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Didactics

Mathematical Aspects in an Architectural Design Course: The Concept, Design Assignments, and Follow-up

Abstract. This paper considers a Mathematical Aspects in Architectural Design course in a college of architecture. The course is based on experiential learning activities in the design studio. It focuses on designing architectural objects, when the design process is tackled from three geometrical complexity directions: tessellations, curve surfaces, and solids intersections. The students perform seminars, exercises, and projects in which they analyse and develop geometrical forms and implement them in design solutions. Students achievements in design and mathematics are assessed. The course follow-up indicated that the students used mathematics as a source of complex geometrical forms and a tool for designing efficient solutions.

Introduction

Architecture education, from the pedagogical perspective, is grounded on the methodology of constructivism considering learning as an active process in which the learner constructs knowledge through practice and interaction with the environment. Teymer [1996] explicitly articulates these roots, stating the need to educate architecture students toward self directed, holistic, profound, and reflective reasoning of the environment.

When discussing the architecture curriculum, Banerjee and De Graaf [1996] point out that it consists of two main blocks, namely, the block of preparatory disciplines and the block of problem oriented integrated disciplines. The preparatory disciplines are traditionally taught with frontal lessons, while the integrated disciplines are given through the project based learning in the architectural design studio. The authors focus on the preparatory structures course and discuss students' difficulties in understanding concepts of statics. These difficulties cause lack of ability to apply the concepts in their studio projects.

Mathematics is one of the preparatory disciplines and is traditionally delivered through frontal teaching. Architecture educators [Salingaros 1999] notice students' difficulties in understanding mathematics concepts and in solving mathematical problems of structures design, and there is a call for change.

Our longitudinal study examines the way to close the gap between the two blocks of disciplines and overcome the disharmony between learning in frontal classrooms and in design studios. Our previous papers [Verner and Maor 2003; Verner and Maor 2006] discuss two main approaches to teaching mathematics with applications in various contexts: Realistic Mathematics Education (RME) and Mathematics as a Service Subject (MSS). From the epistemological view both approaches are grounded on the methodology of constructivism.

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By involvement in solving mathematical problems related to architecture structures, and in geometrical design projects, the students gradually build their mathematical knowledge and develop ability to use it in architecture design. Looking metaphorically, the role of teaching in supporting this constructivist learning is similar to the function of scaffolding as a temporary framework providing stability and efficiency during the building construction.

The RME approach is implemented in our introductory calculus course [Verner and Maor 2003] in which learning mathematical concepts is supported by solving applied problems relevant to architecture studies. Our second, more advanced course Mathematical Aspects in Architectural Design (MAAD) implements the MSS approach [Verner and Maor 2006] when the focus is on developing the ability to apply mathematical methods in performing architecture design assignments. In this paper we summarize the six-year experience of teaching the introductory course and the three-year practice of the MAAD course. During this period the college has become an academic institution educating students for a Bachelor of Science degree in architecture and the two courses continue to be a part of the curriculum.

Curriculum development principles

When developing the MAAD course curriculum we followed the planning and evaluation model for a project-based curriculum in architecture proposed by Teymur [1992]. Accordingly, the didactical principles, practice, outcomes, and evaluation are developed through the following design activities:

A. Define the concept (answering the question – Why?)

B. Design projects assignments (answering the question – What?)

C. Develop a learning environment (answering the questions – How and Where?)

D. Plan the course framework and management way (answering the questions – How and When?).

A. When defining the concept of our course, the main question was to find a way of observing different complex geometrical forms in a systematic way. Our approach [Verner and Maor 2006] is to consider the three directions of geometrical complexity in architectural objects: tessellations, curved surfaces, solids intersections (fig. 1).

