = 0$ and radius r. For $0 < r_1 < r_2 \leq 4r_1$, let $\eta(x) \in C_0^\infty(B_{r_2})$ be such that $\eta \equiv 1$ in B_{r_1}, $0 \leq \eta \leq 1$ in $B_{r_2} \setminus B_{r_1}$ and $|\nabla \eta| \leq 2/(r_2 - r_1)$. Multiplying (6.2.3) by $\phi^2/(1-u)$, where $\phi = \eta^\alpha$ and $\alpha \geq 1$ is to be determined later, and integrating by parts we have

$$\int_{B_{r_2}} \frac{m\phi^2}{(1-u)^3} \leq \int_{B_{r_2}} \frac{\phi^2 \Delta u}{1-u} = -\int_{B_{r_2}} \frac{\phi^2|\nabla u|^2}{(1-u)^2} - \int_{B_{r_2}} \frac{2\phi\nabla\phi\cdot\nabla u}{1-u}\,. \tag{6.2.4}$$

From the fact,

$$\int_{B_{r_2}} \frac{2\phi\nabla\phi\cdot\nabla u}{1-u} \leq \int_{B_{r_2}} \phi^2|\nabla u|^2 + 4\int_{B_{r_2}} \frac{|\nabla\phi|^2}{(1-u)^2} \leq \int_{B_{r_2}} \frac{\phi^2|\nabla u|^2}{(1-u)^2} + 4\int_{B_{r_2}} \frac{|\nabla\phi|^2}{(1-u)^2}\,,$$

(6.2.4) gives that

$$\int_{B_{r_2}} \frac{m\phi^2}{(1-u)^3} \leq 4\int_{B_{r_2}} \frac{|\nabla\phi|^2}{(1-u)^2}\,.$$

Now choose $\phi = \eta^{2\beta}$ with $\beta = \frac{3}{2}$. Then Hölder's inequality implies that

$$m\int_{B_{r_2}} \frac{\eta^{4\beta}}{(1-u)^3} \leq 16\beta^2\Big[\int_{B_{r_2}} |\nabla\eta|^{4\beta}\Big]^{\frac{1}{2\beta}}\Big[\int_{B_{r_2}} \frac{\eta^{4\beta}}{(1-u)^3}\Big]^{\frac{2\beta-1}{2\beta}}\,.$$

This shows that

$$\int_{B_{r_1}} \frac{1}{(1-u)^3} \leq \int_{B_{r_2}} \frac{\eta^{4\beta}}{(1-u)^3} < C(m, r_1). \tag{6.2.5}$$

By virtue of the one-sided Harnack inequality, we have

$$\Big\| \frac{1}{1-u} \Big\|_{L^\infty(B_{\frac{r_1}{2}})} = \| v \|_{L^\infty(B_{\frac{r_1}{2}})} \leq C(r_1) \| v \|_{L^3(B_{r_1})} < C(r_1, m)\,.$$

The rest follows from a standard compactness argument. ■

Proof of Theorem 6.2.1: Set $v = u_t$, then we have for any $t_1 < T$ that

$$v_t = \Delta v + \frac{2\lambda f(x)}{(1-u)^3} v \quad (x,t) \in \Omega \times (0, t_1)\,; \tag{6.2.6a}$$

$$v(x,t) = 0 \quad (x,t) \in \partial\Omega \times (0, t_1)\ ; \quad v(x,0) \geq 0 \quad x \in \Omega. \tag{6.2.6b}$$

Note that the term $\frac{2\lambda f}{(1-u)^3}$ is locally bounded in $\Omega \times (0, t_1)$, so that by the strong maximum principle, we may conclude

$$u_t = v > 0 \text{ for } (x,t) \in \Omega \times (0, t_1) \tag{6.2.7}$$

and therefore, $u_t > 0$ holds for all $(x,t) \in \Omega \times (0,T)$. Since K is an isolated set of touchdown points, there exists an open set U such that $K \subset U \subset \bar{U} \subset \Omega$ with no touchdown points in $\bar{U} \setminus K$. Consider now $0 < t_0 < T$ such that $\inf_{x \in \bar{U}} u_t(x, t_0) = C_1 > 0$. We claim that there exists $\varepsilon > 0$ such that

$$J^\epsilon(x,t) = u_t - \frac{\varepsilon}{(1-u)^2} \geq 0 \quad \text{for all} \quad (x,t) \in U \times (t_0, T)\,, \tag{6.2.8}$$

Indeed, there exists $C_2 > 0$ such that $u_t(x,T) \geq C_2 > 0$ on U, and since ∂U has no touchdown points, there exists $\varepsilon > 0$ such that $J^\epsilon \geq 0$ on the parabolic boundary of $U \times (t_0, T)$. Also, direct calculations imply that

$$J^\epsilon_t - \Delta J^\epsilon = \frac{2\lambda f}{(1-u)^3} J^\epsilon + \frac{6\varepsilon |\nabla u|^2}{(1-u)^4} \geq \frac{2\lambda f}{(1-u)^3} J^\epsilon\,.$$

Since $\frac{\varepsilon}{(1-u)^2}$ is locally bounded on $U \times (t_0, T)$, we can apply the maximum principle to obtaining (6.2.8).

If now $\inf_{x \in K} f(x) = 0$, then we may combine (6.2.8) and (6.0.1), to deduce that for a small neighborhood $B \subset U$ of some point $x_0 \in K$ where $f(x) \leq \varepsilon/2$, we have

$$\Delta u \geq \frac{\varepsilon}{2} \frac{1}{(1-u)^2} \quad \text{for } (x,t) \in B \times (t_0, T)\,.$$

In view of Lemma 6.2.2, this contradicts the assumption that x_0 is a touchdown point.

For the second part, recall from Theorem 2.4.3 that the unique extremal solution for the stationary problem on the ball in the case $N \geq 8$ and for a permittivity profile $f(x) = |x|^\alpha$, is $u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ as long as α is small enough. Theorem 6.1.1 then implies that the origin 0 is a touchdown point of the solution even though it is also a root for the permittivity profile (i.e., $f(0) = 0$). This complements the statement of Theorem 6.2.1 above. In other words, zero points of f in Ω cannot be on the isolated set of touchdown points in finite time (which occur when $\lambda > \lambda^*$) but can very well be touchdown points in infinite time of (6.0.1), which can only happen when $\lambda = \lambda^*$. The proof of Theorem 6.2.1 fails for touchdowns in infinite time, simply because the maximum principle cannot be applied in the infinite cylinder $\Omega \times (0, \infty)$. ∎

6.3 Estimates for finite touchdown times

In this section we give comparison results and explicit estimates on finite touchdown times of dynamic deflections $u = u(x,t)$ whenever $\lambda > \lambda^*$. This often translates into useful information concerning the speed of the operation for many MEMS devices such as RF switches or micro-valves.

6.3.1 Comparison results for finite touchdown time

We start by comparing the effect on finite touchdown time of two different but comparable permittivity profiles $f(x)$, at a given voltage λ.

Theorem 6.3.1. *Suppose $u_1 = u_1(x,t)$ (resp., $u_2 = u_2(x,t)$) is a touchdown solution for* (6.0.1) *associated to a fixed voltage λ and permittivity profiles f_1 (resp., f_2) with a corresponding finite touchdown time $T_\lambda(\Omega, f_1)$ (resp., $T_\lambda(\Omega, f_2)$). If $f_1(x) \geq f_2(x)$ on Ω and if $f_1(x) > f_2(x)$ on a set of positive measure, then necessarily $T_\lambda(\Omega, f_1) < T_\lambda(\Omega, f_2)$.*

Proof: By making a change of variable $v = 1 - u$, we can assume to be working with solutions of the following equation:

$$\frac{\partial v}{\partial t} - \Delta v = -\frac{\lambda f(x)}{v^2} \qquad \text{for} \quad x \in \Omega\,, \tag{6.3.1a}$$

$$v(x,t) = 1 \qquad \text{for} \quad x \in \partial\Omega\,, \tag{6.3.1b}$$

$$v(x,0) = 1 \qquad \text{for} \quad x \in \Omega\,, \tag{6.3.1c}$$

where f is either f_1 or f_2. Suppose now that $T_\lambda(\Omega, f_1) > T_\lambda(\Omega, f_2)$ and let $\Omega_0 \subset \Omega$ be the set of touchdown points of u_2 at finite time $T_\lambda(\Omega, f_2)$. Setting $w = u_2 - u_1$, we get that

$$w_t - \Delta w - \frac{\lambda(f_2 u_1 + f_1 u_2)}{u_1^2 u_2^2} w = \frac{\lambda(f_1 - f_2)}{u_1 u_2} \geq 0 \quad (x,t) \in \Omega \times (0, T_\lambda(\Omega, f_2))\,. \tag{6.3.2}$$

Since $w = 0$ at $t = 0$ as well as on $\partial\Omega \times (0, T_\lambda(\Omega, f_2))$, we get from the maximum principle that w cannot attain a negative minimum in $\Omega \times (0, T_\lambda(\Omega, f_2))$, and therefore $w \geq 0$ in $\Omega \times (0, T_\lambda(\Omega, f_2))$. Since $u_2 \to 0$ in Ω_0 as $t \to T_\lambda(\Omega, f_2)$, and since our assumption is that $T_\lambda(\Omega, f_1) > T_\lambda(\Omega, f_2)$, we then have $u_1 > 0$ in Ω_0 as $t \to T_\lambda(\Omega, f_2)$. Therefore, $w < 0$ in Ω_0 as $t \to T_\lambda(\Omega, f_2)$, which is a contradiction and therefore $T_\lambda(\Omega, f_1) \leq T_\lambda(\Omega, f_2)$.

To prove the strict inequality, we note that the above proof shows that $w \geq 0$ in $\Omega \times (0, T_\lambda(\Omega, f_2))$, which once combined with (6.3.2) gives that

$$w_t - \Delta w \geq 0\,, \quad \textit{in} \quad \Omega \times (t_1, T_\lambda(\Omega, f_2))\,,$$

where $t_1 > 0$ is chosen so that $w(x, t_1) \not\equiv 0$ in Ω. Now we compare w with the solution z of

$$z_t - \Delta z = 0 \quad \textit{in} \quad \Omega \times (t_1, T_\lambda(\Omega, f_2))$$

subject to $z(x, t_1) = w(x, t_1)$ and $z(x,t) = 0$ on $\partial\Omega \times (t_1, T_\lambda(\Omega, f_2))$. Clearly, $w \geq z$ in $\Omega \times (t_1, T_\lambda(\Omega, f_2))$. On the other hand, for any $t_0 > t_1$ we have $z > 0$ in $\Omega \times (t_0, T_\lambda(\Omega, f_2))$. Consequently, $w > 0$ which means that $u_2 > u_1$ in $\Omega \times (t_0, T_\lambda(\Omega, f_2))$ and therefore $T_\lambda(\Omega, f_1) < T_\lambda(\Omega, f_2)$. ■

The second comparison result deals with different applied voltages but identical permittivity profiles.

Theorem 6.3.2. *Suppose $u_1 = u_1(x,t)$ (resp., $u_2 = u_2(x,t)$) is a solution for* (6.0.1) *associated to a voltage λ_1 (resp., λ_2) and which has a finite touchdown time $T_{\lambda_1}(\Omega, f)$ (resp., $T_{\lambda_2}(\Omega, f)$). If $\lambda_1 > \lambda_2$ then necessarily $T_{\lambda_1}(\Omega, f) < T_{\lambda_2}(\Omega, f)$.*

Proof: It is similar to the proof of Theorem 6.3.1, except that for $w = u_2 - u_1$, (6.3.2) is replaced by

$$w_t - \Delta w - \frac{\lambda_1(u_1+u_2)f}{u_1^2 u_2^2} w = \frac{(\lambda_1 - \lambda_2)f}{u_2^2} \geq 0 \quad (x,t) \in \Omega \times (0,T)\,.$$

The details are left for the interested reader. ■

Remark 6.3.1. A reasoning similar to the one found in Proposition 2.5 of [51], gives some information on the dependence on the shape of the domain. Indeed, for any bounded domain Γ in $\mathbb{R}^N$ and any non-negative continuous function f on Γ, we have

$$\lambda^*(\Gamma, f) \geq \lambda^*(B_R, f^*) \text{ and } T_\lambda(\Gamma, f) \geq T_\lambda(B_R, f^*),$$

where $B_R = B_R(0)$ is the Euclidean ball in $\mathbb{R}^N$ with radius $R > 0$ and with volume $|B_R| = |\Gamma|$, where f^* is the Schwarz symmetrization of f.

We now present numerical results comparing finite touchdown times in a slab domain.

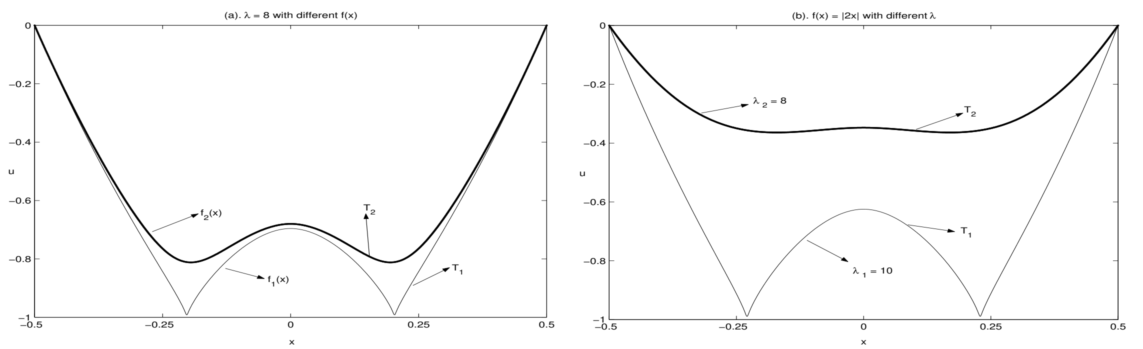

Figure 6.3: *Left Figure: plots of u versus x for different $f(x)$ at $\lambda = 8$ and $t = 0.185736$. Right Figure: plots of u versus x for different λ with $f(x) = |2x|$ and $t = 0.1254864$.*

Figure 6.3(a): Dependence on the dielectric permittivity profiles f
We consider (6.2.1) for the cases where

$$f_1(x) = |2x| \quad \text{and} \quad f_2(x) = \begin{cases} |2x| & \text{if} \quad |x| \leq \frac{1}{8}\,, \\ 1/4 + 2\sin(|x| - 1/8) & \text{otherwise.} \end{cases} \tag{6.3.3}$$

Using $N = 1000$ meshpoints, we plot u versus x with $\lambda = 8$ at the time $t = 0.185736$ in Figure 6.3(a). The numerical results show that the finite touchdown time $T_\lambda(\Omega, f_1)$ for the case $f_1(x)$ and $T_\lambda(\Omega, f_2)$ for the case $f_2(x)$ are 0.185736 and 0.186688, respectively.

Figure 6.3(b): Dependence on the applied voltage λ
Using $N = 1000$ meshpoints and the profile $f(x) = |2x|$, we plot u of (6.2.1) versus x with different values of λ at the time $t = 0.1254864$. The numerical results show that finite touchdown time $T_{\lambda_1}(\Omega, f)$ for applied voltage $\lambda_1 = 10$ and $T_{\lambda_2}(\Omega, f)$ for applied voltage $\lambda_2 = 8$ are 0.1254864 and 0.185736, respectively.

6.3.2 Explicit bounds on touchdown times

For the analytic bounds of touchdown times, we have the following results. Note that here $\bar\lambda_1 \geq \lambda^*$ and $\bar\lambda_2 \geq \lambda^*$ are as in Theorem 2.2.4.

Proposition 6.3.3. Suppose f is a non-negative continuous function on a bounded domain Ω, and let u be a solution of (6.0.1) corresponding to a voltage λ. Then,

1. For any $\lambda > 0$, we have $T_\lambda(\Omega, f) \geq T_* := \frac{1}{3\lambda \sup_{x\in\Omega} f(x)}$.

2. If $\inf_\Omega f > 0$, and if $\lambda > \bar\lambda_1 := \frac{4\mu_\Omega}{27 \inf_{x\in\Omega} f(x)}$, then
$$T_\lambda(\Omega, f) \leq T_{1,\lambda} := \int_0^1 \big[\frac{\lambda \inf_{x\in\Omega} f(x)}{(1-s)^2} - \mu_\Omega s\big]^{-1} ds. \tag{6.3.4}$$

3. If $\inf_{x\in\Omega} f(x) > 0$, then the following upper estimate holds for any $\lambda > \lambda^*$:
$$T_\lambda(\Omega, f) \leq T_{0,\lambda}(\Omega, f) := \frac{8(\lambda+\lambda^*)^2}{3 \inf_{x\in\Omega} f(x)(\lambda-\lambda^*)^2(\lambda+3\lambda^*)} \Big[1 + \Big(\frac{\lambda+3\lambda^*}{2\lambda+2\lambda^*}\Big)^{1/2}\Big]. \tag{6.3.5}$$

4. If $f > 0$ on a set of positive measure, and if $\lambda > \bar\lambda_2 := \frac{\mu_\Omega}{3\int_\Omega f\phi_\Omega dx}$, then
$$T_\lambda(\Omega, f) \leq T_{2,\lambda} := -\frac{1}{\mu_\Omega} \log \big[1 - \frac{\mu_\Omega}{3\lambda} \big(\int_\Omega f\phi_\Omega \, dx\big)^{-1}. \tag{6.3.6}$$

Here μ_Ω and ϕ_Ω are the first eigenpair of $-\Delta$ on $H_0^1(\Omega)$ with normalized $\int_\Omega \phi_\Omega dx = 1$.

Proof: 1) Consider the initial value problem:
$$\begin{aligned} \frac{d\eta(t)}{dt} &= \frac{\lambda M}{(1-\eta(t))^2}\,, \\ \eta(0) &= 0\,, \end{aligned} \tag{6.3.7}$$

where $M = \sup_{x\in\Omega} f(x)$. From (6.3.7) one has $\frac{1}{\lambda M}\int_0^{\eta(t)}(1-s)^2 ds = t$. If T_* is the time where $\lim_{t\to T_*}\eta(t) = 1$, then we have $T_* = \frac{1}{\lambda M}\int_0^1 (1-s)^2 ds = \frac{1}{3\lambda M}$. Obviously, $\eta(t)$ is now a super-function of $u(x,t)$ near touchdown, and thus we have

$$T \geq T_* = \frac{1}{3\lambda M} = \frac{1}{3\lambda \sup_{x\in\Omega} f(x)}\,,$$

which completes the proof of 1).

2) is proved in Theorem 6.1.4. The following analytic upper bounds of finite touchdown time T were established in Theorem 3.1 and 3.2 of [64].

3) Without loss of generality we assume that $\phi_\Omega > 0$ in Ω. Multiplying (6.0.1a) by ϕ_Ω, and integrating over the domain, we obtain

$$\frac{d}{dt}\int_\Omega \phi_\Omega u\,dx = \int_\Omega \phi_\Omega \Delta u\,dx + \int_\Omega \frac{\lambda\phi_\Omega f(x)}{(1-u)^2}\,dx\,. \tag{6.3.8}$$

Using Green's theorem, together with the lower bound C_0 of f, we get

$$\frac{d}{dt}\int_\Omega \phi_\Omega u\,dx \geq -\mu_\Omega \int_\Omega \phi_\Omega u\,dx + \lambda C_0 \int_\Omega \frac{\phi_\Omega}{(1-u)^2}\,dx\,. \tag{6.3.9}$$

Next, we define an energy-like variable $E(t)$ by $E(t) = \int_\Omega \phi_\Omega u\,dx$ so that

$$E(t) = \int_\Omega \phi_\Omega u\,dx \leq \sup_\Omega u \int_\Omega \phi_\Omega\,dx = \sup_\Omega u\,. \tag{6.3.10}$$

Moreover, $E(0) = 0$ since $u = 0$ at $t = 0$. Then, using Jensen's inequality on the second term on the right-hand side of (6.3.9), we obtain

$$\frac{dE}{dt} + \mu_\Omega E \geq \frac{\lambda C_0}{(1-E)^2}\,, \qquad E(0) = 0\,. \tag{6.3.11}$$

We then compare $E(t)$ with the solution $F(t)$ of

$$\frac{dF}{dt} + \mu_\Omega F = \frac{\lambda C_0}{(1-F)^2}\,, \qquad F(0) = 0\,. \tag{6.3.12}$$

Standard comparison principles yield that $E(t) \geq F(t)$ on their domains of existence. Therefore,

$$\sup_\Omega u \geq E(t) \geq F(t)\,. \tag{6.3.13}$$

Next, we separate variables in (6.3.12) to determine t in terms of F. The touchdown time $\bar{T}_1$ for F is obtained by setting $F = 1$ in the resulting formula. In this way, we get

$$\bar{T}_1 \equiv \int_0^1 \Big[\frac{\lambda C_0}{(1-s)^2} - \mu_\Omega s\Big]^{-1} ds\,. \tag{6.3.14}$$

The touchdown time $\bar{T}_1$ is finite when the integral in (6.3.14) converges. A simple calculation shows that this occurs when $\lambda > \bar{\lambda}_1 \equiv \frac{4\mu_\Omega}{27C_0}$. Hence if $\bar{T}_1$ is finite, then (6.3.13) implies that the touchdown time T of (6.0.1) must also be finite. Therefore, when $\lambda > \bar{\lambda}_1 = \frac{4\mu_\Omega}{27C_0}$, we have that T satisfies

$$T \leq \bar{T}_1 \equiv \int_0^1 \left[\frac{\lambda C_0}{(1-s)^2} - \mu_\Omega s\right]^{-1} ds\,. \tag{6.3.15}$$

4) Multiply now (6.0.1a) by $\phi_\Omega(1-u)^2$, and integrate the resulting equation over Ω to get

$$\frac{d}{dt}\int_\Omega \frac{\phi_\Omega}{3}(1-u)^3\,dx = -\int_\Omega \phi_\Omega(1-u)^2\Delta u\,dx - \int_\Omega \lambda f\phi_\Omega\,dx\,. \tag{6.3.16}$$

We calculate the first term on the right-hand side of (6.3.16) to get

$$\frac{d}{dt}\int_\Omega \frac{\phi_\Omega}{3}(1-u)^3\,dx \tag{6.3.17a}$$

$$= \int_\Omega \nabla u \cdot \nabla\Big[\phi_\Omega(1-u)^2\Big]\,dx + \int_{\partial\Omega}(1-u)^2\phi_\Omega \nabla u\cdot\hat{n}\,dS - \int_\Omega \lambda f\phi_\Omega\,dx \tag{6.3.17b}$$

$$= -\int_\Omega 2(1-u)\phi_\Omega|\nabla u|^2\,dx - \int_\Omega \frac{1}{3}\nabla\phi_\Omega\cdot\nabla\Big[(1-u)^3\Big]\,dx - \int_\Omega \lambda f\phi_\Omega\,dx \tag{6.3.17c}$$

$$\leq -\frac{1}{3}\int_{\partial\Omega}\nabla\phi_\Omega\cdot\nu\,dS - \frac{\mu_\Omega}{3}\int_\Omega (1-u)^3\phi_\Omega\,dx - \int_\Omega \lambda f\phi_\Omega\,dx\,, \tag{6.3.17d}$$

where ν is the unit outward normal to $\partial\Omega$. Since $\int_{\partial\Omega}\nabla\phi_\Omega\cdot\nu\,dS = -\mu_\Omega$, we further estimate from (6.3.17d) that

$$\frac{dE}{dt} + \mu_\Omega E \leq R\,, \qquad R \equiv \frac{\mu_\Omega}{3} - \lambda\int_\Omega f\phi_\Omega\,dx\,, \tag{6.3.18}$$

where $E(t)$ is defined by

$$E(t) \equiv \frac{1}{3}\int_\Omega \phi_\Omega(1-u)^3\,dx\,, \qquad E(0) = \frac{1}{3}\,. \tag{6.3.19}$$

Next, we compare $E(t)$ with the solution $F(t)$ of

$$\frac{dF}{dt} + \mu_\Omega F = R\,, \qquad F(0) = \frac{1}{3}\,. \tag{6.3.20}$$

Again, comparison principles and the definition of E yield

$$\frac{1}{3}\inf_\Omega (1-u)^3 \leq E(t) \leq F(t)\,. \tag{6.3.21}$$

For $\lambda > \bar{\lambda}_2$ we have that $R < 0$ in (6.3.18) and (6.3.20). For $R < 0$, we have that $F = 0$ at some finite time $t = \bar{T}_2$. From (6.3.21), this implies that $E = 0$ at finite time. Thus, u has touchdown at some finite-time $T < \bar{T}_2$. By calculating $\bar{T}_2$ explicitly, and by using (6.3.21), the touchdown time T for (6.0.1) is found to satisfy

$$T \leq \bar{T}_2 \equiv -\frac{1}{\mu_\Omega} \log \big[1 - \frac{\mu_\Omega}{3\lambda}\big(\int_\Omega f\phi_\Omega \, dx\big)^{-1}\big] . \tag{6.3.22}$$

■

Remark 6.3.2. It follows from the above that if $\lambda > \max\{\bar{\lambda}_1 , \bar{\lambda}_2\}$, then

$$T \leq \min\{T_{0,\lambda}, T_{1,\lambda}, T_{2,\lambda}\} . \tag{6.3.23}$$

where $T_{0,\lambda}$ is given by Theorem 6.1.4. We note that the three estimates on the touchdown times are not comparable. Indeed, it is clear that $T_{0,\lambda}$ is the better estimate when $\lambda^* < \lambda < \min\{\bar{\lambda}_1 , \bar{\lambda}_2\}$ since $T_{1,\lambda}$ and $T_{2,\lambda}$ are not finite. On the other hand, our numerical simulations show that $T_{0,\lambda}$ can be much worse than the others, for $\lambda > \max\{\bar{\lambda}_1 , \bar{\lambda}_2\}$.

