

# MATHEMATICAL FOUNDATIONS OF SUPERSYMMETRY

Lauren Caston and Rita Fioresi <sup>1</sup>

*Dipartimento di Matematica, Università di Bologna*

*Piazza di Porta S. Donato, 5*

*40126 Bologna, Italia*

e-mail: caston@dm.UniBo.it, fioresi@dm.UniBo.it

---

<sup>1</sup>Investigation supported by the University of Bologna

## Introduction

Supersymmetry (SUSY) is the machinery mathematicians and physicists have developed to treat two types of elementary particles, *bosons* and *fermions*, on the same footing. Supergeometry is the geometric basis for supersymmetry; it was first discovered and studied by physicists Wess, Zumino [25], Salam and Strathde [20] (among others) in the early 1970's. Today supergeometry plays an important role in high energy physics. The objects in super geometry generalize the concept of smooth manifolds and algebraic schemes to include anticommuting coordinates. As a result, we employ the techniques from algebraic geometry to study such objects, namely A. Grothendiek's theory of schemes.

Fermions include all of the material world; they are the building blocks of atoms. Fermions do not like each other. This is in essence the Pauli exclusion principle which states that two electrons cannot occupy the same quantum mechanical state at the same time. Bosons, on the other hand, can occupy the same state at the same time.

Instead of looking at equations which just describe either bosons or fermions separately, supersymmetry seeks out a description of both simultaneously. Transitions between fermions and bosons require that we allow transformations between the commuting and anticommuting coordinates. Such transitions are called supersymmetries.

In classical Minkowski space, physicists classify elementary particles by their mass and spin. Einstein's special theory of relativity requires that physical theories must be invariant under the Poincaré group. Since observable operators (e.g. Hamiltonians) must commute with this action, the classification corresponds to finding unitary representations of the Poincaré group. In the SUSY world, this means that mathematicians are interested in unitary representations of the su-

per Poincaré group. A “super” representation gives a “multiplet” of ordinary particles which include both fermions and bosons.

Up to this point, there have been no colliders that can produce the energy required to physically expose supersymmetry. However, the Large Hadron Collider (LHC) in CERN (Geneva, Switzerland) will be operational in 2007. Physicists are planning proton-proton and proton-antiproton collisions which will produce energies high enough where it is believed supersymmetry can be seen. Such a discovery will solidify supersymmetry as the most viable path to a unified theory of all known forces. Even before the boson-fermion symmetry which SUSY presupposes is made physical fact, the mathematics behind the theory is quite remarkable. The concept that space is an object built out of local pieces with specific local descriptions has evolved through many centuries of mathematical thought. Euclidean and non-Euclidean geometry, Riemann surfaces, differentiable manifolds, complex manifolds, algebraic varieties, and so on represent various stages of this concept. In Alexander Grothendieck’s theory of schemes, we find a single structure (a scheme) that encompasses all previous ideas of space. However, the fact that conventional descriptions of space will fail at very small distances (Planck length) has been the driving force behind the discoveries of unconventional models of space that are rich enough to portray the quantum fluctuations of space at these unimaginably small distances. Supergeometry is perhaps the most highly developed of these theories; it provides a surprising continuation of the Grothendieck theory and opens up large vistas. One should not think of it as a mere generalization of classical geometry, but as a deep continuation of the idea of space and its geometric structure.

Out of the first supergeometric objects constructed by the pioneering physicists came mathematical models of superanalysis and supermanifolds indepen-

dently by F. A. Berezin [2], B. Kostant [15], D.A. Leites [17], and De Witt [8]. The idea to treat a supermanifold as a ringed space with a sheaf of  $\mathbf{Z}/2\mathbf{Z}$ -graded algebras was introduced in these early works. Later, Bernstein [7] and Leites made this treatment rigorous and used techniques from algebraic geometry to deepen the study of supersymmetry. In particular, Bernstein and Leites accented the functor of points approach from Grothendieck's theory of schemes. It is this approach (which we call  $T$ -points) that we present and expand upon in our treatment of mathematical supersymmetry. Interest in SUSY has grown in the past decade, and most recently works by V. S. Varadarajan [23] among others, have continued the exploration of the beautiful area of physics and mathematics and have inspired this work. Given the interest and the number of people who have contributed greatly to this field from various perspectives, it is impossible to give a fair and accurate account of works related to ours.

In our exposition of mathematical SUSY, we use the language of  $T$ -points to build supermanifolds up from their foundations in  $\mathbf{Z}/2\mathbf{Z}$ -graded linear algebra (superalgebra). This treatment is similar to that given by Varadarajan in [23], however we prove some deeper results related to the Frobenius theorem as well as give a full treatment of superschemes in chapters 3-4. Recently the book by G. Tuynman [21] has been brought to our attention. The main results from chapters 5-6 can be found in [21], however we obtained our results independently of this work, moreover, our method of  $T$ -points remains fresh in light of this and other recent works.

Here is a brief description of our work.

In chapter 1 we begin by studying  $\mathbf{Z}/2\mathbf{Z}$ -graded linear objects. We define super vector spaces and superalgebras, then generalize some classical results and ideas from linear algebra to the super setting. For example, we define a super Lie

algebra, discuss supermatrices, and formulate the super trace and determinant (the Berezinian).

In chapter 2 we introduce the most basic geometric structure: a superspace. We present some general properties of superspaces which leads into two key examples of superspaces, supermanifolds and superschemes. Here we also introduce the notion of  $T$ -points which treats our geometric objects as functors; it is a fundamental tool to gain geometric intuition in supergeometry.

Chapters 3-4 lay down the full foundations of  $C^\infty$ -supermanifolds over  $\mathbf{R}$ . We give special attention to super Lie groups and their associated Lie algebras, as well as look at how group actions translate infinitesimally. In chapter 4 we prove the local and global Frobenius theorem on supermanifolds, then use the infinitesimal actions from chapter 3 to build the super Lie subgroup, subalgebra correspondence.

Chapters 5-6 expand upon the notion of a superscheme which we introduce in chapter 2. We immediately adopt the language of  $T$ -points and give criterion for representability: in supersymmetry it is often most convenient to describe an object functorially, then show it is representable. In chapter 5, we explicitly construct the Grassmanian superscheme functorially, then use the representability criterion to show it is representable. Chapter 5 concludes with an examination of the infinitesimal theory of superschemes. We continue this exploration in chapter 6 from the point of view of algebraic supergroups and their Lie algebras. We discuss the linear representations of affine algebraic supergroups; in particular we show that all affine super groups are realized as subgroups of the general linear supergroup.

This work is self-contained; we try to keep references to a minimum in the body of our work so that the reader can proceed without the aid of other texts. We

assume a working knowledge of sheaves, differential geometry, and categories and functors. We suggest that the reader begin with chapters 1 and 2, but chapters 3-4 and chapters 5-6 are somewhat disjoint and may be read independently of one another.

We wish to thank professor V. S. Varadarajan for introducing us to this beautiful part of mathematics. He has truly inspired us through his insight and deep understanding of the subject. We also wish to thank Prof. M. A. Lledo, Prof. A. Vistoli and Prof. M. Duflo for many helpful remarks. R. Fioresi thanks the UCLA Department of Mathematics for its kind hospitality during the realization of this work. L. Caston thanks the Dipartimento di Matematica, Universita' di Bologna for support and hospitality during the realization of this work.

# TABLE OF CONTENTS

<b>1</b>	<b><math>\mathbb{Z}/2\mathbb{Z}</math>-Graded Linear Algebra</b>	<b>9</b>
1.1	Super Vector Spaces and Superalgebras	9
1.2	Lie Algebras	13
1.3	Modules	14
1.4	Matrices	16
<b>2</b>	<b>Supergeometry</b>	<b>22</b>
2.1	Superspaces	22
2.2	Supermanifolds	24
2.3	Superschemes	26
2.4	T-Points	27
<b>3</b>	<b><math>C^\infty</math>-supermanifolds</b>	<b>30</b>
3.1	Charts	30
3.2	Vector Fields	33
3.3	Differential Calculus	36
3.4	Super Lie Groups	39
3.5	Left Invariant Vector Fields	41
3.6	Infinitesimal Action	45
<b>4</b>	<b>The Frobenius Theorem</b>	<b>50</b>
4.1	The Local Frobenius Theorem	50

4.2	The Global Frobenius Theorem on Supermanifolds . . . . .	59
4.3	Lie Subalgebra, Subgroup Correspondence . . . . .	60
<b>5</b>	<b>Supervarieties and Superschemes . . . . .</b>	<b>64</b>
5.1	Basic definitions . . . . .	64
5.2	The functor of points . . . . .	70
5.3	A representability criterion . . . . .	73
5.4	The Grassmannian superscheme . . . . .	80
5.5	The infinitesimal theory . . . . .	82
<b>6</b>	<b>Algebraic Supergroups and Lie Superalgebras . . . . .</b>	<b>91</b>
6.1	Supergroup functors and algebraic supergroups . . . . .	91
6.2	Lie superalgebras . . . . .	94
6.3	$\mathrm{Lie}(G)$ as tangent superspace to a supergroup scheme . . . . .	97
6.4	The Lie superalgebra of a supergroup scheme . . . . .	101
6.5	Affine algebraic supergroups. Linear representations. . . . .	106
<b>7</b>	<b>Appendix . . . . .</b>	<b>113</b>
7.1	Categories . . . . .	113
7.2	SuperNakayama's Lemma and Projective Modules . . . . .	116
	<b>References . . . . .</b>	<b>121</b>



# CHAPTER 1

## $\mathbf{Z}/2\mathbf{Z}$ -Graded Linear Algebra

We first build the foundations of linear algebra in the super context. This is an important starting point as we later build super geometric objects from sheaves of super linear spaces. Let us fix a ground field  $k$ ,  $\text{char}(k) \neq 2, 3$ .

### 1.1 Super Vector Spaces and Superalgebras

**Definition 1.1.1.** A *super vector space* is a  $\mathbf{Z}/2\mathbf{Z}$ -graded vector space

$$V = V_0 \oplus V_1$$

where elements of  $V_0$  are called “even” and elements of  $V_1$  are called “odd”.

**Definition 1.1.2.** The *parity* of  $v \in V$ , denoted  $p(v)$  or  $|v|$ , is defined only on nonzero *homogeneous* elements, that is elements of either  $V_0$  or  $V_1$ :

$$p(v) = |v| = \begin{cases} 0 & \text{if } v \in V_0 \\ 1 & \text{if } v \in V_1 \end{cases}$$

Since any element may be expressed as the sum of homogeneous elements, it suffices to only consider homogeneous elements in the statement of definitions, theorems, and proofs.

**Definition 1.1.3.** The *super dimension* of a super vector space  $V$  is the pair  $(p, q)$  where  $\dim(V_0)=p$  and  $\dim(V_1)=q$  as ordinary vector spaces. We simply write  $\dim(V) = p|q$ .

From now on we will simply refer to the superdimension as the dimension when the category is clear. If  $\dim(V) = p|q$ , then we can find a basis  $\{e_1, \dots, e_p\}$  of  $V_0$  and a basis  $\{\epsilon_1, \dots, \epsilon_q\}$  of  $V_1$  so that  $V$  is canonically isomorphic to the free  $k$ -module generated by the  $\{e_1, \dots, e_p, \epsilon_1, \dots, \epsilon_q\}$ . We denote this  $k$ -module by  $k^{p|q}$ .

**Definition 1.1.4.** A *morphism* from a super vector space  $V$  to a super vector space  $W$  is a  $\mathbf{Z}/2\mathbf{Z}$ -grading preserving linear map from  $V$  to  $W$ . Let  $\text{Hom}(V, W)$  denote the set of morphisms  $V \longrightarrow W$ .

Thus we have formed the abelian category of super vector spaces. It is important to note that the category of super vector spaces also admits an “inner Hom”, which we denote  $\underline{\text{Hom}}(V, W)$ ; it consists of *all* linear maps from  $V$  to  $W$ :

$$\begin{aligned}\underline{\text{Hom}}(V, W)_0 &= \{T : V \longrightarrow W \mid T \text{ preserves parity}\} \quad (= \text{Hom}(V, W)); \\ \underline{\text{Hom}}(V, W)_1 &= \{T : V \longrightarrow W \mid T \text{ reverses parity}\}.\end{aligned}$$

In the category of super vector spaces we have the *parity reversing functor*  $\Pi$  defined by

$$(\Pi V)_0 = V_1 \quad (\Pi V)_1 = V_0.$$

The category of super vector spaces is in fact a tensor category, where  $V \otimes W$  is given the  $\mathbf{Z}/2\mathbf{Z}$ -grading as follows:

$$\begin{aligned}(V \otimes W)_0 &= (V_0 \otimes W_0) \oplus (V_1 \otimes W_1) \\ (V \otimes W)_1 &= (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).\end{aligned}$$

The tensor functor  $\otimes$  is additive and exact in each variable as in the ordinary vector space category; it has a unit object (i.e.  $k$ ) and is associative. Moreover,  $V \otimes W \cong W \otimes V$  by the *commutativity map*

$$c_{V,W} : V \otimes W \longrightarrow W \otimes V$$

where  $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ . This is the so called “sign rule” that one finds in some physics and math literature. In any tensor category with an inner Hom, the *dual*,  $V^*$ , of  $V$  is

$$V^* =_{def} \underline{\text{Hom}}(V, k).$$

**Remark 1.1.5.** We understand completely the object  $V^{\otimes n} = V \otimes \cdots \otimes V$  ( $n$  times) for a super vector space  $V$ . We can extend this notion to make sense of  $V^{\otimes n|m}$  via the parity reversing functor  $\Pi$ . Define

$$V^{n|m} := \underbrace{V \times V \times \cdots \times V}_{n \text{ times}} \times \underbrace{\Pi(V) \times \Pi(V) \times \cdots \times \Pi(V)}_{m \text{ times}},$$

from which the definition of  $V^{\otimes n|m}$  follows by the universal property.