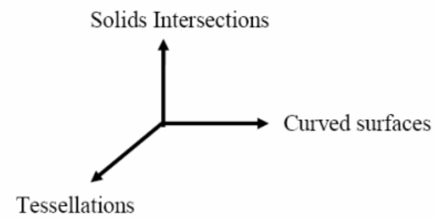

Fig. 1. The three directions of geometrical complexity

We believe that performing geometrical design projects in each of these directions facilitates students' ability and motivation to integrate complex geometrical solutions in their architecture design projects. At the next step mathematical concepts relevant to geometrical design for different directions of geometrical complexity were selected:

- Arranging regular shapes to cover the plane (tessellations).
  Related mathematical concepts: proportions, symmetry, harmonic dimensions, golden section, Fibonacchi numbers, logarithmic spirals, polygons, modularity, curve smoothness, shape displacements, rotations, reflections, and combinations.
- Shaping curve lines and surfaces.
  Related mathematical concepts: folded plates, barrel vaults, domes, and shells.
- Analyzing solids' intersections.
  Related mathematical concepts: polyhedron, vertex, edge, envelop, facet, solids' composition and intersection.

B. To achieve the goals formulated at the first stage, the MAAD course includes exercises, seminars, and projects. They are partly discussed in [Verner and Maor, 2006] which presents the pilot course experience. The projects performed in the subsequent MAAD courses are presented below "Design Projects" section.

C. We found that the architectural design studio fits the requirements of the MAAD course because of the following reasons:

- The studio is familiar to students as an authentic environment for architectural design practices;
- The studio supports experiential learning and learning-by-doing processes [Schoen, 1988];
- The studio is suitable for experimentation with materials and physical modelling;
- The students in the studio learn theoretical concepts which they need to apply in their design practice.

D. The methods of teaching mathematics and geometrical design were integrated in the way presented in Table 1.

Didactical methods	Instructional objectives
Student seminars in geometrical analysis of structures, guided by the teacher	1. Acquiring mathematical concepts and linking them to architecture design concepts 2. Understanding the connection between architecture design and technology 3. Formal defining of mathematical concepts 4. Identifying mathematical concepts in architectural objects
Geometrical problem solving	1. Acquiring competences of applying geometrical concepts and methods 2. Geometrical analysis of physical models 3. Interpreting geometrical objects in the architecture context 4. Acquiring skills of building physical models
Project design and analysis	1. Training of divergent thinking through developing design alternatives. 2. Identifying, solving, and applying mathematical problems related to designing the product. 3. Gaining experience of creating and presenting the product
Peer and self-evaluation of the projects	1. Developing evaluation criteria 2. Evaluation and self-evaluation practice

Table 1. Didactical methods and instructional objectives

As follows from the table, the course applies four different didactical methods each of which has its instructional objectives. These instructional objectives were of three types: mathematical, architectural and integrated objectives. In the next section we present examples of the three design projects and the ways of achieving their instructional objectives.

Design Projects

Project 1. Arranging regular shapes to cover the plane (tessellations)

Tessellations are designed in architecture in order to cover floors, walls, roofs, and other architectural elements. They consist of flat geometrical objects formed by translation, rotation and intersection of basic figures. The coverings should be designed without overlapping and have minimal tailings. Frederickson [1997] pointed out two methods of dissecting geometrical figures into pieces and arranging tessellations: inscribing a figure in a certain tessellation module, and combining figures by joining vertices to compose a module. The additional geometrical concept central for these design was the proportion.