Here are now some numerical estimates of touchdown times for several choices of the domain Ω given by (6.2.2) and the exponential profile $f(x)$ satisfying

$$\text{(slab)}: \quad f(x) = e^{\alpha(x^2-1/4)} \quad \text{(exponential)}, \tag{6.3.24a}$$

$$\text{(unit disk)}: \quad f(x) = e^{\alpha(|x|^2-1)} \quad \text{(exponential)}, \tag{6.3.24b}$$

where $\alpha \geq 0$.

[htb]

Ω	α	$\underline{\lambda}$	λ^*	$\bar{\lambda}_1$	$\bar{\lambda}_2$
(Slab)	0	1.185	1.401	1.462	3.290
(Slab)	1.0	1.185	1.733	1.878	4.023
(Slab)	3.0	1.185	2.637	3.095	5.965
(Slab)	6.0	1.185	4.848	6.553	10.50
(unit disk)	0	0.593	0.789	0.857	1.928
(unit disk)	0.5	0.593	1.153	1.413	2.706
(unit disk)	1.0	0.593	1.661	2.329	3.746
(unit disk)	3.0	0.593	6.091	17.21	11.86

Table 6.1: *Numerical values for pull-in voltage λ^* with the bounds $\underline{\lambda}$, $\bar{\lambda}_1$ and $\bar{\lambda}_2$ given in Theorem 2.2.4. Here the exponential permittivity profile is chosen as (6.3.24).*

In Table 2.1 of §2.2 we give numerical results for the saddle-node value λ^* with the bounds $\underline{\lambda}$, $\bar{\lambda}_1$ and $\bar{\lambda}_2$ given in Theorem 2.2.4, for the exponential permittivity profile chosen as (6.3.24). Following the numerical results of Table 2.1 of §2.2, here we can compute in Table 6.1 the values of finite touchdown time T at $\lambda = 20$, with the bounds T_*, $T_{0,\lambda}$, $T_{1,\lambda}$ and $T_{2,\lambda}$ given in Theorem 6.1.4 and Proposition 6.3.3. Using the meshpoints $N = 800$ we

[htb]

Ω	α	T_*	T	$T_{0,\lambda}$	$T_{1,\lambda}$	$T_{2,\lambda}$
slab	0	1/60	0.01668	0.2555	0.0175	0.01825
slab	1.0	1/60	0.02096	$\leq$ 0.3383	0.0229	0.02275
slab	3.0	1/60	0.03239	$\leq$ 0.6121	0.0395	0.03588
slab	6.0	1/60	0.06312	$\leq$ 1.7033	0.0973	0.07544
unit disk	0	1/60	0.01667	0.2420	0.0172	0.01745
unit disk	0.5	1/60	0.02241	$\leq$ 0.4103	0.0289	0.02507
unit disk	1.0	1/60	0.02927	$\leq$ 0.7123	0.0492	0.03579
unit disk	3.0	1/60	0.09563	$\leq$ 8.9847	1.1614	0.15544

Table 6.2: *Computations for finite touchdown time T with the bounds T_*, $T_{0,\lambda}$, $T_{1,\lambda}$ and $T_{2,\lambda}$ given in Proposition 6.3.3. Here the applied voltage $\lambda = 20$ and the profile is chosen as (6.3.24).*

[htb]

Ω	$T(\lambda = 5)$	$T(\lambda = 10)$	$T(\lambda = 15)$	$(\lambda = 20)$
slab	0.07495	0.03403	0.02239	0.01668
unit disk	0.06699	0.03342	0.02235	0.01667

Table 6.3: *Numerical values for finite touchdown time T at different applied voltages $\lambda = 5$, 10, 15 and 20, respectively. Here the constant permittivity profile $f(x) \equiv 1$ is chosen.*

compute finite touchdown time T with error less than 0.00001. The numerical results in Table 6.1 show that the bounds $T_{1,\lambda}$ and $T_{2,\lambda}$ for T are much better than $T_{0,\lambda}$. Further the bound $T_{1,\lambda}$ is better than $T_{2,\lambda}$ for smaller values of α, and however the bound $T_{2,\lambda}$ is better than $T_{1,\lambda}$ for larger values of α. In fact, for $\alpha \gg 1$ and λ large enough we can deduce from (2.2.20) that

$$T_{1,\lambda} \sim \frac{1}{3\lambda} e^{d_1 \alpha}, \qquad T_{2,\lambda} \sim \frac{d_2}{\lambda} \alpha^2 .$$

Here $d_1 = 1/4$, $d_2 = 1/3\pi^2$ for the slab domain, and $d_1 = 1$, $d_2 = 4/3z_0^2$ for the unit disk, where z_0 is the first zero of $J_0(z) = 0$. Therefore, for $\alpha \gg 1$ and fixed λ large enough, the bound $T_{2,\lambda}$ is better than $T_{1,\lambda}$. Table 6.1 also shows that for fixed applied voltage λ, the touchdown time is seen to increase once α is increased or equivalently the spatial extent where $f(x) \ll 1$ is increased. However, Theorem 6.3.2 tells us that for fixed permittivity profile f, by increasing the applied voltage λ within the available power supply, the touchdown time can be decreased and consequently the operating speed of MEMS devices can be improved. In Table 6.2 we give numerical values for finite touchdown time T with error less than 0.00001, at different applied voltages $\lambda = 5$, 10, 15 and 20, respectively. Here the constant permittivity profile $f(x) \equiv 1$ is chosen and the meshpoints $N = 800$ again.

6.4 Asymptotic analysis of touchdown profiles

In this section, we discuss touchdown profiles by the method of asymptotic analysis, which provide some information on the refined touchdown rate studied in next Chapter.

6.4.1 Touchdown profile: $f(x) \equiv 1$

We first construct a local expansion of the solution near the touchdown time and touchdown location by adapting the method of [77] used for blow-up behavior. In the analysis of this subsection we assume that $f(x) \equiv 1$ and touchdown occurs at $x = 0$ and $t = T$. In the absence of diffusion, the time-dependent behavior of (6.0.1) is given by $u_t = \lambda(1-u)^{-2}$. Integrating this differential equation and setting $u(T) = 1$, we get $(1-u)^3 = -3\lambda(t-T)$. This solution motivates the introduction of a new variable $v(x,t)$ defined in terms of $u(x,t)$ by

$$v = \frac{1}{3\lambda}(1-u)^3\,. \tag{6.4.1}$$

A simple calculation shows that (6.0.1) transforms exactly to the following problem for v:

$$v_t = \Delta v - \frac{2}{3v}|\nabla v|^2 - 1\,, \quad x \in \Omega\,, \tag{6.4.2a}$$

$$v = \frac{1}{3\lambda}\,, \quad x \in \partial\Omega\,; \quad v = \frac{1}{3\lambda}\,, \quad t = 0\,. \tag{6.4.2b}$$

Notice that $u = 1$ maps to $v = 0$. We will find a formal power series solution to (6.4.2a) near $v = 0$.

As in [77] we look for a locally radially symmetric solution to (6.4.2) in the form

$$v(x,t) = v_0(t) + \frac{r^2}{2!}v_2(t) + \frac{r^4}{4!}v_4(t) + \cdots\,, \tag{6.4.3}$$

where $r = |x|$. We then substitute (6.4.3) into (6.4.2a) and collect coefficients in r. In this way, we obtain the following coupled ordinary differential equations for v_0 and v_2:

$$v_0' = -1 + Nv_2\,, \qquad v_2' = -\frac{4}{3v_0}v_2^2 + \frac{(N+2)}{3}v_4\,. \tag{6.4.4}$$

We are interested in the solution to this system for which $v_0(T) = 0$, with $v_0' < 0$ and $v_2 > 0$ for $T - t > 0$ with $T - t \ll 1$. The system (6.4.4) has a closure problem in that v_2 depends on v_4. However, we will assume that $v_4 \ll v_2^2/v_0$ near the singularity. With this assumption, (6.4.4) reduces to

$$v_0' = -1 + Nv_2\,, \qquad v_2' = -\frac{4}{3v_0}v_2^2\,. \tag{6.4.5}$$

We now solve the system (6.4.5) asymptotically as $t \to T^-$ in a similar manner as was done in [77]. We first assume that $Nv_2 \ll 1$ near $t = T$. This leads to $v_0 \sim T - t$, and the following differential equation for v_2:

$$v_2^{'} \sim \frac{-4}{3(T-t)} v_2^2 \,, \quad \text{as } t \to T^- \,. \tag{6.4.6}$$

By integrating (6.4.6), we obtain that

$$v_2 \sim -\frac{3}{4\big[\log(T-t)\big]} + \frac{B_0}{\big[\log(T-t)\big]^2} + \cdots \,, \quad \text{as } t \to T^- \,, \tag{6.4.7}$$

for some unknown constant B_0. From (6.4.7), we observe that the consistency condition that $Nv_2 \ll 1$ as $t \to T^-$ is indeed satisfied. Substituting (6.4.7) into the equation (6.4.5) for v_0, we obtain for $t \to T^-$ that

$$v_0^{'} = -1 + N\Big(- \frac{3}{4\big[\log(T-t)\big]} + \frac{B_0}{\big[\log(T-t)\big]^2} + \cdots \Big) \,. \tag{6.4.8}$$

Using the method of dominant balance, we look for a solution to (6.4.8) as $t \to T^-$ in the form

$$v_0 \sim \big(T-t\big) + \big(T-t\big)\Big[\frac{C_0}{\big[\log(T-t)\big]} + \frac{C_1}{\big[\log(T-t)\big]^2} + \cdots\Big] \,, \tag{6.4.9}$$

for some C_0 and C_1 to be found. A simple calculation yields that

$$v_0 \sim \big(T-t\big) + \frac{-3N(T-t)}{4|\log(T-t)|} + \frac{-N\big(B_0 - 3/4\big)(T-t)}{|\log(T-t)|^2} + \cdots \,, \quad \text{as } t \to T^- \,. \tag{6.4.10}$$

The local form for v near touchdown is $v \sim v_0 + r^2 v_0/2$. Using the leading term in v_2 from (6.4.7) and the first two terms in v_0 from (6.4.10), we obtain the local form

$$v \sim \Big(T-t\Big)\Big[1 - \frac{3N}{4|\log(T-t)|} + \frac{3r^2}{8(T-t)|\log(T-t)|} + \cdots\Big] \,, \tag{6.4.11}$$

for $r \ll 1$ and $t - T \ll 1$. Finally, using the nonlinear mapping (6.4.1) relating u and v, we conclude that

$$u \sim 1 - \Big[3\lambda(T-t)\Big]^{1/3}\Big(1 - \frac{3N}{4|\log(T-t)|} + \frac{3r^2}{8(T-t)|\log(T-t)|} + \cdots\Big)^{1/3} \,. \tag{6.4.12}$$

We note, as in [77], that if we use the local behavior $v \sim (T-t) + 3r^2/[8|\log(T-t)|]$, we get that

$$\frac{|\nabla v|^2}{v} \sim \Big[\frac{2}{3}|\log(T-t)| + \frac{16(T-t)|\log(T-t)|^2}{9r^2}\Big]^{-1} \,. \tag{6.4.13}$$

Hence, the term $|\nabla v|^2/v$ in (6.4.2a) is bounded for any r, even as $t \to T^-$. This allows us to use a simple finite-difference scheme to compute numerical solutions to (6.4.2). With this observation, we now perform a few numerical experiments on the transformed problem

(6.4.2). For the slab domain, we define v_j^m for $j = 1, \ldots, N+2$ to be the discrete approximation to $v(m\Delta t, -1/2 + (j-1)h)$, where $h = 1/(N+1)$ and Δt are the spatial and temporal mesh sizes, respectively. A second order accurate in space, and first order accurate in time, discretization of (6.4.2) is

$$v_j^{m+1} = v_j^m + \Delta t\Big(\frac{\big(v_{j+1}^m - 2v_j^m + v_{j-1}^m\big)}{h^2} - 1 - \frac{\big(v_{j+1}^m - v_{j-1}^m\big)^2}{6v_j^m h^2}\Big), \qquad j = 2, \ldots, N+1, \tag{6.4.14}$$

with $v_1^m = v_{N+2}^m = \Big(3\lambda\Big)^{-1}$ for $m \geq 0$. The initial condition is $v_j^0 = \Big(3\lambda\Big)^{-1}$ for $j = 1, \ldots, N+2$. The time-step Δt is chosen to satisfy $\Delta t < h^2/4$ for the stability of the discrete scheme. Using this argument, one can compute numerical results of dynamic deflection u, see Figures 6.1-6.3 of this Chapter.

6.4.2 Touchdown profile: variable permittivity

In this subsection we obtain some formal asymptotic results for touchdown behavior associated with a spatially variable permittivity profile in a slab domain. Suppose u is a touchdown solution of (6.0.1) at finite time T, and let $x = x_0$ be a touchdown point of u. With the transformation

$$v = \frac{1}{3\lambda}(1-u)^3, \tag{6.4.15}$$

the problem (6.0.1) for u in the slab domain transforms exactly to

$$v_t = v_{xx} - \frac{2}{3v}v_x^2 - f(x), \quad -1/2 < x < 1/2, \tag{6.4.16a}$$

$$v = \frac{1}{3\lambda}, \quad x = \pm 1/2; \qquad v = \frac{1}{3\lambda}, \quad t = 0, \tag{6.4.16b}$$

where $f(x)$ is the permittivity profile.

In order to discuss the touchdown profile of u near (x_0, T), we use the formal power series method of §6.4.1 to locally construct a power series solution to (6.4.16) near touchdown point x_0 and touchdown time T. For this purpose, we look for a touchdown profile for (6.4.16), near $x = x_0$, in the form

$$v(x,t) = v_0(t) + \frac{(x-x_0)^2}{2!}v_2(t) + \frac{(x-x_0)^3}{3!}v_3(t) + \frac{(x-x_0)^4}{4!}v_4(t) + \cdots. \tag{6.4.17}$$

In order for v to be a touchdown profile, it is clear that we must require that

$$\lim_{t\to T^-} v_0 = 0, \quad v_0 > 0, \quad \text{for } t < T; \quad v_2 > 0, \quad \text{for } t - T \ll 1. \tag{6.4.18}$$

We first discuss the case where $f(x)$ is analytic at $x = x_0$ with $f(x_0) > 0$. Therefore, for $x - x_0 \ll 1$, $f(x)$ has the convergent power series expansion

$$f(x) = f_0 + f_0'(x-x_0) + \frac{f_0''(x-x_0)^2}{2} + \cdots, \tag{6.4.19}$$

where $f_0 \equiv f(x_0)$, $f_0' \equiv f'(x_0)$, and $f_0'' \equiv f''(x_0)$. Substituting (6.4.17) and (6.4.19) into (6.4.16), we equate powers of $x - x_0$ to obtain

$$v_0' = -f_0 + v_2 \,, \tag{6.4.20a}$$

$$v_2' = -\frac{4v_2^2}{3v_0} + v_4 - f_0'' \,, \tag{6.4.20b}$$

$$v_3 = f_0' \,. \tag{6.4.20c}$$

We now assume that $v_2 \ll 1$ and $v_4 \ll 1$ as $t \to T^-$. This yields that $v_0 \sim f_0(T-t)$, and

$$v_2' \sim -\frac{4v_2^2}{3f_0(T-t)} - f_0'' \,. \tag{6.4.21}$$

For $t \to T^-$, we obtain from a simple dominant balance argument that

$$v_2 \sim -\frac{3f_0}{4\big[\log(T-t)\big]} + \cdots \,, \quad \text{as} \;\; t \to T^- \,. \tag{6.4.22}$$

Substituting (6.4.22) into (6.4.20a), and integrating, we obtain that

$$v_0 \sim f_0\big(T-t\big) + \frac{-3f_0(T-t)}{4|\log(T-t)|} + \cdots \,, \quad \text{as} \;\; t \to T^- \,. \tag{6.4.23}$$

Next, we substitute (6.4.22), (6.4.23) and (6.4.20c) into (6.4.17), to obtain the local touchdown behavior

$$v \sim f_0\big(T-t\big)\Big[1 - \frac{3}{4|\log(T-t)|} + \frac{3(x-x_0)^2}{8(T-t)|\log(T-t)|} + \frac{f_0'(x-x_0)^3}{6f_0(T-t)} + \cdots\Big] \,, \tag{6.4.24}$$

for $(x - x_0) \ll 1$ and $t - T \ll 1$. Finally, using the nonlinear mapping (6.4.15) relating u and v, we conclude that

$$u \sim 1 - \Big[3f_0\lambda(T-t)\Big]^{1/3}\Big(1 - \frac{3}{4|\log(T-t)|} + \frac{3(x-x_0)^2}{8(T-t)|\log(T-t)|} + \frac{f_0'(x-x_0)^3}{6f_0(T-t)} + \cdots\Big)^{1/3} . \tag{6.4.25}$$

Here $f_0 \equiv f(x_0)$ and $f_0' \equiv f'(x_0)$.

In the following, we exclude the possibility of $f(x_0) = 0$ by using a formal power series analysis. We discuss the case where $f(x)$ is analytic at $x = x_0$, with $f(x_0) = 0$ and $f'(x_0) = 0$, so that $f(x) = f_0(x-x_0)^2 + O\big((x-x_0)^3\big)$ as $x \to x_0$ with $f_0 > 0$. We then look for a power series solution to (6.4.16) as in (6.4.17). In place of (6.4.20) for v_3, we get $v_3 = 0$, and

$$v_0' = v_2 \,, \qquad v_2' = -\frac{4v_2^2}{3v_0} + v_4 - 2f_0 \,. \tag{6.4.26}$$

Assuming that $v_4 \ll 1$ as before, we can combine the equations in (6.4.26) to get

$$v_0'' = -\frac{4\left(v_0'\right)^2}{3v_0} - 2f_0 \,. \tag{6.4.27}$$

By solving (6.4.27) with $v_0(T) = 0$, we obtain the exact solution

$$v_0 = -\frac{3f_0}{11}(T-t)^2 < 0\,, \qquad v_2 = \frac{6f_0}{11}(T-t)\,. \tag{6.4.28}$$

Since the criteria (6.4.18) are not satisfied, the form (6.4.28) does not represent a touchdown profile centered at $x = x_0$. Therefore, the above asymptotical analysis also shows that the point $x = x_0$ satisfying $f(x_0) = 0$ is not a touchdown point of u.

6.5 Pull-in distance

One of the primary goals in the design of MEMS devices is to maximize the pull-in distance over a certain allowable voltage range that is set by the power supply. Here pull-in distance refers to as the maximum stable deflection of the elastic membrane before touchdown occurs. In this section, we provide numerical results of pull-in distance with some explicit examples, from which one can observe that both larger pull-in distance and pull-in voltage can be achieved by properly tailoring the permittivity profile.

Following from [64], we focus on the dynamic solution u satisfying

$$\frac{\partial u}{\partial t} - \Delta u = -\frac{\lambda f(x)}{(1+u)^2} \quad \text{for} \quad x \in \Omega\,, \tag{6.5.1a}$$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega\,; \quad u(x,0) = 0 \quad \text{for} \quad x \in \Omega\,, \tag{6.5.1b}$$

One can apply Theorem 6.1.1 that for $\lambda \le \lambda^*$, the dynamic solution $u(x,t)$ of (6.5.1) globally converges to its unique maximal negative steady-state $u_\lambda(x)$. On the other hand, Theorem 2.4.3 implies that the unique maximal negative steady-state $u_\lambda(x)$ is strictly increasing in λ. Therefore, we can deduce that pull-in distance of (6.5.1) is achieved exactly at $\lambda = \lambda^*$. Since the space dimension N of MEMS devices is 1 or 2, Theorems 2.4.3 & 6.1.1 give that the pull-in distance $\mathcal{D}$ of MEMS devices exactly satisfies

$$\mathcal{D} := \lim_{t\to\infty} \| u^*(x,t) \|_{L^\infty(\Omega)} = \| u^*(x) \|_{L^\infty(\Omega)} \le C(N) < 1\,, \quad N = 1,2, \tag{6.5.2}$$

where $u^*(x,t)$ is the unique global solution of (6.5.1) at $\lambda = \lambda^*$, and while $u^*(x)$ is the unique extremal steady-state of (6.5.1).

In order to understand the relationship between pull-in distance $\mathcal{D}$ and permittivity profile $f(x)$, we first consider the steady-state of (6.5.1) satisfying

$$\begin{cases} \Delta u = \dfrac{\lambda f(x)}{(1+u)^2} & \text{in } \Omega, \\ -1 < u < 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{6.5.3}$$

where the domain Ω is considered to be a slab or an unit disk defined by (6.2.2). Here we still choose the following permittivity profile $f(x)$ as before:

$$(\text{Slab}): \quad f(x) = |2x|^\alpha \quad (\text{power-law})\,; \quad f(x) = e^{\alpha(x^2-1/4)} \quad (\text{exponential})\,, \tag{6.5.4a}$$

$$(\text{Unit Disk}): \quad f(x) = |x|^\alpha \quad (\text{power-law})\,; \quad f(x) = e^{\alpha(|x|^2-1)} \quad (\text{exponential})\,, \tag{6.5.4b}$$

with $\alpha \geq 0$. For above choices of domain Ω and profile $f(x)$, since the extremal solution $u^*(x)$ of (6.5.3) is unique, Lemma 2.3.6 shows that $u^*(x)$ must be radially symmetric. Therefore, the pull-in distance $\mathcal{D}$ of (6.5.3) satisfies $\mathcal{D} = |u^*(0)|$.

(a). Exponential Profiles:

Ω	α	λ^*
slab	0	1.401
slab	3	2.637
slab	6	4.848
slab	10	10.40
unit disk	0	0.789
unit disk	3	6.096
unit disk	4.8	15.114
unit disk	5.6	20.942

(b). Power-Law Profiles:

Ω	α	λ^*
slab	0	1.401
slab	1	4.388
slab	3	15.189
slab	6	43.087
unit disk	0	0.789
unit disk	1	1.775
unit disk	5	9.676
unit disk	20	95.66

Table 6.4: *Numerical values for pull-in voltage λ^*: Table (a) corresponds to exponential profiles, while Table (b) corresponds to power-law profiles.*

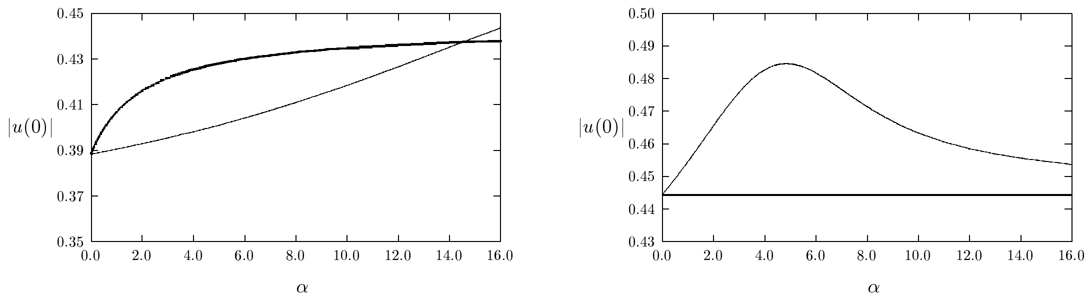

Figure 6.4: *Plots of the pull-in distance $|u(0)| = |u^*(0)|$ versus α for the power-law profile (heavy solid curve) and the exponential profile (solid curve). Left figure: the slab domain. Right figure: the unit disk.*

As in §2.2, using Newton's method and COLSYS [6] to solve the boundary value problem (6.5.3), we first numerically calculate λ^* of (6.5.3) as the saddle-node point. We give numerical values of λ^* in Table 6.3(a) for exponential profiles and in Table 6.3(b) for power-law profiles, respectively. For the slab domain, in Figure 6.4(a) we plot $\mathcal{D} = |u(0)| = |u^*(0)|$ versus α for both the power-law and the exponential conductivity profile $f(x)$ in the slab domain, which show that the pull-in distance $\mathcal{D}$ can be increased by increasing the value of α (and hence by increasing the range of $f(x) \ll 1$). A similar plot of $\mathcal{D} = |u(0)| = |u^*(0)|$ versus α is shown in Figure 6.4(b) for the unit disk. For the power-law profile in the unit disk we observe that $|u(0)| \approx 0.444$ for any $\alpha > 0$. Therefore, rather curiously, the power-law profile does not increase the pull-in distance for the unit disk. For the exponential profile we observe from Figure 6.4(b) that the pull-in distance is not a monotonic function of α. The maximum value occurs at $\alpha \approx 4.8$ where $\lambda^* \approx 15.11$ (see Figure 2.1(b)) and $\mathcal{D} = |u(0)| = 0.485$. For $\alpha = 0$, we have $\lambda^* \approx 0.789$ and $|u(0)| = 0.444$. Therefore, since λ^* is proportional to V^2 (cf. §1.1.2) we conclude that the exponential permittivity profile for the unit disk can increase the pull-in distance by roughly 9% if the voltage is increased by roughly a factor of four.

