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MATHEMATICAL FOUNDATION OF GENERAL COOPERATIVE FUZZY GAMES

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1. Introduction. Aumann and Shapley [4] developed the theory of non-atomic games for coalitions of players, Lebesgue measurable sets in [0, 1], which was originated by Aumann [2], [3]. Aubin [1] worked on n-person games on fuzzy coalitions. Fuzzy coalitions are called ideal coalitions by Aumann and Shapley.

In this paper we consider a general σ-finite measure space $(X, \mathscr{F}, \mu)$ as the set of all players, a family $\mathscr{M}$ of bounded measurable functions on X as the set of fuzzy coalitions and a superadditive functional on $\mathscr{M}$ as a game. First of all we construct a signed measure associated with a given game which enables us to clarify various properties of fuzzy games, and prove finally that the core $\mathscr{C}_v$ defined by the excess coincides with a core $\mathscr{D}_v$ defined by the set of all undominated allocations.

We give here some comments on the game-theoretic background of the present paper. The functional v of the game represents a certain profit of the coalitions of players in the game. The superadditivity of v reflects the property that the cooperation of players in the game has an effect on profits of players, while the additive game v is called inessential because the cooperation is of no effect. In the study of the games in the functional formulation, the emphasis is laid on the investigation of dominance structures on the set $\mathscr{A}_v$ of all allocations rather than the determination of the strategy. For this purpose, we treat strategically equivalent classes of the games. The solutions of the game in this treatment are some special subsets of $\mathscr{A}_v$ which are stable in a sense under the dominance relation. In the present paper we consider only two subsets of $\mathscr{A}_v$. Let $e(m)$ be the difference $v(m) - a(m)$ between the profit $v(m)$ and an imputed value $a(m)$ where $a \in \mathscr{A}_v$. Then, $e(m)$ is the excess of the profit of the coalition m which is not imputed to the members of m if it is possible for us to imagine them. The set $\mathscr{C}_v$ of the allocations a which satisfy $e(m) \leqq 0$ for any $m \in \mathscr{M}$ is acceptable for the players because each coalition gains at least its own earnings. We call the subset $\mathscr{C}_v$ of $\mathscr{A}_v$ the core of the game and an allocation in $\mathscr{C}_v$

a core-allocation. Let $f(B)$ be the set of the all allocations which are not dominated by any allocation belonging to a subset B of $\mathscr{A}_v$. We call the mapping $f: 2^{\mathscr{A}_v} \to 2^{\mathscr{A}_v}$ "undom" where $2^{\mathscr{A}_v}$ stands for the class of all subsets of $\mathscr{A}_v$. Then, $\mathscr{D}_v$ mentioned in the previous paragraph is the image of $\mathscr{A}_v$ under the undom mapping f. Generally $\mathscr{C}_v \subset \mathscr{D}_v$ but these two subsets of $\mathscr{A}_v$ coincide in the superadditive games considered hitherto. This coincidence will be proved also in the present one.

2. Game and associated signed measure. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space. A measurable set is called a *coalition*. A measurable function m is called a *fuzzy coalition* or an *ideal coalition* if $0 \leqq m \leqq 1$ a.e. Let $\mathscr{M}$ denote the set of all fuzzy coalitions. Let Π_A be the set of all partitions of a coalition A to finite or countably infinite coalitions.

A real valued functional v on $\mathscr{M}$ is called a *game* if it satisfies (V1)-(V4):

(V1) [Superadditivity] $v(m_1) + v(m_2) \leqq v(m_1 + m_2)$ whenever $m_1, m_2, m_1 + m_2 \in \mathscr{M}$.

(V2) [Continuity] $\lim_k v(m_k) = v(m)$ whenever $m, m_k \in \mathscr{M} (k=1, 2, \cdots)$ and $\lim_k m_k = m$ a.e.

(V3) [Total boundedness] $\sup_{\{X_i\} \in \Pi_X} \sum_i |v(1_{X_i})| < \infty$, where 1_{X_i} denotes the indicator function of X_i.