Let us now define a super  $k$ -algebra:

**Definition 1.1.6.** We say that a super vector space  $A$  is a *superalgebra* if there is a multiplication morphism  $\tau : A \otimes A \longrightarrow A$ .

We then say that a superalgebra  $A$  is *commutative* if

$$\tau \circ c_{A,A} = \tau,$$

that is, if the product of homogeneous elements obeys the rule

$$ab = (-1)^{|a||b|} ba.$$

Similarly we say that  $A$  is *associative* if  $\tau \circ \tau \otimes id = \tau \circ id \otimes \tau$  on  $A \otimes A \otimes A$ , and that  $A$  has a *unit* if there is an even element  $1$  so that  $\tau(1 \otimes a) = \tau(a \otimes 1) = a$  for all  $a \in A$ .

From now on we will assume all superalgebras are associative and commutative with unit unless specified.

An *even derivation* of a superalgebra  $A$  is a super vector space homomorphism  $D : A \longrightarrow A$  such that for  $a, b \in A$ ,  $D(ab) = D(a)b + aD(b)$ . We may of course extend this definition to include odd linear maps:

**Definition 1.1.7.** Let  $D \in \underline{\text{Hom}}_k(A, A)$  be a  $k$ -linear map. Then  $D$  is a *derivation* of the superalgebra  $A$  if for  $a, b \in A$ ,

$$D(ab) = D(a)b + (-1)^{|D||a|}aD(b). \quad (1.1)$$

Those derivations in  $\text{Hom}_k(A, A)$  are even (as above) while those in  $\underline{\text{Hom}}_k(A, A)_1$  are odd. The set of all derivations of a superalgebra  $A$ , denoted  $\text{Der}(A)$ , is itself a special type of superalgebra called a *super Lie algebra* which we describe in the following section.

**Example 1.1.8.** *Grassmann coordinates.* Let

$$A = k[t_1, \dots, t_p, \theta_1, \dots, \theta_q]$$

where the  $t_1, \dots, t_p$  are ordinary indeterminates and the  $\theta_1, \dots, \theta_q$  are *odd indeterminates*, i.e. they behave like Grassmannian coordinates:

$$\theta_i \theta_j = -\theta_j \theta_i.$$

(This of course implies that  $\theta_i^2 = 0$  for all  $i$ .) We claim that  $A$  is a supercommutative algebra. In fact,

$$A_0 = \{f_0 + \sum_{|I| \text{ even}} f_I \theta_I \mid I = \{i_1 < \dots < i_r\}\}$$

where  $\theta_I = \theta_{i_1} \theta_{i_2} \dots \theta_{i_r}$  and  $f_0, f_I \in k[t_1, \dots, t_p]$ , and

$$A_1 = \{\sum_{|J| \text{ odd}} f_J \theta_J \mid J = \{i_1 < \dots < i_s\}\}$$

for  $s$  odd ( $|J| = 2m + 1$ ,  $m = 1, 2, \dots$ ) and  $f_J \in k[t_1, \dots, t_q]$ . Note that although the  $\{\theta_j\} \in A_1$ , there are plenty of nilpotents in  $A_0$ ; take for example  $\theta_1\theta_2 \in A_0$ .

Consider the  $k$ -linear operators  $\{\partial/\partial t_i\}$  and  $\{\partial/\partial \theta_j\}$  of  $A$  to itself where

$$\begin{aligned} \partial/\partial t_i(t_k) &= \delta_k^i & \partial/\partial t_i(\theta_l) &= 0; \\ \partial/\partial \theta_j(t_k) &= 0 & \partial/\partial \theta_j(\theta_l) &= \delta_l^j. \end{aligned} \tag{1.2}$$

It is easy to verify that  $\{\partial/\partial t_i, \partial/\partial \theta_j\} \in \text{Der}(A)$ , and we leave it as an exercise to check that

$$\text{Der}(A) = \text{Span}_A \left\{ \frac{\partial}{\partial t_i}, \frac{\partial}{\partial \theta_j} \right\}.$$

## 1.2 Lie Algebras

An important object in supersymmetry is the super Lie algebra.

**Definition 1.2.1.** A *super Lie algebra*  $L$  is an object in the category of super vector spaces together with a morphism  $[\cdot, \cdot] : L \otimes L \longrightarrow L$  which categorically satisfies the usual conditions.

It is important to note that in the super category, these conditions are slightly different to accomodate the odd variables. The bracket  $[\cdot, \cdot]$  must satisfy

1. Anti-symmetry

$$[\cdot, \cdot] + [\cdot, \cdot] \circ c_{L,L} = 0$$

which may be interpreted as  $[x, y] + (-1)^{|x||y|}[y, x] = 0$  for  $x, y \in L$  homogeneous.

2. The Jacobi identity

$$[\cdot, [\cdot, \cdot]] + [\cdot, [\cdot, \cdot]] \circ \sigma + [\cdot, [\cdot, \cdot]] \circ \sigma^2 = 0$$

where  $\sigma \in S_3$  is a three-cycle, i.e. it takes the first entry of  $[\cdot, [\cdot, \cdot]]$  to the second, the second to the third, and the third to the first. So for  $x, y, z \in L$  homogeneous,

this reads:

$$[x, [y, z]] + (-1)^{|x||y|+|x||z|}[y, [z, x]] + (-1)^{|y||z|+|x||z|}[z, [x, y]] = 0.$$

**Remark 1.2.2.** We can immediately extend this definition to the case where  $L$  is an  $A$ -module.

**Example 1.2.3.** In the Grassmannian example above (1.1.8),

$$\text{Der}(A) = \text{Span}_A \left\{ \frac{\partial}{\partial t_i}, \frac{\partial}{\partial \theta_j} \right\}$$

is a super Lie algebra where the bracket is taken for  $D_1, D_2 \in \text{Der}(A)$  to be  $[D_1, D_2] = D_1 D_2 - (-1)^{|D_1||D_2|} D_2 D_1$ .

In fact, we can make any associative algebra  $A$  into a Lie algebra by taking the bracket to be

$$[a, b] = ab - (-1)^{|a||b|} ba,$$

i.e. we take the bracket to be the difference  $\tau - \tau \circ c_{A,A}$  where we recall  $\tau$  is the multiplication morphism on  $A$ . We will discuss other examples of super Lie algebras after the following discussion of superalgebra modules. In particular we want to examine the SUSY-version of a matrix algebra.

**Remark 1.2.4.** If the ground field has characteristic 2 or 3 in addition to the antisymmetry and Jacobi conditions one requires that  $[x, x] = 0$  for  $x$  even if the characteristic is 2 or  $[y, [y, y]] = 0$  for  $y$  odd if the characteristic is 3. For more details on superalgebras over fields with positive characteristic see [3].

### 1.3 Modules

Let  $A$  be a superalgebra, not necessarily commutative in this section.

**Definition 1.3.1.** A *left A-module* is a super vector space  $M$  with a morphism  $A \otimes M \longrightarrow M$  obeying the usual identities found in the ordinary category.

A *right A-module* is defined similarly. Note that if  $A$  is commutative, a left  $A$ -module is also a right  $A$ -module using the sign rule

$$m \cdot a = (-1)^{|m||a|} a \cdot m$$

for  $m \in M, a \in A$ . Morphisms of  $A$ -modules are also obviously defined, and so we have the category of  $A$ -modules. For  $A$  commutative, the category of  $A$ -modules is a tensor category: for  $M_1, M_2$   $A$ -modules,  $M_1 \otimes M_2$  is taken as the tensor of  $M_1$  as a right module with  $M_2$  as a left module.

Let us now turn our attention to *free*  $A$ -modules. We already have the notion of the vector space  $k^{p|q}$  over  $k$ , and so we define  $A^{p|q} := A \otimes k^{p|q}$  where

$$\begin{aligned} (A^{p|q})_0 &= A_0 \otimes (k^{p|q})_0 \oplus A_1 \otimes (k^{p|q})_1 \\ (A^{p|q})_1 &= A_1 \otimes (k^{p|q})_0 \oplus A_0 \otimes (k^{p|q})_1. \end{aligned}$$

**Definition 1.3.2.** We say that an  $A$ -module  $M$  is *free* if it is isomorphic (in the category of  $A$ -modules) to  $A^{p|q}$  for some  $(p, q)$ .

This definition is equivalent to saying that there are *even* elements  $\{e_1, \dots, e_p\}$  and *odd* elements  $\{\epsilon_1, \dots, \epsilon_q\}$  which generate  $M$  over  $A$ .

Let  $T : A^{p|q} \longrightarrow A^{r|s}$  be a morphism of free  $A$ -modules and write  $e_{p+1}, \dots, e_{p+q}$  for the odd variables  $\epsilon_1, \dots, \epsilon_q$ . Then  $T$  is defined on the basis elements  $\{e_1, \dots, e_{p+q}\}$  by

$$T(e_j) = \sum_{i=1}^{p+q} e_i t_j^i. \quad (1.3)$$

Hence  $T$  can be represented as a matrix of size  $(r + s) \times (p + q)$ :

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \quad (1.4)$$

where  $T_1$  is an  $r \times p$  matrix consisting of even elements of  $A$ ,  $T_2$  is an  $r \times q$  matrix of odd elements,  $T_3$  is an  $s \times p$  matrix of even elements, and  $T_4$  is an  $s \times q$  matrix of odd elements. We say that  $T_1$  and  $T_4$  are *even blocks* and that  $T_2$  and  $T_3$  are *odd blocks*. Because  $T$  is a morphism of super  $A$ -modules, it must preserve parity, and therefore the parity of the blocks is determined. Note that when we define  $T$  on the basis elements, in the expression (1.3) the basis element *preceeds* the coordinates  $t_j^i$ . This is important to keep the signs in order and comes naturally from composing morphisms. For any  $x \in A^{p|q}$ , we can express  $x$  as the column vector  $x = \sum e_i x^i$  and so  $T(x)$  is given by the matrix product  $Tx$ . Similarly the composition of morphisms is given by a matrix product.

## 1.4 Matrices

Let us now consider all endomorphisms of  $M = A^{p|q}$ , i.e.  $\text{Hom}(M, M)$ . This is an ordinary algebra (i.e. *not* super) of matrices of the same type as  $T$  above. Even though in matrix form each morphism contains blocks of odd elements of  $A$ , each morphism is an even linear map from  $M$  to itself since a morphism in the super category must preserve parity. In order to get a truly SUSY-version of the ordinary matrix algebra, we must consider *all* linear maps  $M$  to  $M$ , i.e. we are interested in  $\underline{\text{Hom}}(M, M)$ . Now we can talk about even and odd matrices. An even matrix  $T$  takes on the block form from above. But the parity of the blocks is reversed for an odd matrix  $S$ ; we get

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$$

where  $S_1$  is a  $p \times p$  odd block,  $S_4$  is a  $q \times q$  odd block,  $S_2$  is a  $p \times q$  even block, and  $S_3$  is a  $q \times p$  even block. Note that in the case where  $M = k^{p|q}$ , the odd blocks are just zero blocks. We will denote this super algebra of even and odd



$(p+q) \times (p+q) = p|q \times p|q$  matrices by  $\text{Mat}(A^{p|q})$ . This super algebra is in fact a super Lie algebra where we define the bracket  $[\cdot, \cdot]$  as in Example 1.2.3:

$$[T, S] = TS - (-1)^{|T||S|}ST$$

for  $S, T \in \text{Mat}(A^{p|q})$ .

**Remark 1.4.1.** Note that  $\text{Mat}(A^{p|q}) = \underline{\text{Hom}}(A^{p|q}, A^{p|q})$ . We do not want to confuse this with what we will later denote as  $M_{p|q}(A)$ , which will functorially only include the *even part* of  $\text{Mat}(A^{p|q})$ , i.e.

$$\text{Mat}(A^{p|q})_0 = M_{p|q}(A) = \text{Hom}(A^{p|q}, A^{p|q})$$

(see chapter 3).

We now turn to the SUSY-extensions of the trace and determinant. Let  $T : A^{p|q} \longrightarrow A^{p|q}$  be a morphism (i.e.  $T \in (\text{Mat}(A^{p|q}))_0$ ) with block form (1.4).

**Definition 1.4.2.** We define the *super trace* of  $T$  to be:

$$\text{Tr}(T) := \text{tr}(T_1) - \text{tr}(T_4) \tag{1.5}$$

where “tr” denotes the ordinary trace.

This negative sign is actually forced upon us when we take a categorical view of the trace. We will not discuss this here, but we later give motivation to this definition when we explore the SUSY-extension of the determinant.