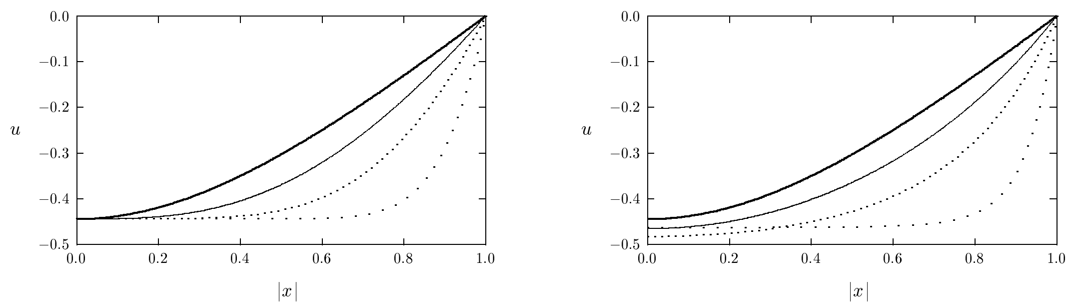

Figure 6.5: *Left figure: plots of* u *versus* $|x|$ *at* $\lambda = \lambda^*$ *for* $\alpha = 0$, $\alpha = 1$, $\alpha = 3$, *and* $\alpha = 10$, *in the unit disk for the power-law profile. Right figure: plots of* u *versus* $|x|$ *at* $\lambda = \lambda^*$ *for* $\alpha = 0$, $\alpha = 2$, $\alpha = 4$, *and* $\alpha = 10$, *in the unit disk for the exponential profile. In both figures the solution develops a boundary-layer structure near* $|x| = 1$ *as* α *is increased.*

For the unit disk, in Figure 6.5(a) we plot u versus $|x|$ at $\lambda = \lambda^*$ with four values of α for the power-law profile. Notice that $u(0)$ is the same for each of these values of α. A similar plot is shown in Figure 6.5(b) for the exponential permittivity profile. From these figures, we observe that u has a boundary-layer structure when $\alpha \gg 1$. In this limit, $f(x) \ll 1$ except in a narrow zone near the boundary of the domain. For $\alpha \gg 1$ the pull-in distance $\mathcal{D} = |u(0)|$ also reaches some limiting value (see Figures 6.4 & 6.5). For the slab domain with an exponential permittivity profile, we remark that the limiting asymptotic behavior of $|u(0)|$ for $\alpha \gg 1$ is beyond the range shown in Figure 6.4(a).

For $\alpha \gg 1$, we now use a boundary-layer analysis to determine a scaling law of λ^* for both types of permittivity profiles and for either a slab domain or the unit disk. We illustrate the analysis for a power-law permittivity profile in the unit disk. For $\alpha \gg 1$, there is an outer region defined by $0 \le r \ll 1 - O(\alpha^{-1})$, and an inner region where $r - 1 = O(1/\alpha)$. In the outer region, where $\lambda r^\alpha \ll 1$, (6.5.3) reduces asymptotically to $\Delta u = 0$. Therefore, the leading-order outer solution is a constant $u = A$. In the inner region, we introduce new variables w and ρ by

$$w(\rho) = u\,(1 - \rho/\alpha)\,, \qquad \rho = \alpha(1 - r)\,. \tag{6.5.5}$$

Substituting (6.5.5) into (6.5.3) with $f(r) = r^\alpha$, using the limiting behavior $(1-\rho/\alpha)^\alpha \to e^{-\rho}$

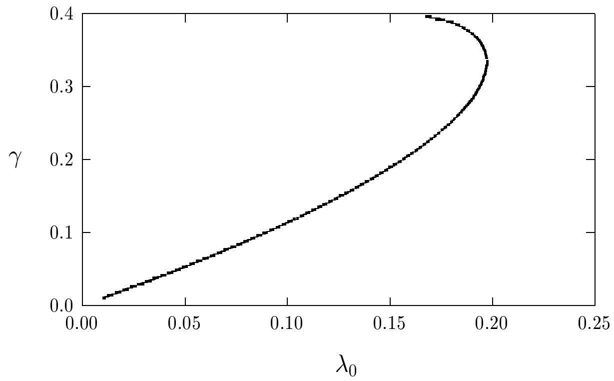

Figure 6.6: *Bifurcation diagram of* $w'(0) = -\gamma$ *versus* λ_0 *from the numerical solution of (6.5.6).*

as $\alpha \to \infty$, and defining $\lambda = \alpha^2 \lambda_0$, we obtain the leading-order boundary-layer problem

$$w'' = \frac{\lambda_0 e^{-\rho}}{(1+w)^2}, \quad 0 \le \rho < \infty; \quad w(0) = 0, \quad w'(\infty) = 0, \quad \lambda = \alpha^2 \lambda_0. \tag{6.5.6}$$

In terms of the solution to (6.5.6), the leading-order outer solution is $u = A = w(\infty)$.

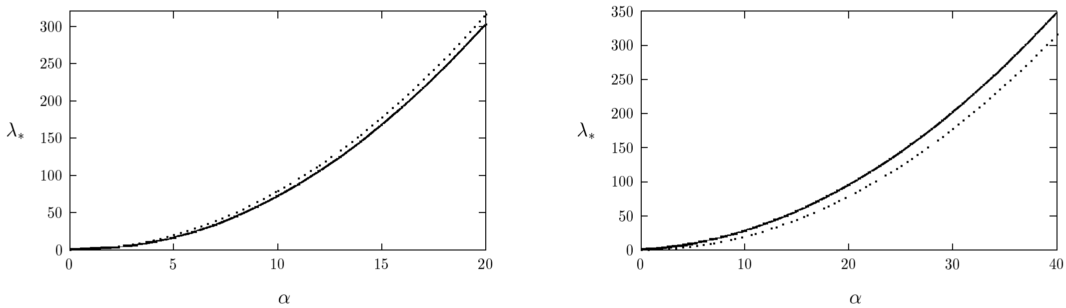

Figure 6.7: *Comparison of numerically computed* λ^* *(heavy solid curve) with the asymptotic result (dotted curve) from (6.5.7) for the unit disk. Left figure: the exponential profile. Right figure: the power-law profile.*

We define γ by $w'(0) = -\gamma$ for $\gamma > 0$, and we solve (6.5.6) numerically using COLSYS [6] to determine $\lambda_0 = \lambda_0(\gamma)$. In Figure 6.6 we plot $\lambda_0(\gamma)$ and show that this curve has a saddle-node point at $\lambda_0 = \lambda_0^* \equiv 0.1973$. At this value, we compute $w(\infty) \approx 0.445$, which sets the limiting membrane deflection for $\alpha \gg 1$. Therefore, (6.5.6) shows that for $\alpha \gg 1$, the saddle-node value has the scaling law behavior $\lambda^* \sim 0.1973\alpha^2$ for a power-law profile in the unit disk. A similar boundary-layer analysis can be done to determine the scaling law of λ^* when $\alpha \gg 1$ for other cases. In each case we can relate λ^* to the saddle-node value of the boundary-layer problem (6.5.6). In this way, for $\alpha \gg 1$, we obtain

$$\lambda^* \sim 4(0.1973)\alpha^2\,, \quad \bar{\lambda}_2 \sim \frac{4\alpha^2}{3}\,, \quad \text{(power-law, slab), (exponential, unit disk)}\,, \qquad (6.5.7a)$$

$$\lambda^* \sim (0.1973)\alpha^2\,, \quad \bar{\lambda}_2 \sim \frac{\alpha^2}{3}\,, \quad \text{(power-law, unit disk), (exponential, slab)}\,, \qquad (6.5.7b)$$

Notice that $\bar{\lambda}_2 = O(\alpha^2)$, with a factor that is about 5/3 times as large as the multiplier of α^2 in the asymptotic formula for λ^*. In Figure 6.7, we compare the computed λ^* as a saddle-node point with the asymptotic result of λ^* from (6.5.7).

Next we present a few of numerical results for pull-in distance of dynamic problem (6.2.1) by applying the implicit Crank-Nicholson scheme again. Here we always consider the domain and the profile defined by (6.2.2) and (6.5.4), respectively. We choose the mesh-points $N = 4000$ and the applied voltage $\lambda = \lambda^*$ given in Table 6.3:

Figure 6.8: Case of exponential profiles
We consider pull-in distance of (6.2.1) for exponential profiles in the slab or unit disk domain. In Figure 6.8(a) we plot u versus x at the time $t = 80$ in the slab domain, with $\alpha = 0$ (solid line), $\alpha = 3$ (dashed line), $\alpha = 6$ (dotted line) and $\alpha = 10$ (dash-dot line), respectively. This figure and Figure 6.4(a) show that pull-in distance is increasing in α. In Figure 6.8(b) we plot u versus $|x|$ at the time $t = 80$ in the unit disk domain, with $\alpha = 0$ (dash-dot line), $\alpha = 3$ (dashed line), $\alpha = 4.8$ (dotted line) and $\alpha = 5.6$ (solid line), respectively. In this figure we observe that the solution develops a boundary-layer structure near the boundary of the domain as α is increased, and pull-in distance is not a monotonic function of α. Actually from Figure 6.4(b) we know that pull-in distance is first increasing and then decreasing in α. The maximum value of pull-in distance occurs at $\alpha \approx 4.8$ and $\lambda^* \approx 15.114$.

Figure 6.9: Case of power-law profiles
We consider pull-in distance of the membrane for power-law profiles in the slab or unit disk domain. In Figure 6.9(a) we plot u versus x at the time $t = 80$ in the slab domain, with $\alpha = 0$ (solid line), $\alpha = 1$ (dashed line), $\alpha = 3$ (dash-dot line) and $\alpha = 6$ (dotted line), respectively. This figure and Figure 6.4(a) show that pull-in distance is increasing in α. In Figure 6.9(b) we plot u versus $|x|$ at the time $t = 80$ in the unit disk domain, with $\alpha = 0$ (dotted line), $\alpha = 1$ (dash-dot line), $\alpha = 5$ (dashed line) and $\alpha = 20$ (solid line), respectively. For the power-law profiles in the unit disk domain, we observe that pull-in distance is a constant for any $\alpha \geq 0$. Therefore, with Figure 6.4(b), it is rather curious that power-law profile does not change pull-in distance in the unit disk domain. In both figures, the solution develops a boundary-layer structure near the boundary of the domain as α in increased.

Since one of the primary goals of MEMS design is to maximize the pull-in distance over a certain allowable voltage range that is set by the power supply, it would be interesting

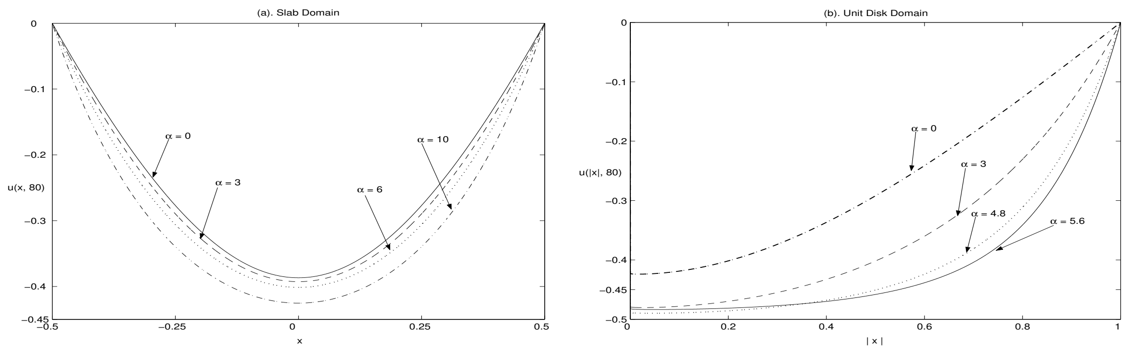

Figure 6.8: *Left figure: plots of u versus x at $\lambda = \lambda^*$ in the slab domain. Right figure: plots of u versus $|x|$ at $\lambda = \lambda^*$ in the unit disk domain.*

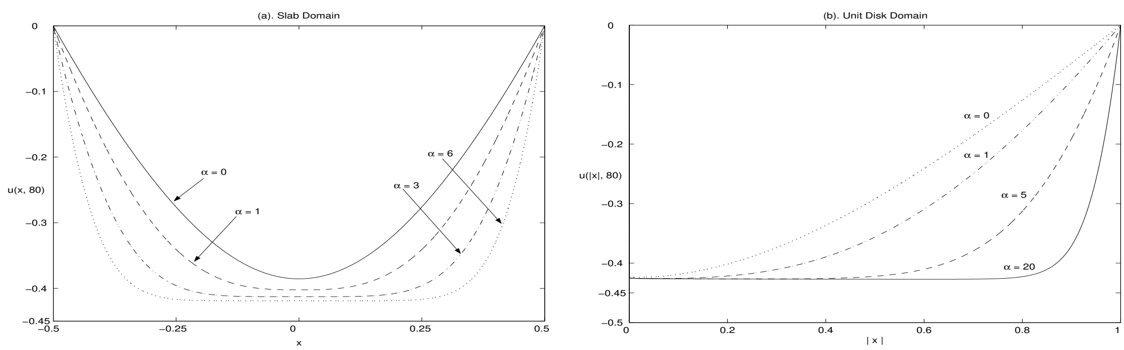

Figure 6.9: *Left figure: plots of u versus x at $\lambda = \lambda^*$ in the slab domain. Right figure: plots of u versus $|x|$ at $\lambda = \lambda^*$ in the unit disk domain.*

to formulate an optimization problem that computes a dielectric permittivity $f(x)$ that maximizes the pull-in distance for a prescribed range of the saddle-node threshold λ^*.

6.6 Some comments

Main results of this Chapter are available in [52, 64]. The discussion of dynamic solutions for (6.0.1) at $\lambda = \lambda^*$ is due to H. Brezis and etc. in [13], where they dealt with parabolic problems with regular nonlinearities. The analysis result has shown that for dimension $N \geq 8$, the unique solution of (6.0.1) can touchdown at infinite time at the zero point of $f(x)$. Our proof of finite-time touchdown for solutions of (6.0.1) at $\lambda > \lambda^*$ is motivated by H. Bellout's work [11], where the concavity method was used to discuss finite-time blow-up behavior with regular nonlinearities. However, in our situation we have developed a delicate trick to deal with the assumption that the profile $f(x)$ can vanish at somewhere. There are many open problems for arbitrary nonzero initial data.

Due to Guo, Pan and Ward [64], the upper bounds of finite touchdown time were established only for sufficiently large λ, *i.e*,

$$\lambda > \bar{\lambda} := min\{\frac{4\mu_\Omega}{27\inf_{x\in\Omega} f(x)}, \frac{\mu_\Omega}{3\int_\Omega f\phi_\Omega dx}\},$$

see Proposition 6.3.3 for more details. Since numerical results were shown that $\bar{\lambda} > \lambda^*$, it is desirable to fill this gap by establishing upper bounds of finite touchdown time in the case $\lambda^* < \lambda \leq \bar{\lambda}$. The finite touchdown time is expected to go to infinity as λ decreases to λ^*.

The asymptotic analysis of blow-up profiles can be traced back to [77]. And it was first applied in [64] to our singular nonlinearity case, where the following interesting phenomenon was first observed and discussed: zero point of profile $f(x)$ can not be a touchdown point. As far as we know, such a phenomenon has not been completely proved, which we shall further discuss in next Chapter.

The pull-in distance discussed in §6.5 follows from [64], where some interesting phenomena have been first observed too. For example, for the case where $f(x) = |x|^\alpha \geq 0$ and Ω is a unit disk in $\mathbb{R}^2$, Figure 6.4 shows that pull-in distance of (6.0.1) is independent of the value α; While Figure 6.8 & 6.9 show that the solution of (6.0.1) develops a boundary-layer structure near the boundary of the domain as α is increased.

Chapter 7

Refined Touchdown Behavior

In this Chapter, we continue the study of dynamic solutions of (1.2.17) in the form

$$u_t - \Delta u = -\frac{\lambda f(x)}{u^2} \qquad \text{for} \quad x \in \Omega\,, \tag{7.0.1a}$$

$$u(x,t) = 1 \qquad \text{for} \quad x \in \partial\Omega\,, \tag{7.0.1b}$$

$$u(x,0) = 1 \qquad \text{for} \quad x \in \Omega\,. \tag{7.0.1c}$$

where the permittivity profile $f(x)$ is allowed to vanish somewhere, and will be assumed to satisfy

$$\begin{aligned} &f \in C^{\alpha}(\bar{\Omega}) \text{ for some } \alpha \in (0,1],\ 0 \le f \le 1 \text{ and} \\ &f > 0 \text{ on a subset of } \Omega \text{ of positive measure.} \end{aligned} \tag{7.0.2}$$

We consider the case $\lambda > \lambda^*$ such that a unique solution u of (7.0.1) must touchdown at finite time $T = T(\lambda, \Omega, f)$ in the sense

Definition 7.0.1. *A solution $u(x,t)$ of (7.0.1) is said to touchdown at finite time $T = T(\lambda, \Omega, f)$ if the minimum value of u reaches 0 at the time $T < \infty$.*

We shall give a refined description of finite-time touchdown behavior for u satisfying (7.0.1), including some touchdown estimates, touchdown rates, as well as some information on the properties of touchdown set –such as compactness, location and shape.

This Chapter is organized as follows: the purpose of §7.1 is mainly to derive some a priori estimates of touchdown profiles under the assumption that touchdown set of u is a compact subset of Ω. Note that whether the compactness of touchdown set holds for any $f(x)$ satisfying (7.0.2) is a quite challenging problem. In §7.1 we first prove in Proposition 7.1.1 that the compactness of touchdown set holds for the case where the domain Ω is convex and $f(x)$ satisfies the additional condition

$$\frac{\partial f}{\partial \nu} \le 0 \quad \text{on} \quad \Omega_\delta^c := \{x \in \Omega : dist(x, \partial\Omega) \le \delta\} \text{ for some } \delta > 0. \tag{7.0.3}$$

Here ν is the outward unit norm vector to $\partial\Omega$.

Under the compactness assumption of touchdown set, in §7.1.1 we establish the lower bound estimate of touchdown profiles and we also prove an interesting phenomenon: finite-time touchdown point of u is not the zero point of $f(x)$, see Theorem 7.1.3. In §7.1.2 we estimate the derivatives of touchdown solution u, see Lemma 7.1.5; and as a byproduct, an integral estimate is also given in Theorem 7.1.6 of §7.1.2.

Motivated by Theorem 7.1.3, the key point of studying touchdown profiles is a similarity variable transformation of (7.0.1). For the touchdown solution $u = u(x,t)$ of (7.0.1) at finite time T, we use the associated similarity variables

$$y = \frac{x-a}{\sqrt{T-t}}, \quad s = -log(T-t), \quad u(x,t) = (T-t)^{\frac{1}{3}} w_a(y,s), \tag{7.0.4}$$

where a is any interior point of Ω. Then $w_a(y,s)$ is defined in $W_a := \{(y,s) : a + ye^{-s/2} \in \Omega, s > s' = -logT\}$, and it solves

$$\rho(w_a)_s - \nabla\cdot(\rho\nabla w_a) - \frac{1}{3}\rho w_a + \frac{\lambda\rho f(a+ye^{-\frac{s}{2}})}{w_a^2} = 0,$$

where $\rho(y) = e^{-|y|^2/4}$. Here $w_a(y,s)$ is always strictly positive in W_a. The slice of W_a at a given time s^1 is denoted by $\Omega_a(s^1) := W_a \cap \{s = s^1\} = e^{s^1/2}(\Omega - a)$. Then for any interior point a of Ω, there exists $s_0 = s_0(a) > 0$ such that $B_s := \{y : |y| < s\} \subset \Omega_a(s)$ for $s \ge s_0$. We introduce the frozen energy functional

$$E_s[w_a](s) = \frac{1}{2}\int_{B_s}\rho|\nabla w_a|^2 dy - \frac{1}{6}\int_{B_s}\rho w_a^2 dy - \int_{B_s}\frac{\lambda\rho f(a)}{w_a}dy. \tag{7.0.5}$$

By estimating the energy $E_s[w_a](s)$ in B_s, in §7.1.3 we shall prove the upper bound estimate of w_a, see Theorem 7.1.11.

In order to study touchdown behavior of solutions for (7.0.1), it is quite necessary to study the asymptotic behavior of positive entire solutions for the semilinear elliptic problem

$$\Delta w - \frac{1}{2}y\cdot\nabla w + \frac{w}{3} - \frac{1}{w^2} = 0 \quad \text{in} \quad \mathbb{R}^N, \; N \ge 1. \tag{7.0.6}$$

This is the main goal of §7.2, where for generality we shall discuss whether every non-constant positive entire solution of (7.2.2) must be strictly increasing for all $|y|$ sufficiently large.

Applying the results of §§7.1 & 7.2, we shall establish refined touchdown profiles in §7.3, where self-similar method and center manifold analysis will be applied. Note that the uniqueness of solutions for (7.0.1) gives the radial symmetry of u in Theorem 7.3.5. It should remark from Theorem 7.3.5 that for $N \ge 2$, we are only able to discuss the refined touchdown profiles for special touchdown point $x = 0$ in the radial situation, and it seems unknown for the general case.

Adapting various analytical and numerical techniques, §7.4 will be focused on the set of touchdown points. This may provide useful information on the design of MEMS devices.

In §7.4.1 we discuss the radially symmetric case of (7.0.1), and we prove there that suppose $f(r) = f(|x|)$ satisfies (7.0.2) and $f'(r) \leq 0$ in a bounded ball $B_R(0) \subset \mathbb{R}^N$ with $N \geq 1$, then $r = 0$ is the unique touchdown point of u, which is the maximum value point of $f(r) = f(|x|)$, see Theorem 7.4.2 and Remark 7.4.1.

For one dimensional case, Theorem 7.4.2 already implies that touchdown points must be unique when permittivity profile $f(x)$ is uniform. In §7.4.2 we further discuss one dimensional case of (7.0.1) for varying profile $f(x)$, where numerical simulations show that touchdown points may be composed of finite points or finite compact subsets of the domain.

7.1 A priori estimates of touchdown behavior

Under the assumption that touchdown set of u is a compact subset of Ω, in this section we study some a priori estimates of touchdown behavior, and establish the claims in Theorems 7.1.3 and 7.1.11. In §7.1.1 we establish a lower bound estimate, from which we complete the proof of Theorem 7.1.3. Using the lower bound estimate, in subsection §7.1.2 we shall prove some estimates for the derivatives of touchdown solution u, and an integration estimate will be also obtained as a byproduct. In subsection §7.1.3 we shall study the upper bound estimate by energy methods, which gives Theorem 7.1.11.

We first prove the following compactness result for a large class of profiles $f(x)$ satisfying (7.0.2) and

$$\frac{\partial f}{\partial \nu} \leq 0 \quad \text{on} \quad \Omega_\delta^c := \{x \in \Omega : \, dist(x, \partial\Omega) \leq \delta\} \text{ for some } \delta > 0. \tag{7.1.1}$$

Proposition 7.1.1. *Assume f satisfies* (7.0.2) *and* (7.1.1) *on a bounded convex domain Ω, and suppose u is a touchdown solution of* (7.0.1) *at finite time T. Then, the set of touchdown points for u is a compact subset of Ω.*

Proof: We prove Proposition 7.1.1 by adapting moving plane method from Theorem 3.3 in [48], where it is used to deal with blow-up problems. Take any point $y_0 \in \partial\Omega$, and assume for simplicity that $y_0 = 0$ and that the half space $\{x_1 > 0\}$ ($x = (x_1, x')$) is tangent to Ω at y_0. Let $\Omega_\alpha^+ = \Omega \cap \{x_1 > \alpha\}$ where $\alpha < 0$ and $|\alpha|$ is small, and also define $\Omega_\alpha^- = \{(x_1, x') : (2\alpha - x_1, x') \in \Omega_\alpha^+\}$, the reflection of Ω_α^+ with respect to the plane $\{x_1 = \alpha\}$, where $x' = (x_2, \cdots, x_N)$.