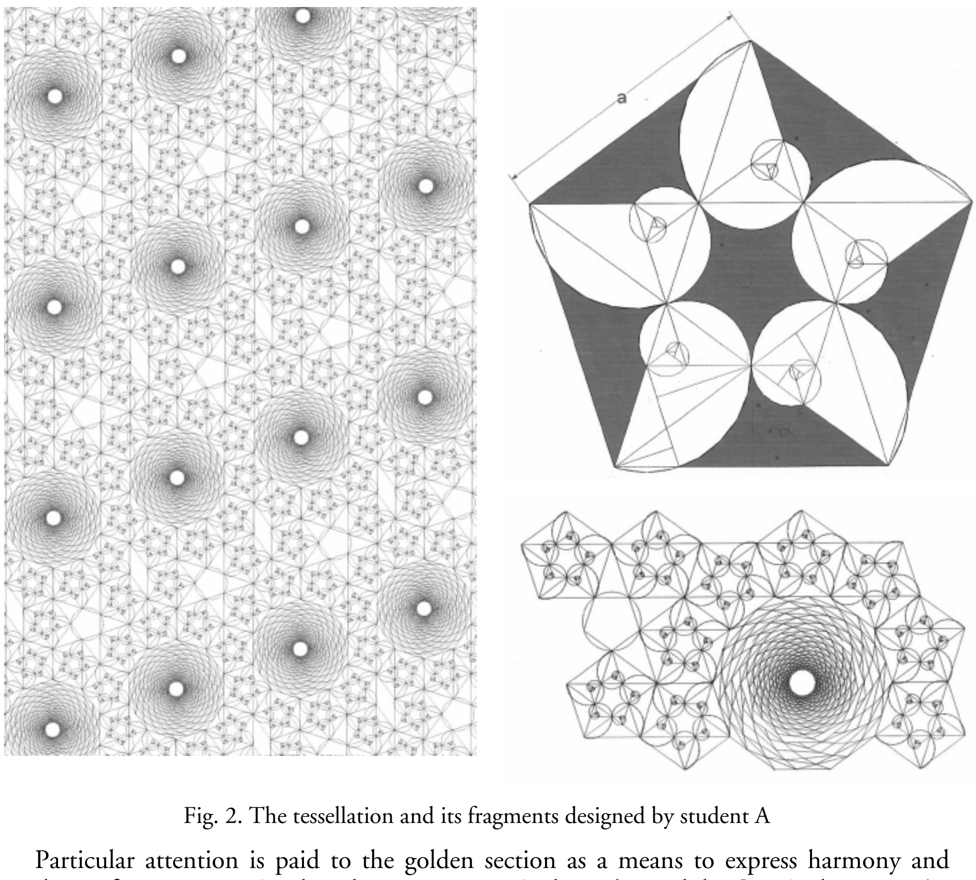

Fig. 2. The tessellation and its fragments designed by student A

Particular attention is paid to the golden section as a means to express harmony and aesthetics from ancient Greek architecture to Le Corbusier's Modulor [Le Corbusier 1968; Huylebrouck and Labarque 2002]. Ranucci [1974] studied these mathematical ideas and procedures of tessellation design implemented in Escher's artworks.