**Remark 1.4.3.** The trace is actually defined for *all* linear maps. For  $S \in \text{Mat}(A^{p|q})_1$  an odd matrix,

$$\text{Tr}(S) = \text{tr}S_1 + \text{tr}S_4.$$

Note the sign change. Note also that the trace is commutative, meaning that for even matrices  $A, B \in \text{Mat}(A^{p|q})_0$ , we have the familiar formula

$$\text{Tr}(AB) = \text{Tr}(BA).$$

**Definition 1.4.4.** Again let  $M = A^{p|q}$ , the free  $A$ -module generated by  $p$  even and  $q$  odd variables. Then  $\mathrm{GL}(A^{p|q})$  denotes the *super general linear group of automorphisms of  $M$* ; we may also use the notation  $\mathrm{GL}_{p|q}(A) = \mathrm{GL}(A^{p|q})$ .

**Remark 1.4.5.** If  $M$  is an  $A$ -module, then  $\mathrm{GL}(M)$  is defined as the group of automorphisms of  $M$ . If  $M = A^{p|q}$ , then we write  $\mathrm{GL}(M) = \mathrm{GL}_{p|q}(A)$  as above.

Next we define the generalization of the determinant, called the *Berezinian*, on elements of  $\mathrm{GL}(A^{p|q})$ .

**Definition 1.4.6.** Let  $T \in \mathrm{GL}(A^{p|q})$  have the standard block form (1.4) from above. Then we formulate  $\mathrm{Ber}$ :

$$\mathrm{Ber}(T) = \det(T_1 - T_2 T_4^{-1} T_3) \det(T_4)^{-1} \quad (1.6)$$

where “ $\det$ ” is the usual determinant.

**Remark 1.4.7.** The first thing we notice is that in the super category, we only define the Berezinian for *invertible* transformations. We immediately see that it is necessary that the block  $T_4$  be invertible for the formula (1.6) to make sense, however one can actually define the Berezinian on all matrices with *only* the  $T_4$  block invertible (i.e. the matrix itself may not be invertible, but the  $T_4$  block is). There is a similar formulation of the Berezinian which requires that only the  $T_1$  block be invertible:

$$\mathrm{Ber}(T) = \det(T_4 - T_3 T_1^{-1} T_2) \det(T_1)^{-1}$$

So we can actually define the Berezinian on all matrices with *either* the  $T_1$  *or* the  $T_4$  block invertible. Note that in the case where both blocks are invertible (i.e. when the matrix  $T$  is invertible), both formulae of the Berezinian give the same answer.

We leave the following proposition as an exercise.

**Proposition 1.4.8.** *Let  $T : A^{p|q} \longrightarrow A^{p|q}$  be a morphism with the usual block form (1.4). Then  $T$  is invertible if and only if  $T_1$  and  $T_4$  are invertible.*

**Proposition 1.4.9.** *The Berezinian is multiplicative: For  $S, T \in GL(A^{p|q})$ ,*

$$\text{Ber}(ST) = \text{Ber}(S)\text{Ber}(T).$$

*Proof.* We will only briefly sketch the proof here and leave the details to the reader. First note that any  $T \in GL(A^{p|q})$  with block form (1.4) may be written as the product of the following “elementary matrices”:

$$T_+ = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}, \quad T_0 = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}, \quad T_- = \begin{pmatrix} 1 & 0 \\ Z & 1 \end{pmatrix}. \quad (1.7)$$

If we equate  $T = T_+T_0T_-$ , we get a system of equations which lead to the solution

$$\begin{aligned} X &= T_2T_4^{-1}, \\ Y_1 &= T_1 - T_2T_4^{-1}T_3, \\ Y_2 &= T_4, \\ Z &= T_4^{-1}T_3. \end{aligned}$$

It is also easy to verify that  $\text{Ber}(ST) = \text{Ber}(S)\text{Ber}(T)$  for  $S$  of type  $\{T_+, T_0\}$  or  $T$  of type  $\{T_-, T_0\}$ . The last case to verify is for

$$S = \begin{pmatrix} 1 & 0 \\ Z & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}.$$

We may assume that both  $X$  and  $Z$  each have only one non-zero entry since the product of two matrices of type  $T_+$  results in the sum of the upper right blocks, and likewise with the product of two type  $T_-$  matrices. Let  $x_{ij}, z_{kl} \neq 0$ . Then

$$ST = \begin{pmatrix} 1 & X \\ Z & 1 + ZX \end{pmatrix}$$

and  $\text{Ber}(ST) = \det(1 - X(1 + ZX)^{-1}Z) \det(1 + ZX)^{-1}$ . Because all the values within the determinants are either upper triangular or contain an entire column of zeros ( $X, Z$  have at most one non-zero entry), the values  $x_{ij}, z_{kl}$  contribute to the determinant only when the product  $ZX$  has its non-zero term on the diagonal, i.e. only when  $i = j = k = l$ . But then  $\det(1 - X(1 + ZX)^{-1}Z) = 1 + x_{ii}z_{ii}$ , and it is clear that  $\text{Ber}(ST) = 1$ . A direct calculation shows that  $\text{Ber}(S) = \text{Ber}(T) = 1$ . ■

**Corollary 1.4.10.** *The Berezinian is a homomorphism*

$$\text{Ber} : GL(A^{p|q}) \longrightarrow GL_{1|0}(A) = A_0^\times$$

*into the invertible elements of  $A$ .*

*Proof.* This follows immediately from above proposition. ■

The usual determinant on the general linear group  $GL_n$  induces the trace on its Lie algebra, namely the matrices  $M_n$  (see Remark 1.4.1). The extension to the Berezinian gives

$$\text{Ber}(I + \epsilon T) = 1 + \epsilon \text{Tr}(T)$$

where  $I$  is the  $p|q \times p|q$  identity matrix (ones down the diagonal, zeros elsewhere) and  $\epsilon^2 = 0$ . An easy calculation then exposes the super trace formula with the negative sign. This of course leads to the question of how the formula for the Berezinian arises. The answer lies in the SUSY-version of integral forms on supermanifolds called *densities*. In F.A. Berezin's pioneering work in superanalysis, Berezin calculated the change of variables formula for densities on isomorphic open submanifolds of  $\mathbf{R}^{p|q}$  ([2]). This lead to an extension of the Jacobian in ordinary differential geometry; the Berezinian is so named after him.

We finish our summary of superlinear algebra by giving meaning to the *rank* of a endomorphism of  $A^{p|q}$ .

**Definition 1.4.11.** Let  $T \in \text{End}(A^{p|q})$ . Then the *rank* of  $T$ ,  $\text{rank}(T)$ , is the superdimension of the largest submatrix of  $T$  (obtained by removing columns and rows).

**Proposition 1.4.12.** *Again, let  $T \in \text{End}(A^{p|q})$  with block form (1.4). Then  $\text{rank}(T) = \text{rank}(T_1)|\text{rank}(T_4)$ .*

*Proof.* Assume that  $\text{rank}(T) = r|s$ . Then there is an invertible  $r|s \times r|s$  submatrix of  $T$  and it is clear that  $r \leq \text{rank}(T_1), s \leq \text{rank}(T_4)$ . Conversely, if  $\text{rank}(T_1) = r', \text{rank}(T_4) = s'$  it is also clear that there exists an invertible  $r'|s' \times r'|s'$  submatrix of  $T$ . Therefore we must have  $r = r', s = s'$ . ■

## CHAPTER 2

### Supergeometry

In this chapter we discuss the foundations of supergeometric objects. We begin by defining the most basic object, the *super ringed space* and build some basic concepts from this definition.

#### 2.1 Superspaces

**Definition 2.1.1.** As in ordinary algebraic geometry, a *super ringed space* is a topological space  $|S|$  endowed with a sheaf of supercommuting rings which we denote by  $\mathcal{O}_S$ . Let  $S$  denote the super ringed space  $(|S|, \mathcal{O}_S)$ .

**Definition 2.1.2.** A *superspace* is a super ringed space  $S$  with the property that the stalk  $\mathcal{O}_{S,x}$  is a local ring for all  $x \in |S|$ .

Given an open subset  $U \subset |S|$ , we get an induced *subsuperspace* given by restriction:  $(U, \mathcal{O}_S|_U)$ . For a closed superspace, we make the following definition:

**Definition 2.1.3.** Let  $S$  be a superspace. Then we say that  $S'$  is a *closed sub-superspace* of  $S$  if

- (i)  $|S'| \subset |S|$  is a closed subset;

(ii) The structure sheaf on  $S'$  is obtained by taking the quotient of  $\mathcal{O}_S$  by a quasi-coherent sheaf of ideals  $\mathcal{I}$  in  $\mathcal{O}_S$ :

$$\mathcal{O}_{S'}(U) = \mathcal{O}_S(U)/\mathcal{I}(U)$$

for all open subsets  $U$ .

Next we define a morphism of superspaces so that we can talk about the category of superspaces.

**Definition 2.1.4.** Let  $S$  and  $T$  be superspaces. Then a morphism  $S \longrightarrow T$  is a continuous map  $|\varphi| : |S| \longrightarrow |T|$  together with a sheaf map  $\varphi^* : \mathcal{O}_T \longrightarrow \varphi_*\mathcal{O}_S$  so that  $\varphi_x^*(\mathfrak{m}_{|\varphi|(x)}) \subset \mathfrak{m}_x$  where  $\mathfrak{m}_x$  is the maximal ideal in  $\mathcal{O}_{S,x}$  and  $\varphi_x^*$  is the stalk map. We denote the pair  $(|\varphi|, \varphi^*)$  by  $\varphi : S \longrightarrow T$ .

**Remark 2.1.5.** The sheaf map  $\varphi^* : \mathcal{O}_T \longrightarrow \varphi_*\mathcal{O}_S$  corresponds to the system of maps  $\varphi^*|_U : \mathcal{O}_T(U) \longrightarrow \mathcal{O}_S(|\varphi|^{-1}(U))$  for all open sets  $U \subset T$ . To ease notation, we also refer to the maps  $\varphi^*|_U$  as  $\varphi^*$ .

Essentially the condition  $\varphi_x^*(\mathfrak{m}_{|\varphi|(x)}) \subset \mathfrak{m}_x$  means that the sheaf homomorphism is local. Note also that  $\varphi^*$  is a morphism of supersheaves, so it preserves parity. The main point to make here is that the sheaf map must be specified along with the continuous topological map since sections are not necessarily genuine functions on the topological space as in ordinary differential geometry. An arbitrary section cannot be viewed as a function because supercommutative rings have many nilpotent elements, and nilpotent sections are identically zero as functions on the underlying topological space. Therefore we employ the methods of algebraic geometry to study such objects. We will address this in more detail later. Now we introduce two types of superspaces that we examine in detail in the forthcoming chapters: supermanifolds and superschemes.

## 2.2 Supermanifolds

A supermanifold is a specific type of “smooth” superspace which we describe via a local model. Because we always keep an eye on the physics literature from which supersymmetry springs, the supermanifolds of interest to us are the  $C^\infty$ -supermanifolds over  $\mathbf{R}$ .

Let  $C_U^\infty$  be the sheaf of  $C^\infty$ -functions on the domain  $U \subset \mathbf{R}^p$ . We define the *superdomain*  $U^{p|q}$  to be the super ringed space  $(U, C_U^\infty[\theta^1, \dots, \theta^q])$  where  $C_U^\infty[\theta^1, \dots, \theta^q]$  is the sheaf of supercommutative  $\mathbf{R}$ -algebras given by (for  $V \subset U$  open)

$$V \mapsto C_U^\infty(V)[\theta^1, \dots, \theta^q].$$

The  $\theta^j$  are odd (anti-commuting) global sections which we restrict to  $V$ . Most immediately, the superspaces  $\mathbf{R}^{p|q}$  are superdomains with sheaf  $C_{\mathbf{R}^p}^\infty[\theta^1, \dots, \theta^q]$ .

**Definition 2.2.1.** A *supermanifold* of dimension  $p|q$  is a superspace which is locally isomorphic to  $\mathbf{R}^{p|q}$ . Given any point  $x \in |M|$ , there exists a neighborhood  $V \subset |M|$  of  $x$  with  $p$  even functions  $(t^i)$  and  $q$  odd functions  $\theta^j$  on  $V$  so that

$$\mathcal{O}_M|_V = \underbrace{C^\infty(t^1, \dots, t^p)}_{C_M^\infty(V)}[\theta^1, \dots, \theta^q]. \quad (2.1)$$

Morphisms of supermanifolds are morphisms of the underlying superspaces. For  $M, N$  supermanifolds, a morphism  $\varphi : M \longrightarrow N$  is a continuous map  $|\varphi| : |M| \longrightarrow |N|$  together with a (local) morphism of sheaves of superalgebras  $\varphi^* : \mathcal{O}_N \longrightarrow \varphi_* \mathcal{O}_M$ . Note that in the purely even case of ordinary  $C^\infty$ -manifolds, the above notion of a morphism agrees with the ordinary one. We may now talk about the category of supermanifolds. The difficulty in dealing with  $C^\infty$ -supermanifolds



arises when one tries to think of “points” or “functions” in the traditional sense. The ordinary points only account for the topological space and the underlying sheaf of ordinary  $C^\infty$ -functions, and one may truly only talk about the “value” of a section  $f \in \mathcal{O}_M(U)$  for  $U \subset |M|$  an open subset; the value of  $f$  at  $x \in U$  is the unique real number  $c$  so that  $f - c$  is not invertible in any neighborhood of  $x$ . What this says is that we cannot reconstruct a section by knowing only its values at topological points. Such sections are then not truly functions in the ordinary sense, however, now that we have clarified this point, we may adhere to the established notation and call such sections *“functions on  $U$ ”*.