Consider the function

$$w(x,t) = u(2\alpha - x_1, x', t) - u(x_1, x', t)$$

for $x \in \Omega_\alpha^-$, then w satisfies

$$w_t - \Delta w = \frac{\lambda(u(x_1, x', t) + u(2\alpha - x_1, x', t))f(x)}{u^2(x_1, x', t)u^2(2\alpha - x_1, x', t)} w \, .$$

It is clear that $w = 0$ on $\{x_1 = \alpha\}$. Since $u(x,t) = 1$ along $\partial\Omega$ and since the maximum principle gives $u_t < 0$ for $0 < t < T$, we may choose a small $t_0 > 0$ such that

$$\frac{\partial u(x, t_0)}{\partial \nu} > 0 \quad \text{along} \quad \partial\Omega \, , \tag{7.1.2}$$

where ν is the outward unit norm vector to $\partial\Omega$. Then for sufficiently small $|\alpha|$, (7.1.2) implies that $w(x,t_0) \geq 0$ in Ω_α^- and also $w = 1 - u(x_1, x', t) > 0$ on $(\partial\Omega_\alpha^- \cap \{x_1 < \alpha\}) \times (t_0, T)$. Applying the maximal principle we now conclude that $w > 0$ in $\Omega_\alpha^- \times (t_0, T)$ and $\frac{\partial w}{\partial x_1} = -2\frac{\partial u}{\partial x_1} < 0$ on $\{x_1 = \alpha\}$. Since α is arbitrary, it follows by varying α that

$$\frac{\partial u}{\partial x_1} > 0\,, \quad (x,t) \in \Omega_{\alpha_0}^+ \times (t_0, T) \tag{7.1.3}$$

provided $|\alpha_0| = |\alpha_0(t_0)| > 0$ is sufficiently small.

Fix $0 < |\alpha_0| \leq \delta$, where δ is as in (7.1.1), we now consider the function

$$J = u_{x_1} - \varepsilon_1(x_1 - \alpha_0) \quad \text{in} \quad \Omega_{\alpha_0}^+ \times (t_0, T)\,,$$

where $\varepsilon_1 = \varepsilon_1(\alpha_0, t_0) > 0$ is a constant to be determined later. The direct calculations show that

$$J_t - \Delta J = \frac{2\lambda f}{u^3} u_{x_1} - \frac{\lambda f_{x_1}}{u^2} = \frac{2\lambda f}{u^3} u_{x_1} - \frac{\lambda}{u^2}\frac{\partial f}{\partial \nu}\frac{\partial \nu}{\partial x_1} \geq 0 \quad \text{in} \quad \Omega_{\alpha_0}^+ \times (t_0, T) \tag{7.1.4}$$

due to (7.1.1). Therefore, J can not attain negative minimum in $\Omega_{\alpha_0}^+ \times (t_0, T)$. Next, $J > 0$ on $\{x_1 = \alpha_0\}$ by (7.1.3). Since (7.1.2) gives $\frac{\partial u(x,t_0)}{\partial x_1} \geq C > 0$ along $(\partial\Omega_{\alpha_0}^+ \cap \partial\Omega)$ for some $C > 0$, we have $J > 0$ on $\{t = t_0\}$ provided $\varepsilon_1 = \varepsilon_1(\alpha_0, t_0) > 0$ is sufficiently small. We now claim that for small $\varepsilon_1 > 0$,

$$J > 0 \quad \text{on} \quad (\partial\Omega_{\alpha_0}^+ \cap \partial\Omega) \times (t_0, T)\,. \tag{7.1.5}$$

To prove (7.1.5), we compare the solution $U := 1 - u$ satisfying

$$\begin{aligned} U_t - \Delta U &= \frac{\lambda f(x)}{(1-U)^2} \quad (x,t) \in \Omega \times (t_0, T), \\ U(x,t_0) &= 1 - u(x,t_0)\,; \quad U(x,t) = 0 \quad x \in \partial\Omega \end{aligned}$$

with the solution v of the heat equation

$$v_t = \Delta v\,, \quad (x,t) \in \Omega \times (t_0, T)\,,$$

where $0 \leq v(x,t_0) = U(x,t_0) < 1$ and $v = 0$ on $\partial\Omega$. Then we have $U \geq v$ in $\Omega \times (t_0, T)$. Consequently,

$$\frac{\partial U}{\partial \nu} \leq \frac{\partial v}{\partial \nu} \leq -C_0 < 0 \quad \text{on} \quad (\partial\Omega_{\alpha_0}^+ \cap \partial\Omega) \times (t_0, T)\,,$$

and hence $\frac{\partial u}{\partial \nu} \geq C_0 > 0$ on $(\partial\Omega_{\alpha_0}^+ \cap \partial\Omega) \times (t_0, T)$. It then follows that $J \geq C_0 \frac{\partial \nu}{\partial x_1} - \varepsilon_1(x_1 - \alpha_0) > 0$ provided $\varepsilon_1 = \varepsilon_1(\alpha_0, t_0)$ is small enough, which gives (7.1.5).

The maximum principle now yields that there exists $\varepsilon_1 = \varepsilon_1(\alpha_0, t_0) > 0$ so small that $J \geq 0$ in $\Omega_{\alpha_0}^+ \times (t_0, T)$, *i.e.*,

$$u_{x_1} \geq \varepsilon_1(x_1 - \alpha_0)\,, \tag{7.1.6}$$

if $x' = 0$ and $\alpha_0 \le x_1 < 0$. Integrating (7.1.6) with respect to x_1 on $[\alpha_0, y_1]$, where $\alpha_0 < y_1 < 0$, yields that

$$u(y_1, 0, t) - u(\alpha_0, 0, t) \ge \frac{\varepsilon_1}{2}|y_1 - \alpha_0|^2.$$

It follows that

$$\underline{\lim}_{t\to T^-} u(0,t) = \underline{\lim}_{t\to T^-} \lim_{y_1\to 0^-} u(y_1, 0, t) \ge \varepsilon_1\alpha_0^2/2 > 0,$$

which shows that $y_0 = 0$ can not be a touchdown point of $u(x,t)$.

The proof of (7.1.3) can be slightly modified to show that $\frac{\partial u}{\partial \nu} > 0$ in $\Omega^+_{\alpha_0} \times (t_0, T)$ for any direction ν close enough to the x_1-direction. Together with (7.1.1), this enables us to deduce that any point in $\{x' = 0, \alpha_0 < x_1 < 0\}$ can not be a touchdown point. Since above proof shows that α_0 can be chosen independently of initial point y_0 on $\partial\Omega$, by varying y_0 along $\partial\Omega$ we deduce that there is an $\Omega-$Neighborhood Ω' of $\partial\Omega$ such that each point $x \in \Omega'$ can not be a touchdown point. This completes the proof of Proposition 7.1.1. ■

Remark 7.1.1. When $f(x)$ does not satisfy (7.1.1), the compactness of touchdown set is numerically observed, see numerical simulations in Chapter 5 or in §6.4 of the present paper. Therefore, it is our conjecture that under the convexity of Ω, the compactness of touchdown set may hold for any $f(x)$ satisfying (7.0.2).

7.1.1 Lower bound estimate

Define for $\eta > 0$,

$$\Omega_\eta := \{x \in \Omega : dist(x, \partial\Omega) > \eta\}, \quad \Omega^c_\eta := \{x \in \Omega : dist(x, \partial\Omega) \le \eta\}. \tag{7.1.7}$$

Since touchdown set of u is assumed to be a compact subset of Ω, in the rest of this section we may choose a small $\eta > 0$ such that any touchdown point of u must lie in Ω_η. Our first aim of this subsection is to prove that any point $x_0 \in \bar{\Omega}_\eta$ satisfying $f(x_0) = 0$ can not be a touchdown point of u at finite time T, which then leads to the following proposition.

Proposition 7.1.2. *Assume f satisfies* (7.0.2) *on a bounded domain Ω, and suppose $u(x,t)$ is a touchdown solution of* (7.0.1) *at finite time T. If touchdown set of u is a compact subset of Ω, then any point $x_0 \in \bar{\Omega}$ satisfying $f(x_0) = 0$ cannot be a touchdown point of $u(x,t)$.*

Proof: Since touchdown set of u is assumed to be a compact subset of Ω, it now suffices to discuss the point x_0 lying in the interior domain Ω_η for some small $\eta > 0$, such that there is no touchdown point on Ω^c_η.

For any $t_1 < T$, we first recall that the maximum principle gives $u_t < 0$ for all $(x,t) \in \Omega \times (0, t_1)$. Further, the boundary point lemma shows that the outward normal derivative of $v = u_t$ on $\partial\Omega$ is positive for $t > 0$. This implies that for taking small $0 < t_0 < T$, there exists a positive constant $C = C(t_0, \eta)$ such that $u_t(x, t_0) \le -C < 0$ for all $x \in \bar{\Omega}_\eta$. For any $0 < t_0 < t_1 < T$, we next claim that there exists $\varepsilon = \varepsilon(t_0, t_1, \eta) > 0$ such that

$$J^\epsilon(x,t) = u_t + \frac{\varepsilon}{u^2} \le 0 \quad \text{for all} \quad (x,t) \in \Omega_\eta \times (t_0, t_1)\,. \tag{7.1.8}$$

Indeed, it is now clear that there exists $C_\eta = C_\eta(t_0, t_1, \eta) > 0$ such that $u_t(x,t) \le -C_\eta$ on the parabolic boundary of $\Omega_\eta \times (t_0, t_1)$. And further, we can choose $\varepsilon = \varepsilon(t_0, t_1, \eta) > 0$ so small that $J^\epsilon \le 0$ on the parabolic boundary of $\Omega_\eta \times (t_0, t_1)$, due to the local boundedness of $\frac{1}{u^2}$ on $\partial\Omega_\eta \times (t_0, t_1)$. Also, direct calculations imply that

$$J^\epsilon_t - \Delta J^\epsilon = \frac{2\lambda f}{u^3} J^\epsilon - \frac{6\varepsilon|\nabla u|^2}{u^4} \le \frac{2\lambda f}{u^3} J^\epsilon\,.$$

Now (7.1.8) follows again from the maximum principle.

Combining (7.1.8) and (7.0.1) we deduce that for a small neighborhood B of x_0 where $\lambda f(x) \le \varepsilon/2$ is in $B \subset \bar{\Omega}_\eta$, we have for $v := 1 - u$,

$$\Delta v \ge \frac{\varepsilon}{2} \frac{1}{(1-v)^2}\,, \quad (x,t) \in B \times (t_0, t_1)\,.$$

Now Proposition 7.1.2 is a direct result of Lemma 6.2.2, since $t_1 < T$ is arbitrary. ■

Essentially, the claim (7.1.8) is ready to give a lower bound estimate, from which we obtain the following theorem.

Theorem 7.1.3. *Assume f satisfies* (7.0.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (7.0.1) *at finite time T. If touchdown set of u is a compact subset of Ω, then*

1. *any point $a \in \bar{\Omega}$ satisfying $f(a) = 0$ is not a touchdown point for $u(x,t)$;*
2. *there exists a bounded positive constant M such that*

$$M(T-t)^{\frac{1}{3}} \le u(x,t) \quad in \quad \Omega \times (0,T)\,. \tag{7.1.9}$$

Proof: In view of Proposition 7.1.2, it now needs only to prove the lower bound estimate (7.1.9).

Given any small $\eta > 0$, applying the same argument used for (7.1.8) yields that for any $0 < t_0 < t_1 < T$, there exists $\varepsilon = \varepsilon(t_0, t_1, \eta) > 0$ such that

$$u_t \le -\frac{\varepsilon}{u^2} \quad \text{in} \quad \Omega_\eta \times (t_0, t_1).$$

This inequality shows that $u_t \to -\infty$ as u touchdown, and there exists $M > 0$ such that

$$M_1(T-t)^{\frac{1}{3}} \le u(x,t) \quad \text{in} \quad \Omega_\eta \times (0,T) \tag{7.1.10}$$

due to the arbitrary of t_0 and t_1, where M_1 depends only on λ, f and η. Furthermore, one can obtain (7.1.9) because of the boundedness of u on $\Omega^c_{\eta}.$, and the theorem is proved. ■

7.1.2 Gradient estimates

As a preliminary of next section, it is now important to know a priori estimates for the derivatives of touchdown solution u, which are the contents of this subsection. Following the analysis in [48], our first lemma is about the derivatives of first order without the compactness assumption of touchdown set.

Lemma 7.1.4. *Assume f satisfies (7.0.2) on a bounded convex domain Ω, and suppose u is a touchdown solution of (7.0.1) at finite time T. Then for any $0 < t_0 < T$, there exists a bounded constant $C > 0$ such that*

$$\frac{1}{2}|\nabla u|^2 \le \frac{C}{\underline{u}} - \frac{C}{u} \quad in \quad \Omega \times (0, t_0)\,, \tag{7.1.11}$$

where $\underline{u} = \underline{u}(t_0) = \min_{x\in\Omega} u(x, t_0)$, and C depends only on λ, f and Ω.

Proof: Fix any $0 < t_0 < T$ and treat $\underline{u}(t_0)$ as a fixed constant. Let $w = u - \underline{u}$, then w satisfies

$$\begin{aligned} w_t - \Delta w &= -\frac{\lambda f(x)}{(w+\underline{u})^2} && \text{in} \quad \Omega \times (0, t_0)\,, \\ w &= 1 - \underline{u} && \text{in} \quad \partial\Omega \times (0, t_0)\,, \\ w(x, 0) &= 1 - \underline{u} && \text{in} \quad \Omega\,. \end{aligned}$$

We introduce the function

$$P = \frac{1}{2}|\nabla w|^2 + \frac{C}{w+\underline{u}} - \frac{C}{\underline{u}}\,, \tag{7.1.12}$$

where the bounded constant $C \ge 2\lambda \sup_{x\in\bar\Omega} f$ will be determined later. Then we have

$$\begin{aligned} P_t - \Delta P &= \frac{C\lambda f(x)}{(w+\underline{u})^4} - \frac{\lambda \nabla f(x)\nabla w}{(w+\underline{u})^2} + \frac{2(\lambda f(x) - C)|\nabla w|^2}{(w+\underline{u})^3} - \sum_{i,j=1}^N w_{ij}^2 \\ &\le \frac{\lambda C \sup_{x\in\bar\Omega} f}{(w+\underline{u})^4} + \frac{-2\lambda|\nabla w|^2 \sup_{x\in\bar\Omega} f + \lambda|\nabla w| \sup_{x\in\bar\Omega}|\nabla f|}{(w+\underline{u})^3} - \sum_{i,j=1}^N w_{ij}^2 \\ &\le \frac{\lambda(C\sup_{x\in\bar\Omega} f + C_1)}{(w+\underline{u})^4} - \sum_{i,j=1}^N w_{ij}^2\,, \end{aligned} \tag{7.1.13}$$

where $C_1 := \frac{(\sup_{x\in\bar\Omega}|\nabla f|)^2}{8\sup_{x\in\bar\Omega} f} \ge 0$ is bounded. Since (7.1.12) gives

$$\sum_{i=1}^N \big(P_i + \frac{C}{(w+\underline{u})^2} w_i\big)^2 = \sum_{i,j=1}^N (w_j w_{ij})^2 \le |\nabla w|^2 \sum_{i,j=1}^N w_{ij}^2\,, \tag{7.1.14}$$

we now take

$$C := \max\Big\{2\lambda \sup_{x\in\bar\Omega} f,\ \frac{\lambda \sup_{x\in\bar\Omega} f + \lambda\sqrt{(\sup_{x\in\bar\Omega} f)^2 + 4C_1}}{2}\Big\} \ge 2\lambda \sup_{x\in\bar\Omega} f$$

so that $C^2 \geq \lambda(C \sup_{x\in\bar\Omega} f + C_1)$, where C clearly depends only on λ, f and Ω. From the choice of C, a combination of (7.1.13) and (7.1.14) gives that

$$P_t - \Delta P \leq \overrightarrow{b} \cdot \nabla P \,,$$

where $\overrightarrow{b} = -|\nabla w|^{-2}(\nabla P + \frac{2C\nabla w}{(w+\underline{u})^2})$ is a locally bounded when $\nabla w \not\equiv 0$. Therefore, P can only attain positive maximum either at the point where $\nabla w = 0$, or on the parabolic boundary of $\Omega \times (0, t_0)$. But when $\nabla w = 0$, we have $P \leq 0$.

On the initial boundary, $P = \frac{C}{1+\underline{u}} - \frac{C}{\underline{u}} < 0$. Let (y, s) be any point on $\partial\Omega \times (0, t_0)$, if we can prove that

$$\frac{\partial P}{\partial \nu} \leq 0 \quad \text{at} \quad (y, s) \,, \tag{7.1.15}$$

it then follows from the maximum principle that $P \leq 0$ in $\Omega \times (0, t_0)$. And therefore, the assertion (7.1.11) is reduced from (7.1.12) together with $w = u - \underline{u}$.

To prove (7.1.15), we recall the fact that since $w = const.$ on $\partial\Omega$ (for $t = s$), we have

$$\Delta w = w_{\nu\nu} + (N-1)\kappa w_\nu \quad \text{at} \quad (y, s) \,,$$

where κ is the non-negative mean curvature of $\partial\Omega$ at y. It then follows that

$$\begin{aligned} \frac{\partial P}{\partial \nu} = w_\nu w_{\nu\nu} - \frac{C w_\nu}{(w+\underline{u})^2} &\leq w_\nu \Big[\Delta w - (N-1)\kappa w_\nu - \frac{\lambda f(x)}{(w+\underline{u})^2}\Big] \\ &= w_\nu [w_t - (N-1)\kappa w_\nu] = -(N-1)\kappa w_\nu^2 \leq 0 \end{aligned}$$

at (y, s), and we are done. ■

The following lemma is dealt with the derivatives of higher order, and the idea of its proof is similar to Proposition 1 of [56].

Lemma 7.1.5. *Assume f satisfies* (7.0.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (7.0.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, and $x = a$ is any point of Ω_η for some small $\eta > 0$. Then there exists a positive constant M' such that*

$$|\nabla^m u(x,t)|(T-t)^{-\frac{1}{3}+\frac{m}{2}} \leq M' \,, \quad m = 1, 2 \tag{7.1.16}$$

holds for $|x - a| \leq R$.

Proof: It suffices to consider the case $a = 0$ by translation, and we may focus on $\frac{1}{2}R^2 < r^2 < R^2$ and denote $Q_r = B_r \times (T[1 - (\frac{r}{R})^2], T)$.

Our first task is to show that $|\nabla u|$ and $|\nabla^2 u|$ are uniformly bounded on compact subsets of Q_R. Indeed, since $f(x)/u^2$ is bounded on any compact subset D of Q_R, standard L^p estimates for heat equations (cf. [80]) gives

$$\int\int_D (|\nabla^2 u|^p + |u_t|^p) dxdt < C \,, \quad 1 < p < \infty \,.$$

Choosing p to be large enough, we then conclude from Sobolev's inequality that $f(x)/u^2$ is Hölder continuous on D. Therefore, Schauder's estimates for heat equations (cf. [80]) show that $|\nabla u|$ and $|\nabla^2 u|$ are uniformly bounded on compact subsets of D. In particular, there exists M_1 such that

$$|\nabla u| + |\nabla^2 u| \le M_1 \quad \text{for} \quad (x,t) \in B_r \times \big(T[1-(\frac{r}{R})^2], T[1-\frac{1}{2}(1-\frac{r}{R})^2]\big)\,, \tag{7.1.17}$$

where M_1 depends only on R, N and M given in (7.1.9).

We next prove (7.1.16) for $|x| < r$ and $T[1-\frac{1}{2}(1-\frac{r}{R})^2] \le t < T$. Fix such a point (x,t), let $\mu = [\frac{2}{T}(T-t)]^{\frac{1}{2}}$ and consider

$$v(z,\tau) = \mu^{-\frac{2}{3}} u(x+\mu z, T-\mu^2(T-\tau))\,. \tag{7.1.18}$$

For above given point (x,t), we now define $O := \{z : (x+\mu z) \in \Omega\}$ and $g(z) := f(x+\mu z) \ge 0$ on O. One can verify that $v(z,\tau)$ is a solution of

$$\begin{aligned} v_\tau - \Delta_z v &= -\frac{\lambda g(z)}{v^2} \qquad z \in O, \\ v(z,0) = v_0(z) > 0\,; &\qquad v(z,\tau) = \mu^{-\frac{2}{3}} \quad z \in \partial O\,, \end{aligned} \tag{7.1.19}$$

where Δ_z denotes the Laplacian operator with respect to z, and $v_0(z) = \mu^{-\frac{2}{3}} u(x+\mu z, T-\mu^2 T) > 0$ satisfies $\Delta_z v_0 - \frac{\lambda g(z)}{v_0^2} \le 0$ on O. The formula (7.1.18) implies that T is also the finite touchdown time of v, and the domain of v includes Q_{r_0} for some $r_0 = r_0(R) > 0$. Since touchdown set of u is assumed to be a compact subset of Ω, one can observe that touchdown set of v is also a compact subset of O. Therefore, the argument of Theorem 7.1.3(2) can be applied to (7.1.19), yielding that there exists a constant $M_2 > 0$ such that

$$v(z,\tau) \ge M_2 (T-\tau)^{\frac{1}{3}}$$

where M_2 depends only on R, λ, f and Ω again. The argument used for (7.1.17) then yields that there exists $M_1' > 0$, depending on R, N and M_2, such that

$$|\nabla_z v| + |\nabla_z^2 v| \le M_1' \quad \text{for} \quad (z,\tau) \in B_r \times \big(T[1-(\frac{r}{r_0})^2], T[1-\frac{1}{2}(1-\frac{r}{r_0})^2]\big)\,, \tag{7.1.20}$$

where we assume $\frac{1}{2} r_0^2 < r^2 < r_0^2$. Applying (7.1.18) and taking $(z,\tau) = (0, \frac{T}{2})$, this estimate reduces to

$$\mu^{-\frac{2}{3}+1} |\nabla u| + \mu^{-\frac{2}{3}+2} |\nabla^2 u| \le M_1'\,.$$

Therefore, (7.1.16) follows since $\mu = [\frac{2}{T}(T-t)]^{\frac{1}{2}}$. ■

Before concluding this subsection, we now apply gradient estimates to establishing integral estimates.

Theorem 7.1.6. *Assume f satisfies* (7.0.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (7.0.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, then for $\gamma > \frac{3}{2}N$ we have*

$$\underline{\lim}_{t\to T^-}\int_\Omega f(x)u^{-\gamma}(x,t)dx = +\infty\,.$$

Proof: For any given $t_0 \in (0,T)$ close to T, Lemma 7.1.4 implies that

$$\frac{1}{2}|\nabla u|^2 \le \frac{C}{\underline{u}^2}(u-\underline{u}) \quad \text{in} \quad \Omega\times(0,t_0) \tag{7.1.21}$$

for some bounded constant $C > 0$, where $\underline{u} = u(x_0,t_0) = min_{x\in\Omega}u(x,t_0)$. Considering any t sufficiently close to t_0, we now introduce polar coordinates (r,θ) about the point x_0. Then in any direction θ, there is a smallest value of $r_0 = r_0(\theta,t)$ such that $u(r_0,t) = 2\underline{u}$. Note that r_0 is very small as $t < t_0$ sufficiently approach to T. Furthermore, since x_0 approaches to one of touchdown points of u as $t \to T^-$, Proposition 7.1.2 shows that as $t < t_0$ sufficiently approach to T, we have $f(x) \ge C_0 > 0$ in $\{r < r_0\}$ for some $C_0 > 0$. Since (7.1.21) and the definition of $\underline{u}$ imply that $\frac{u_r}{\sqrt{u-\underline{u}}} \le \frac{\sqrt{2C}}{\underline{u}}$, which is $2\sqrt{u-\underline{u}} \le \frac{\sqrt{2C}}{\underline{u}}r$, we attain $\sqrt{\frac{2}{C}}\underline{u}^{3/2} \le r_0$ by taking $r = r_0$. Therefore, for $\gamma > \frac{3}{2}N$ we have

$$\begin{aligned}\int_\Omega u^{-\gamma}dx \ge C\int_\Omega f(x)u^{-\gamma}dx \ge CC_0\int_{\{r\le r_0\}}u^{-\gamma}dx \ge C\int_\theta dS_\theta\int_{\{r\le r_0\}}u^{-\gamma}r^{N-1}dr\\ \ge C\int_\theta dS_\theta\int_{\{r\le r_0\}}(2\underline{u})^{-\gamma}r^{N-1}dr\\ \ge C\int_\theta dS_\theta\underline{u}^{-\gamma}r_0^N \ge C\int_\theta dS_\theta\underline{u}^{-\gamma+\frac{3}{2}N} = +\infty\end{aligned}$$

as $t \to T^-$, which completes the proof of Theorem 7.1.6. ■

7.1.3 Upper bound estimate

In this subsection, we discuss the upper bound estimate of touchdown solution u by applying energy methods, see Theorem 7.1.11.