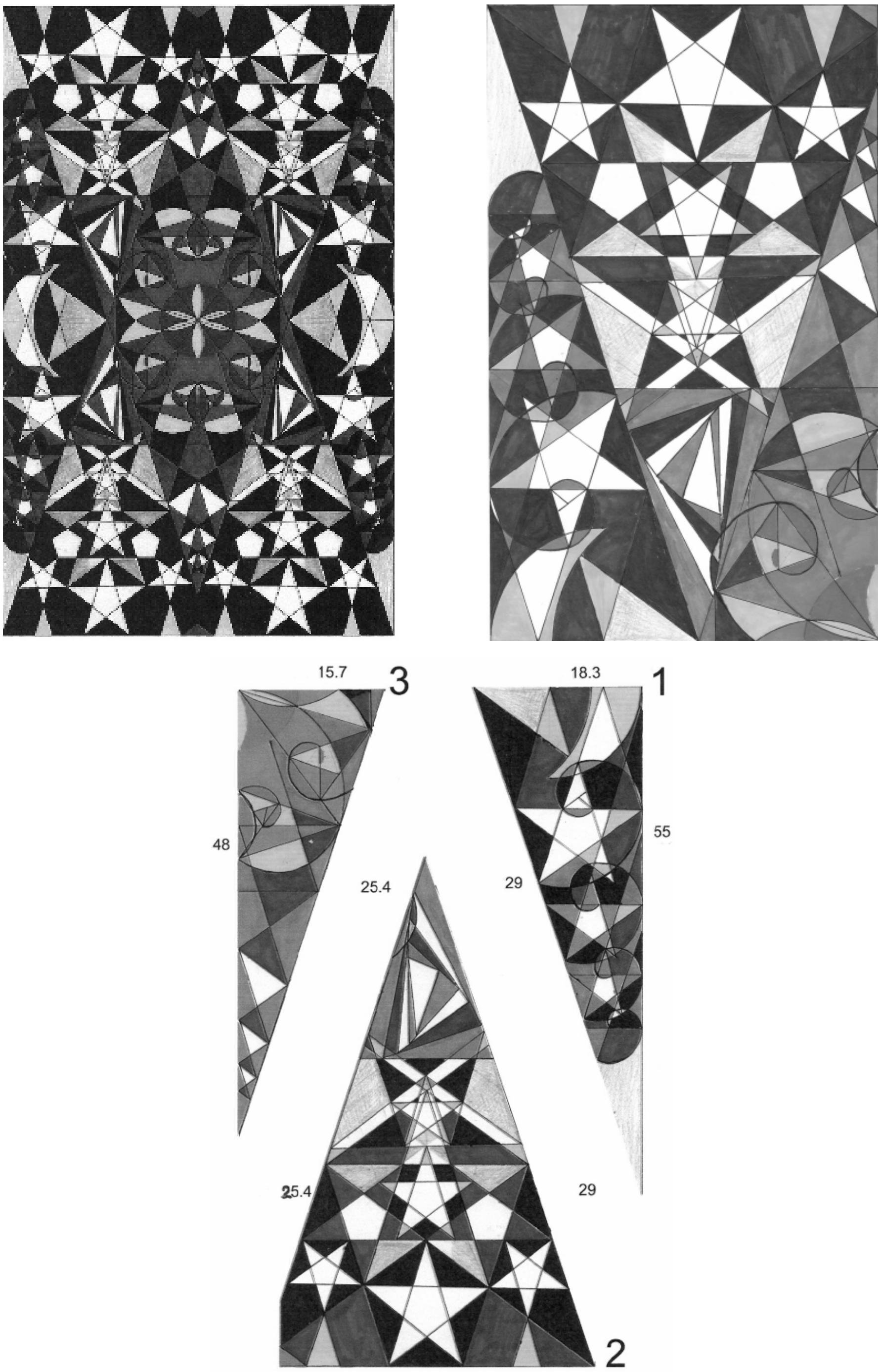

Fig. 3. The tessellation and its fragments designed by student B

The value of these geometrical concepts is recognized in the mathematics education. Boles and Newman [1990] developed a curriculum which studied plain tessellations arranged by basic geometrical shapes with focus on proportions and symmetry. Applications of Fibonacci numbers and golden section in designing tessellations were emphasized. Following [Salingaros 2001], tessellation studies focus on designing one module and then replicating it to compose the covering.

In our course the tessellation project assignment was as follows:

> Design a tessellation of a floor surface of 34×55 m by means of identical rectangular modules. The module should be a periodic combination of various geometrical figures. Define proportions and dimensions of the figures using golden section ratio and Fibonacci numbers. Develop a concept of the designed module choosing one of the following metaphoric subjects: a temple, kindergarten, political message, harmony with nature, and musical impression.

Below we consider how the ideas outlined above were implemented in project work performed by four students. The tessellation designed by student A is shown in fig. 2. It is based on regular pentagons with five logarithmic spirals in each of them. The spirals are drawn by means of a sequence of 72°-72°-36° triangles related to the golden section [Boles and Newman, 1990, p. 186]: $2 \cdot \cos(36°) = \varphi = 1.618\ldots$.