(V4) [Positive homogeneity] $v(\alpha 1_A) = \alpha v(1_A)$ whenever α is a constant with $\alpha 1_A \in \mathscr{M}$.

Let $\mathscr{V}$ be the set of all games.

For any game v, it is easily seen that

(V5) $v(1_A) = 0$ whenever $\mu(A) = 0$,

(V6) $\sup_{\{A_i\} \in \Pi_A} \sum_i |v(1_{A_i})| < \infty$,

(V7) $v(1_A) \geqq \sum_i v(1_{A_i})$ for any $\{A_i\} \in \Pi_A$.

In fact, (V5), (V6) and (V7) follow from (V4) + (V2), from (V3) and from (V1) + (V6), respectively.

A game v is called *inessential* if it satisfies

(A) [Additivity] $v(m_1) + v(m_2) = v(m_1 + m_2)$ whenever $m_1, m_2, m_1 + m_2 \in \mathscr{M}$.

Let $\mathscr{A}$ be the set of all inessential games.

THEOREM 2.1. (i) *For any $v \in \mathscr{V}$, there exists a μ-absolutely continuous signed measure λ_v on $\mathscr{F}$ such that*

$$v(m) \geqq \int_X m d\lambda_v$$

for all $m \in \mathscr{M}$ and λ_v is maximal in thes ense that if λ is a μ-absolutely *continuous signed measure on* $\mathscr{F}$ *and if* $v(1_A) \geqq \lambda(A)$ *for all* $A \in \mathscr{F}$, $\lambda_v \geqq \lambda$ *on* $\mathscr{F}$.

If $v \geqq 0$ *on* $\mathscr{M}$, $\lambda_v \geqq 0$ *on* $\mathscr{F}$. (λ_v *is called a signed measure associated with* v.)

(ii) *For any* $v \in \mathscr{V}$, $v \in \mathscr{A}$ *iff there exists a* μ*-absolutely continuous signed measure* λ_v *on* $\mathscr{F}$ *such that*

$$v(m) = \int_X m d\lambda_v$$

for all $m \in \mathscr{M}$.

PROOF. (i) Define

$$\lambda_v(A) = \inf_{\{S_i\} \in \Pi_A} \sum_i v(1_{S_i})$$

for $A \in \mathscr{F}$. Consider $\{A_k\}_k \in \Pi_A$ and $\varepsilon > 0$. Then there exists $\{S_{ki}\}_i \in \Pi_{A_k}$ such that $\lambda_v(A_k) + \varepsilon/2^k \geqq \sum_i v(1_{S_{ki}})$, and so $\sum_k \lambda_v(A_k) + \varepsilon \geqq \sum_{k,i} v(1_{S_{ki}}) \geqq \lambda_v(A)$. Hence $\sum_k \lambda_v(A_k) \geqq \lambda_v(A)$.

On the other hand, there exists $\{S_i\}_i \in \Pi_A$ such that $\lambda_v(A) + \varepsilon \geqq \sum_i v(1_{S_i})$. Since $\{S_i \cap A_k\}_k \in \Pi_{S_i}$, we have $v(1_{S_i}) = v(\sum_k 1_{S_i \cap A_k}) \geqq \sum_k v(1_{S_i \cap A_k})$. Hence $\lambda_v(A) + \varepsilon \geqq \sum_{i,k} v(1_{S_i \cap A_k}) \geqq \sum_k \lambda_v(A_k)$, and so $\lambda_v(A) \geqq \sum_k \lambda_v(A_k)$.

Consequently $\lambda_v(A) = \sum_k \lambda_v(A_k)$.

Therefore λ_v is a signed measures. λ_v is clearly μ-absolutely continuous. Since $v(1_A) \geqq \lambda_v(A)$ for all $A \in \mathscr{F}$, by virtue of (V1), (V4) and (V2), we have $v(m) \geqq \int_X m d\lambda_v$ for all $m \in \mathscr{M}$.