First, we note the following local upper bound estimate.

Proposition 7.1.7. *Suppose u is a touchdown solution of* (7.0.1) *at finite time T. Then, there exists a bounded constant $C = C(\lambda, f, \Omega) > 0$ such that*

$$\min_{x\in\Omega} u(x,t) \le C(T-t)^{\frac{1}{3}} \quad for \quad 0<t<T\,. \tag{7.1.22}$$

Proof: Set

$$U(t) = \min_{x\in\Omega} u(x,t), \quad 0<t<T\,,$$

and let $U(t_i) = u(x_i, t_i)$ $(i = 1, 2)$ with $h = t_2 - t_1 > 0$. Then,

$$U(t_2) - U(t_1) \le u(x_1, t_2) - u(x_1, t_1) = hu_t(x_1, t_1) + o(h),$$

$$U(t_2) - U(t_1) \ge u(x_2, t_2) - u(x_2, t_1) = hu_t(x_2, t_2) + o(h).$$

It follows that $U(t)$ is lipschitz continuous. Hence, for $t_2 > t_1$ we have

$$\frac{U(t_2) - U(t_1)}{t_2 - t_1} \ge u_t(x_2, t_2) + o(1).$$

On the other hand, since $\Delta u(x_2, t_2) \ge 0$ we obtain,

$$u_t(x_2, t_2) \ge -\frac{\lambda f(x_2)}{u^2(x_2, t_2)} = -\frac{\lambda f(x_2)}{U^2(t_2)} \ge -\frac{C}{U^2(t_2)} \quad \text{for} \quad 0 < t_2 < T\,.$$

Consequently, at any point of differentiability of $U(t)$, it deduces from above inequalities that

$$U^2 U_t \ge -C \quad \text{a.e.} \quad t \in (0, T)\,. \tag{7.1.23}$$

Integrating (7.1.23) from t to T we obtain (7.1.22). ■

For the touchdown solution $u = u(x, t)$ of (7.0.1) at finite time T, we now introduce the associated similarity variables

$$y = \frac{x - a}{\sqrt{T - t}}\,, \quad s = -log(T - t)\,, \quad u(x, t) = (T - t)^{\frac{1}{3}} w_a(y, s)\,, \tag{7.1.24}$$

where a is any point of Ω_η for some small $\eta > 0$. Then $w_a(y, s)$ is defined in

$$W_a := \{(y, s) :\, a + ye^{-s/2} \in \Omega, s > s' = -logT\}\,,$$

and it solves

$$\frac{\partial}{\partial s} w_a - \Delta w_a + \frac{1}{2} y \cdot \nabla w_a - \frac{1}{3} w_a + \frac{\lambda f(a + ye^{-\frac{s}{2}})}{w_a^2} = 0\,. \tag{7.1.25}$$

Here $w_a(y, s)$ is always strictly positive in W_a. Note that the form of w_a defined by (7.1.24) is motivated by Theorem 7.1.3 and Proposition 7.1.7. The slice of W_a at a given time s^1 will be denoted by $\Omega_a(s^1)$:

$$\Omega_a(s^1) := W_a \cap \{s = s^1\} = e^{s^1/2}(\Omega - a)\,.$$

Then for any $a \in \Omega_\eta$, there exists $s_0 = s_0(\eta, a) > 0$ such that

$$B_s := \{y : |y| < s\} \subset \Omega_a(s) \quad \text{for} \quad s \ge s_0\,. \tag{7.1.26}$$

From now on, we often suppress the subscript a, writing w for w_a, etc.

In view of (7.1.24), one can combine Theorem 7.1.3 and Lemma 7.1.5 to reaching the following estimates on $w = w_a$:

Corollary 7.1.8. *Assume f satisfies* (7.0.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (7.0.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, then the rescaled solution $w = w_a$ satisfies*

$$M \le w \le e^{\frac{s}{3}}\,, \quad |\nabla w| + |\Delta w| \le M' \quad in \quad W,$$

where M is a constant as in Theorem 7.1.3 and while M' is a constant as in Lemma 7.1.5. Moreover, it satisfies

$$M \le w(y_1, s) \le w(y_2, s) + M'|y_2 - y_1|$$

for any $(y_i, s) \in W$, $i = 1, 2$.

We now rewrite (7.1.25) in divergence form:

$$\rho w_s - \nabla \cdot (\rho \nabla w) - \frac{1}{3}\rho w + \frac{\lambda \rho f(a + ye^{-\frac{s}{2}})}{w^2} = 0\,, \tag{7.1.27}$$

where $\rho(y) = e^{-|y|^2/4}$. We also introduce the frozen energy functional

$$E_s[w](s) = \frac{1}{2}\int_{B_s} \rho |\nabla w|^2 dy - \frac{1}{6}\int_{B_s} \rho w^2 dy - \int_{B_s} \frac{\lambda \rho f(a)}{w} dy\,, \tag{7.1.28}$$

which is defined in the compact set B_s of $\Omega_a(s)$ for $s \ge s_0$.

Lemma 7.1.9. *Assume f satisfies* (7.0.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (7.0.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, then the rescaled solution $w = w_a$ satisfies*

$$\frac{1}{2}\int_{B_s} \rho |w_s|^2 dy \le -\frac{d}{ds} E_s[w](s) + g_\eta(s) \quad for \quad s \ge s_0\,, \tag{7.1.29}$$

where $g_\eta(s)$ is positive and satisfies $\int_{s_0}^{\infty} g_\eta(s) ds < \infty$.

Proof: Multiply (7.1.27) by w_s and use integration by parts to get

$$
\begin{aligned}
\int_{B_s}\rho|w_s|^2dy &= \int_{B_s} w_s\nabla(\rho\nabla w)dy+\frac{1}{3}\int_{B_s}\rho w w_s dy-\int_{B_s}\frac{\lambda\rho w_s f(a+ye^{-\frac{s}{2}})}{w^2}dy\\
&= -\frac{1}{2}\int_{B_s}\frac{d}{ds}|\nabla w|^2\rho dy+\int_{B_s}\frac{d}{ds}\Big(\frac{1}{6}w^2+\frac{\lambda f(a)}{w}\Big)\rho dy\\
&\quad +\int_{\partial B_s}\rho w_s\frac{\partial w}{\partial \nu}dS+\int_{B_s}\frac{\lambda\rho w_s[f(a)-f(a+ye^{-\frac{s}{2}})]}{w^2}dy\\
&= -\frac{d}{ds}E_s[w](s)+\int_{\partial B_s}\rho w_s\frac{\partial w}{\partial \nu}dS+\frac{1}{2s}\int_{\partial B_s}\rho|\nabla w|^2(y\cdot\nu)dS\\
&\quad -\frac{1}{s}\int_{\partial B_s}\rho\Big(\frac{1}{6}w^2+\frac{\lambda f(a)}{w}\Big)(y\cdot\nu)dS+\int_{B_s}\frac{\lambda\rho w_s[f(a)-f(a+ye^{-\frac{s}{2}})]}{w^2}dy\\
&\le -\frac{d}{ds}E_s[w](s)+\int_{\partial B_s}\rho w_s\frac{\partial w}{\partial \nu}dS+\frac{1}{2s}\int_{\partial B_s}\rho|\nabla w|^2(y\cdot\nu)dS\\
&\quad +\int_{B_s}\frac{\lambda\rho w_s[f(a)-f(a+ye^{-\frac{s}{2}})]}{w^2}dy\\
&:= -\frac{d}{ds}E_s[w](s)+I_1+I_2+I_3\,,
\end{aligned}
\tag{7.1.30}
$$

where ν is the exterior unit norm vector to $\partial\Omega$ and dS is the surface area element. The following formula is applied in the third equality of (7.1.30): if $g(y,s):W\mapsto R$ is a smooth function, then

$$
\begin{aligned}
\frac{d}{ds}\int_{B_s}g(y,s)dy &= \frac{d}{ds}\int_{B_1}g(sz,s)s^N dz\\
&= N\int_{B_1}g(sz,s)s^{N-1}dz+\int_{B_1}g_s(sz,s)s^N dz+\int_{B_1}\big(\nabla_y g\cdot z\big)s^N dz\\
&= \int_{B_s}g_s(y,s)dy+N\int_{B_s}g(y,s)\frac{dy}{s}+\int_{B_s}\big(\nabla g\cdot\frac{y}{s}\big)dy\\
&= \int_{B_s}g_s(y,s)dy+\frac{1}{s}\int_{\partial B_s}g(y,s)(y\cdot\nu)dS\,.
\end{aligned}
$$

For $s\ge s_0$, we next estimate integration terms I_1, I_2 and I_3 as follows:

Considering $|y|\le S$ in B_s, Corollary 7.1.8 gives

$$|w_s|=|\Delta w-\frac{1}{2}y\cdot\nabla w+\frac{1}{3}w-\frac{\lambda f(a+ye^{-\frac{s}{2}})}{w^2}|\le C(1+|y|)+\frac{1}{3}w\le C_1 s+\frac{1}{3}e^{\frac{s}{3}}\,,$$

which implies

$$I_1\le Cs^{N-1}e^{-\frac{s^2}{4}}\big(C_1 s+\frac{1}{3}e^{\frac{s}{3}}\big)\le C_2 s^N e^{-\frac{s^2}{4}+\frac{s}{3}}\,. \tag{7.1.31}$$

It is easy to observe that

$$I_2\le C_3 s^{N-1}e^{-\frac{s^2}{4}}\,. \tag{7.1.32}$$

As for I_3, since w has a lower bound and since $f(x) \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1]$, we apply Young's inequality to deduce

$$I_3 \le Ce^{-\frac{\alpha}{2}s} \int_{B_s} \rho |y|^\alpha w_s dy \le Ce^{-\frac{\alpha}{2}s} \Big[\varepsilon \int_{B_s} \rho w_s^2 dy + C(\varepsilon) \int_{B_s} \rho |y|^{2\alpha} dy \Big] ,$$

where the constant $\varepsilon > 0$ is arbitrary. Because $e^{-\frac{\alpha}{2}s} < \infty$, one can take sufficiently small ε such that

$$I_3 \le \frac{1}{2} \int_{B_s} \rho w_s^2 dy + C_4 e^{-\frac{\alpha}{2}s} . \tag{7.1.33}$$

Combining (7.1.30) – (7.1.33) then yields

$$\begin{aligned} \frac{1}{2} \int_{B_s} \rho |w_s|^2 dy \le & -\frac{d}{ds} E_s[w](s) + \bar{C}_1 s^N e^{-\frac{s^2}{4} + \frac{s}{3}} + \bar{C}_2 e^{-\frac{\alpha}{2}s} \\ := & -\frac{d}{ds} E_s[w](s) + g_\eta(s) , \end{aligned}$$

where $g_\eta(s)$ is positive and satisfies $\int_{s_0}^\infty g_\eta(s) ds < \infty$, and we are done. ■

Remark 7.1.2. Supposing the convexity of Ω, one can establish an energy estimate in the whole domain $\Omega_a(s)$:

$$\int_{\Omega_a(s)} \rho |w_s|^2 dy \le -\frac{d}{ds} E_{\Omega_a(s)}[w](s) + K_\eta(s) \quad \text{for} \quad s \ge s_0 , \tag{7.1.34}$$

where $K_\eta(s)$ is positive and satisfies $\int_{s_0}^\infty K_\eta(s) ds < \infty$, and $E_{\Omega_a(s)}[w](s)$ is defined by

$$E_{\Omega_a(s)}[w](s) = \frac{1}{2} \int_{\Omega_a(s)} \rho |\nabla w|^2 dy - \frac{1}{6} \int_{\Omega_a(s)} \rho w^2 dy - \int_{\Omega_a(s)} \frac{\lambda \rho f(a)}{w} dy . \tag{7.1.35}$$

However, by estimating the energy functional $E_s[w](s)$ in B_s, instead of $\Omega_a(s)$, it is sufficient to obtain the desirable upper bound estimate of w, see Theorem 7.1.11 below.

The following lemma is also necessary for establishing the desirable upper bound estimate.

Lemma 7.1.10. *Assume f satisfies* (7.0.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (7.0.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, and a is any point of Ω_η for some $\eta > 0$. Then there exists a constant $\varepsilon > 0$, depending only on λ, f and Ω, such that if*

$$u(x,t)(T-t)^{-\frac{1}{3}} \ge \varepsilon \tag{7.1.36}$$

for all $(x,t) \in Q_\delta := \{(x,t) : |x-a| < \delta, T-\delta < t < T\}$, then a is not a touchdown point for u. Here $\delta > 0$ is an arbitrary constant.

Proof: Setting $v(x,t) = \frac{1}{u(x,t)}$, then $v(x,t)$ blows up at finite time T, and v satisfies

$$v_t - \Delta v = -\frac{2|\nabla v|^2}{v} + \lambda f(x) v^4 \leq K(1+v^4) \quad \text{in} \quad Q_\delta\,, \tag{7.1.37}$$

where $K := \lambda \sup_{x\in\bar\Omega} f(x) > 0$. We now apply Theorem 2.1 of [58] to (7.1.37), which gives that there exists a constant $\frac{1}{\varepsilon} > 0$, depending only on λ, f and Ω, such that if

$$v(x,t) \leq \frac{1}{\varepsilon}(T-t)^{-\frac{1}{3}} \quad \text{in} \quad Q_\delta\,,$$

then a is not a blow-up point for v, and hence (7.1.36) follows. ■

Theorem 7.1.11. *Assume f satisfies* (7.0.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (7.0.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, and a is any point of Ω_η for some $\eta > 0$. If $w_a(y,s) \to \infty$ as $s \to \infty$ uniformly for $|y| \leq C$, where C is any positive constant, then a is not a touchdown point for u.*

Proof: We first claim that if $w_a(y,s) \to \infty$ as $s \to \infty$ uniformly for $|y| \leq C$, then

$$E_s[w_a](s) \to -\infty \quad \text{as} \quad s \to \infty\,. \tag{7.1.38}$$

Indeed, it is obvious from Corollary 7.1.8 that the first term and the third term in $E_s[w_a](s)$ are uniformly bounded. As for the second term, we can write

$$\int_{B_s} \rho w^2 dy = \int_{B_C} \rho w^2 dy + \int_{B_s\setminus B_C} \rho w^2 dy \geq \int_{B_C} \rho w^2 dy\,.$$

Since $w_a \to \infty$ as $s \to \infty$ uniformly on B_C, we have $\int_{B_C} \rho w^2 dy \to \infty$ as $s \to \infty$, which gives $-\frac{1}{6}\int_{B_C} \rho w^2 dy \to -\infty$ as $s \to \infty$, and hence (7.1.38) follows.

Let K be a large positive constant to be determined later. Then (7.1.38) implies that there exists an $\bar s$ such that $E_{\bar s}[w_a](\bar s) \leq -4K$. Using the same argument as in [57], it is easy to show that for any fixed s, $E_s[w_a](s)$ varies smoothly with $a \in \Omega$. Therefore, there exists an $r_0 > 0$ such that

$$E_{\bar s}[w_b](\bar s) \leq -3K \quad \text{for} \quad |b-a| < r_0\,.$$

Since touchdown set of u is assumed to be a compact subset of Ω, we have $dist(a, \partial\Omega) > \eta$ for some $\eta > 0$. Therefore, it now follows from Lemma 7.1.9 that

$$E_s[w_b](s) \leq -2K \quad \text{for} \quad |b-a| < r_0\,,\ s \geq \bar s$$

provided $K \geq M_1 := \int_{s_0}^\infty g_\eta(s)ds$, where $g_\eta(s)$ is as in Lemma 7.1.9. Since the first term and the third term in $E_s[w_b](s)$ are uniformly bounded, we have

$$\int_{B_s} \rho w_b^2 dy \geq 6K \quad \text{for} \quad |b-a| < r_0\,,\ s \geq \bar s\,. \tag{7.1.39}$$

Recalling from Corollary 7.1.8,

$$w_b^2(y,s) \le 2\big(w_b^2(0,s) + M'^2|y|^2\big)\,,$$

we obtain from (7.1.39) that

$$3K \le w_b^2(0,s)\int_{B_s}\rho dy + M'^2\int_{B_s}\rho|y|^2 dy \le C_1 w_b^2(0,s) + C_2\,.$$

We now choose $K \ge max\{M_1, \frac{2}{3}C_2\}$ so large that

$$w_b(0,s) \ge \sqrt{\frac{3K}{2C_1}} := \varepsilon\,. \tag{7.1.40}$$

Setting $\bar t := T - e^{-\bar s}$, it reduces from (7.1.40) that

$$u(b,t)(T-t)^{-\frac{1}{3}} \ge \varepsilon \quad \text{for} \quad |b-a| < r_0\,,\ \bar t < t < T\,.$$

Applying Lemma 7.1.10 with a small r_0, we finally conclude that a is not a touchdown point for u, and the theorem is proved. ■

7.2 Entire solutions of a semilinear elliptic problem

As another preliminary of next section, it is quite necessary to study the asymptotic behavior of positive entire solutions for a semilinear elliptic problem

$$\Delta w - \frac{1}{2}y\cdot\nabla w + \frac{w}{3} - \frac{1}{w^2} = 0 \quad \text{in} \quad \mathbb{R}^N,\ N \ge 1\,. \tag{7.2.1}$$

For generality, we focus on positive entire solutions of the following problem

$$\Delta w - \frac{1}{2}y\cdot\nabla w = F(w) := \frac{1}{w^\beta} - \frac{w}{1+\beta} \quad \text{in} \quad \mathbb{R}^N, \tag{7.2.2}$$

where $N \ge 1$ and $\beta \ge 1$ is a parameter. We are interested in the asymptotic behavior whether every non-constant solution w of (7.2.2) must be strictly increasing for all $|y|$ sufficiently large.

7.2.1 Asymptotic behavior for $N = 1$

In this subsection, we consider positive entire solutions of (7.2.2) with $N = 1$ satisfying

$$w''(y) - \frac{yw'(y)}{2} = F\big(w(y)\big) := \frac{1}{w^\beta(y)} - \frac{w(y)}{1+\beta} \quad \text{in} \quad (-\infty,+\infty), \tag{7.2.3}$$

where $\beta \ge 1$ is a parameter. For this case, Fila & Hulshof in [43] established the following theorem.

Theorem 7.2.1. *Suppose $\beta \geq 1$, then as $y \searrow -\infty$ and/or $y \nearrow +\infty$, every nonconstant solution of (7.2.3) is eventually strictly convex and tends to $+\infty$.*

As proved in [43], Theorem 7.2.1 is a consequence of several lemmas.

Lemma 7.2.2. *A solution of (7.2.3) cannot be nonincreasing near $y = +\infty$ unless it is identically equal to*

$$k = k(\beta) := (1+\beta)^{\frac{1}{1+\beta}}, \tag{7.2.4}$$

which is the unique positive root of F.

Proof: By contradiction, we suppose that there exists a nonconstant solution w which is decreasing for large positive values of y. It is easily seen from (7.2.3) that w then has to drop below the value k defined by (7.2.4). Using the variation of constant formula, we obtain

$$w'(y) = e^{\frac{1}{4}y^2}\Big(w'(p) + \int_p^y e^{-\frac{1}{4}\eta^2} g(\eta)d\eta\Big), \tag{7.2.5}$$

where $g(y) := F\big(w(y)\big)$ and $p \geq 0$ is arbitrary. Since $w(y)$ decreases to a nonnegative limit as $y \nearrow +\infty$, we have $w'(+\infty) = 0$ and therefore, (7.2.5) can be rewritten as

$$w'(y) = -e^{\frac{1}{4}y^2} \int_y^{+\infty} e^{-\frac{1}{4}\eta^2} g(\eta)d\eta\,. \tag{7.2.6}$$

However, since the function $g(y)$ is positive and bounded away from zero for sufficiently large y, it gives that the integral

$$\int_0^{+\infty} e^{\frac{1}{4}y^2} \int_y^{+\infty} e^{-\frac{1}{4}\eta^2} g(\eta)d\eta dy$$

diverges to $+\infty$. Together with (7.2.6), this implies that $w(y)$ cannot remain positive as $y \nearrow +\infty$, a contradiction. ∎

For the solution $w(y)$ of (7.2.3), inspired by [47] we define a function $J(y)$

$$J(y) = e^{-\frac{1}{4}y^2}\Big(w'(y)w''(y) + \frac{w(y)w'''(y)}{\beta}\Big)\,.$$

Then one can deduce that

$$J'(y) = \frac{1}{2\beta} e^{-\frac{1}{4}y^2}\Big((1+\beta)yw'(y)w''(y) + (\beta-1)w'(y)^2\Big)\,. \tag{7.2.7}$$

Indeed, as in [47] one can differentiate (7.2.3) twice and then multiply by w/β to get that

$$\frac{ww^{(4)}}{\beta} - \frac{ww''}{\beta} - \frac{yww''}{2\beta} = -\frac{ww''}{\beta(1+\beta)} - \frac{w''}{w^\beta} + (\beta+1)\frac{(w')^2}{w^{\beta+1}}\,. \tag{7.2.8}$$

Multiply (7.2.3) by w'' to obtain

$$(w'')^2 - \frac{yw'w''}{2} = \frac{w''}{w^\beta} - \frac{ww''}{\beta+1}\,. \tag{7.2.9}$$

Combining (7.2.8) and (7.2.9) now yields that

$$(w'')^2 - \frac{yw'w''}{2} + \frac{ww^{(4)}}{\beta} - \frac{yww''}{2\beta} = (\beta+1)\frac{(w')^2}{w^{\beta+1}}\,. \tag{7.2.10}$$

On the other hand, using the first derivative of (7.2.3), we have

$$w''' = \frac{w'}{2} + \frac{yw''}{2} - \beta\frac{w'}{w^{\beta+1}} - \frac{w'}{\beta+1}\,,$$

which gives

$$\frac{w'w'''}{\beta} + w'w''' = \frac{\beta+1}{\beta}w'w''' = \frac{(\beta+1)w'}{\beta}\Big[\frac{w'}{2} + \frac{yw''}{2} - \beta\frac{w'}{w^{\beta+1}} - \frac{w'}{\beta+1}\Big]\,. \tag{7.2.11}$$

By adding (7.2.10) to (7.2.11), the identity (7.2.7) is then obtained after multiplication by $e^{-\frac{y^2}{4}}$.

Lemma 7.2.3. *Let $w(y)$ be a solution of (7.2.3) with $w'(0) = 0$ and $0 < w(0) < k$. Then w is a strictly convex function.*

Proof: First, we observe from (7.2.3) that $w''(y) > 0$ for small values of y. We also note from (7.2.3) that

$$w''' = \frac{yw''}{2} + \frac{w'}{2} - \frac{\beta w'}{w^{\beta+1}} - \frac{w'}{\beta+1}\,. \tag{7.2.12}$$

Since $w'(0) = 0$, we get from (7.2.12) that $w'''(0) = 0$ and hence $J(0) = 0$. By (7.2.7), if $w''(y) > 0$ then we have $J'(y) > 0$ and $w'(y) > 0$ since $w'(0) = 0$. Now suppose that there is some $y_0 > 0$ such that $w''(y_0) = 0$. We may assume that y_0 is the first value at which this happens. But then $w'''(y_0) \le 0$ and hence $J(y_0) \le 0$, which contradicts the facts that $J(0) = 0$ and $J'(y) > 0$ for all $0 < y < y_0$. This completes the proof of Lemma 7.2.3. ■

Lemma 7.2.4. *Let $w(y)$ be a nonconstant solution of (7.2.3) which is decreasing on some subinterval of the positive reals. Then, $w(y)$ attains a positive minimum in a unique positive y_1 and it is strictly convex for $y \ge y_1$.*

Proof: By Lemma 7.2.2 there has to be a minimal positive value y_1 such that $w'(y_1) = 0$ and $w''(y_1) \ge 0$. Since $w(y)$ is not a constant, it follows that $w''(y_1) > 0$. We see from (7.2.12) that consequently $w'''(y_1) > 0$, and therefore, $J(y_1) > 0$. The remainder of the proof is identical to the proof of Lemma 7.2.3 and it is therefore left to the interested reader. ■

Proof of Theorem 7.2.1: Suppose that $w(y)$ is a nonconstant solution of (7.2.3) for which the statement is false. By Lemma 7.2.4, it has to be nondecreasing on the positive reals, and by symmetry, it has to be nonincreasing on the negative reals. Therefore $w'(0) = 0$, and obviously $0 < w(0) < k$. But then we can reach a contradiction by applying Lemma 7.2.3. ■

7.2.2 Asymptotic behavior for $N \geq 2$

It seems difficult to study the asymptotic behavior of positive entire solutions for (7.2.2) with $N \geq 2$. To our knowledgement, the available results is due to J. S. Guo [62], where the author discussed positive radially symmetric solution $w(r)$ of (7.2.2) with $N \geq 2$, in the sense that there exists an $\eta > 0$ such that $w(0) = \eta$, $w'(0) = 0$, and that w satisfies the equation

$$w'' + \Big(\frac{N-1}{r} - \frac{r}{2}\Big)w' + f_1(w) = 0 \quad r > 0 \tag{7.2.13}$$

with $f_1(w) = \frac{w}{1+\beta} - \frac{1}{w^\beta}$, where $\beta > 1$ is a parameter again. For convenience, we define

$$k = k(\beta) := (1+\beta)^{\frac{1}{1+\beta}}\,, \quad F_1(w) = \int_k^w f_1(t)dt\,;$$

$$\rho(r) = e^{-\frac{r^2}{4}}\,, \quad \sigma(r) = r^{N-1}\rho(r)\,,$$

and therefore, we have

$$F_1 \geq 0\,, \quad F_1(w) \sim w^2 \quad \text{as} \quad w \to \infty \quad \text{and} \quad F_1(w) \sim w^{1-\beta} \quad \text{as} \quad w \to 0\,;$$

$$\rho'(r) = -\frac{r}{2}\rho(r)\,, \quad \sigma'(r) = (\frac{N-1}{r} - \frac{r}{2})\sigma(r)\,.$$

J. S. Guo in [62] established the following asymptotic behavior of positive radially symmetric solution $w(r)$ for (7.2.13).