**Remark 2.2.2.** Let  $M$  be a supermanifold,  $U$  an open subset in  $|M|$ , and  $f$  a function on  $U$ . If  $\mathcal{O}_M(U) = C^\infty(t^1, \dots, t^p)[\theta^1, \dots, \theta^q]$  as in (2.1), there exist even functions  $f_I \in C^\infty(t)$  ( $t = t^1, \dots, t^p$ ) so that

$$f(t, \theta) = f_0(t) + \sum_i f_i(t) \theta^i + \sum_{i < j} f_{ij}(t) \theta^i \theta^j + \dots = f_0(t) + \sum_{|I|=1}^q f_I(t) \theta^I \quad (2.2)$$

where  $I = \{i_1 < i_2 < \dots < i_r\}_{r=1}^q$ .

Let us establish the following notation. Let  $M$  be a supermanifold, then we write the nilpotent sections as

$$J_M = \mathcal{O}_{M,1} + \mathcal{O}_{M,1}^2 = \langle \mathcal{O}_{M,1} \rangle_{\mathcal{O}_M}. \quad (2.3)$$

This is an ideal sheaf in  $\mathcal{O}_M$  and thus defines a natural subspace of  $M$  we shall call  $M_{\text{red}}$ , or  $\widetilde{M}$ , where

$$\widetilde{M} = (|M|, \mathcal{O}_M/J_M). \quad (2.4)$$

Note that  $\widetilde{M}$  is a completely even superspace, and hence lies in the ordinary category of ordinary  $C^\infty$ -manifolds, i.e. it is locally isomorphic to  $\mathbf{R}^p$ . The quotient map from  $\mathcal{O}_M \longrightarrow \mathcal{O}_M/J_M$  defines the inclusion morphism  $\widetilde{M} \hookrightarrow M$ . The subspace  $\widetilde{M}$  should not be confused with the purely even superspace  $(|M|, \mathcal{O}_{M,0})$  which is *not* a  $C^\infty$ -manifold since the structure sheaf still contains nilpotents.

**Observation 2.2.3.** Here we examine closed submanifolds in the super category. Let  $M$  be a supermanifold. Then a submanifold of  $M$  is a supermanifold  $N$  together with an immersion, that is a map  $i : N \longrightarrow M$  so that  $i$  induces an imbedding of  $\widetilde{N}$  onto a closed (locally closed) ordinary submanifold of  $\widetilde{M}$  and  $i_U^* : \mathcal{O}_M(U) \longrightarrow \mathcal{O}_N(i^{-1}(U))$  is surjective for all open  $U \subset |M|$ .

Locally, this means that we can find a system of coordinates  $(t^1, \dots, t^p, \theta^1, \dots, \theta^q)$  in any open neighborhood of  $M$  so that  $N$  restricted to this neighborhood is described by the vanishing of some of the coordinates:

$$t^1 = \dots = t^r = \theta^1 = \dots = \theta^s = 0.$$

One can check that this explanation of submanifolds agrees with the definition of a closed sub superspace given earlier.

## 2.3 Superschemes

A *superscheme* is an object in the category of superspaces which generalizes the notion of a scheme.

**Definition 2.3.1.** A superspace  $S = (|S|, \mathcal{O}_S)$  is a superscheme if  $(|S|, \mathcal{O}_{S,0})$  is an ordinary scheme and  $\mathcal{O}_{S,1}$  is a quasi-coherent sheaf of  $\mathcal{O}_{S,0}$ -modules.

Because any non-trivial supercommutative ring has non-zero nilpotents, we need to redefine what we mean by a reduced superscheme.

**Definition 2.3.2.** We say that a superscheme  $S$  is *super reduced* if  $\mathcal{O}_S/J_S$  is reduced. In other words, in a super reduced superscheme, we want that the odd sections generate all the nilpotents.

**Example 2.3.3.** Let  $\mathbf{A}^m$  be the ordinary affine space of dimension  $m$  over  $\mathbf{C}$  given with the Zariski topology. On  $\mathbf{A}^m$  we define the following sheaf  $\mathcal{O}_{\mathbf{A}^{m|n}}$  of superalgebras. Given  $U \subset \mathbf{A}^m$  open,

$$\mathcal{O}(U) = \mathcal{O}_{\mathbf{A}^m}(U)[\xi_1, \dots, \xi_n] \quad (2.5)$$

where  $\mathcal{O}_{\mathbf{A}^m}$  is the ordinary sheaf on  $\mathbf{A}^m$  and the  $\xi_1, \dots, \xi_n$  are anti-commuting (or *odd*) variables. One may readily check that  $(\mathbf{A}^m, \mathcal{O}_{\mathbf{A}^{m|n}})$  is a reduced superscheme which we hereon denote by  $\mathbf{A}^{m|n}$ .

**Remark 2.3.4.** The superscheme  $\mathbf{A}^{m|n}$  is more than reduced; it is a smooth superscheme. The difference being the local splitting in (2.5). We do not further explore the notion of smoothness in these notes.

Morphisms of superschemes are just morphisms of superspaces, so we may talk about the subcategory of superschemes. The category of superschemes is larger than the category of schemes; any scheme is a superscheme if we take a trivial odd component in the structure sheaf. We will complete our exposition of the category of superschemes in chapters 5-6.

## 2.4 T-Points

The presence of odd coordinates steals some of the geometric intuition away from the language of supergeometry. For instance, we cannot see an “odd point” – they are invisible both topologically and as classical functions on the underlying

topological space. We see the odd points only as sections of the structure sheaf. To bring some of the intuition back, we turn to the functor of points approach from algebraic geometry.

**Definition 2.4.1.** Let  $S$  and  $T$  be superspaces. Then a  $T$ -point of  $S$  is a morphism  $T \longrightarrow S$ . We denote the set of all  $T$ -points by  $S(T)$ . Equivalently,

$$S(T) = \text{Hom}(T, S).$$

Let us recall an important lemma.

**Lemma 2.4.2.** (*Yoneda's Lemma*) *There is a bijection from the set of morphisms  $\varphi : M \longrightarrow N$  to the set of maps  $\varphi_T : M(T) \longrightarrow N(T)$ , functorial in  $T$ .*

*Proof.* Given a map  $\varphi : M \longrightarrow N$ , for any morphism  $t : T \longrightarrow M$ ,  $\varphi \circ t$  is a morphism  $T \longrightarrow N$ . Conversely, we attach to the system  $(\varphi_T)$  the image of the identity map from  $\varphi_M : M(M) \longrightarrow N(M)$ .

■

Yoneda's lemma allows us to replace a superspace  $S$  with its set of  $T$ -points,  $S(T)$ . We can now think of a superspace  $S$  as a representable functor from the category of superspaces to the category of sets. In fact, when constructing a superspace, it is often most convenient to construct the functor of points, then prove that the functor is *representable*. Let us give a couple examples of  $T$ -points from the category of supermanifolds.

**Example 2.4.3.** (i) If  $T$  is just an ordinary topological point (i.e.  $T = (\mathbf{R}^{0|0}, \mathbf{R})$ ), then a  $T$ -point of  $M$  is an ordinary topological point of  $|M|$ .

(ii) If  $M = \mathbf{R}^{p|q}$ , then a  $T$ -point of  $M$  is a system of  $p$  even and  $q$  odd functions on  $T$  by definition of a superspace morphism. This is made more clear in chapter

5 by Proposition 3.1.2. Thus  $\mathbf{R}^{p|q}(T) = \mathcal{O}_{T,0}^p \oplus \mathcal{O}_{T,1}^q = (\mathcal{O}_T^{p|q})_0$ .

We already see the power of  $T$ -points in these two examples. The first example ( $T = \mathbf{R}^{0|0}$ ) gives us complete topological information while the second ( $M = \mathbf{R}^{p|q}$ ) will allow us to talk about coordinates on supermanifolds. We fully explore these topics in the next chapter.

In chapter 5 we give a criterion for the representability of functors from the category of superschemes to the category of sets. As in the ordinary case, it turns out that representable functors must be *local*, i.e. they should admit a cover by open affine subfunctors which glue together in some sense.

## CHAPTER 3

### $C^\infty$ -supermanifolds

We have already described a supermanifold in chapter 2 as a superspace which is locally isomorphic to  $\mathbf{R}^{p|q}$ . Recall also that given a supermanifold  $M$ , there is a surjection  $\mathcal{O}_M \longrightarrow \mathcal{O}_M/J_M$  which corresponds to the natural inclusion  $\widetilde{M} \hookrightarrow M$ . For local functions  $f$  on  $\mathcal{O}_M$ , this means  $f \mapsto \widetilde{f} = f_0$  from the decomposition (2.2).

#### 3.1 Charts

Let us begin studying supermanifold morphisms in detail through the following example.

**Example 3.1.1.** Consider the supermanifold  $\mathbf{R}^{1|2}$  with a morphism  $\varphi : \mathbf{R}^{1|2} \longrightarrow \mathbf{R}^{1|2}$ . On  $\mathbf{R}^{1|1}$  we have global coordinates  $t, \theta^1, \theta^2$  and so we may express any function  $f$  as in (2.2):

$$f = f(t, \theta^1, \theta^2) = f_0(t) + f_1(t)\theta^1 + f_2(t)\theta^2 + f_{12}(t)\theta^1\theta^2.$$

Then  $\widetilde{f} = f_0(t) \in C^\infty(\mathbf{R})$  which sits as a function on the reduced  $C^\infty$ -manifold  $\widetilde{\mathbf{R}^{1|2}} = \mathbf{R}$ . The morphism  $\varphi$  is described by a continuous map  $|\varphi|$  and a sheaf map  $\varphi^*$ .

We first prescribe the global coordinates under  $\varphi^*$ :

$$\begin{aligned} t &\mapsto t^* = t + \theta^1 \theta^2 \\ \theta^1 &\mapsto \theta^{1*} = \theta^1 \\ \theta^2 &\mapsto \theta^{2*} = \theta^2. \end{aligned} \tag{3.1}$$

We claim that knowing  $\varphi^*$  on only these global coordinates is enough to completely describe  $\varphi$ . Indeed, we first see that  $t \mapsto t + \theta^1 \theta^2$  tells us that  $|\varphi|$  is just the identity map. Next, let  $f \in C^\infty(t)[\theta^1, \theta^2]$  be as above. Then  $f \mapsto f^*$ ;

$$f^* = f(t^*, \theta^{1*}, \theta^{2*}) = f_0(t + \theta^1 \theta^2) + f_1(t + \theta^1 \theta^2) \theta^1 + f_2(t + \theta^1 \theta^2) \theta^2 + f_{12}(t + \theta^1 \theta^2) \theta^1 \theta^2. \tag{3.2}$$

And so we must only make sense of  $f_I(t + \theta^1 \theta^2)$ . The key is that we take a *Taylor series expansion*; the series of course terminates thanks to the nilpotence of the the odd coordinates:

$$f_I(t + \theta^1 \theta^2) = f_I(t) + \theta^1 \theta^2 f'_I(t). \tag{3.3}$$

It is easy to check that this in fact gives a homomorphism of superalgebras. For  $g, h \in C^\infty(\mathbf{R})$ ,  $(gh)^* = gh + \theta^1 \theta^2 (gh)' = g^* h^*$ . The global sections are enough since the full sheaf map is given by restriction. So in this example, it is enough to know  $\varphi$  on only the coordinates. In fact, the morphism induced by the equations (3.1) is unique via the above construction.

That a morphism is determined by local coordinates is true in general; we summarize this fact in the following *Chart Theorem*.

**Theorem 3.1.2.** (*Chart*) *Let  $U \subset \mathbf{R}^{p|q}$  be an open submanifold of  $\mathbf{R}^{p|q}$  ( $U = U^{p|q}$  is a superdomain). There is a bijection between*

- (i) *the set of morphisms  $\varphi : M \longrightarrow U$  and*
- (ii) *the set of systems of  $p$  even functions  $t^{i*}$  and  $q$  odd functions  $\theta^{j*}$  on  $M$  so that  $\tilde{t}^{i*}(m) \in |U|$  for all  $m \in |M|$ .*

*Proof.* We sketch the proof of this well-known result here (for more details, see for example [17]). The key point is that given a system of  $p$  even functions  $t^{i*}$  and  $q$  odd functions  $\theta^{j*}$  on  $M$ , we can define a sheaf map. As in the example (3.1.1) it is enough to define the sheaf map for  $f \in C^\infty(U)$  since the expansion of an arbitrary section is linear in the odd coordinates over  $C^\infty(U)$  and since we can restrict to an open  $V \subset U$ .