We shall show the maximality of λ_v. Let λ be a signed measure as in the theorem and let $A \in \mathscr{F}$. Then $\lambda(A) = \sum_i \lambda(S_i) \leqq \sum_i v(1_{S_i})$ for any $\{S_i\} \in \Pi_A$, and so $\lambda(A) \leqq \inf_{\{S_i\} \in \Pi_A} \sum_i v(1_{S_i}) = \lambda_v(A)$.

It is clear that if $v \geqq 0$ on $\mathscr{M}$, then $\lambda_v \geqq 0$ on $\mathscr{F}$.

(ii) Suppose $v \in \mathscr{A}$. Then λ_v defined as in the proof of (i) satisfies $v(1_A) = \lambda_v(A)$ for any $A \in \mathscr{F}$, and so, by virtue of (A), (V4) and (V2), $v(m) = \int_X m d\lambda_v$ for any $m \in \mathscr{M}$.

The converse is clear.

COROLLARY 2.2. *Let* $v \in \mathscr{V}$. *Then* $v \notin \mathscr{A}$ *iff* $v(1) > \lambda_v(X)$.

3. Strategical equivalence. Let $u, v \in \mathscr{V}$. If there exists an $a \in \mathscr{A}$ and a constant $c > 0$ with $u = cv + a$, v is said to be *strategically equivalent* to u and is denoted by $v \sim u$.

Let $u, v \in \mathscr{V}$. u is called a $[0, 1]$-*standard* of v provided that $v \sim u$ and, in the case of $v \in \mathscr{A}$,

$$u = 0$$

and, in the case of $v \notin \mathscr{A}$,

$$u \geqq 0 \,, \quad u(1) = 1 \,, \qquad b \in \mathscr{A} \,, \quad 0 \leqq b \leqq u \quad \text{implies} \quad b = 0 \,.$$

Note that if u is standardized as above, then $\lambda_u = 0$ on $\mathscr{F}$. In fact, put $b(m) = \int_X m d\lambda_u$ for $m \in \mathscr{M}$. Then $b \in \mathscr{A}$, $0 \leqq b \leqq u$ and so $b = 0$ or, equivalently, $\lambda_u = 0$.

THEOREM 3.1. *For any $v \in \mathscr{V}$, there exists a unique* $[0, 1]$*-standard of v.*

PROOF. It is enough only to consider the case of $v \notin \mathscr{A}$. Since $v(1) - \lambda_v(X) > 0$ by Corollary 2.2, we can define

$$u(m) = \left(v(m) - \int_X m d\lambda_v\right) \Big/ (v(1) - \lambda_v(X))$$

for $m \in \mathscr{M}$. Then it is clear that $u \in \mathscr{V}$, $v \sim u$, $u \geqq 0$ and $u(1) = 1$. Suppose $b \in \mathscr{A}$ and $0 \leqq b \leqq u$. Then

$$v(1_A) \geqq (\lambda_v + (v(1) - \lambda_v(X)) \cdot \lambda_b)(A)$$

for $A \in \mathscr{F}$. Hence, by the maximality of λ_v, $\lambda_b = 0$ and so $b = 0$. Therefore u is a $[0, 1]$-standard of v.

Now we shall show the uniqueness of standardization. Suppose that u and u' are $[0, 1]$-standards of v. Since $u' \sim u$, $u = cu' + a$ for some $a \in \mathscr{A}$ and some $c > 0$. Then $u \geqq a$ and $u' \geqq -(1/c)a$. Consider the supporting set D of the Hahn decomposition of the signed measure λ_a and put $a_1(m) = a(m \cdot 1_D)$, $a_2(m) = -a(m \cdot 1_{D^c})$. Then $a = a_1 - a_2$, $a_1, a_2 \in \mathscr{A}$ and, for any $m \in \mathscr{M}$,

$$u(m) \geqq u(m \cdot 1_D) \geqq a_1(m) = \int_D m d\lambda_a \geqq 0 \,,$$

$$u'(m) \geqq u'(m \cdot 1_{D^c}) \geqq (1/c)a_2(m) = -(1/c)\int_{D^c} m d\lambda_a \geqq 0 \,.$$

Hence $a_1 = 0$, $(1/c)a_2 = 0$ and so $a = 0$. Then, since $1 = u(1) = u'(1)$, we have $c = 1$ and $u = u'$.