Theorem 7.2.5. *Every nonconstant radial solution w of (7.2.13) must be strictly increasing for all r sufficiently large, and $w(r) \to \infty$ as $r \to \infty$.*

In order to prove Theorem 7.2.5, we need to establish some lemmas. The first one is motivated by [55].

Lemma 7.2.6. *For any $\eta > 0$, there is a unique positive entire solution $w(r) = w(r;\eta)$ of (7.2.13) such that $w(0) = \eta$ and $w'(0) = 0$.*

Proof: Since $r = 0$ is a regular singular point and $f_1(w)$ is real analytic at $w = \eta$, the local existence and uniqueness follow. Let $[0, a)$ be the maximal interval on which w is well-defined. Multiplying (7.2.13) by $\rho^2 w'$, we obtain

$$(\frac{\rho^2 w'^2}{2})' + \frac{N-1}{r}\rho^2 w'^2 + \rho^2 F_1'(w) = 0\,. \tag{7.2.14}$$

Integrating (7.2.14) over $[0,r]$ with any $r < a$, we have

$$\frac{\rho^2(r)w'^2(r)}{2} + \int_0^r \frac{N-1}{s}\rho^2(s)w'^2(s)ds + \int_0^r \rho^2(s)F_1'\big(w(s)\big)ds = 0\,. \tag{7.2.15}$$

Using integration by parts for the last integral of (7.2.15), we have

$$\frac{\rho^2(r)w'^2(r)}{2} + \int_0^r \frac{N-1}{s}\rho^2(s)w'^2(s)ds + \int_0^r s\rho^2(s)F_1\big(w(s)\big)ds + \rho^2(r)F_1\big(w(r)\big) - F_1(\eta) = 0\,.$$

Therefore, we get that

$$\frac{1}{2}\rho^2(r)w'^2(r) + \rho^2(r)F_1\big(w(r)\big) \le F_1(\eta)\,. \tag{7.2.16}$$

Therefore, (7.2.16) and the standard continuation theorem give that $a = \infty$. Moreover, (7.2.16) implies $w > 0$ for any $r \ge 0$. ■

Lemma 7.2.7. *Any critical point r_0 of a radial solution w for (7.2.13) is a local maximal point if $w(r_0) > k$, and a local minimal point if $w(r_0) < k$. Moreover, there cannot exist a point r with $w(r) = k$ and $w'(r) = 0$ except when $w \equiv k$.*

Proof: From the uniqueness of initial value problems for ordinary differential equations, it follows that there cannot exist a point r with $w(r) = k$ and $w'(r) = 0$ except when $w \equiv k$. Let r_0 be any critical point of w, then we have $w''(r_0) = -f_1\big(w(r_0)\big)$. This shows that r_0 is a local maximal point if $w(r_0) > k$, and while r_0 is a local minimal point if $w(r_0) < k$. ■

Lemma 7.2.8. *Any nonconstant radial solution w of (7.2.13) which takes the value k only finite times must be strictly increasing for all r sufficiently large. Moreover, $w(r) \to \infty$ as $r \to \infty$.*

Proof: If w is a nonconstant radial solution of (7.2.13) which takes the value k only finite times, then there exists a number $\bar{r}$ such that either $w(r) > k$ or $w(r) < k$ for all $r \ge \bar{r}$. Consider the case where $w(r) > k$ for all $r \ge \bar{r}$. We claim that w is monotone for all r sufficiently large. Indeed, otherwise then there exist a local maximum and a local minimum which are both greater than k, a contradiction to Lemma 7.2.7. Therefore, w must be monotone for all r sufficiently large. The other case can be dealt similarly. This shows that there is $r_0 > \bar{r}$ such that either $w'(r) > 0$ or $w'(r) < 0$ for all $r \ge r_0$. We claim that the second case cannot happen.

Suppose $w'(r) < 0$ for all $r \ge r_0$, then the limit $\lim_{r\to\infty} w(r) = l$ exists. We shall show that $l < k$. Indeed, if $l \ge k$ then $w(r) \ge k$ for all $r \ge r_0$. Then

$$(\sigma w')' = -\sigma f_1(w) \le 0 \quad \text{for all} \quad r \ge r_0\,. \tag{7.2.17}$$

It then follows that the limit $\lim_{r\to\infty}(\sigma w')(r) = l_1$ exists and $l_1 \le 0$. If $l_1 < 0$ then w' is unbounded and hence w is also unbounded which is a contradiction. It now follows that $l_1 = 0$. Therefore, we deduce from (7.2.17) that

$$\int_{r_0}^\infty \sigma f_1(w)dr = \sigma(r_0)w'(r_0) < 0\,,$$

which contradicts the fact that $\sigma f_1(w) \geq 0$ for all $r \geq r_0$. Therefore, we conclude that $l < k$.

We now rewrite (7.2.13) as

$$\frac{w''}{r} + \Big(\frac{N-1}{r^2} - \frac{1}{2}\Big)w' = -\frac{1}{r} f_1(w)\,. \tag{7.2.18}$$

Since $\int_{r_0}^{\infty} w'(r)dr = l - w(r_0) < \infty$, then there exists a sequence $r_j \to \infty$ such that $w'(r_j) \to 0$ as $j \to \infty$. Integrating (7.2.18) from r_0 to r_j, we get that

$$\int_{r_0}^{r_j} \frac{w''}{r} dr + \int_{r_0}^{r_j} \Big(\frac{N-1}{r^2} - \frac{1}{2}\Big) w' dr = -\int_{r_0}^{r_j} \frac{1}{r} f_1(w) dr\,. \tag{7.2.19}$$

Note that

$$\int_{r_0}^{r_j} \frac{w''}{r} dr = \int_{r_0}^{r_j} \frac{w'}{r^2} dr + \frac{w'(r_j)}{r_j} - \frac{w'(r_0)}{r_0} < \infty \quad \text{as} \quad j \to \infty\,.$$

The second integral in (7.2.19) is also finite. On the other hand, since $l < k$ then there is a positive number R such that $w(r) \leq w(R) < k$ for all $r \geq R$. Thus we get

$$\int_{R}^{r_j} \frac{1}{r} f_1(w) dr \leq f_1\big(w(R)\big) \int_{R}^{r_j} \frac{dr}{r} \to -\infty \quad \text{as} \quad j \to \infty\,,$$

a contradiction, and the claim is proved.

It now concludes that $w'(r) > 0$ for all $r \geq r_0$, and moreover, $w(r) \to \infty$ as $r \to \infty$. This completes the proof of Lemma 7.2.8. ■

Consider now the phase plane of the system

$$w' = v\,, \quad v' = \Big(\frac{r}{2} - \frac{N-1}{r}\Big)v - f_1(w)\,. \tag{7.2.20}$$

Then we have the following positive invariant lemma.

Lemma 7.2.9. *For any $\alpha > 0$, there exists a number $R = R(\alpha, \beta)$ such that if $r_1 \geq R$ and $\big(w(r_1), v(r_1)\big) \in A_\alpha := \{(w, v); w \geq k, v \geq \alpha w\}$, then $\big(w(r), v(r)\big) \in A_\alpha$ for all $r \geq r_1$.*

Proof: For any given $\alpha > 0$, choose $R = R(\alpha, \beta)$ so that $\frac{R}{2} - \frac{N-1}{R} = \alpha + \frac{1}{\alpha(\beta+1)}$. In the phase plane of (7.2.20), we have $w' > 0$ and

$$\begin{aligned} v' &= \big(\frac{r}{2} - \frac{N-1}{r}\big)v \geq \big(\alpha + \frac{1}{\alpha(\beta+1)}\big)v > 0\,; \\ \frac{v'}{w'} &= \big(\frac{r}{2} - \frac{N-1}{r}\big) \geq \alpha + \frac{1}{\alpha(\beta+1)} > \alpha \end{aligned} \tag{7.2.21}$$

on the line $\{w = k, v \geq \alpha w\}$ for $r \geq R$. Similarly, we have $w' > 0$ and

$$\begin{aligned} v' &= \big(\frac{r}{2} - \frac{N-1}{r}\big)\alpha w - f_1(w) \geq \alpha^2 w > 0\,; \\ \frac{v'}{w'} &= \big(\frac{r}{2} - \frac{N-1}{r}\big) - \frac{1}{\alpha(\beta+1)} + \frac{1}{v w^\beta} > \alpha \end{aligned} \tag{7.2.22}$$

on the line $\{w > k, v = \alpha w\}$ for $r \geq R$. Therefore, Lemma 7.2.9 follows from (7.2.21) and (7.2.22). ∎

We say that a nonconstant radial solution w of (7.2.13) is oscillatory about k if w takes the value k infinite times.

Lemma 7.2.10. *Let w be a nonconstant radial solution of (7.2.13), then it is impossible that w is oscillatory about k.*

Proof: Let w be a nonconstant radial solution of (7.2.13). We first note that (7.2.13) can be rewritten as

$$\big(\frac{w'^2}{2} + F_1(w)\big)' = \big(\frac{r}{2} - \frac{N-1}{r}\big)w'^2\,. \tag{7.2.23}$$

If $r \geq R_0 := \sqrt{2(N-1)}$, then the right hand side of (7.2.23) is nonnegative, and hence the limit

$$\lim_{r\to\infty}\big(\frac{w'^2}{2} + F_1(w)\big) = l \geq 0$$

exists. We now claim that $l = \infty$.

If $l = 0$, then $w'(r) \to 0$ and $w(r) \to k$ as $r \to \infty$. Integrating (7.2.23) from r to ∞, we obtain for $r \geq R_0$ that

$$-\frac{w'^2}{2} - F_1(w) = \int_r^\infty \big(\frac{s}{2} - \frac{N-1}{s}\big)w'^2(s)ds\,. \tag{7.2.24}$$

Since the right hand side of (7.2.24) is nonnegative, the left hand side of (7.2.24) is nonpositive, and r is arbitrary, we conclude that $w' \equiv 0$ and $w \equiv k$, a contradiction.

Suppose now $l \in (0, \infty)$. Then w must be oscillatory about k in view of Lemma 7.2.8. For any sequence of extremal points $r_m \to \infty$, we have $F_1\big(w(r_m)\big) \to l$ as $m \to \infty$. In particular, for any sequence of maximal points $r_m \to \infty$, we have

$$w(r_m) \to F^{-1}(l) \cap \{w > k\} \quad \text{as} \quad m \to \infty\,,$$

and hence w is bounded from above. Similarly, we can prove that w is bounded from below and away from zero. Therefore, there exist positive constants C_1 and C_2 such that

$$0 < C_1 \leq w \leq C_2 < \infty \quad \text{for all} \quad r > 0\,. \tag{7.2.25}$$

Setting $v = w'$, then v satisfies

$$v'' + \Big(\frac{N-1}{r} - \frac{r}{2}\Big)v' + \Big(\frac{1}{1+\beta} - \frac{1}{2} - \frac{N-1}{r^2} + \beta w^{-\beta-1}\Big)v = 0\,.$$

Define $u(\tau) = v(r)$, where $\tau = \tau(r) = \int_1^r \frac{ds}{\sigma(s)}$ for $r \geq 1$. Notice that τ strictly increases to ∞ as r increases to ∞. Then u satisfies the equation

$$\frac{d^2u}{d\tau^2} + \sigma^2(r)\Big(\frac{1}{1+\beta} - \frac{1}{2} - \frac{N-1}{r^2} + \beta w^{-\beta-1}\Big)u = 0\,, \qquad \tau > 0\,.$$

Applying L' Hôpital's rule, we get

$$\lim_{r\to\infty}[\tau^2\sigma^2(r)] = \lim_{r\to\infty}\Big[\frac{\big(\int_1^r \frac{ds}{\sigma(s)}\big)^2}{\sigma^{-2}(r)}\Big] = 0\,.$$

Since (7.2.25) gives

$$\Big|\frac{1}{1+\beta} - \frac{1}{2} - \frac{N-1}{r^2} + \beta w^{-\beta-1}\Big| < \infty\,,$$

we conclude that

$$\lim_{r\to\infty}\tau^2\sigma^2(r)\Big[\frac{1}{1+\beta} - \frac{1}{2} - \frac{N-1}{r^2} + \beta w^{-\beta-1}\Big] = 0\,.$$

Applying a non-oscillatory criterion of Hartman (cf. [73], p. 362), we obtain that v can only have finitely many zeros. This contradicts the fact that w is oscillatory about k. This proves the claim

$$\lim_{r\to\infty}\big(\frac{w'^2}{2} + F_1(w)\big) = l = \infty\,. \tag{7.2.26}$$

Suppose now w is oscillatory about k. Then we can choose $R_m \to \infty$ so that $w(R_m) = k$ and $w'(R_m) > 0$. Thus (7.2.26) gives $w'(R_m) \to \infty$ as $m \to \infty$. Given $\alpha > 0$ and choose m_0 so large that $R_{m_0} \geq R(\alpha,\beta)$ and $w'(R_{m_0}) > \alpha k$. Then we obtain $\big(w(R_{m_0}), v(R_{m_0})\big) \in A_\alpha$. Now Lemma 7.2.9 shows that $\big(w(r), v(r)\big) \in A_\alpha$ for all $r \geq R_{m_0}$. In particular, we have $w'(r) > 0$ for all $r \geq R_{m_0}$, a contradiction. This completes the proof of Lemma 7.2.10. ■

We now conclude Theorem 7.2.5 by directly applying Lemmas 7.2.8 and 7.2.10.

7.3 Refined touchdown profiles

In this section we first establish touchdown rates by applying self-similar method [56]. Then the refined touchdown profiles for $N = 1$ and $N = 2$ will be separately derived by using center manifold analysis of a PDE [45], which will be discussed for $N = 1$ in §7.3.1 and for $N \geq 2$ in §7.3.2, respectively. It should be pointed out that for $N = 1$ we may establish the refined touchdown profiles for any touchdown point, see Theorem 7.3.3; while for $N \geq 2$, we are only able to deal with the refined touchdown profiles in the radial situation for the special touchdown point $r = 0$, see Theorem 7.3.5. Throughout this section and unless mentioned otherwise, touchdown set for u is assumed to be a compact subset of Ω, and a is always assumed to be any touchdown point of u. Therefore, all a priori estimates of last section can be adapted here.

Our starting point of studying touchdown profiles is a similarity variable transformation of (7.0.1). For the touchdown solution $u = u(x,t)$ of (7.0.1) at finite time T, as before we use the associated similarity variables

$$y = \frac{x-a}{\sqrt{T-t}}\,, \quad s = -log(T-t)\,, \quad u(x,t) = (T-t)^{\frac{1}{3}}w(y,s)\,, \tag{7.3.1}$$

where a is any touchdown point of u. Then $w(y,s)$ is defined in $W = \{(y,s) : |y| < Re^{s/2}, s > s' = -logT\}$, where $R = \max\{|x-a| : x \in \Omega\}$, and it solves

$$w_s - \frac{1}{\rho}\nabla(\rho\nabla w) - \frac{1}{3}w + \frac{\lambda f(a + ye^{-\frac{s}{2}})}{w^2} = 0 \tag{7.3.2}$$

with $\rho(y) = e^{-|y|^2/4}$, where $f(a) > 0$ since a is assumed to be a touchdown point. Therefore, studying touchdown behavior of u is equivalent to studying large time behavior of w.

Lemma 7.3.1. *Suppose w is a solution of (7.3.2). Then, $w(y,s) \to w_\infty(y)$ as $s \to \infty$ uniformly on $|y| \le C$, where $C > 0$ is any bounded constant, and $w_\infty(y)$ is a bounded positive solution of*

$$\Delta w - \frac{1}{2}y \cdot \nabla w + \frac{1}{3}w - \frac{\lambda f(a)}{w^2} = 0 \quad in \quad \mathbb{R}^N\,, \tag{7.3.3}$$

where $f(a) > 0$.

Proof: We adapt the arguments from the proofs of Propositions 6 and 7 in [56]: let $\{s_j\}$ be a sequence such that $s_j \to \infty$ and $s_{j+1} - s_j \to \infty$ as $j \to \infty$. We define $w_j(y,s) = w(y, s+s_j)$. According to Theorem 7.1.3, Corollary 7.1.8 and Arzela-Ascoli theorem, there is a subsequence of $\{w_j\}$, still denoted by w_j, such that

$$w_j(y,s) \to w_\infty(y,s)$$

uniformly on compact subsets of W, and

$$\nabla w_j(y,m) \to \nabla w_\infty(y,m)$$

for almost all y and for each integer m. We obtain from Corollary 7.1.8 that either $w_\infty \equiv \infty$ or $w_\infty < \infty$ in $\mathbb{R}^{N+1}$. Since a is a touchdown point for u, the case $w_\infty \equiv \infty$ is ruled out by Theorem 7.1.11, and hence $w_\infty < \infty$ in $\mathbb{R}^{N+1}$. Therefore, we conclude again from Corollary 7.1.8 that

$$w \le C_1(1 + |y|) \tag{7.3.4}$$

for some constant $C_1 > 0$.

Define the associated energy of w at time s,

$$E_R[w](s) = \frac{1}{2}\int_{B_R} \rho|\nabla w|^2 dy - \frac{1}{6}\int_{B_R} \rho w^2 dy - \int_{B_R} \frac{\lambda\rho f(a)}{w} dy\,. \tag{7.3.5}$$

Taking $R(s) = s$, the same calculations as in (7.1.30) give

$$-\frac{d}{ds}E_s[w](s) = \int_{B_s} \rho(y)|w_s|^2 dy - K(s) \tag{7.3.6}$$

with

$$K(s) = \int_{\partial B_s} \rho w_s \frac{\partial w}{\partial \nu} dS + \frac{1}{2s} \int_{\partial B_s} \rho |\nabla w|^2 (y \cdot \nu) dS - \frac{1}{s} \int_{\partial B_s} \rho \Big(\frac{1}{6} w^2 + \frac{\lambda f(a)}{w} \Big) (y \cdot \nu) dS$$
$$+ \lambda \int_{B_s} \frac{\rho w_s \big[f(a) - f(a + y e^{-s/2}) \big]}{w^2} dy \,.$$

We note that the expression $K(s)$ can be estimated as $s \gg 1$. Essentially, since $f(x) \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0, 1]$, using (7.3.4) and applying the same estimates as in Lemma 7.1.9 one can deduce that

$$K(s) - \frac{1}{2} \int_{B_s} \rho w_s^2 dy \le G(s) := C_1 s^N e^{-\frac{s^2}{4}} + C_2 e^{-\frac{\alpha}{2} s} \quad \text{for } s \gg 1. \tag{7.3.7}$$

Together with (7.3.7), integrating (7.3.6) in time yields an energy inequality

$$\frac{1}{2} \int_a^b \int_{B_s} \rho |w_s|^2 dy ds \le E_a[w](a) - E_b[w](b) + \int_a^b G(s) ds \,, \tag{7.3.8}$$

whenever $a < b$.

We now use (7.3.8) to prove that w_∞ is independent of s. We set $a = s_j + m$ and $b = s_{j+1} + m$ in (7.3.8) to obtain

$$\frac{1}{2} \int_m^{m + s_{j+1} - s_j} \int_{B_{s_j + s}} \rho |w_{js}|^2 dy ds \le E_{s_j + m}[w_j](m) - E_{s_{j+1} + m}[w_{j+1}](m) + \int_{s_j + m}^{s_{j+1} + m} G(s) ds \tag{7.3.9}$$

for any integer m, where we use $w_j(y, s) = w(y, s + s_j)$. Since $\nabla w_j(y, m)$ is bounded and independent of j, and since we have assumed that $\nabla w_j(y, m) \to \nabla w_\infty(y, m)$ *a.e.* as $j \to \infty$, the dominated convergence theorem shows that

$$\int \rho(y) |\nabla w_j(y, m)|^2 dy \to \int \rho(y) |\nabla w_\infty(y, m)|^2 dy \quad as \quad j \to \infty \,.$$

Arguing similarly for the other terms we can deduce that

$$\lim_{j \to \infty} E_{s_j + m}[w_j](m) = \lim_{j \to \infty} E_{s_{j+1} + m}[w_{j+1}](m) := E[w_\infty] \,. \tag{7.3.10}$$

On the other hand, because $m + s_j \to \infty$ as $j \to \infty$, (7.3.7) assures that the term involving G in (7.3.9) tends to zero as $j \to \infty$. Therefore, the right side of (7.3.9) tends to zero as $j \to \infty$. It now follows from $s_{j+1} - s_j \to \infty$ that

$$\lim_{j \to \infty} \int_m^M \int_{B_{s_j + s}} \rho |w_{js}|^2 dy ds = 0 \tag{7.3.11}$$

for each pair of integers $m < M$. Further, since (7.3.4) implies $|w_{js}(y, s)| \le C(1 + |y|)$ with C independently of j, one can deduce that w_{js} converges weakly to $w_{\infty s}$. Because ρ

decreases exponentially as $|y| \to \infty$, the integral of (7.3.11) is lower semi-continuous, and hence

$$\int_m^M \int_{\mathbb{R}^N} \rho |w_{\infty s}|^2 dy ds = 0\,,$$

where m and M are arbitrary, which shows that w_∞ is independent of the choice of s.

We now notice from (7.3.5) that (7.3.10) defines $E[w_\infty]$ by

$$E[v] = \frac{1}{2}\int_{\mathbb{R}^N} \rho |\nabla_y v|^2 dy - \frac{1}{6}\int_{\mathbb{R}^N} \rho |v|^2 dy - \int_{\mathbb{R}^N} \frac{\lambda \rho f(a)}{v} dy\,.$$

We claim that $E[w_\infty]$ is independent of the choice of the sequence $\{s_j\}$. If this is not the case, then there is another $\{\bar{s}_j\}$ such that $E[w_\infty] \neq E[\bar{w}_\infty]$, where $\bar{w}_\infty = \lim_{j\to\infty} \bar{w}_j$ with $\bar{w}_j(y,s) = w(y, s+\bar{s}_j)$. Relabeling and passing to a sequence if necessary, we may suppose that $E[w_\infty] < E[\bar{w}_\infty]$ with $s_j < \bar{s}_j$. Now the energy inequality (7.3.8), with $a = s_j$ and $b = \bar{s}_j$, gives that

$$\frac{1}{2}\int_{s_j}^{\bar{s}_j} \int_{B_s} \rho |w_s|^2 dy ds \le E_{s_j}[w_j](0) - E_{\bar{s}_j}[\bar{w}_j](0) + \int_{s_j}^{\bar{s}_j} G(s) ds\,. \tag{7.3.12}$$

Since $E_{s_j}[w_j](0) - E_{\bar{s}_j}[\bar{w}_j](0) \to E[w_\infty] - E_{\bar{w}_\infty}[\bar{w}_\infty] < 0$ and $\int_{s_j}^{\bar{s}_j} G(s)ds \to 0$ as $j \to \infty$, the right side of (7.3.12) is negative for sufficiently large j. This leads to a contradiction, because the left side of (7.3.12) is non-negative. Hence $E[w_\infty] = E[\bar{w}_\infty]$, which implies that $E[w_\infty]$ is independent of the choice of the sequence $\{s_j\}$.

Therefore, we conclude that $w(y,s) \to w_\infty(y)$ as $s \to \infty$ uniformly on $|y| \le C$, where C is any bounded constant, and $w_\infty(y)$ is a bounded positive solution of (7.3.3). ■

7.3.1 Refined touchdown profiles for $N = 1$

In this subsection, we establish refined touchdown profiles for the deflection $u = u(x,t)$ in one dimensional case. We begin with the discussions on the solution $w_\infty(y)$ of (7.3.3). For one dimensional case, due to Fila and Hulshof [43], Theorem 7.2.1 shows that every non-constant solution $w(y)$ of

$$w_{yy} - \frac{1}{2} y w_y + \frac{1}{3} w - \frac{1}{w^2} = 0 \quad \text{in} \quad (-\infty, \infty)$$

must be strictly increasing for all $|y|$ sufficiently large, and $w(y)$ tends to ∞ as $|y| \to \infty$. So it reduces from Lemma 7.3.1 that it must have $w_\infty(y) \equiv const.$. Therefore, by scaling we conclude that

$$\lim_{s\to\infty} w(y,s) \equiv \big(3\lambda f(a)\big)^{\frac{1}{3}}$$

uniformly on $|y| \le C$ for any bounded constant C. This gives the following touchdown rate.