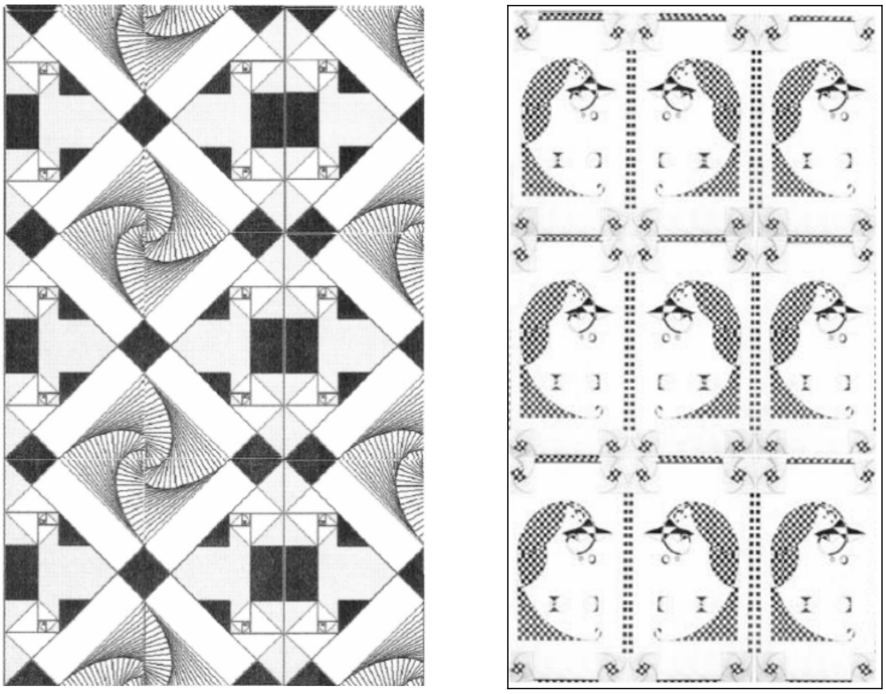

Fig. 4. The tessellations designed by students C and D

The module consists of these pentagons and the decagon created by joining ten pentagons. The decagon is dissected by the segments connecting the mid-points of its sides. Thus a smaller decagon is created and dissected in the same way, and so on creating also a Baravelle spiral [Boles and Newman 1990, 197]. Finally, the modules are combined composing the floor surface rectangle.

A design created by student B is presented in fig. 3. This student also used 72°-72°-36° triangles and the same procedure for drawing logarithmic spirals. But the original idea of this student was to dissect the triangles into triangle pieces (fig. 3, bottom) which constitute pentagons and pentagonal stars of different dimensions, composing the module (fig. 3, upper right), while the tessellation (fig. 3, upper left) is the combination of the modules.

Tessellations created by students C and D are shown in fig. 4. Student C created squares of two types. Squares of the first type are compositions of four straight line spirals [Boles and Newman 1990, 199], while squares of the second type have in the corners spirals, based on the Fibonacci rectangles. The tessellation of student D is based on images in which spirals are used to limit shading areas.

Project 2. Shaping curve lines and surfaces

Mathematical curved surfaces are used in architectural design as "the close link between form and structure, between geometry and the flow of forces in the structure" [Hanaor 1998, 147-148]. Curved surfaces that minimize deformation of structures under distributed loads are implemented in solutions existing in nature [Grosjean and Rassias 1992].

There is a long tradition of using mathematical surfaces in architecture. Gaudi systematically applied mechanical modeling to create geometrical forms and to examine their properties [Alsina and Gomes-Serrano 2002]. He also created 3D surfaces such as paraboloids, helicoids, and conoids by moving generator lines "in a dynamic manner" [Alsina, 2002, 89]. In architecture education, by studying mathematical surfaces the students are exposed to the construction of optimal structural elements.

In our course the second project focuses on using mathematical surfaces for roof design. The project assignment was described in our previous paper [Verner and Maor 2006] which presented results of the course given for the first time. Since then, the course has become part of the academic curriculum. Here we present some advanced ideas implemented in students' projects. Some of these ideas the students found in modern architecture such as Buckminster Fuller's Geodesic Dome, Tully Daniel's saddles (Hypar surfaces), and Santiago Calatrava's geometrical surfaces.

For the reader's convenience we repeat the project assignment definition from our previous paper:

> Design a plan of a gas station. Start from a zero level plan including access roads, parking, pumps, car wash, coffee shop, and an office. Design a top covering for the pumps area, or the roof of the coffee shop and office building. Find a design solution answering the stability, constructive efficiency, complexity, and aesthetics criteria.