We define  $\varphi$  formally by

$$f = f(t^1, \dots, t^p) \mapsto f^* = f(t^{1*}, \dots, t^{p*}).$$

We can write  $t^{i*} = \widetilde{t}^{i*} + n^i$  where the  $n^i$  are nilpotent, and we are set up to take a Taylor series expansion, just as in the above example:

$$f(\widetilde{t}^{1*} + n^1, \dots, \widetilde{t}^{p*} + n^p) := \sum_k \frac{\partial^k}{\partial t^k} f(\widetilde{t}^{1*}, \dots, \widetilde{t}^{p*}) \frac{n^k}{k!} \quad (3.4)$$

where  $n^k$  is  $k$ -tuples of  $\{n^i\}$ . This series terminates again thanks to the nilpotent  $n_i$ . The  $C^\infty$ -functions  $\widetilde{t}^{i*}$  completely determine the topological map  $|\varphi|$ .  $\blacksquare$

**Remark 3.1.3.** Note that because the expansion (3.4) involves an arbitrary number of derivatives, there is no way to make sense of  $C^k$ -supermanifolds. We may, however, talk about the category of analytic (over  $\mathbf{R}$  or  $\mathbf{C}$ ) supermanifolds. We refer to theorem (3.1.2) as the *Chart Theorem* because it equates the definition of a supermanifold to giving an atlas of local charts. These local charts glue together isomorphic copies of open subsets of  $\mathbf{R}^{p|q}$ .

As in the category of superschemes, products exist in the category. Let  $M$  be a dimension  $p|q$  supermanifold and  $N$  be a dimension  $r|s$  supermanifold, then we describe  $M \times N$  by

$$M \times N = (|M| \times |N|, \mathcal{O}_{M \times N}).$$



We define the sheaf  $\mathcal{O}_{M \times N}$  as follows. For coordinate neighborhoods  $U(= (x, \theta)) \subset |M|, V(= (t, \eta)) \subset |N|$ ,  $\mathcal{O}_{M \times N}(U \times V) = C^\infty(x, t)[\theta, \eta]$ . One must show that gluing conditions are satisfied, but this calculation mimics that in the ordinary category and is left for the reader. So  $M \times N$  is a  $(p+r)|(q+s)$ -dimensional supermanifold with  $\widetilde{M \times N} = \widetilde{M} \times \widetilde{N}$ . As in the ordinary category,  $\mathcal{O}_{M \times N} \neq \mathcal{O}_M \times \mathcal{O}_N$ ; instead we must take the completion of the tensor product to get an equality.

**Remark 3.1.4.** We cannot think of a supermanifold simply as a fiber space over an ordinary manifold. Morphisms between supermanifolds mix both even and odd coordinates and thus for an open neighborhood  $U$  of a supermanifold  $M$ ,  $C^\infty$  cannot be realized as a subsheaf of  $\mathcal{O}_M$ ; it follows that there is no natural morphism  $M \longrightarrow \widetilde{M}$ . The symmetries of interest in these extensions of classical manifolds are those which place even and odd on the same level. Such symmetries are called *supersymmetries* and are at the foundation of the physical supersymmetry theory which aims to treat bosons and fermions on the same footing.

## 3.2 Vector Fields

Many concepts and results from ordinary differential geometry extend naturally to the category of supermanifolds. If we keep the categorical language we have developed, there is hardly any difference in fundamental differential geometry between the ordinary and the super categories. For example, a vector bundle on a supermanifold  $M$  is a locally free sheaf of (super)modules over  $\mathcal{O}_M$ . This leads to the notion of a tangent bundle on  $M$ , where we find super extensions of the inverse and implicit function theorem (see [17]), and the local and global Frobenius theorem which we will prove in the next chapter.

**Definition 3.2.1.** A *vector field*  $V$  on a supermanifold  $M$  is an  $\mathbf{R}$ -linear deriva-

tion of  $\mathcal{O}_M$ , i.e. it is a family of derivations  $\mathcal{O}_M(U) \longrightarrow \mathcal{O}_M(U)$  that is compatible with restrictions.

The vector fields form a sheaf of modules over  $\mathcal{O}_M$ , the *tangent sheaf* which we denote by  $\text{Vec}_M$ . The sheaf  $\text{Vec}_M$  is actually locally free over  $\mathcal{O}_M$  which we establish with the following lemma. The lemma also helps us understand the local structure of a vector field.

**Lemma 3.2.2.** *Let  $(t, \theta)$  be coordinates on some open subsupermanifold  $U \subset \mathbf{R}^{p|q}$ . Then the  $\mathcal{O}_U$ -module of  $\mathbf{R}$ -linear derivations of  $\mathcal{O}_U$  is a rank  $p|q$  free sheaf over  $\mathcal{O}_U$  with basis  $\{\partial/\partial t^i, \partial/\partial \theta^j\}$  where*

$$\frac{\partial}{\partial t^i}(f_I(t)\theta^I) = \frac{\partial f_I(t)}{\partial t^i}\theta^I, \quad \frac{\partial}{\partial \theta^j}(f_I(t)\theta^j\theta^I) = f_I(t)\theta^I \quad (3.5)$$

where  $j \notin I$ .

*Proof.* The proof is the same as in the classical case since the  $\theta$ -variables are polynomial (in fact, they are linear). ■

Since  $U \subset \mathbf{R}^{p|q}$  is the local model for any dimension  $p|q$  supermanifold  $M$ , the lemma implies that  $\text{Vec}_M$  is a vector bundle of rank  $p|q$ . If  $V$  is a vector field on  $U$ , then in a coordinate neighborhood  $U' \subset U$  with coordinates  $(t, \theta)$ , there exist functions  $f_i, g_j$  on  $U'$ , so that  $V$  has the unique expression

$$V|_{U'} = \sum_{i=1}^p f_i(t, \theta) \frac{\partial}{\partial t^i} + \sum_{j=1}^q g_j(t, \theta) \frac{\partial}{\partial \theta^j}. \quad (3.6)$$

We similarly define the tangent space at a point  $m$  of  $M$ , which we denote  $T_m(M)$ . We think of tangent vectors as  $\mathbf{R}$ -linear derivations  $\mathcal{O}_m \longrightarrow \mathbf{R}$  of the stalk at  $m$ ; we may think of a tangent vector  $v \in T_m(M)$  as a vector field on  $U$ , a neighborhood of  $m$ , composed with evaluation at  $m$ . If the open subset  $U$  from

definition 3.2.1 is a coordinate neighborhood around  $m$ , the vector  $v$  takes the expression

$$v = \sum a_i \frac{\partial}{\partial x^i} |_m + b_j \frac{\partial}{\partial \theta^j} |_m \quad (3.7)$$

for  $a_i, b_j \in \mathbf{R}$ .

For  $M$  and  $N$  supermanifolds, we can extend a vector on  $M$  to a  $\mathcal{O}_N$ -linear derivation on  $M \times N$ , and likewise we may trivially treat any vector field on  $M$  as a vector field on  $M \times N$ . We will call these extensions *extended vectors* and *extended vector fields* respectively.

**Definition 3.2.3.** Let  $v$  be a tangent vector of  $M$  at  $m$ ,  $U_m \subset |M|$  an open neighborhood of  $m$ . We view  $v$  as a derivation  $\mathcal{O}_M(U_m) \rightarrow \mathbf{R}$  and identify  $\mathcal{O}_N$  with  $\mathcal{O}_{\mathbf{R} \times N}$ . Then  $v$  extends uniquely to a  $\mathcal{O}_N$ -linear derivation:

$$\begin{array}{ccc} v_N : \mathcal{O}_{M \times N}(U_m \times V) & \longrightarrow & \mathcal{O}_N(V) \\ & \searrow & \nearrow \\ & \mathcal{O}_{\mathbf{R} \times N}(\mathbf{R} \times V) & \end{array}$$

for any open  $V \subset |N|$  (this is easily seen locally by using coordinates, and then by patching using local uniqueness) so that

$$v_N(a \otimes b) = v(a)b \quad (3.8)$$

where  $a$  and  $b$  are local functions of  $M$  and  $N$  respectively.

One may similarly “extend” vector fields: let  $V$  be a vector field on  $M$ . Then we extend  $V$  to a derivation  $(V \otimes id)$  on  $M \times N$  by forcing  $V$  to act trivially on  $N$ . If  $(t, \theta)$  and  $(x, \xi)$  are local coordinates on  $M$  and  $N$  respectively,  $V$  has the coordinate expression as in (3.6). Then the extension  $(V \otimes id)$  has the same coordinate expression on  $M \times N$  described by coordinates  $(t, x, \theta, \xi)$ , i.e. it is identically zero on  $(x, \xi)$ . Again the extension is unique by patching using local uniqueness.

### 3.3 Differential Calculus

In this section we discuss the notion of differential of a morphism of supermanifolds. In this context the theory of supermanifolds resembles very closely the classical theory. For completeness, we give a summary of the well known results, sketching only briefly the proofs or leaving them as exercises.

**Definition 3.3.1.** Let  $\alpha : M \longrightarrow N$  be a morphism of supermanifolds. We define *differential* of  $\alpha$  at a topological point  $m \in |M|$  the map  $(d\alpha)_m : T_m M \longrightarrow T_{|\alpha|(m)} N$  given by:

$$(d\alpha)_m(X)(f) = X(\alpha_m^*(f)), \quad \alpha_m^* : \mathcal{O}_{N,|\alpha|(m)} \longrightarrow \mathcal{O}_{M,m},$$

where  $X \in T_{|\alpha|(m)} N \cong \text{Der}(\mathcal{O}_{M,m}, \mathbf{R})$ , as we have seen in the previous section.

In local coordinates one can readily check that  $(d\alpha)_m$  has the usual jacobian expression. In fact, let's choose suitable open submanifolds  $U \subset M$  and  $V \subset N$  such that  $m \in |U|$  and  $|\alpha|(m) \in |V|$ , homeomorphic respectively to open submanifolds in  $\mathbf{R}^{r|s}$  and  $\mathbf{R}^{u|v}$ , and let  $(t^i, \theta^j)$  be local coordinates in  $U$ . We have that:

$$\alpha(t^i, \theta^j) = (f^k, \phi^l) \in V.$$

Then

$$(d\alpha)_m = \begin{pmatrix} \frac{\partial f^k}{\partial t^i} & \frac{\partial \phi^l}{\partial t^i} \\ \frac{\partial f^k}{\partial \theta^j} & \frac{\partial \phi^l}{\partial \theta^j} \end{pmatrix}_m,$$

where the subscript  $m$  means evaluation at  $m$ .

As an example let's compute the differential of the morphism described in example 3.1.1.

**Example 3.3.2.** Let  $\alpha : \mathbf{R}^{1|2} \longrightarrow \mathbf{R}^{1|2}$  be the morphism given locally (and globally) by:

$$\alpha(t^1, \theta^1, \theta^2) = (t^1 + \theta^1 \theta^2, \theta^1, \theta^2)$$

Then the differential at a generic topological point  $m = (t_0, 0, 0)$  is:

$$(d\alpha)_m = \begin{pmatrix} 1 & 0 & 0 \\ \theta^2 & 1 & 0 \\ -\theta^1 & 0 & 1 \end{pmatrix}_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As in the ordinary theory there are classes of morphisms that play a key role: immersions, submersions and diffeomorphisms.

**Definition 3.3.3.** Let  $\alpha : M \longrightarrow N$  be a supermanifold morphism and  $\tilde{\alpha} : \tilde{M} \longrightarrow \tilde{N}$  the underlying classical morphism on the reduced spaces.  $\alpha$  is an *immersion* at  $m \in |M|$  if  $\tilde{\alpha}$  is an ordinary immersion at  $m$  and  $(d\alpha)_m$  is injective. Likewise  $\alpha$  is a *submersion* at  $m$  if  $\tilde{\alpha}$  is a submersion at  $m$  and  $(d\alpha)_m$  is surjective. Finally  $\alpha$  is a *diffeomorphism* at  $m$  if it is a submersion and an immersion. When we say  $\alpha$  is a immersion (resp. submersion or diffeomorphism) we mean  $\alpha$  is such at all points of  $|M|$ .

As in the classical setting submersions and immersions have the usual local models.

**Proposition 3.3.4.** Let  $\alpha : M \longrightarrow N$  be a supermanifold morphism. Let  $m \in |M|$  and let  $U \subset M$  and  $V \subset N$  two suitable open sets homeomorphic respectively to open sets in  $\mathbf{R}^{r|s}$  and  $\mathbf{R}^{u|v}$ ,  $m \in |U|$ ,  $\alpha(m) \in |V|$ .

1. If  $\alpha$  is an immersion at  $m$ , after suitable changes of coordinate in  $U$  and  $V$ , locally we have that

$$\alpha(t^i, \theta^k) = (t^i, 0, \theta^k, 0)$$

2. If  $\alpha$  is an submersion at  $m$ , after suitable changes of coordinate in  $U$  and  $V$ , locally we have that

$$\alpha(t^i, s^j, \theta^k, \sigma^l) = (t^i, \theta^k)$$

The proof of this result is essentially the same as in the ordinary setting and can be found for example in [23] chapter 4.

**Remark 3.3.5.** A *closed* sub supermanifold  $N$  of  $M$  can be equivalently defined as a supermanifold such that  $\widetilde{N}$  is a closed supermanifold of  $\widetilde{M}$  and  $N \subset M$  is an immersion. We leave as an exercise to the reader the check that this definition is equivalent to the one given in observation 2.2.3 in chapter 2.

The Submersion Theorem in the supercontext plays an important role in proving that certain closed subset of a supermanifold admit a supermanifold structure.