4. Allocation and core-allocation.

Let $v \in \mathscr{V}$ and $a \in \mathscr{A}$. a is called an *allocation for v* if

$$a(1) = v(1) \,, \quad a(m) \geqq \int_X m d\lambda_v$$

for any $m \in \mathscr{M}$.

Let $\mathscr{A}_v$ denote the set of all allocations for v.

PROPOSITION 4.1. (i) *$\mathscr{A}_v \neq \varnothing$ for any $v \in \mathscr{V}$.* (ii) *$\mathscr{A}_v = \{v\}$ for any*

$v \in \mathscr{A}$.

PROOF. (i) Let u be the [0, 1]-standard of v and $v = cu + a$ for some $a \in \mathscr{A}$ and for some $c > 0$. Consider a mapping f of $\mathscr{A}_u$ to $\mathscr{A}_v$ defined by $f(x) = cx + a$. Take a μ-absolutely continuous probability measure p on $\mathscr{F}$ and put $a(m) = \int_X mdp$ for $m \in \mathscr{M}$. Then $a \in \mathscr{A}_u$, because $u(1) = 1$ and $\lambda_u = 0$. Since $\mathscr{A}_v$ is equipollent to $\mathscr{A}_u$ by the mapping f, we have $\mathscr{A}_v \neq \emptyset$.

(ii) Clearly $v \in \mathscr{A}_v$. Let $a \in \mathscr{A}_v$. Then $\lambda_a(X) = a(1) = v(1) = \lambda_v(X)$ and moreover $\lambda_a \geqq \lambda_v$ since $\int_X md\lambda_a = a(m) \geqq \int_X md\lambda_v$. Hence $\lambda_a = \lambda_v$ and so $a = v$.

Let $v \in \mathscr{V}$ and $a \in \mathscr{A}$. a is called a *core-allocation for* v if

$$a(1) = v(1)\ , \quad a(m) \geqq v(m)$$

for any $m \in \mathscr{M}$.

Let $\mathscr{C}_v$ denote the set of all core-allocations for v. $\mathscr{C}_v$ is called the *core for* v.

Then we have clearly:

PROPOSITION 4.2. (i) *$\mathscr{C}_v \subset \mathscr{A}_v$ for any $v \in \mathscr{V}$.*

(ii) *$\mathscr{C}_v = \{v\}$ for any $v \in \mathscr{A}$.*

Let $v \in \mathscr{V} \setminus \mathscr{A}$. v is called an *essential constant-sum game* if $v(1_A) + v(1_{A^c}) = v(1)$ for any $A \in \mathscr{F}$.

PROPOSITION 4.3. *If v is an essential constant-sum game, then $\mathscr{C}_v = \emptyset$.*

PROOF. Suppose $\mathscr{C}_v \neq \emptyset$ and choose $a \in \mathscr{C}_v$. Then, for any $A \in \mathscr{F}$, $a(1_A) \geqq v(1_A)$, $a(1_{A^c}) \geqq v(1_{A^c})$, $v(1) = a(1) = a(1_A) + a(1_{A^c}) \geqq v(1_A) + v(1_{A^c}) = v(1)$, so that it must be $a(1_A) = v(1_A)$. Hence $v(1_A) \geqq \lambda_a(A)$ for all $A \in \mathscr{F}$ and so, by the maximality of λ_v, we get $\lambda_v \geqq \lambda_a$. Since $a \in \mathscr{A}_v$ and so $\lambda_a \geqq \lambda_v$, we have $\lambda_v = \lambda_a$. Hence $v(1) = a(1) = \lambda_a(X) = \lambda_v(X)$ which contradicts $v \notin \mathscr{A}$ by Corollary 2.2. Therefore $\mathscr{C}_v = \emptyset$.

5. Domination. Let $x, y \in \mathscr{A}_v$. If there exists an $m \in \mathscr{M}$ such that $v(m) \geqq x(m) > y(m)$, we say x *dominates* y and use the notation "$x \succ y$".