Lemma 7.3.2. *Assume f satisfies* (7.0.2) *on a bounded domain $\Omega \subset \mathbb{R}^1$, and suppose u is a unique touchdown solution of* (7.0.1) *at finite time T. Assume touchdown set for u is a compact subset of Ω. If $x = a$ is a touchdown point of u, then we have*

$$\lim_{t \to T^-} u(x,t)(T-t)^{-\frac{1}{3}} \equiv \big(3\lambda f(a)\big)^{\frac{1}{3}}$$

uniformly on $|x-a| \le C\sqrt{T-t}$ for any bounded constant C.

We next determine the refined touchdown profiles for one dimensional case. Our method is based on the center manifold analysis of a PDE that results from a similarity group transformation of (7.0.1). Such an approach was used in [64] for the uniform permittivity profile $f(x) \equiv 1$. A closely related approach was used in [45] to determine the refined blow-up profile for a semilinear heat equation. We now briefly outline this method and the results that can be extended to the varying permittivity profile $f(x)$:

Continuing from (7.3.2) with touchdown point $x = a$, for $s \gg 1$ and $|y|$ bounded we have $w \sim w_\infty + v$, where $v \ll 1$ and $w_\infty \equiv (3\lambda f(a))^{1/3} > 0$. Keeping the quadratic terms in v, we obtain for $N = 1$ that

$$\begin{aligned} v_s - v_{yy} + \frac{y}{2}v_y - v = {} & \frac{w_\infty}{3}\big[1 - \frac{f(a+ye^{-s/2})}{f(a)}\big] + \frac{2\big[f(a+ye^{-s/2}) - f(a)\big]}{3f(a)}v \\ & - \frac{3\lambda f(a+ye^{-s/2})}{w_\infty^4}v^2 + O(v^3) \\ \approx {} & -\big(3\lambda f(a)\big)^{-\frac{1}{3}}v^2 + O\big(v^3 + e^{-\frac{\alpha}{2}s}\big)\,, \end{aligned} \tag{7.3.13}$$

for $s \gg 1$ and bounded $|y|$, due to the assumption (7.0.2) that $f(x) \in C^\alpha(\bar{\Omega})$ for some $0 < \alpha \le 1$. As shown in [45] (see also [64]), the linearized operator in (7.3.13) has a one-dimensional nullspace when $N = 1$. By projecting the nonlinear term in (7.3.13) against the nullspace of the linearized operator, the following far-field behavior of v for $s \to +\infty$ and $|y|$ bounded is obtained (see (1.7) of [45]):

$$v \sim -\frac{\big(3\lambda f(a)\big)^{\frac{1}{3}}}{4s}\Big(1 - \frac{|y|^2}{2}\Big)\,, \quad N = 1\,. \tag{7.3.14}$$

The refined touchdown profile is then obtained from $w \sim w_\infty + v$, (7.3.1) and (7.3.14), which is for $t \to T^-$,

$$u \sim [3\lambda f(a)(T-t)]^{1/3}\left(1 - \frac{1}{4|\log(T-t)|} + \frac{|x-a|^2}{8(T-t)|\log(T-t)|} + \cdots\right)\,, \quad N = 1\,. \tag{7.3.15}$$

Combining Lemma 7.3.2 and (7.3.15) directly gives the following refined touchdown profile.

Theorem 7.3.3. *Assume f satisfies* (7.0.2) *on a bounded domain Ω in $\mathbb{R}^N$, and suppose u is a touchdown solution of* (7.0.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, and suppose $N = 1$ and $x = a$ is a touchdown point of u. Then we have*

$$\lim_{t\to T^-} u(x,t)(T-t)^{-\frac{1}{3}} \equiv \big(3\lambda f(a)\big)^{\frac{1}{3}} \tag{7.3.16}$$

uniformly on $|x-a| \le C\sqrt{T-t}$ for any bounded constant C. Moveover, when $t \to T^-$,

$$u \sim [3\lambda f(a)(T-t)]^{1/3}\left(1 - \frac{1}{4|\log(T-t)|} + \frac{|x-a|^2}{8(T-t)|\log(T-t)|} + \cdots\right), \quad N = 1\,. \tag{7.3.17}$$

We finally remark that applying formal asymptotic methods, when $N = 1$ the refined touchdown profile of (7.0.1) was also established in (6.4.12). By making a binomial approximation, it is easy to compare that (7.3.15) agrees asymptotically with (6.4.12).

7.3.2 Refined touchdown profiles for $N \ge 2$

For obtaining refined touchdown profiles in higher dimension, in this subsection we assume that $f(r) = f(|x|)$ is radially symmetric and $\Omega = B_R(0)$ is a bounded ball in $\mathbb{R}^N$ with $N \ge 2$. Then the uniqueness of solutions for (7.0.1) implies that the solution u of (7.0.1) must be radially symmetric. We study the refined touchdown profile for the special touchdown point $r = 0$ of u at finite time T. In this situation, the fact that the solution u of (7.0.1) is radially symmetric implies the radial symmetry of $w(y,s)$ in y, and hence the radial symmetry of $w_\infty(y)$ (cf. [53]). Note that $w_\infty(y)$ is a radially symmetric solution of

$$w_{yy} + \big(\frac{N-1}{y} - \frac{y}{2}\big)w_y + \frac{1}{3}w - \frac{\lambda f(0)}{w^2} = 0 \quad \text{for} \quad y > 0\,, \tag{7.3.18}$$

where $w_y(0) = 0$ and $f(0) > 0$. For this case, due to [62], Theorem 7.2.5 yields that every non-constant radial solution $w(y)$ of (7.3.18) must be strictly increasing for all y sufficiently large, and $w(y)$ tends to ∞ as $y \to \infty$. It now reduces again from Lemma 7.3.1 that

$$\lim_{s\to\infty} w(y,s) \equiv \big(3\lambda f(0)\big)^{\frac{1}{3}}$$

uniformly on $|y| \le C$ for any bounded constant C. This gives the following touchdown rate.

Lemma 7.3.4. *Assume $f(r) = f(|x|)$ satisfies* (7.0.2) *on a bounded ball $B_R(0) \subset \mathbb{R}^N$ with $N \ge 2$, and suppose u is a unique touchdown solution of* (7.0.1) *at finite time T. Assume touchdown set for u is a compact subset of Ω. If $r = 0$ is a touchdown point of u, then we have*

$$\lim_{t\to T^-} u(r,t)(T-t)^{-\frac{1}{3}} \equiv \big(3\lambda f(0)\big)^{\frac{1}{3}}$$

uniformly for $r \le C\sqrt{T-t}$ for any bounded constant C.

We next derive a refined touchdown profile (7.3.20). Similar to one dimensional case, indeed we can establish the refined touchdown profiles for varying permittivity profile $f(|x|)$ defined in higher dimension $N \geq 2$. Specially, applying a result from [45], the refined touchdown profile for $N = 2$ is given by

$$u \sim [3\lambda f(0)(T-t)]^{1/3}\left(1 - \frac{1}{2|\log(T-t)|} + \frac{|x-a|^2}{4(T-t)|\log(T-t)|} + \cdots\right), \quad N = 2\,.$$

This leads to the following refined touchdown profile for higher dimensional case.

Theorem 7.3.5. *Assume f satisfies* (7.0.2) *on a bounded domain Ω in $\mathbb{R}^N$, and suppose u is a touchdown solution of* (7.0.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, and suppose $\Omega = B_R(0) \subset \mathbb{R}^N$ is a bounded ball with $N \geq 2$ and $f(r) = f(|x|)$ is radially symmetric. If $r = 0$ is a touchdown point of u, then we have*

$$\lim_{t \to T^-} u(r,t)(T-t)^{-\frac{1}{3}} \equiv \big(3\lambda f(0)\big)^{\frac{1}{3}} \tag{7.3.19}$$

uniformly on $r \leq C\sqrt{T-t}$ for any bounded constant C. Moveover, when $t \to T^-$,

$$u \sim [3\lambda f(0)(T-t)]^{1/3}\left(1 - \frac{1}{2|\log(T-t)|} + \frac{r^2}{4(T-t)|\log(T-t)|} + \cdots\right), \quad N = 2\,. \tag{7.3.20}$$

Remark 7.3.1. Applying analytical and numerical techniques, next section we shall show that Theorem 7.3.5 does hold for a larger class of profiles $f(r) = f(|x|)$.

Before concluding this section, it is interesting to compare the solution of (7.0.1) with that of the ordinary differential equation obtained by omitting Δu. For that we focus on one dimensional case, and we compare the solutions of

$$u_t - u_{xx} = -\frac{\lambda f(x)}{u^2} \quad \text{in} \quad (-a,a)\,, \tag{7.3.21a}$$

$$u(\pm a,t) = 1\,; \quad u(x,0) = 1\,, \tag{7.3.21b}$$

and

$$v_t = -\frac{\lambda f(x)}{v^2} \quad \text{in} \quad (-a,a)\,, \tag{7.3.22a}$$

$$v(\pm a,t) = 1\,; \quad v(x,0) = 1\,, \tag{7.3.22b}$$

where f is assumed to satisfy (7.0.2) and (7.1.1). The ordinary differential equation (7.3.22) is explicitly solvable, and the solution touches down at finite time

$$v(x,t) = \big(1 - 3\lambda f(x)t\big)^{\frac{1}{3}}\,, \tag{7.3.23}$$

which shows that touchdown point of v is the maximum value point of $f(x)$. In the partial differential equation (7.3.21), there is a contest between the dissipating effect of the Laplacian u_{xx} and the singularizing effect of the nonlinearity $f(x)/u^2$; when u touches down at $x = x_0$ in finite time T, then the nonlinear term dominates (essentially, for some special cases, touchdown point x_0 of u is also the maximum value point of $f(x)$, see Theorem 7.4.2 for details).

However, we claim that a smoothing effect of the Laplacian can be still observed in the different character of touchdown. Indeed, letting $f(y_0) = \max\{f(x) : x \in (-a, a)\}$, then $f'(y_0) = 0$ and $f''(y_0) \leq 0$. And (7.3.23) gives the finite touchdown time T_0 for v satisfying $T_0 = 1/[3\lambda f(y_0)]$. Furthermore, we can get from (7.3.23), together with the Taylor series of $f(x)$,

$$\lim_{t\to T_0^-}(T_0-t)^{-\frac{1}{3}}v\big(y_0+(T_0-t)^{\frac{1}{2}}y,t\big) = \big(3\lambda f(y_0)\big)^{\frac{1}{3}}\big[1-\frac{f''(y_0)}{2f^2(y_0)}|y|^2\big]^{\frac{1}{3}} \geq \big(3\lambda f(x_0)\big)^{\frac{1}{3}}. \quad (7.3.24)$$

And our Theorem 7.3.3 says that for such u we have

$$\lim_{t\to T^-}(T-t)^{-\frac{1}{3}}u\big(x_0+(T-t)^{\frac{1}{2}}y,t\big) = \big(3\lambda f(x_0)\big)^{\frac{1}{3}}. \quad (7.3.25)$$

Comparing (7.3.24) with (7.3.25), we see that the touchdown of the partial differential equation (7.3.21) is "flatter" than that of the ordinary differential equation (7.3.22).

7.4 Set of touchdown points

This section is focussed on the set of touchdown points for (7.0.1), which may provide useful information on the design of MEMS devices. In subsection §7.4.1, we consider the radially symmetric case where $f(r) = f(|x|)$ with $r = |x|$ is a radial function and Ω is a ball $B_R = \{|x| \leq R\} \subset \mathbb{R}^N$ with $N \geq 1$. In subsection §7.4.2, numerically we compute some simulations for one dimensional case, from which we discuss the compose of touchdown points for some explicit permittivity profiles $f(x)$.

7.4.1 Radially symmetric case

In this subsection, $f(r) = f(|x|)$ is assumed to be a radial function and Ω is assumed to be a ball $B_R = \{|x| \leq R\} \subset \mathbb{R}^N$ with any $N \geq 1$. For this radially symmetric case, the uniqueness of solutions for (7.0.1) implies that the solution $u = u(x,t)$ of (7.0.1) must be radially symmetric. We begin with the following lemma for proving Theorem 7.4.2:

Lemma 7.4.1. *Suppose $f(r)$ satisfies (7.0.2) and $f'(r) \leq 0$ in B_R, and let $u = u(r,t)$ be a touchdown solution of (7.0.1) at finite time T. Then $u_r > 0$ in $\{0 < r < R\} \times (t_0, T)$ for some $0 < t_0 < T$.*

Proof: Setting $w = r^{N-1}u_r$, then (7.0.1) gives

$$u_t - \frac{1}{r^{N-1}}w_r = -\frac{\lambda f(r)}{u^2}\,, \quad 0 < t < T\,. \tag{7.4.1}$$

Differentiating (7.4.1) with respect to r, we obtain

$$w_t - w_{rr} + \frac{N-1}{r}w_r - \frac{2\lambda f}{u^3}w = -\frac{\lambda f' r^{N-1}}{u^2} \geq 0\,, \quad 0 < t < T\,, \tag{7.4.2}$$

since $f'(r) \leq 0$ in B_R. Therefore, w can not attain negative minimum in $\{0 < r < R\}\times(0,T)$. Since $w(0,t) = w(r,0) = 0$ and $u_t < 0$ for all $t \in (0,T)$, we have $w = r^{N-1}u_r > 0$ on $\partial B_R \times (0,T)$. So the maximum principle shows that $w \geq 0$ in $\{0 < r < R\} \times (0,T)$. This gives

$$w_t - w_{rr} + \frac{N-1}{r}w_r \geq 0 \quad in \quad \{0 < r < R\} \times (t_1, T)\,,$$

where $t_1 > 0$ is chosen so that $w(r,t_1) \not\equiv 0$ in $\{0 < r < R\}$.

Now compare w with the solution z of

$$z_t - z_{rr} + \frac{N-1}{r}z_r = 0 \quad in \quad \{0 < r < R\} \times (t_1, T)$$

subject to $z(r,t_1) = w(r,t_1)$ for $0 \leq r \leq R$, $z(R,t) = w(R,t) > 0$ and $z(0,t) = 0$ for $t_1 \leq t < T$. The comparison principle yields $w \geq z$ in $\{0 < r < R\} \times (t_1,T)$. On the other hand, for any $t_0 > t_1$ we have $z > 0$ in $\{0 < r < R\} \times (t_0,T)$. Consequently we conclude that $w > 0$, *i.e* $u_r > 0$ in $\{0 < r < R\} \times (t_0,T)$. ■

Theorem 7.4.2. *Assume $f(r) = f(|x|)$ satisfies* (7.0.2) *and $f'(r) \leq 0$ in a bounded ball $B_R(0) \subset \mathbb{R}^N$ with $N \geq 1$, and suppose u is a touchdown solution of* (7.0.1) *at finite time T. Then, $r = 0$ is the unique touchdown point of u.*

Proof: For $w = r^{N-1}u_r$, we set $J(r,t) = w - \varepsilon \int_0^{r^\theta} f(s)ds$, where $\theta \geq N$ and $\varepsilon = \varepsilon(\theta) > 0$ are constants to be determined. We calculate from (7.4.1) and (7.4.2) that

$$\begin{aligned} J_t - J_{rr} + \frac{N-1}{r}J_r &= b_1 J + \frac{2\lambda\varepsilon f \int_0^{r^\theta} f(s)ds}{u^3} - \frac{\lambda f' r^{N-1}}{u^2} + \theta\varepsilon r^{\theta-1} f' \\ &\geq b_1 J - r^{N-1}\big(\lambda - \theta\varepsilon r^{\theta-N}\big) f' \geq b_1 J\,, \end{aligned}$$

provided ε is sufficiently small, where b_1 is a locally bounded function. Here we have applied the assumption $f'(r) \leq 0$ and the relations $u_r = w/r^{N-1}$ and $w = J + \varepsilon \int_0^{r^\theta} f(s)ds$. Note that $J(0,t) = 0$, and hence it follows that J can not obtain negative minimum in $B_R \times (0,T)$.

We next observe that J can not obtain negative minimum on $\{r = R\}$ provided ε is sufficiently small, which comes from the fact

$$J_r(R,t) = w_r - \theta\varepsilon R^{\theta-1} f(R) = \frac{\lambda R^{N-1} f(R)}{u^2} - \theta\varepsilon R^{\theta-1} f(R) \geq R^{N-1} f(R)\big[\lambda - \theta\varepsilon R^{\theta-N}\big] \geq 0$$

for sufficiently small $\varepsilon > 0$, where (7.4.1) is applied. We now choose some $0 < t_0 < T$ such that $w(r,t_0) > 0$ for $0 < r \le R$ in view of Lemma 7.4.1. This gives $u_r(r,t_0) > 0$ for $0 < r \le R$. Since $u_r(0,t_0) = 0$, there exists some $\alpha > 0$ such that

$$u_{rr}(0,t_0) = \lim_{r\to 0} \frac{u_r(r,t_0)}{r^\alpha} = \lim_{r\to 0} \frac{w(r,t_0)}{r^{N+\alpha-1}} > 0\,.$$

We now choose $\theta = \max\{N, N+\alpha-1\}$, from which one can further deduce that $J(r,t_0) \ge 0$ for $0 \le r < R$ provided $\varepsilon = \varepsilon(t_0,\theta) > 0$ is sufficiently small.

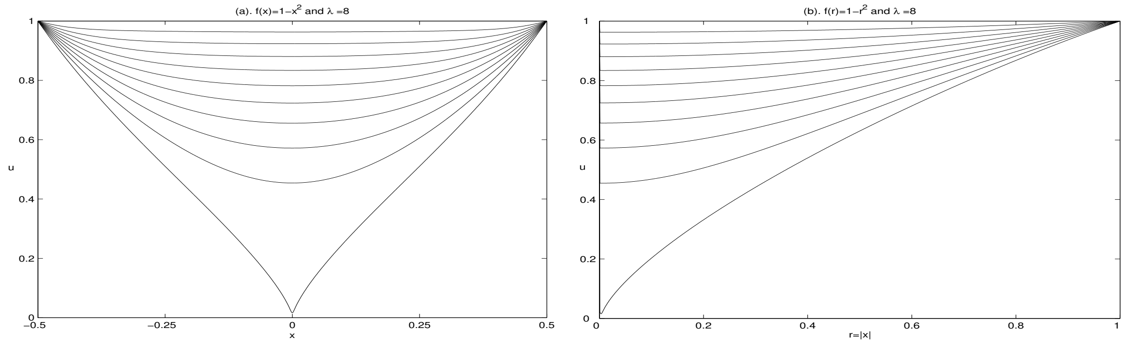

Figure 7.1: *Left figure: plots of u versus x at different times with $f(x) = 1 - x^2$ in the slab domain, where the unique touchdown point is $x = 0$. Right figure: plots of u versus $r = |x|$ at different times with $f(r) = 1 - r^2$ in the unit disk domain, where the unique touchdown point is $r = 0$ too.*

It now concludes from the maximum principle that $J \ge 0$ in $B_R \times (t_0, T)$ provided $\varepsilon = \varepsilon(t_0) > 0$ is sufficiently small. This leads to

$$u(r,t) \ge u(r,t) - u(0,t) \ge \varepsilon \int_0^r \frac{\int_0^{s^\theta} f(\mu)d\mu}{s^{N-1}} ds. \tag{7.4.3}$$

Given small $C_0 > 0$, then the assumption of $f(r)$ implies that there exists $0 < r_0 = r_0(C_0) \le R$ such that $f(r) \ge C_0$ on $[0, r_0]$. Denote $r_m = min\{r_0, r\}$, and then (7.4.3) gives

$$u(r,t) \ge \varepsilon \int_0^{r_m} \frac{\int_0^{s^\theta} f(\mu)d\mu}{s^{N-1}} ds \ge \varepsilon \int_0^{r_m} \frac{C_0 s^\theta}{s^{N-1}} ds = \frac{1}{\theta - N + 2} \varepsilon C_0 r_m^{\theta-N+2}\,, \quad \text{where} \quad \theta - N + 2 \ge 2\,,$$

which implies that $r = 0$ must be the unique touchdown point of u. ■

Remark 7.4.1. Assume $f(r) = f(|x|)$ satisfies (7.0.2) and $f'(r) \le 0$ in a bounded ball $B_R(0) \subset \mathbb{R}^N$ with $N \ge 1$. Together with Proposition 7.1.1 below, Theorems 7.1.3 and 7.4.2 show an interesting phenomenon: finite-time touchdown point is not the zero point of $f(x)$, but the maximum value point of $f(x)$.

Remark 7.4.2. Numerical simulations in subsection §4.1 show that the assumption $f'(r) \leq 0$ in Theorem 7.4.2 is sufficient, but not necessary. This gives that Theorem 7.3.5 does hold for a larger class of profiles $f(r) = f(|x|)$.

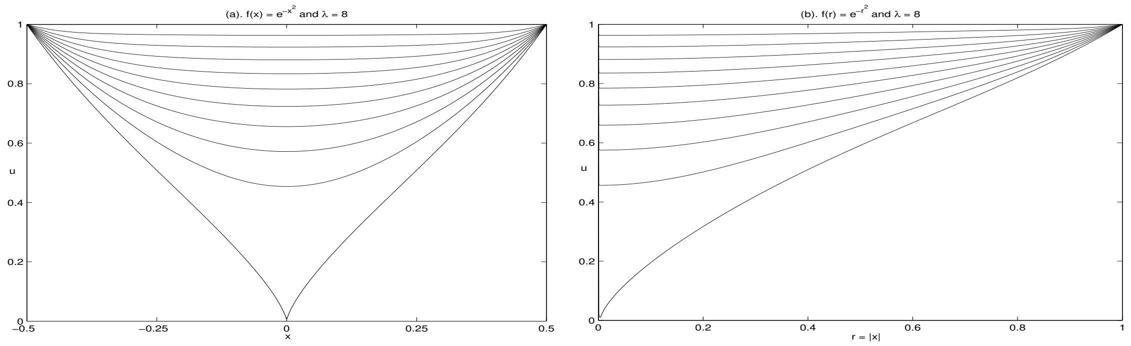

Figure 7.2: *Left figure: plots of u versus x at different times with $f(x) = e^{-x^2}$ in the slab domain, where the unique touchdown point is $x = 0$. Right figure: plots of u versus $r = |x|$ at different times with $f(r) = e^{-r^2}$ in the unit disk domain, where the unique touchdown point is $r = 0$ too.*

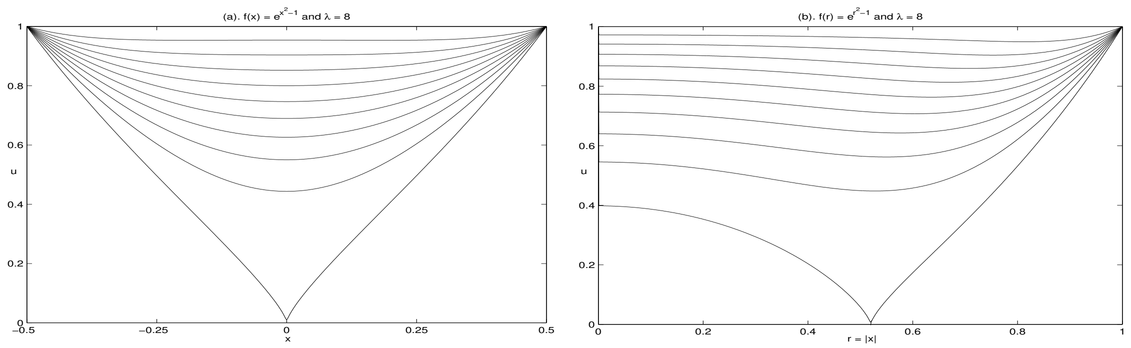

Figure 7.3: *Left figure: plots of u versus x at different times with $f(x) = e^{x^2-1}$ in the slab domain, where the unique touchdown point is still at $x = 0$. Right figure: plots of u versus $r = |x|$ at different times with $f(r) = e^{r^2-1}$ in the unit disk domain, where the touchdown points satisfy $r = 0.51952$.*

Before ending this subsection, we now present a few numerical simulations on Theorem 7.4.2. Here we apply the implicit Crank-Nicholson scheme. In the following simulations $1 \sim 3$, we always take $\lambda = 8$ and the number of meshpoints $N = 1000$, and consider (7.0.1) in the following symmetric slab or unit disk domains:

$$\Omega : [-1/2, 1/2] \quad (\text{Slab})\,; \qquad \Omega : x^2 + y^2 \leq 1 \quad (\text{Unit Disk})\,. \tag{7.4.4}$$

Simulation 1: $f(|x|) = 1 - |x|^2$ is chosen as a permittivity profile. In Figure 7.1(a), u versus x is plotted at different times for (7.0.1) in the symmetric slab domain. For this touchdown behavior, touchdown time is $T = 0.044727$ and the unique touchdown point is $x = 0$. In Figure 7.1(b), u versus $r = |x|$ is plotted at different times for (7.0.1) in the unit disk domain. For this touchdown behavior, touchdown time is $T = 0.0455037$ and the unique touchdown point is $r = 0$.