**Theorem 3.3.6.** *Submersion Theorem.* Let  $f : M \longrightarrow N$  be a submersion at  $n \in |N|$ , and let  $|P| = |g|^{-1}(n)$ . Then  $|P|$  admits a supermanifold structure, i. e. there exists a supermanifold  $P = (|P|, \mathcal{O}_P)$  where  $\mathcal{O}_P = \mathcal{O}_M|_{|P|}$ . Moreover:

$$\dim P = \dim M - \dim N$$

*Proof.* (Sketch). Locally we can define for  $p \in |P|$ ,  $\mathcal{O}_{P,p} = \mathcal{O}_{M,p}/f^*(I_n)$ , where  $I_n$  is the ideal in  $\mathcal{O}_{N,n}$  of elements vanishing at  $n$ . For  $W \in |P|$  open, sections in  $\mathcal{O}_P(W)$  are defined as maps

$$\begin{aligned} u : W &\longrightarrow \coprod_{p \in W} \mathcal{O}_{P,p} \\ q &\longmapsto u(q) \in \mathcal{O}_{P,q} \end{aligned}$$

This gives  $P$  the structure of a superspace. By the previous proposition we know that locally  $f(t, s, \theta, \sigma) = (t, \theta)$ . Hence locally we have coordinates  $(s, \sigma)$  and  $P$  is a supermanifold of the prescribed dimension. ■

**Example 3.3.7.** Let  $X$  be the open submanifold of  $\mathbf{R}^{2|2}$  whose topological space consists of the points of plane  $\mathbf{R}^2$  with the coordinate axis,  $x = 0, y = 0$  removed. Let  $f : X \longrightarrow \mathbf{R}^{1|0}$  be the morphism  $\alpha(x, y, \xi, \eta) = y^{-1}(x - \xi y^{-1} \eta)$ . One can check that at the topological point  $1 \in \mathbf{R}$ ,  $f$  is a submersion, hence  $|P| = |f|^{-1}(1)$  admits a sub supermanifold structure. If we identify  $\mathbf{R}^{2|2}$  with the  $1|1 \times 1|1$  supermatrices  $P$  corresponds to the supermatrices with Berezinian equal to 1 and we denote  $P$  with  $SL(1|1)$ , the *special linear supergroup*.

We next turn our attention to a special type of supermanifold, a *super Lie group*.

### 3.4 Super Lie Groups

A Lie group is a group object in the category of manifolds. Likewise a super Lie group is a group object in the category of supermanifolds. This means that there are appropriate morphisms which correspond to the group operations: product  $\mu : G \times G \longrightarrow G$ , unit  $e : 1 \longrightarrow G$  (where  $1 \in |G|$  may be equated to  $\mathbf{R}^{0|0}$ , a single topological point), and inverse  $i : G \longrightarrow G$  so that the necessary diagrams commute (these are, in fact, the same diagrams as in the ordinary setting). We may of course interpret all these maps and diagrams in the language of  $T$ -points, which gives us (for any supermanifold  $T$ ) morphisms  $\mu_T : G(T) \times G(T) \longrightarrow G(T)$ , etc. that obey the same commutative diagrams. In other words, Yoneda's Lemma says that the set  $G(T)$  is in fact a group for all  $T$ . This leads us to our working definition of a super Lie group.

**Definition 3.4.1.** A supermanifold  $G$  is a *super Lie group* if for any supermanifold  $T$ ,  $G(T)$  is a group, and for any supermanifold  $S$  and morphism  $T \longrightarrow S$ , the corresponding map  $G(S) \longrightarrow G(T)$  is a group homomorphism.

In other words,  $T \mapsto G(T)$  is a functor into the category of groups.

**Example 3.4.2.** Let us consider the super Lie group  $\mathbf{R}^{1|1}$  through the symbolic language of  $T$ -points. The product morphism  $\mu : \mathbf{R}^{1|1} \times \mathbf{R}^{1|1} \longrightarrow \mathbf{R}^{1|1}$  is given by

$$(t, \theta) \cdot (t', \theta') = (t + t' + \theta\theta', \theta + \theta') \quad (3.9)$$

where the coordinates  $(t, \theta)$  and  $(t', \theta')$  represent two distinct  $T$ -points for some supermanifold  $T$ . It is then clear by the formula (3.9) that the group axioms inverse, identity, and associativity are satisfied.

Also in the language of  $T$ -points, the definition given above is equivalent to saying that a super Lie group is a functor from the category of supermanifolds to the category of groups which is representable. In this vein, let us further examine the  $\mathrm{GL}_{p|q}$  example.

**Example 3.4.3.** Let's first construct the supermanifold  $\mathrm{GL}_{p|q}$ . The reduced space  $\widetilde{\mathrm{GL}}_{p|q} = \mathrm{GL}_p \times \mathrm{GL}_q$  is an open subset of  $\mathbf{R}^{p^2} \times \mathbf{R}^{q^2}$ . We build the sheaf on  $\mathrm{GL}_{p|q}$  from the smooth functions on  $\mathrm{GL}_p \times \mathrm{GL}_q$  and the restriction of the global odd coordinates  $\theta^1, \dots, \theta^{2pq}$  on  $\mathbf{R}^{p^2+q^2|2pq}$ , i.e. for an open  $U \subset \mathrm{GL}_p \times \mathrm{GL}_q$ ,

$$\mathcal{O}_{\mathrm{GL}_{p|q}}(U) = (\mathrm{GL}_p \times \mathrm{GL}_q)(U) \otimes [\theta^1, \dots, \theta^{2pq}]|_U.$$

Now we can examine the  $T$ -points of  $\mathrm{GL}_{p|q}$ . Let  $T$  be a supermanifold,  $t \in \mathrm{GL}_{p|q}(T)$  a  $T$ -point, then  $t : T \longrightarrow \mathrm{GL}_{p|q}$  is a morphism. By proposition 3.1.2,  $t$  is equivalent to giving the pullbacks of coordinates. By taking into account the determinant identities which must be satisfied, we see that  $t$  is then equivalent to an invertible matrix with coefficients in  $\mathcal{O}_T$ , and so  $t$  corresponds to an automorphism of  $\mathcal{O}_T^{p|q}$ . Thus  $\mathrm{GL}_{p|q}(T)$  is the group of automorphisms of  $\mathcal{O}_T^{p|q}$ .

**Example 3.4.4.** Let us consider another example of a super Lie group,  $\mathrm{SL}_{p|q}$ . We define  $\mathrm{SL}_{p|q}$  in a way which mimics the classical construction. For each



supermanifold  $T$ , the Berezinian gives a morphism from the  $T$ -points of  $\mathrm{GL}_{p|q}$  to the  $T$ -points of  $\mathrm{GL}_{1|0}$ :

$$\mathrm{Ber}_T : \mathrm{GL}_{p|q}(T) \longrightarrow \mathrm{GL}_{1|0}(T).$$

The super special linear group  $\mathrm{SL}_{p|q}$  is the kernel of  $\mathrm{Ber}_T$ .

Using a similar argument as in example 3.3.7 one can show that the functor  $\mathrm{SL}_{p|q}$  is the functor of points of a super Lie group, closed sub supermanifold of  $\mathrm{GL}_{p|q}$ . In fact  $|\mathrm{SL}_{p|q}| = |\mathrm{Ber}|^{-1}(1)$ , where  $\mathrm{Ber}$  is the map  $\mathrm{Ber} : \mathrm{GL}_{p|q} \longrightarrow \mathrm{GL}_{1|0}$  between supermanifold corresponding by Yoneda's lemma to the family of maps  $\mathrm{Ber}_T$  given above.

**Example 3.4.5.** In our last example we extend the classical orthogonal group to the super category. Let  $\Phi$  be an even nondegenerate bilinear form on  $\mathbf{R}^{p|2q}$  with values in  $\mathbf{R}^{1|0}$ . The form  $\Phi$  is equivalent to giving nondegenerate symmetric bilinear form on  $\mathbf{R}^p$  and a nondegenerate alternating form on  $\mathbf{R}^{2q}$ . Then for any supermanifold  $T$ , define  $\mathrm{OSp}_{p|2q}(T)$  as the subgroup of  $\mathrm{GL}_{p|2q}(T)$  which preserves  $\Phi$ .

**Remark 3.4.6.** A word of caution. In the above two examples, we only give  $\mathrm{SL}_{p|q}$  and  $\mathrm{OSp}_{p|2q}$  in terms of their  $T$ -points. It is clear that each is a functor from supermanifolds to groups. However, it is not clear without a further argument, that the functors defined above are *representable*.

### 3.5 Left Invariant Vector Fields

In the remainder of this chapter, we discuss left invariant vector fields on a super Lie group, then examine the infinitesimal interpretation of a super Lie group acting on a supermanifold, which will be most relevant when we examine the super Lie group/algebra, super Lie subgroup/subalgebra pairing.

Let  $G$  be a super Lie group with group law  $\mu : G \times G \rightarrow G$ . Via  $T$ -points, we can symbolically understand this group law as  $(x, \xi) \cdot (x', \xi') = (t, \theta)$  where  $t = t(x, x', \xi, \xi')$  are even functions and  $\theta = \theta(x, x', \xi, \xi')$  are odd functions. Again, all this really says is that  $\mu^*(t^i) = t^{i*} = t_i(x, x', \xi, \xi')$  for some even section  $t_i$  of  $\mathcal{O}_{G \times G}$  and  $\mu^*(\theta^j) = \theta^{j*} = \theta_j(x, x', \xi, \xi')$  for some odd section.

Recall classically that for an ordinary Lie group  $H$ , we could define a map  $\ell_h$ , “left multiplication by  $h$ ” ( $h \in H$ ):

$$H \xrightarrow{\ell_h} H; \quad a \mapsto ha \quad (3.10)$$

(for  $a \in H$ ). The differential of this map gives

$$T_e(H) \xrightarrow{d\ell_h} T_h(H) \quad (3.11)$$

and for a vector field  $X$  on  $H$ , we say that  $X$  is *left invariant* if

$$d\ell_h \cdot X = X \cdot \ell_h. \quad (3.12)$$

We interpret this in the super category by saying that a left invariant vector field on  $G$  is invariant with respect to the group law  $\mu^*$  “on the left”. What this amounts to in making a formal definition is that we replace the ordinary group law  $\mu$  with the anti-group law  $\iota$  given (via  $T$ -points) by:

$$\iota(g, g') = \mu(g', g) = g' \cdot g.$$

Since  $V$  is a vector field,  $V|_U : \mathcal{O}_G(U) \rightarrow \mathcal{O}_G(U)$  is a derivation for all open  $U \subset |G|$ , the expression  $\iota^* \circ V$  makes sense. Now in the spirit of (3.12) we need to understand “ $V \circ \iota^*$ .” We trivially extend the derivation  $V$  to  $\mathcal{O}_{G \times G}$ , and the expression  $(V \otimes id) \circ \iota^*$  is formal. We can now make a definition.

**Definition 3.5.1.** If  $V$  is a vector field on the super Lie group  $G$ , we say that  $V$  is left invariant if  $(V \otimes id)\iota^* = \iota^*V$ .

As in the classical theory, we have the following theorem.

**Theorem 3.5.2.** *There is a bijection between*

- i. left-invariant vector fields on  $G$  and*
- ii.  $T_e(G)$ .*

Before we prove theorem (3.5.2), let us first establish some useful notation and a technical lemma. Recall that for a supermanifold  $X$  and any  $v \in T_x X$ , we have the extended  $\mathcal{O}_T$ -linear derivation  $v_T$  for any supermanifold  $T$ . Moreover, let  $\varphi : X \rightarrow Y$  be a morphism of supermanifolds. Then  $\varphi$  induces the morphism  $\varphi \times id_T : X \times T \rightarrow Y \times T$  and we denote the pullback by

$$(\varphi \times id_T)^* = \varphi^* \otimes id_T. \quad (3.13)$$

Similarly we can define  $id_T \otimes \varphi^*$ .

*Proof.* (Theorem 3.5.2)

Since  $G$  is a super Lie group, there is a map  $id : \{e\} \rightarrow G$  which gives  $\epsilon : \mathcal{O}_G \rightarrow k$ , “evaluation at  $e$ ”. If  $V$  is a left invariant vector field on  $G$ , then  $\epsilon V = v$  is a tangent vector at the origin of  $G$ . We claim that this  $v$  in fact determines  $V$ :

$$V = (v_G)\iota^*.$$

Let us first show that given any tangent vector  $v$ ,  $(v_G)\iota^*$  is a left invariant vector field on  $G$ .

Heuristically we are doing the same thing as in the classical setting; we are infinitesimally pushing the vector  $v$  with the group law. It is clear that  $(v_G)\iota^*$  is locally a derivation on  $\mathcal{O}_G$ ; we next show it is left invariant, i.e. we must show that

$$((v_G)\iota^* \otimes id_G)\iota^* = \iota^*(v_G)\iota^*. \quad (3.14)$$

A direct check on local coordinates (one can always choose coordinates of the form  $a \otimes b$  on  $G \times G$ ) shows that

$$((v_G)\iota^* \otimes id_G) = v_{G \times G}(\iota^* \otimes id_G).$$

But  $v_{G \times G}(\iota^* \otimes id_G) = v_{G \times G}(id_G \otimes \iota^*)$  by the coassociativity of  $G$ , and another direct check shows that

$$v_{G \times G}(id_G \otimes \iota^*) = \iota^*(v_G)$$

Hence the claimed equality (3.14).