Four examples of project works performed by students in the Spring 2006 course are presented in fig. 5. These examples implement different variations of mathematical surfaces of two types: Sarger and Conoid segments. Thee segments are described by the following equations:

$$Z = y\frac{f}{L}\left(1 - \frac{4x^2}{l^2}\right) \qquad Z = d\left(1 - \frac{y^2}{L^2}\right) + f \cdot \frac{y^2}{L^2}\cos\left(\frac{\pi \cdot L \cdot x}{l \cdot y}\right)$$

(Conoid segment) (Sarger segment)

In these equations parameters l and f are the width and height of the generator line, L is the segment length, d is the end-point height.

The physical models were constructed by the students after precise calculation. Student E composed her solution from three Sarger segments combined in the form of flower petals. The roof created by student F consists of six Conoid segments, three at each side. Student G also uses six Conoids but combines them in a "wavy" form. Student H creates the roof from two Conoid segments of different dimensions.

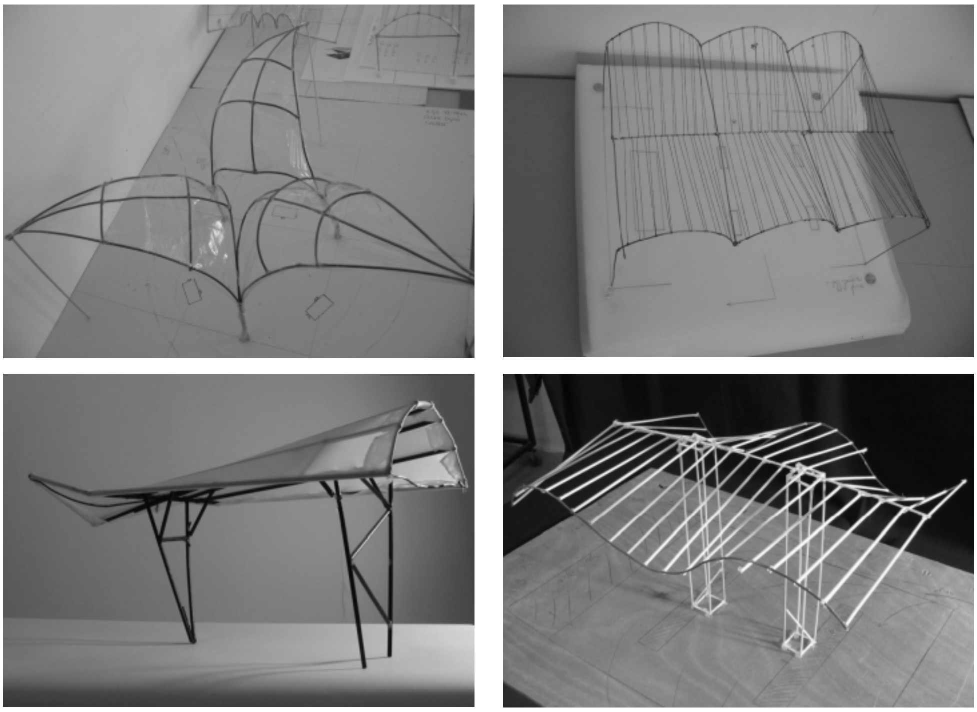

Fig. 5. The roof physical models designed by four students (E – H)

Project 3. Analyzing the intersections of solids

Many architectural buildings combine solid elements of different forms answering diverse functional needs. Combining these elements requires solutions of solid intersection problems.

Alsina [2002, 119-126] considered the design of complex three-dimensional forms by intersecting various geometrical forms. He analyzed the use of these forms in Gaudi's creations in order to achieve functional purposes such as light effects or symbolic expressions. Burt [1996] examined integrating and subdividing space by different types of polyhedral elements. He emphasized that this design method can provide efficient architectural solutions.