If no x dominates y, we say y is an *undominated allocation for* v. Let $\mathscr{D}_v$ denote the set of all undominated allocations for v.

If there exists a bijection f of $\mathscr{A}_u$ to $\mathscr{A}_v$ such that

$$x \succ y \quad \text{iff} \quad f(x) \succ f(y)$$

for any $x, y \in \mathscr{A}_u$, we say u is *isomorphic* to v or $\mathscr{A}_u$ is *isomorphic* to

$\mathscr{A}_v$ and use the notation "$u \cong v$".

THEOREM 5.1. *Let $u, v \in \mathscr{V}$. If u is strategically equivalent to v, u is isomorphic to v.*

PROOF. Let v be expressed as $v = cu + a$ for some $a \in \mathscr{A}$ and some $c > 0$.

First we shall show

(1) $\lambda_v = c\lambda_u + \lambda_a$.

Since $v(m) \geqq \int_X md(c\lambda_u + \lambda_a)$, the maximality of λ_v implies $\lambda_v \geqq c\lambda_u + \lambda_a$. On the other hand, since $u(m) \geqq \int_X md((1/c)\lambda_v - (1/c)\lambda_a)$, the maximality of λ_u implies $\lambda_u \geqq (1/c)\lambda_v - (1/c)\lambda_a$. Hence (1) holds.

We shall now prove $u \cong v$. Consider a mapping f of $\mathscr{A}_u$ to $\mathscr{A}_v$ defined by $f(x) = cx + a$ for $x \in \mathscr{A}_u$. Suppose $x \in \mathscr{A}_u$. Then $f(x) \in \mathscr{A}$, $f(x)(1) = v(1)$ and, by (1), $f(x)(m) \geqq \int_X md(c\lambda_u + \lambda_a) = \int_X md\lambda_v$ for any m. Hence $f(x) \in \mathscr{A}_v$. Similarly, we can show that if $y \in \mathscr{A}_v$, then $f^{-1}(y) = (1/c)y - (1/c)a \in \mathscr{A}_u$. It is now easily seen that f is an isomorphism.

THEOREM 5.2. $\mathscr{D}_v = \mathscr{C}_v$.

PROOF OF $\mathscr{C}_v \subset \mathscr{D}_v$. Let $a \in \mathscr{A}_v \setminus \mathscr{D}_v$. Then there exists an $x \in \mathscr{A}_v$ which dominates a. Hence $v(m) \geqq x(m) > a(m)$ for some m, and so $a \in \mathscr{A}_v \setminus \mathscr{C}_v$. Therefore $\mathscr{C}_v \subset \mathscr{D}_v$.

For the proof of $\mathscr{D}_v \subset \mathscr{C}_v$, we need lemmas.

LEMMA 5.3. *Let $v \in \mathscr{V} \setminus \mathscr{A}$ and let u be the* [0, 1]*-standard of v. Then $\mathscr{D}_u = \mathscr{C}_u$ implies $\mathscr{D}_v = \mathscr{C}_v$.*

The proof is easy and so omitted.

LEMMA 5.4. *Let $u \in \mathscr{V}$, $u \geqq 0$, $u(1) = 1$ and let λ_0 be a μ-absolutely continuous probability measure on $\mathscr{F}$ such that*

$$u(m_0) > \int_X m_0 d\lambda_0$$

for some $m_0 \in \mathscr{M}$. Then there exists a μ-absolutely continuous probability measure λ_1 on $\mathscr{F}$ such that

$$u(m_1) > \int_X m_1 d\lambda_1 > \int_X m_1 d\lambda_0$$

for some $m_1 \in \mathscr{M}$.

PROOF. Let S be the support of λ_0, that is, $S = \{d\lambda_0/d\mu > 0\}$, where $d\lambda_0/d\mu$ is the Radon-Nikodym derivative of λ_0 with respect to μ. Then $\mu(S) > 0$.

The case of $\mu(S^c) = 0$. In this case, S is not an atom. In fact, if S is an atom, m_0 is constant μ-a.e. on X. Denote its constant value by c. Then clearly $c > 0$, and

$$1 = u(1) = (1/c)u(m_0) > (1/c)\int_X m_0 d\lambda_0 = 1 ,$$

which is a contradiction. Hence S is not atom.