The only item left to show is that  $V = (v_G)\iota^*$ . Note that we have the equality  $id_G = (\epsilon \otimes id_G)\iota^*$  from the “identity” group axiom. Then  $V = (\epsilon \otimes id_G)\iota^*V = (\epsilon \otimes id_G)(V \otimes id_G)\iota^*$  by left-invariance of  $V$ , but the right hand side of the last equality is precisely  $(v_G)\iota^*$  by evaluation on local coordinates. ■

**Remark 3.5.3.** A *right invariant* vector field is similarly defined; we need only replace  $\iota$  by  $\mu$  in the above definitions and theorems. There is a natural anti-homomorphism from left invariant vector fields to right invariant vector fields induced by the inverse map  $i : G \longrightarrow G$ .

The left invariant vector fields are a subsuper Lie algebra of  $\text{Vec}_G$  which we denote by

$$\mathfrak{g} = \{V \in \text{Vec}_G \mid (V \otimes id)\iota^* = \iota^*V\}.$$

Since the bracket of left invariant vector fields is left invariant, in fact,  $\mathfrak{g}$  is the super Lie algebra associated to the super Lie group  $G$ , and we write  $\mathfrak{g} = \text{Lie}(G)$  as usual.

**Example 3.5.4.** We will calculate the left invariant vector fields on  $\mathbf{R}^{1|1}$  with the group law from example (3.4.2)

$$(t, \theta) \cdot_{\mu} (t', \theta') = (t + t' + \theta\theta', \theta + \theta'). \quad (3.15)$$

From theorem (3.5.2), we know that the Lie algebra of left invariant vector fields can be extracted from  $T_e G = \text{span}\{\frac{\partial}{\partial t}|_e, \frac{\partial}{\partial \theta}|_e\}$ . We use the identity  $V = (v_G)\iota^*$  from the proof of Theorem 3.5.2 to calculate the corresponding left invariant vector fields:

$$(\frac{\partial}{\partial t}|_e)_G \circ \iota^*, \quad (\frac{\partial}{\partial \theta}|_e)_G \circ \iota^*. \quad (3.16)$$

To get coordinate representations of (3.16), we apply them to coordinates  $(t, \theta)$ :

$$\begin{aligned} (\frac{\partial}{\partial t}|_e)_G \circ \iota^*(t) &= (\frac{\partial}{\partial t}|_e)_G(t' + t + \theta'\theta) = 1 \\ (\frac{\partial}{\partial t}|_e)_G \circ \iota^*(\theta) &= (\frac{\partial}{\partial t}|_e)_G(\theta' + \theta) = 0; \end{aligned} \quad (3.17)$$

$$\begin{aligned} (\frac{\partial}{\partial \theta}|_e)_G \circ \iota^*(t) &= (\frac{\partial}{\partial \theta}|_e)_G(t' + t + \theta'\theta) = -\theta' \\ (\frac{\partial}{\partial \theta}|_e)_G \circ \iota^*(\theta) &= (\frac{\partial}{\partial \theta}|_e)_G(\theta' + \theta) = 1. \end{aligned} \quad (3.18)$$

Thus the left invariant vector fields on  $(\mathbf{R}^{1|1}, \mu)$  are

$$\frac{\partial}{\partial t}, \quad -\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}. \quad (3.19)$$

A quick check using the definition shows that (3.19) are in fact left invariant.

### 3.6 Infinitesimal Action

In this section we discuss the infinitesimal interpretation of a super Lie group acting on a supermanifold. We later use the results from this section in chapter 4 to build the super Lie group/algebra super Lie subgroup/subalgebra correspondence.

Let  $G$  be a super Lie group,  $M$  a supermanifold, and

$$\varphi : G \times M \longrightarrow M$$

be a morphism. If  $v \in T_e G$  ( $e \in G$  the identity), then the composition

$$\mathcal{O}_M(U) \xrightarrow{\varphi^*} \mathcal{O}_{G \times M}(U_e \times U) \xrightarrow{v_M} \mathcal{O}_M(U) \quad (3.20)$$

is then a derivation of  $\mathcal{O}_M(U)$  for any open  $U \subset |M|$  ( $U_e \subset |G|$  is some open neighborhood of the identity  $e$ ). The Leibniz property can be verified directly by calculating on local coordinates. Then the composition  $v_M \circ \varphi^*$  defines a vector field on  $M$  which we denote by  $V_M(v, \varphi)$ :

$$V_M(v, \varphi)(f) = v_M(\varphi^*(f)) \quad (3.21)$$

for  $f$  local function on  $M$ . It is clear that as  $v$  varies we get a map of super vector spaces from  $T_x(X)$  into the super vector space of vector fields on  $M$ . Let  $S$  be another supermanifold and consider the morphism

$$\varphi \times id_S : G \times M \times S \longrightarrow M \times S$$

and we see that

$$v_{M \times S}(\varphi^*(f) \otimes s) = v_M(\varphi^*(f)) \otimes s \quad (3.22)$$

for  $f$  again a function on  $M$  and  $s$  a function on  $S$ . We thus obtain the equality:

$$V_{M \times S}(v, \varphi \times id_S) = V_M(v, \varphi) \otimes id_S. \quad (3.23)$$

Note that it is enough to verify (3.23) on sections of the form  $f \otimes s$ . The equation establishes an equality of vector fields, and so it is enough to check it on coordinates. We can always write coordinates in the form  $f \otimes s$ , and so the calculation (3.22) is enough.

**Example 3.6.1.** Let  $M = G$  and let the map  $\varphi = \iota$  be the anti-group law. Then

$$V_G(v, \iota) =: V^\ell$$

is the unique left invariant vector field on  $G$  which defines the tangent vector  $v$  at  $e$ . If we take  $\varphi = \mu$  where  $\mu(gg') = gg'$  is the ordinary group law, then

$$V_G(v, \mu) = V^r$$

where  $V^r$  is the unique right invariant vector field defining the tangent vector  $v$  at  $e$ . We know that  $v \mapsto V^\ell$  is a linear isomorphism of  $T_e(G)$  with  $\text{Lie}(G)$  and one can check that  $V^\ell \mapsto V^r$  is an anti-isomorphism of super Lie algebras.

**Definition 3.6.2.** Let  $\varphi = \sigma$  be an action of  $G$  on  $M$

$$\sigma : G \times M \longrightarrow M.$$

We define a linear map  $\rho$  by

$$\rho(v) =_{\text{def}} V_M(v, \sigma), \quad \rho(v)(f) = v_M(\sigma^*(f)) \quad (3.24)$$

for  $f$  a section on  $\mathcal{O}_M$ .

In fact, the next theorem asserts that the association

$$V^\ell \longmapsto \rho(v)$$

is a linear map from  $\text{Lie}(G)$  to  $\text{Vec}_M$ .

The definition of action of  $G$  on  $M$  gives rise to a commutative diagram

$$\begin{array}{ccc} G \times G \times M & \xrightarrow{\mu \times id_M} & G \times M \\ id_G \times \sigma \downarrow & & \downarrow \sigma \\ G \times M & \xrightarrow{\sigma} & M. \end{array} \quad (3.25)$$

**Theorem 3.6.3.** *The map  $\rho$  (3.6.2) is an antimorphism of super Lie algebras  $\text{Lie}(G) \longrightarrow \text{Vec}_M$ . It moreover satisfies the property*

$$(V^r \otimes id_M)(\sigma^* f) = \sigma^*(\rho(v)f) \quad (3.26)$$

for  $v \in T_e(G)$ ,  $V^r$  its corresponding right-invariant vector field, and  $f$  a function on  $M$ .

*Proof.* It is enough to prove (3.26) to prove both assertions. Indeed, suppose we have proved (3.26). Then we see that the image of  $\mathcal{O}_M$  under  $\sigma^*$  is stable under all the vector fields  $V^r \otimes id_M$ , and that  $V^r \otimes id_M$  and  $\rho(v)$  are  $\sigma$ -related. Moreover, as  $f$  varies,  $\sigma^*(f)$  is surjective onto all sections of  $\mathcal{O}_M$  by restriction since  $G$  is a super Lie group (i.e.  $G$  contains an identity element which acts trivially on  $M$ ). It is then immediate that

$$V^r \longmapsto \rho(v)$$

is a morphism of super Lie algebras. Hence  $\rho$  is an antimorphism of  $\text{Lie}(G)$  into  $\text{Vec}_M$ .

It thus remains to prove (3.26). It will come as a consequence of the relation from the commutative diagram (3.25) that

$$(\mu \times id_M)^*(\sigma^*(f)) = (id_G \times \sigma)^*(\sigma^*(f)), \quad (3.27)$$

and the equality we seek will come by evaluating  $v_{G \times M}$  on both sides of (3.27).

By (3.23),  $V_{G \times M}(v, \mu \times id_M) = V_G(v, \mu) \otimes id_M$ . Hence

$$\begin{aligned} v_{G \times M}((\mu \times id_M)^*(\sigma^*(f))) &= V_{G \times M}(v, \mu \times id_M)(\sigma^*(f)) \\ &= (V_G(v, \mu) \otimes id_M)(\sigma^*(f)) \\ &= (V^r \otimes id_M)(\sigma^*(f)). \end{aligned} \quad (3.28)$$

We shall next evaluate  $v_{G \times M}$  on the right side of (3.27). Now for  $u \in \mathcal{O}_{G \times M}$ ,

$$v_{G \times M}((id_G \times \sigma)^*(u)) = V_{G \times M}(v, id_G \times \sigma)(u).$$

Let  $Z$  denote the vector field  $V_{G \times M}(v, id_G \times \sigma)$  on  $G \times M$  for brevity. Let  $a \in \mathcal{O}_G$  and  $b \in \mathcal{O}_M$ ; we get

$$Z(a \otimes b) = v_{G \times M}(a \otimes \sigma^*(b)) = v(a)\sigma^*(b) = \sigma^*(v(a)b).$$

On the other hand,  $v_M(a \otimes b) = v(a)b$ , so we may rewrite the last equation as

$$Z(a \otimes b) = \sigma^*(v_M(a \otimes b)).$$



But  $Z$  and  $\sigma^* \circ v_M$  are both derivations of  $\mathcal{O}_{G \times M}$ , and hence are vector fields on  $G \times M$ . Then the above relation shows that they must be identical as it is enough to check the equality of vector fields on coordinates, and we may always find coordinates of the form  $a \otimes b$ . Hence

$$Z(u) = \sigma^*(v_M(u))$$

for  $u \in \mathcal{O}_{G \times M}$ . If we take  $u = \sigma^*(f)$  again for  $f$  a local function on  $M$ , by definition we have  $v_M(\sigma^*(f)) = \rho(v)(f)$  and so the right side is equal to  $\sigma^*(\rho(v)f)$ . The left side is equal to  $v_{G \times M}((id_G \times \sigma)^*(f))$ . Hence

$$v_{G \times M}((id_G \times \sigma)^*(f)) = \sigma^*(\rho(v)f). \quad (3.29)$$

The equations (3.28) and (3.29) give us our result. ■

**Corollary 3.6.4.** *The anti-morphism  $\rho$  extends to an associative algebra anti-morphism (which we also call  $\rho$ ),*

$$\rho : \mathcal{U}(\text{Lie}(G)) \longrightarrow \mathcal{U}(\text{Vec}_M).$$

*Proof.* We use the universal property of the universal enveloping algebra and extend the anti-morphism by mapping basis to basis. We can characterize the extension also by the relation (3.26): for  $v_1, v_2, \dots, v_k \in T_e(G)$  and  $f$  a local section of  $\mathcal{O}_M$ ,

$$(V_1^r V_2^r \dots V_k^r \otimes id_G)(\sigma^* f) = \sigma^*(\rho(v_1 v_2 \dots v_k)f) = \sigma^*(\rho(v_1)\rho(v_2) \dots \rho(v_k)f).$$

■

## CHAPTER 4

### The Frobenius Theorem

#### 4.1 The Local Frobenius Theorem

We want a mechanism by which we can construct a subsupermanifold of a given supermanifold  $M$ . In this chapter we present a construction from the tangent bundle of  $M$ . We first prove the super extension of the Frobenius theorem on manifolds, then prove a global result.

Let  $M$  be a supermanifold with tangent bundle  $\text{Vec}_M$ .

**Definition 4.1.1.** A *distribution* on  $M$  is an  $\mathcal{O}_M$ -submodule  $\mathcal{D}$  of  $\text{Vec}_M$  which is locally a direct factor.

**Definition 4.1.2.** We say that a distribution  $\mathcal{D}$  is *integrable* if it is stable under the bracket on  $\text{Vec}_M$ , i.e. for  $D_1, D_2 \in \mathcal{D}$ ,  $[D_1, D_2] \in \mathcal{D}$ .

**Lemma 4.1.3.** *Any distribution  $\mathcal{D}$  is locally free.*

*Proof.* By definition, a distribution is a locally direct subsheaf of the tangent sheaf  $\text{Vec}_M$ . Let  $x \in |M|$ , then

$$T_x(M) = \mathcal{D}_x \oplus D'$$

where  $\mathcal{D}_x$  is a subsuper vector space of  $T_x(M)$  and we may say that  $\mathcal{D}_x$  has basis  $\{s_1, s_2, \dots, s_k\}$ . Then by Nakayama's Lemma (7.2.3), the  $\{s_i\}$  correspond

to vector fields which span  $\mathcal{D}$  in a neighborhood of  $x$ , and by the locally direct property of a distribution, these vector fields are linearly independent in this neighborhood. Hence  $\mathcal{D}$  is actually locally free.  $\blacksquare$

We can then define the *rank* of a distribution.