In architecture education solids and their intersections are studied in the morphology course [Haspelmath 2002]. Traditionally, this course does not include analytic calculations of intersections of solids required for precise design. Our course addresses this issue by offering the third project assignment which is formulated as follows:

> Select a known public building which was designed with solids intersections. The project requirements are:
>
> – Seminar on the building design process;
> – Mathematical analysis of the solids intersection;
> – Building a model which accurately presents solids intersection in the building;
> – Designing an additional functional module in the solids intersection area.

A sample project work performed by one of the students is presented in fig. 6. When selecting a public building (a museum in the north of Israel) the student recognized the solids intersection part of the structure (cylinder-pyramid). She measured building's dimensions and extracted geometrical data from the architecture design plans. After mathematical analysis of these data (fig. 6a) the student constructed the precise physical model of the building (fig. 6b).

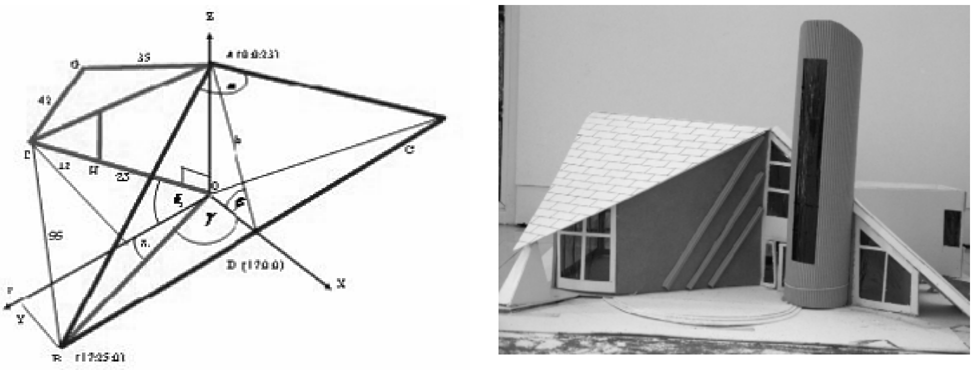

Fig. 6. A building composed of different solid elements:
a) the geometrical scheme; b) the physical model

Project Assessment

The students in the course perform the three projects and report them in project portfolios. Students' achievements in design and in mathematics are assessed using different evaluation criteria which fit specific characteristics of the projects. Our previous paper presents evaluation results only for the second project (curved surfaces). In this paper we present and compare evaluation results for the three course projects. The achievements in mathematics are assessed by the course lecturer (Maor), while the design achievements are assessed by an architect teaching the Design Studio course.

Tables 2A and 2B present mean grades in the three course projects, for design and mathematics criteria mentioned in the tables.

Project 1	Subject expression	Conception & application	Geometrical variety	Graphic representation	Design grade
Mean	83.1	89.0	77.4	64.6	74.7

Project 2	Efficiency	Aesthetics	Functionality	Program quality	Design grade
Mean	76.9	85.6	73.5	87.5	78.9

Project 3	Structure selection	Geometrical form	Model aesthetics	Additional module	Design grade
Mean	90.0	68.0	76.3	77.5	77.0

Table 2A. Design assessment grades

Project 1	Problems perception	Calculus application	Modularity application	Proportion harmony	Calculations	Drawings precision	Parametric solutions	Math grade
Mean	72.1	56.2	78.4	87.5	80.2	79.4	58.1	73.1

Project 2	Dimensions calculation	Surface parameters	Roof calculation	Building a model	Model precision	Model analysis	Geometrical complexity	Math grade
Mean	76.9	68.8	81.3	76.3	81.9	81.9	85.0	78.9

Project 3	Dimensions calculation	Use of parameters	Intersection calculation	Building a model	Model precision	Intersection analysis	Geometrical complexity	Math grade
Mean	73.8	68.8	77.5	75.0	80.0	77.5	82.5	76.4

Table 2B. Mathematics assessment grades

Tables 2A and 2B reveal the following features:

1. The mean grades of the three projects are similar in design (74.7-78.9) and in mathematics (73.1-78.9). The project 1 grades were slightly lower than of projects 2 and 3. A possible reason is that in the project the students dealt with tessellations design for the first time.
2. Mean grades for the use of parameters in the three projects were lower than for other mathematics criteria. These difficulties originated from the students' mathematical background.
3. Close correlation between the individual design and mathematics grades was found in project 1 ($\rho = 0.665$) and in project 2 ($\rho = 0.698$). This result indicates the tight integration of design and mathematical aspects of these course projects. In project 3 the correlation between the grades was lower ($\rho = 0.398$). A possible explanation is that project 3 does not include a design component, but focuses on analyzing existing structures.

Learning Activities in the MAAD Projects

An educational study was conducted in conjunction with the MAAD course development. Its methodology is discussed in our previous paper [Verner and Maor 2006] and presented in detail in the doctoral dissertation [Maor 2005]. One of the main results of this study is characterizing learning activities in the MAAD course projects. The learning activities are characterized at each of the architectural design stages. A summary of mathematical characteristics specific for these learning activities is presented in Table 3.

Design stages	Specific characteristics of learning activities
1. Project idea	Understanding the design and mathematical requirements of the project assignment. Identifying mathematical concepts adequate for expressing the metaphoric, symbolic, and analogical aspects of the project idea.
2. Data presentation	Imagining geometrical forms suitable for representing the design concept. Experimentation with physical and simulative models of geometrical forms and their mathematical description.
3. Analysis of the data and the constraints	Analysis of dimensions, scales, constraints, and properties. Search for suitable geometrical forms.
4. Generation of design alternatives	Identifying mathematical criteria for evaluation of design alternatives. Iterative synthesis, evaluation, and revision of alternative geometrical solutions.
5. Definition of design criteria	Mathematical description of the aesthetics, proportion, efficiency, modularity, symbolism, and accuracy criteria.
6. Selection of the design solution	Mathematical analysis of alternatives, finding the optimal solution, and its substantiation.
7. Presentation of the solution	Drawing the designed architectural object based on calculating dimensions, scales, and functions. Building physical and simulative models. Writing the mathematics-in-design report.
8. Evaluation	Presenting the project to peers and defending the solution.

Table 3. Characteristics of learning at different design stages

Conclusion

The proposed Mathematical Aspects in Architectural Design (MAAD) course has been taught for three years and has become important part of the college architecture curriculum. The course offers mathematics studies grounded on the constructivistic methodology through learning by design activities in the design studio. This second-year course continues and relies on the first-year mathematics course, in which applied problems are integrated following the realistic mathematics education approach.

Addressing geometrical complexity of architectural objects from the three its different directions (tessellations, curved surfaces, and solids' intersections) provides students with the broad perspective of mathematics applications in designing architectural forms.

Students in the course learn to use mathematics as a source of creative solutions and as an instrument to answer design criteria, such as constructive efficiency, functionality, optimization, shape variety, stability, and preciseness.

Each of the three parts of the course includes a study of mathematical concepts and methods with connection to architecture, practice in solving mathematical problems, and a design project. The mathematical learning activities in the projects include: analytic description of metaphoric, symbolic, and analogical aspects of the project idea; search for suitable geometrical forms and their analysis; synthesis, evaluation, and revision of geometrical solutions; mathematical description of the criteria for aesthetics, proportions, efficiency, modularity, and accuracy; finding the optimal solution, and its substantiation; building physical and simulative models based on calculating dimensions, scales, and functions.

The projects in the course are assessed through analysis of activities, design solutions and mathematics applications. Design solutions are assessed following the existing practice of studio evaluation with regards to the following aspects: concept, planning/detailing, and representation/expression. Mathematics applications are assessed using the following criteria: perception of mathematical problems, solving applied problems, precision in drawing or building a physical model of geometrical objects, accuracy of calculations and parametric solutions. Results of the course assessment indicated that the students used various complex geometrical shapes as a source of creative and efficient design solutions of the three project assignments.

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