By the continuity (V2) of v, there exists an ε $(0 < \varepsilon \leqq 1)$ such that

$$u(m_\varepsilon) > \int_X m_\varepsilon d\lambda_0 ,$$

where $m_\varepsilon = m_0 \cdot 1_{\{m_0 \geqq \varepsilon\}} + \varepsilon \cdot 1_{\{m_0 < \varepsilon\}} \in \mathscr{M}$.

Since S is not an atom, it is easily seen that, even if m_ε is constant μ-a.e. on S, there exists a constant ε_0 $(\geqq \varepsilon)$ and two measurable sets S_1, S_2 such that

$$S_1 \subset S \cap \{m_\varepsilon \leqq \varepsilon_0\} , \quad S_2 \subset S \cap \{m_\varepsilon \geqq \varepsilon_0\} ,$$
$$S_1 \cap S_2 = \varnothing , \quad \mu(S_1) > 0 , \quad \mu(S_2) > 0 .$$

Then, note that $\lambda_0(S_1) > 0$, $\lambda_0(S_2) > 0$.

Choose a number γ with $u(m_\varepsilon) > \gamma > \int_X m_\varepsilon d\lambda_0$.

By the continuity (V2) of v, there exists a constant η $(0 < \eta < \varepsilon)$ such that $u(m_1) > \gamma$, where $m_1 = (m_\varepsilon - \eta)\cdot 1_{S_1} + m_\varepsilon \cdot 1_{S_1^c} \in \mathscr{M}$.

There exists a constant α (>1) such that

$$\gamma > \alpha \cdot \int_{S_2} m_\varepsilon d\lambda_0 + \int_{S_2^c} m_\varepsilon d\lambda_0 ,$$
$$(\alpha - 1)\cdot \lambda_0(S_2) < \eta \cdot \lambda_0(S_1) .$$

Take a constant β $(0 < \beta < 1)$ for which $(1 - \beta)\cdot\lambda_0(S_1) = (\alpha - 1)\cdot\lambda_0(S_2)$.

Define

$$\lambda_1(A) = \beta\cdot\lambda_0(A \cap S_1) + \alpha\cdot\lambda_0(A \cap S_2) + \lambda_0(A \cap (S_1 \cup S_2)^c)$$

for $A \in \mathscr{F}$. Then λ_1 is a μ-absolutely continuous probability measure on $\mathscr{F}$. Moreover

$$u(m_1) > \gamma > \alpha\cdot\int_{S_2} m_\varepsilon d\lambda_0 + \int_{S_2^c} m_\varepsilon d\lambda_0$$
$$\geqq \int_{S_2} m_1 d(\alpha\cdot\lambda_0) + \int_{S_1} m_1 d(\beta\cdot\lambda_0) + \int_{(S_1 \cup S_2)^c} m_1 d\lambda_0 = \int_X m_1 d\lambda_1 ,$$

$$\int_X m_1 d\lambda_1 - \int_X m_1 d\lambda_0$$
$$= \left(\int_{S_2} m_\varepsilon d\lambda_1 - \int_{S_2} m_\varepsilon d\lambda_0\right) - \left(\int_{S_1} m_1 d\lambda_0 - \int_{S_1} m_1 d\lambda_1\right) = J_1 - J_2 ,$$

say,

$$J_1 = (\alpha - 1)\int_{S_2} m_\varepsilon d\lambda_0 \geqq (\alpha - 1)\varepsilon_0\cdot\lambda_0(S_2)\ ,$$

$$J_2 = (1 - \beta)\int_{S_1} (m_\varepsilon - \eta)d\lambda_0 \leqq (1 - \beta)(\varepsilon_0 - \eta)\cdot\lambda_0(S_1)\ ,$$

so that

$$\begin{aligned} J_1 - J_2 &\geqq (\alpha - 1)\varepsilon_0\cdot\lambda_0(S_2) - (1 - \beta)(\varepsilon_0 - \eta)\cdot\lambda_0(S_1) \\ &= (1 - \beta)\eta\cdot\lambda_0(S_1) > 0\ . \end{aligned}$$