Simulation 2: $f(|x|) = e^{-|x|^2}$ is chosen as a permittivity profile. In Figure 7.2(a), u versus x is plotted at different times for (7.0.1) in the symmetric slab domain. For this touchdown behavior, touchdown time is $T = 0.044675$ and the unique touchdown point is $x = 0$. In Figure 7.2(b), u versus $r = |x|$ is plotted at different times for (7.0.1) in the unit disk domain. For this touchdown behavior, touchdown time is $T = 0.0450226$ and the unique touchdown point is $r = 0$ too.

Simulation 3: $f(|x|) = e^{|x|^2-1}$ is chosen as a permittivity profile. In Figure 7.3(a), u versus x is plotted at different times for (7.0.1) in the symmetric slab domain. For this touchdown behavior, touchdown time is $T = 0.147223$ and touchdown point is still uniquely at $x = 0$. In Figure 7.3(b), u versus $r = |x|$ is plotted at different times for (7.0.1) in the unit disk domain. For this touchdown behavior, touchdown time is $T = 0.09065363$, but touchdown points are at $r_0 = 0.51952$, which compose into the surface of $B_{r_0}(0)$. This simulation shows that the assumption $f'(r) \leq 0$ in Theorem 7.4.2 is just sufficient, not necessary.

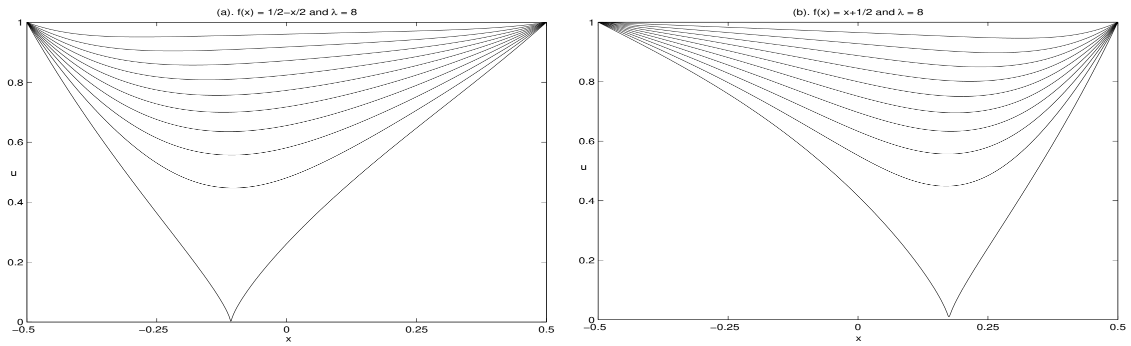

Figure 7.4: *Left figure: plots of* u *versus* x *at different times with* $f(x) = 1/2 - x/2$ *in the slab domain, where the unique touchdown point is* $x = -0.10761$*. Right figure: plots of* u *versus* $r = |x|$ *at different times with* $f(x) = x + 1/2$ *in the slab domain, where the unique touchdown point is* $x = 0.17467$*.*

7.4.2 One dimensional case

For one dimensional case, Theorem 7.4.2 already gives that touchdown points must be unique if the permittivity profile $f(x)$ is uniform. In the following, we choose some explicit varying permittivity profiles $f(x)$ to perform two numerical simulations. Here we apply the implicit Crank-Nicholson scheme again.

Simulation 4: Monotone Function $f(x)$:
We take $\lambda = 8$ and the number of meshpoints $N = 1000$, and we consider (7.0.1) in the slab domain Ω defined in (7.4.4). In Figure 7.4(a), the monotonically decreasing profile $f(x) = 1/2 - x/2$ is chosen, and u versus x is plotted for (7.0.1) at different times. For this touchdown behavior, the touchdown time is $T = 0.09491808$ and the unique touchdown point is $x = -0.10761$. In Figure 7.4(b), the monotonically increasing profile $f(x) = x+1/2$ is chosen, and u versus x is plotted for (7.0.1) at different times. For this touchdown behavior, the touchdown time is $T = 0.0838265$ and the unique touchdown point is $x = 0.17467$. For the general case where $f(x)$ is monotone in a slab domain, it is interesting to look insights into whether the touchdown points must be unique.

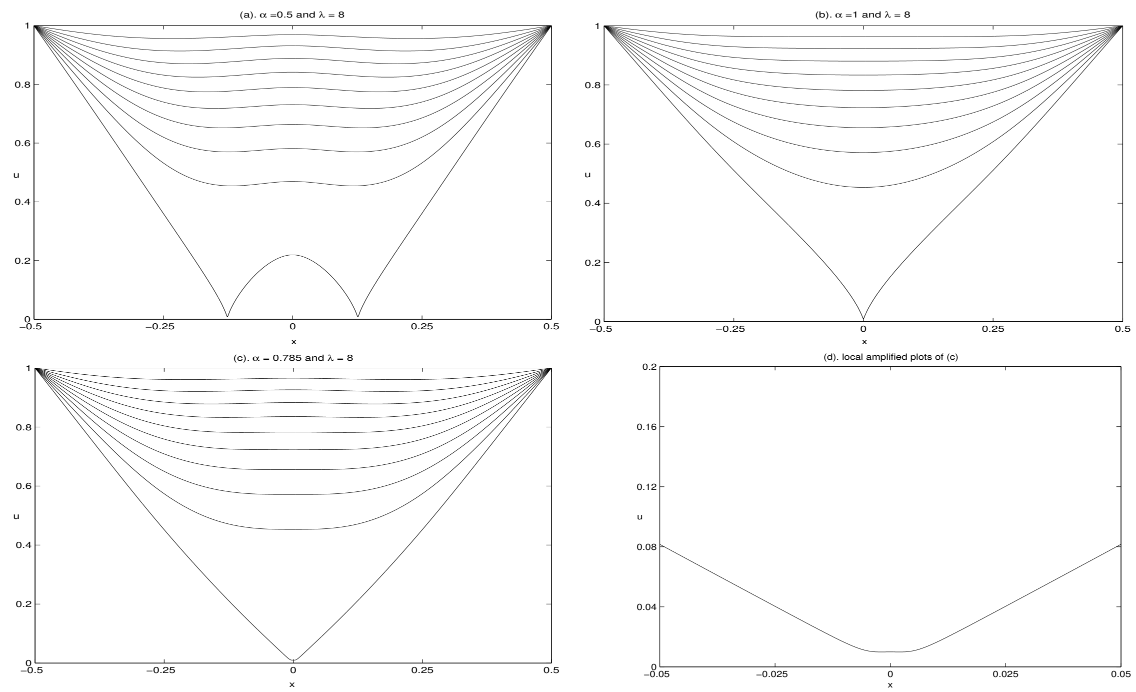

Figure 7.5: *plots of* u *versus* x *at different times in the slab domain, for different permittivity profiles* $f[\alpha](x)$ *given by* (7.4.5). *Top left (a): when* $\alpha = 0.5$*, two touchdown points are at* $x = \pm 0.12631$. *Top right (b): when* $\alpha = 1$*, the unique touchdown point is at* $x = 0$. *Bottom Left (c): when* $\alpha = 0.785$*, touchdown points are observed to consist of a closed interval* $[-0.0021255, 0.0021255]$. *Bottom right (d): local amplified plots of* (c).

Simulation 5: "M"-Form Function $f(x)$:
In this simulation, we consider (7.0.1) in the slab domain Ω defined in (7.4.4). Here we take

$\lambda = 8$ and the number of the meshpoints $N = 2000$, and the varying dielectric permittivity profiles satisfies

$$f[\alpha](x) = \begin{cases} 1 - 16(x + 1/4)^2\,, & if \quad x < -1/4\,; \\ \alpha + (1-\alpha)|sin(2\pi x)|\,, & if \quad |x| \le 1/4\,; \\ 1 - 16(x - 1/4)^2\,, & if \quad x > 1/4 \end{cases} \tag{7.4.5}$$

with $\alpha \in [0, 1]$, which has "M"-form. In Figure 7.5, u versus x is plotted at different times for (7.0.1) for different α, *i.e* for different permittivity profiles $f[\alpha](x)$. In Figure 7.5(a): when $\alpha = 0.5$, the touchdown time is $T = 0.05627054$ and two touchdown points are at $x = \pm 0.12631$. In Figure 7.5(b): when $\alpha = 1$, the touchdown time is $T = 0.0443323$ and the unique touchdown point is at $x = 0$. In Figure 7.5(c): when $\alpha = 0.785$, the touchdown time is $T = 0.04925421$ and touchdown points are observed to compose into a closed interval $[-0.0021255, 0.0021255]$. In Figure 7.5(d): local amplified plots of (c) at touchdown time $t = T$. This simulation shows for dimension $N = 1$ that the set of touchdown points may be composed of finite points or finite compact subsets of the domain, if the permittivity profile is ununiform.

7.5 Some comments

Main results of this Chapter can be found in [63]. Under the additional assumption (7.1.1), the compactness result of touchdown set is established in Proposition 7.1.1 by adapting the method of moving plane, which is due to A. Friedman and B. Mcleod's [48]. Such compactness turns out to be crucial for understanding the refined touchdown behavior, such as touchdown rate, other properties of touchdown set and etc. We strongly expect that the assumption (7.1.1) in Proposition 7.1.1 can be removed. However, it seems difficult to address this interesting problem.

The asymptotic behavior of positive entire solutions for (7.2.1) was studied by M. Fila and J. Hulshof [43] for $N = 1$, and while for $N \ge 2$ it was only addressed for positive radial solutions by J. S. Guo in [62]. Whether any bounded solution of (7.2.1) with $N \ge 2$ must be trivial still remains open, which directly results in that Theorem 7.3.5 is true only for special touchdown point $x = 0$ in radially symmetric case.

In §7.4 we have proved the uniqueness of touchdown points for the case where $f(x) = f(|x|)$ is non-increasing in $|x|$ and $\Omega = B_R$ is a ball in $\mathbb{R}^N$. On the contrary, for the case where $f(x) = f(|x|)$ is increasing in $|x|$ and $\Omega = B_R$ is a ball in $\mathbb{R}^N$, it is still open that whether touchdown points of (7.0.1) must compose into the surface of a ball B_{r_0} for some $0 < r_0 < R$. This phenomenon was first observed in [64] for power-law profile $f(x) = |x|^\alpha$.

Bibliography

[1] J. Ai, K.-S. Chou and J. Wei, *Self-similar solutions for the anisotropic affine curvature shortening problem*, Calc. Var. Partial Differential Equations **13** (2001), no. 3, 311–337.

[2] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), no. 2, 519–543.

[3] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[4] S. Alama and G. Tarantello, *Elliptic problems with nonlinearities indefinite in sign*, J. Funct. Anal. **141** (1996), no. 1, 159–215.

[5] S. Alama and G. Tarantello, *On the solvability of a semilinear elliptic equation via an associated eigenvalue problem*, Math. Z. **221** (1996), no. 3, 467–493.

[6] U. Ascher, J. Christiansen and R.D. Russell, *A collocation solver for mixed order systems of boundary value problems*, Math. Comp. **33** (1979), no. 146, 659–679.

[7] A. Bahri and P.-L. Lions, *Morse index of some min-max critical points. I. Application to multiplicity results*, Comm. Pure Appl. Math. **41** (1988), no. 8, 1027–1037.

[8] A. Bahri and P.-L. Lions, *Solutions of superlinear elliptic equations and their Morse indices*, Comm. Pure Appl. Math. **45** (1992), no. 9, 1205–1215.

[9] C. Bandle, *Isoperimetric inequalities and applications*, Monographs and Studies in Mathematics, 7. Boston, Mass.-London, Pitman, 1980.

[10] R.E. Bank, *PLTMG: a software package for solving elliptic partial differential equations, User's guide 8.0*, Software, Environments, and Tools **5**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (1998).

[11] H. Bellout, *A criterion for blow-up of solutions to semilinear heat equations*, SIAM J. Math. Anal. **18** (1987), no. 3, 722–727.

[12] D. Bernstein, P. Guidotti and J.A. Pelesko, *Analytic and numerical analysis of electrostatically actuated MEMS devices*, Proc. of Modeling and Simulation of Microsystems **2000** (2000), 489–492.

[13] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, *Blow up for* $u_t - \Delta u = g(u)$ *revisited*, Adv. Differential Equations **1** (1996), no. 1, 73–90.

[14] H. Brezis and L. Nirenberg, H^1 *versus* C^1 *local minimizers*, C. R. Acad. Sci. Paris Sr. I Math. **317** (1993), no. 5, 465–472.

[15] H. Brezis, L.A. Peletier and D. Terman, *A very singular solution of the heat equation with absorption*, Arch. Rational Mech. Anal. **95** (1986), no. 3, 185–209.

[16] H. Brezis and J.L. Vazquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), no. 2, 443–469.

[17] C.-M. Brauner and B. Nicolaenko, *On nonlinear eigenvalue problems which extend into free boundaries problems*, Bifurcation and Nonlinear Eigenvalue Problems, Lecture Notes in Math., Springer, Berlin, **782**, 1980, 61–100.

[18] C.-M Brauner and B. Nicolaenko, *Sur une classe de problèmes elliptiques non linéaires*, C. R. Acad. Sci. Paris Sr. A-B **286** (1978), no. 21, A1007–A1010.

[19] B. Buffoni, E.N. Dancer and J.F. Toland, *The sub-harmonic bifurcation of Stokes waves*, Arch. Ration. Mech. Anal. **152** (2000), no. 3, 241–271.

[20] X. Cabré, *Extremal solutions and instantaneous complete blow-up for elliptic and parabolic problems*, Preprint (2005).

[21] X. Cabré and A. Capella, *On the stability of radial solutions of semilinear elliptic equations in all of* $\mathbb{R}^n$, C. R. Math. Acad. Sci. Paris **338** (2004), no. 10, 769–774.

[22] X. Cabré and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal. **238** (2006), no. 2, 709–733.

[23] X. Cabré and Y. Martel, *Weak eigenfunctions for the linearization of extremal elliptic problems*, J. Funct. Anal. **156** (1998), no. 1, 30–56.

[24] L.A. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), 271–297.

[25] D. Castorina, P. Esposito and B. Sciunzi, *Stable and unstable solutions of* $p-$*Laplace equations with singular nonlinearity*, Preprint (2007).

[26] E.K. Chan and R.W. Dutton, *Effects of capacitors, resistors and residual change on the static and dynamic performance of electrostatically actuated devices*, Proceedings of SPIE **3680** (1999), 120–130.

[27] K.-C. Chang, *Infinite-dimensional Morse theory and multiple solution problems*, Progress in Nonlinear Differential Equations and their Applications **6**. Birkauser, Boston, 1993.

[28] K.-C. Chang and J.Q. Liu, *On Nirenberg's problem*, International J. Math. **4**, (1993), no. 1, 35–58.

[29] W. Chen, C. Li, *Classification of some nonlinear elliptic equation*, Duke Math. J. **63** (1991), 615–623.

[30] M.G. Crandall and P.H. Rabinowitz, *Bifurcation, perturbation of simple eigenvalues and linearized stability*, Arch. Rational Mech. Anal. **52** (1973), 161–180.

[31] M.G. Crandall and P.H. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rational Mech. Anal. **58** (1975), no. 3, 207–218.

[32] E.N. Dancer, *Infinitely many turning points for some supercritical problems*, Ann. Mat. Pura Appl. (4) **178** (2000), 225–233.

[33] E.N. Dancer, *Finite Morse index solutions of exponential problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire (2007), doi: 10.1016/j.anihpc.2006.12.001.

[34] J.I. Diaz, J.-M. Morel and L. Oswald, *An elliptic equation with singular nonlinearity*, Comm. Partial Differential Equations **12** (1987), no. 12, 1333–1344.

[35] O. Druet, E. Hebey and F. Robert, *Blow-up theory for elliptic PDEs in Riemannian geometry*, Mathematical Notes **45**. Princeton University Press, Princeton, NJ, 2004.

[36] P. Esposito, *Compactness of a nonlinear eigenvalue problem with a singular nonlinearity*, Preprint (2006).

[37] P. Esposito, *Linear unstability of entire solutions for a class of non-autonomous elliptic equations*, Preprint (2007).

[38] P. Esposito, N. Ghoussoub and Y. Guo, *Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math., to appear.

[39] L. Evans, *Partial differential equations*, Graduate Studies in Mathematics **19**. AMS, Providence, RI, 1998.

[40] A. Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of* $\mathbb{R}^N$, J. Math. Pures Appl., to appear.

[41] A. Farina, , Preprint (2007).

[42] P. Feng and Z. Zhou, *Multiplicity and symmetry breaking for positive radial solutions of semilinear elliptic equations modelling MEMS on annular domains*, Electron. J. Differential Equations **146** (2005), 1–14.

[43] M. Fila and J. Hulshof, *A note on the quenching rate*, Proc. Amer. Math. Soc. **112** (1991), no. 2, 473–477.

[44] M. Fila, J. Hulshof and P. Quittner, *The quenching problem on the N-dimensional ball*, Nonlinear diffusion equations and their equilibrium states **3**, 183–196. Birkhäuser, Boston, MA, 1992.

[45] S. Filippas and R.V. Kohn, *Refined asymptotics for the blowup of $u_t - \Delta u = u^p$*, Comm. Pure Appl. Math. **45** (1992), no. 7, 821–869.

[46] G. Flores, G.A. Mercado and J.A. Pelesko, *Dynamics and touchdown in electrostatic MEMS*, Proceedings of ICMENS **2003** (2003), 182–187.

[47] A. Friedman, J. Friedman and B. McLeod, *Concavity of solutions of nonlinear ordinary differential equations*, J. Math. Anal. Appl. **131** (1988), no. 2, 486–500.

[48] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. **34** (1985), no. 2, 425–447.

[49] J.P. Garcia Azorero and I. Peral Alonso, *Hardy inequalities and some critical elliptic and parabolic problems*, J. Differential Equations **144** (1998), no. 2, 441–476.

[50] F. Gazzola and A. Malchiodi, *Some remarks on the equation $-\Delta u = \lambda(1+u)^p$ for varying λ, p and varying domains*, Comm. Partial Differential Equations **27** (2002), no. 3-4, 809–845.

[51] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices: stationary case*, SIAM J. Math. Anal., to appear.

[52] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices II: dynamic case*, Preprint (2006).

[53] B. Gidas, W.M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243.

[54] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), 525–598.

[55] Y. Giga, *On elliptic equations related to self-similar solutions for nonlinear heat equations*, Hiroshima Math. J. **16** (1986), no. 3, 539–552.

[56] Y. Giga and R.V. Kohn, *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. Pure and Appl. Math. **38** (1985), no. 3, 297–319.

[57] Y. Giga and R.V. Kohn, *Characterizing blowup using similarity variables*, Indiana Univ. Math. J. **36** (1987), no. 1, 1–40.

[58] Y. Giga and R.V. Kohn, *Nondegeneracy of blowup for semilinear heat equations*, Comm. Pure Appl. Math. **42** (1989), no. 6, 845–884.

[59] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order. Reprin of the* 1998 *edition,* Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[60] H. Guo, Z. Guo and K. Li, *Positive solutions of a semilinear elliptic equation with singular nonlinearity*, J. Math. Anal. Appl. **323** (2006), no. 1, 344–359.

[61] J.-S. Guo, *On the quenching behavior of the solution of a semilinear parabolic equation*, J. Math. Anal. Appl. **151** (1990), no. 1, 58–79.

[62] J.-S. Guo, *On the semilinear elliptic equation* $\Delta w - \frac{1}{2} y \cdot \nabla w + \lambda w - w^{-\beta} = 0$ *in* $\mathbb{R}^n$, Chinese J. Math. **19** (1991), no. 4, 355–377.

[63] Y. Guo, *On the partial differential equations of electrostatic MEMS devices III: refined touchdown behavior*, Preprint (2006).

[64] Y. Guo, Z. Pan and M.J. Ward, *Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties*, SIAM J. Appl. Math. **66** (2005), no. 1, 309–338.

[65] Z. Guo and J. Wei, *Hausdorff dimension of ruptures for solutions of a semilinear elliptic equation with singular nonlinearity*, Manuscripta Math. **120** (2006), no. 2, 193–209.

[66] Z. Guo and J. Wei, *Symmetry of nonnegative solutions of a semilinear elliptic equation with singular nonlinearity*, Proc. Roy. Soc. Edinburgh Sect. A, to appear.

[67] Z. Guo and J. Wei, *Infinitely many turning points for an elliptic problem with a singular nonlinearity*, Preprint (2006).

[68] Z. Guo and J. Wei, *On an elliptic problem with a singular nonlinearity*, Preprint (2006).

[69] Q. Han and F. Lin, *Elliptic partial differential equations*, Courant Lecture Notes **1**. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.

[70] Z. Han, *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Analyse non linèaire **8** (1991), no.2, 159–174.

[71] F. Hang and X. Wang, *A new approach to some nonlinear geometric equations in dimension two*, Cal. Var. Partial Differential Equations **26** (2006), no. 1, 119–135.

[72] A. Haraux and F.B. Weissler, *Nonuniqueness for a semilinear initial value problem*, Indiana Univ. Math. J. **31** (1982), no. 2, 167–189.

[73] P. Hartman, *Ordinary differential equations*. John Wiley & Sons, Inc., New York-London-Sydney 1964.

[74] J.D. Jackson, *Classical electrodynamics*, Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1975.

[75] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Ration. Mech. Anal. **49** (1972/73), 241–269.

[76] J.P. Keener and H.B. Keller, *Positive solutions of convex nonlinear eigenvalue problems*, J. Differential Equations **16** (1974), 103–125.

[77] J.B. Keller, J.S. Lowengrub, *Asymptotic and numerical results for blowing-up solutions to semilinear heat equations*, Singularities in Fluids, Plasmas and Optics, 111–129. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 404, Kluwer Acad. Publ., Dordrecht, 1993.

[78] E. Koizumi and K. Schmitt, *Ambrosetti-Prodi-type problems for quasilinear elliptic elliptic problems*, Differential Integral Equations **18** (2005), no. 3, 241–262.

[79] H.A. Levine, *Quenching, nonquenching, and beyond quenching for solution of some parabolic equations*, Ann. Mat. Pura Appl. **155** (1989), no. 4, 243–260.

[80] O.A. Ladyženskaja, V.A. Solonnikov and N.N. Ural′ceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs **23**. American Mathematical Society, Providence, R.I., 1967.

[81] A.M. Meadows, *Stable and singular solutions of the equation* $\Delta u = \frac{1}{u}$, Indiana Univ. Math. J. **53** (2004), no. 6, 1681–1703.

[82] F. Mignot and J.-P. Puel, *Sur une classe de problèmes non linéaires avec non linéarité positive, croissante, convexe*, Comm. Partial Differential Equations **5** (1980), no. 8, 791–836.

[83] H.C. Nathanson, W.E. Newell and R.A. Wickstrom, J.R. Davis, *The resonant gate transistor*, IEEE Trans. on Elect. Devices. **14** (1967), 117–133.

[84] G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations*. C. R. Math. Acad. Sci. Paris **330** (2000), 997–1002.

[85] J.A. Pelesko, *Mathematical modeling of electrostatic MEMS with tailored dielectric properties*, SIAM J. Appl. Math. **62** (2001/2002), no. 3, 888–908.

[86] J.A. Pelesko and D.H. Bernstein, *Modeling MEMS and NEMS*. Chapman & Hall/CRC, Boca Raton, FL, 2003.

[87] J.A. Pelesko, D.H. Bernstein and J. McCuan, *Symmetry and symmetry breaking in electrostatic MEMS*, Proceedings of MSM **2003** (2003), 304–307.

[88] J.A. Pelesko and A.A. Triolo, *Nonlocal problems in MEMS device control*, J. Engrg. Math. **41** (2001), no. 4, 345–366.

[89] J. Prajapat and G. Tarantello, *On a class of elliptic problem in $\mathbb{R}^N$: symmetry and uniqueness results*, Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), 967–985.

[90] J.I. Seeger and S.B. Crary, *Stabilization of electrostatically actuated mechanical devices*, Proc. of 1997 International Conference on Solid-State Sensors and Actuators (1997), 1133–1136.

[91] I. Stakgold, *Green's functions and boundary value problems*. Second edition. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1998.

[92] G.I. Taylor, *The coalescence of closely spaced drops when they are at different electric potentials*, Proc. Roy. Soc. Edinburgh Sect. A. **306** (1968), 423–434.

[93] G. Zheng, *New results on the formation of singularities for parabolic problems*, Ph.D. Thesis, 2005.