Therefore

$$u(m_1) > \int_X m_1 d\lambda_1 > \int_X m_1 d\lambda_0\ .$$

The case of $\mu(S^c) > 0$. Define

$$m_1 = m_0\cdot 1_S + 1_{S^c}\ .$$

Then $m_1 \in \mathscr{M}$ and $u(m_1) \geqq u(m_0) > \int_X m_0 d\lambda_0 = \int_X m_1 d\lambda_0$.

Choose a number γ with $u(m_1) > \gamma > \int_X m_1 d\lambda_0$ and a μ-absolutely continuous measure λ on $\mathscr{F}$ with $\lambda(S^c) = 1$.

Define

$$\lambda_1(A) = \frac{1-\gamma}{1-c}\cdot\lambda_0(A \cap S) + \frac{\gamma - c}{1-c}\cdot\lambda(A \cap S^c)$$

for $A \in \mathscr{F}$, where $c = \int_X m_1 d\lambda_0 < 1$. Then λ_1 is a μ-absolutely continuous probability measure on $\mathscr{F}$ and $\int_X m_1 d\lambda_1 = \gamma$, so that

$$u(m_1) > \int_X m_1 d\lambda_1 > \int_X m_1 d\lambda_0\ .$$

The proof of Lemma 5.4 is completed.

Now we go back to the proof of Theorem 5.2.

PROOF OF $\mathscr{D}_v \subset \mathscr{C}_v$. *The case of* $v \in \mathscr{A}$. By Propositions 4.1 and 4.2, $\mathscr{A}_v = \{v\} = \mathscr{C}_v$. Hence $\mathscr{D}_v = \mathscr{C}_v$.

The case of $v \in \mathscr{V} \setminus \mathscr{A}$. By virtue of Lemma 5.3, it is enough to prove that if u is the [0, 1]-standard of v, $\mathscr{D}_u \subset \mathscr{C}_u$. Let $a \in \mathscr{A}_u \setminus \mathscr{C}_u$. Then λ_a is a μ-absolutely continuous probability measure on $\mathscr{F}$ and $u(m_0) > \int_X m_0 d\lambda_a$ for some $m_0 \in \mathscr{M}$.

Hence, by Lemma 5.4, there exists a μ-absolutely continuous probability measure λ_1 on $\mathscr{F}$ such that

$$u(m_1) > \int_X m_1 d\lambda_1 > \int_X m_1 d\lambda_a$$

for some $m_1 \in \mathscr{M}$. Put $x(m) = \int_X m d\lambda_1$ for $m \in \mathscr{M}$. Then $x \in \mathscr{A}_u$ and x dominates a, and so $a \in \mathscr{A}_u \setminus \mathscr{D}_u$. Therefore $\mathscr{D}_u \subset \mathscr{C}_u$. The proof of Theorem 5.2 is completed.

6. Another formulation. In this section, we shall give another formulation of the theory by changing the definitions of $\mathscr{M}$, core-allocation and domination. It should be noticed here that we have still all theorems and propositions in the same forms as described previously and moreover we have entirely no need of changing of their proofs except that of Theorem 5.2.

Let $\mathscr{M}$ denote either the set of all fuzzy coalitions or the set of all indicator functions of coalitions. In the latter case of $\mathscr{M}$, we note that

(i) (V1), (V2) and (V4) may be restated as follows:

(V1) $v(1_{A_1}) + v(1_{A_2}) \leqq v(1_{A_1 \cup A_2})$ whenever A_1 and A_2 are disjoint mod μ;

(V2) $\lim_k v(1_{A_k}) = v(1_A)$ whenever $\lim_k A_k = A$ mod μ;

(V4) $v(0) = 0$;

and, for example,

(ii) "$v(m) \geqq \int_X m d\lambda_v$ for all $m \in \mathscr{M}$" in Theorem 2.1 is reduced to "$v(1_A) \geqq \lambda_v(A)$ for all $A \in \mathscr{F}$."

Before changing the definition of core-allocation, we note that a is an allocation for v iff

$$a(1) = v(1)\ , \quad a(1_A) \geqq \lambda_v(A)$$

for any $A \in \mathscr{F}$.

Let $v \in \mathscr{V}$ and $a \in \mathscr{A}$. a is called a *core-allocation for* v if

$$a(1) = v(1)\ , \quad a(1_A) \geqq v(1_A)$$

for any $A \in \mathscr{F}$.

Let $x, y \in \mathscr{A}_v$. If there exists an $A \in \mathscr{F}$ such that

$$\mu(A) > 0\ , \quad v(1_A) \geqq x(1_A)\ , \quad d\lambda_x/d\mu > d\lambda_y/d\mu \quad \mu\text{-a.e. on} \quad \mathscr{A}\ ,$$

we say x *dominates* y.

PROOF OF THEOREM 5.2 UNDER THE PRESENT FORMULATION. The implication $\mathscr{C}_v \subset \mathscr{D}_v$ is easily shown as in the proof of the previous Theorem 5.2.

Proof of $\mathscr{D}_v \subset \mathscr{C}_v$. Let $a \in \mathscr{A}_v \setminus \mathscr{C}_v$. Then there exists an $A \in \mathscr{F}$ such that $v(1_A) > \lambda_a(A)$. Then $\mu(A) > 0$, $\mu(A^c) > 0$ and moreover $\lambda_a(A^c) -$

$\lambda_v(A^c) \geqq v(1_A) - \lambda_a(A)$, because

$$(\lambda_a(A^c) - \lambda_v(A^c)) - (v(1_A) - \lambda_a(A)) = \lambda_a(X) - v(1_A) - \lambda_v(A^c)$$
$$= v(1) - v(1_A) - \lambda_v(A^c) \geqq v(1_{A^c}) - \lambda_v(A^c) \geqq 0 .$$

Hence there exists a μ-absolutely continuous measure p on $\mathscr{F}$ such that

$$dp/d\mu > 0 \quad \mu\text{-a.e.} \quad \text{on} \quad A ,$$
$$d\lambda_a/d\mu - d\lambda_v/d\mu \geqq dp/d\mu \geqq 0 \quad \mu\text{-a.e.} \quad \text{on} \quad A^c ,$$
$$p(A) = p(A^c) = v(1_A) - \lambda_a(A) .$$

Define

$$\lambda(E) = \lambda_a(E) + p(E \cap A) - p(E \cap A^c)$$

for $E \in \mathscr{F}$. Then λ is a μ-absolutely continuous signed measure on $\mathscr{F}$. Put $x(m) = \int_X m d\lambda$ for $m \in \mathscr{M}$. Then

$$x(1) = \lambda(X) = \lambda_a(X) = a(1) = v(1) ,$$
$$x(1_E) = \lambda(E) \geqq \lambda_a(E \cap A) + \lambda_a(E \cap A^c) - p(E \cap A^c)$$
$$\geqq \lambda_a(E \cap A) + \lambda_v(E \cap A^c) \geqq \lambda_v(E)$$

for any $E \in \mathscr{F}$, because

$$\lambda_a(E \cap A^c) - p(E \cap A^c) = \int_{E \cap A^c} \left(\frac{d\lambda_a}{d\mu} - \frac{dp}{d\mu}\right) d\mu$$
$$\geqq \int_{E \cap A^c} \frac{d\lambda_v}{d\mu} d\mu = \lambda_v(E \cap A^c) .$$

Hence $x \in \mathscr{A}_v$. Moreover

$$v(1_A) = \lambda_a(A) + p(A) = \lambda(A) = x(1_A) ,$$
$$d\lambda/d\mu = d\lambda_a/d\mu + dp/d\mu > d\lambda_a/d\mu \quad \mu\text{-a.e} .$$

on A.

Hence x dominates a and so $a \in \mathscr{A}_v \setminus \mathscr{D}_v$. Therefore $\mathscr{D}_v \subset \mathscr{C}_v$.

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