**Definition 4.1.4.** Let  $\mathcal{D}$  be a distribution as above. Then  $\text{rank}(\mathcal{D})$  is the dimension of  $\mathcal{D}_x$  for  $x \in |M|$ . This definition is well-defined thanks to Lemma 4.1.3.

Now we prove a series of lemmas before we prove the local Frobenius theorem on supermanifolds.

**Remark 4.1.5.** Note that all the following lemmas which pertain to the local Frobenius theorem are local results. Thus it suffices to consider the case  $M = \mathbf{R}^{p|q}$  in a coordinate neighborhood of the origin.

**Lemma 4.1.6.** *Let  $\mathcal{D}$  be an integrable distribution. Then there exist linearly independent supercommuting vector fields which span  $\mathcal{D}$ .*

*Proof.* Let  $X_1, \dots, X_r, \chi_1, \dots, \chi_s$  be a basis for  $\mathcal{D}$  and let  $(t, \theta) = (t^1, \dots, t^p, \theta^1, \dots, \theta^q)$  be a local set of coordinates. Then we can express the vector fields:

$$\begin{aligned} X_j &= \sum_i a_{ij} \frac{\partial}{\partial t^i} + \sum_l \alpha_{lj} \frac{\partial}{\partial \theta^l} \\ \chi_k &= \sum_i \beta_{ik} \frac{\partial}{\partial t^i} + \sum_l b_{lk} \frac{\partial}{\partial \theta^l}. \end{aligned} \tag{4.1}$$

The coefficients form an  $r|s \times p|q$  matrix  $T$ ;

$$T = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}$$

of rank  $r|s$  since the  $\{X_i, \chi_j\}$  are linearly independent. This is to say that the submatrix  $(a)$  has rank  $r$

and  $\text{rank}(b) = s$ . Then by renummerating coordinates  $(t, \theta)$ , we may assume that

$$T = (T_0 | *)$$

where  $T_0$  is an invertible  $r|s \times r|s$  matrix. Multiplying  $T$  by any invertible matrix on the left does not change the row space of  $T$  (i.e. the distribution  $\mathcal{D}$ ), so we can multiply by  $T_0^{-1}$  and assume that

$$T = \begin{pmatrix} I_r & 0 & * \\ 0 & I_s & * \end{pmatrix},$$

which is to say that we may assume that

$$\begin{aligned} X_j &= \frac{\partial}{\partial t^j} + \sum_{i=r+1}^p a_{ij} \frac{\partial}{\partial t^i} + \sum_{l=s+1}^q \alpha_{lj} \frac{\partial}{\partial \theta^l} \\ \chi_k &= \frac{\partial}{\partial \theta^k} + \sum_{l=s+1}^q b_{lk} \frac{\partial}{\partial \theta^l} + \sum_{i=r+1}^p \beta_{ij} \frac{\partial}{\partial t^i}. \end{aligned} \quad (4.2)$$

We then claim that  $[X_j, X_k] = 0$ . By the involutive property of  $\mathcal{D}$ , we know that

$$[X_j, X_k] = \sum_{i=1}^r f_i X_i + \sum_{l=1}^s \varphi_l \chi_l$$

where the  $f_i$  are even functions and the  $\varphi_l$  are odd functions. Then by (4.2),  $f_i$  is the coefficient of the  $\frac{\partial}{\partial t^i}$  term in the vector field  $[X_j, X_k]$ . However, again by (4.2), it is clear that  $[X_j, X_k]$  has only  $\frac{\partial}{\partial t^i}$  terms for  $i > r$ , and so we have that  $f_i = 0$  for all  $i$ . Similarly,  $[X_j, X_k]$  has only  $\frac{\partial}{\partial \theta^l}$  terms for  $l > s$ , hence also  $\varphi_l = 0$  for all  $l$ .

The cases  $[X_j, \chi_k] = 0$  and  $[\chi_l, \chi_k] = 0$  follow by using the same argument above. ■

**Lemma 4.1.7.** *Let  $X$  be an even vector field. There exist local coordinates so that*

$$X = \frac{\partial}{\partial t^1}.$$

*Proof.* Let  $r = 1$ , i.e. we begin with a single even vector field  $X$ , and we want to show that we may express  $X = \frac{\partial}{\partial t^1}$  in some coordinate system. Let  $\mathcal{J}$  be the ideal generated by the odd functions on  $\mathbf{R}^{p|q}$ . Then since  $X$  is even,  $X$  maps  $\mathcal{J}$  to itself. Thus  $X$  induces a vector field, and hence an integrable distribution, on the reduced space  $\mathbf{R}^p$ . Then we may apply the classical Frobenius theorem to get a coordinate system where  $X = \frac{\partial}{\partial t^1} \pmod{\mathcal{J}}$ . So we may assume

$$X = \frac{\partial}{\partial t^1} + \sum_{i \geq 2} a_i \frac{\partial}{\partial t^i} + \sum_j \alpha_j \frac{\partial}{\partial \theta^j}$$

where the  $a_i$  are even,  $\alpha_j$  are odd, and  $a_i, \alpha_j \in \mathcal{J}$ . That the  $a_i$  are even implies that  $a_i \in \mathcal{J}^2$ . Moreover, we can find an even matrix  $(b_{jk})$  so that  $\alpha_j = \sum_k b_{jk} \theta^k \pmod{\mathcal{J}^2}$ , and so modulo  $\mathcal{J}^2$  we have that

$$X = \frac{\partial}{\partial t^1} + \sum_{j,k} b_{jk} \theta^k \frac{\partial}{\partial \theta^j}.$$

Let  $(t, \theta) \mapsto (y, \eta)$  be a change of coordinates where  $y = t$  and  $\eta = g(t)\theta$  for  $g(t) = g_{ij}(t)$  a suitable invertible matrix of smooth functions, that is,  $\eta_j = \sum_i g_{ij}(t) \theta^i$ . Then

$$X = \frac{\partial}{\partial y^1} + \sum_{jk} \theta^k \left( \frac{\partial g_{jk}}{\partial t^1} + \sum_l g_{jl} b_{lk} \right) \frac{\partial}{\partial \eta^j}, \quad (4.3)$$

and we choose  $g(t)$  so that it satisfies the matrix differential equation and initial condition

$$\frac{\partial g}{\partial t^1} = -gb, \quad g(0) = I.$$

Then from (4.3) we may then assume that modulo  $\mathcal{J}^2$ ,

$$X = \frac{\partial}{\partial y^1}.$$

Next we claim that if  $X = \frac{\partial}{\partial t^1} \pmod{\mathcal{J}^k}$ , then  $X = \frac{\partial}{\partial t^1} \pmod{\mathcal{J}^{k+1}}$ . Since  $\mathcal{J}$  is nilpotent, this claim will imply the result for the  $1|0$ -case.

Again, let  $(t, \theta) \mapsto (y, \eta)$  be a change of coordinates so that  $y^i = t^i + c_i$  and  $\eta^j = \theta^j + \gamma_j$  for  $c_i, \gamma_j \in \mathcal{J}^k$  suitably chosen. In the  $(t, \theta)$  coordinate system, let

$$X = \frac{\partial}{\partial t^1} + \sum_{i \geq 2} h_i \frac{\partial}{\partial t^i} + \sum_u \varphi_u \frac{\partial}{\partial \theta^u}$$

for  $h_i, \varphi_u \in \mathcal{J}^k$ . In the new coordinate system, this becomes

$$X = \frac{\partial}{\partial y^1} + \sum_i (h_i + \frac{\partial c_i}{\partial t^1}) \frac{\partial}{\partial y^i} + \sum_l (\varphi_l + \frac{\partial \gamma_l}{\partial t^1}) \frac{\partial}{\partial \eta^l} + Y$$

for some  $Y = 0 \pmod{\mathcal{J}^{k+1}}$  since  $2k - 1 \geq k + 1$  for  $k \geq 2$ . So choose the  $c_i$  and  $\gamma_l$  so that they satisfy the differential equations

$$\frac{\partial c_i}{\partial t^1} = -h_i, \quad \frac{\partial \gamma_l}{\partial t^1} = -\varphi_l,$$

and we get that  $X = \frac{\partial}{\partial y^1} \pmod{\mathcal{J}^{k+1}}$  as we wanted. ■

The above Lemma 4.1.7 sets us up to prove the following.

**Lemma 4.1.8.** *Let  $\{X_j\}$  be a set of supercommuting even vector fields. Then there exist local coordinates  $(t, \theta)$  so that*

$$X_j = \frac{\partial}{\partial t^j} + \sum_{i=1}^{j-1} a_{ij} \frac{\partial}{\partial t^i}$$

for some even functions  $a_{ij}$ .

*Proof.* Notice that since the  $\{X_j\}$  supercommute, they in fact form a distribution. Now we proceed by induction. The  $r = 1$  case is presented above.

We may now assume that we can find coordinates which work for  $r - 1$  supercommuting vector fields, and we want to prove the lemma for  $r$ . Again, assume there are coordinates so that  $X_j = \frac{\partial}{\partial t^j} + \sum_{i=1}^{j-1} a_{ij} \frac{\partial}{\partial t^i}$  for  $j < r$ . Then

$$X_r = \sum_{i=1}^p f_i \frac{\partial}{\partial t^i} + \sum_{k=1}^q \varphi_k \frac{\partial}{\partial \theta^k}$$

for some even functions  $f_i$  and odd functions  $\varphi_k$ . The assumption  $[X_r, X_j] = 0$  gives

$$\sum f_i \left[ \frac{\partial}{\partial t^i}, X_j \right] + \sum \varphi_k \left[ \frac{\partial}{\partial \theta^k}, X_j \right] - \sum (X_j f_i) \frac{\partial}{\partial t^i} - \sum (X_j \varphi_k) \frac{\partial}{\partial \theta^k} = 0.$$

We know that  $[\frac{\partial}{\partial t^i}, X_j]$  is a linear combination of  $\frac{\partial}{\partial t^l}$  for  $l < r$ , which means that  $X_j f_i = 0$  for all  $j \geq r - 1$ . Because the coefficients of the  $X_j$  are “upper triangular” for  $j \leq r - 1$ , we see that  $f_i$  depends only on  $(t^r, \dots, t^p, \theta^1, \dots, \theta^q)$  for  $i \geq r$ . We also have that  $[\frac{\partial}{\partial \theta^k}, X_j] = 0$  for all  $k$ , and so  $X_j \varphi_k = 0$  for all  $j$  as well. We can then again conclude that the  $\varphi_k$  depend only on  $(t^r, \dots, t^p, \theta^1, \dots, \theta^q)$  as well.

Now we can rewrite  $X_r$  as follows:

$$X_r = \left( \sum_{i=1}^{r-1} f_i \frac{\partial}{\partial t^i} \right) + \underbrace{\sum_{l=r}^p f_l \frac{\partial}{\partial t^l} + \sum_{k=1}^q \varphi_k \frac{\partial}{\partial \theta^k}}_{\parallel X'_r}.$$

Here the  $X'_r$  depends only on  $(t^r, \dots, \theta^q)$ , and so by an application of the 1|0-*lemma* on  $X'_r$ , we may change the coordinates  $(t^r, \dots, \theta^q)$  so that  $X'_r = \frac{\partial}{\partial t^r}$ , and so

$$X_r = \frac{\partial}{\partial t^r} + \sum_{i=1}^{r-1} f'_i \frac{\partial}{\partial t^i}$$

(where the  $f'_i$  are the  $f_i$  above under the change of coordinates prescribed by Lemma 4.1.7).

■

In fact, the above lemma proves the local Frobenius theorem in the case when  $\mathcal{D}$  is a purely even distribution (i.e. of rank  $r|0$ ). For the most general case we need one more lemma.

**Lemma 4.1.9.** *Say  $\chi$  is an odd vector field so that  $\chi^2 = 0$  and that  $\text{Span}\{\chi\}$  is a distribution. Then there exist coordinates so that locally  $\chi = \frac{\partial}{\partial \theta^1}$ .*

*Proof.* As we have previously remarked, since we want a local result, it suffices to assume that  $\chi$  is a vector field on  $\mathbf{R}^{p|q}$  near the origin. Let us say  $(y, \eta)$  are coordinates on  $\mathbf{R}^{p|q}$ . Then

$$\chi = \sum_i \alpha_i(y, \eta) \frac{\partial}{\partial y^i} + \sum_j a_j(y, \eta) \frac{\partial}{\partial \eta^j}$$

where the  $\alpha_i$  are odd, the  $a_\sigma$  are even, and we may assume that  $a_1(0) \neq 0$ .

Now consider the map

$$\pi : \mathbf{R}^{0|1} \times \mathbf{R}^{p|q-1} \longrightarrow \mathbf{R}^{p|q}$$

given by

$$\begin{aligned} y^i &= t^i + \epsilon \alpha_i(t, 0, \hat{\theta}), \\ \eta^1 &= \epsilon a_1(t, 0, \hat{\theta}), \\ \eta^{j \geq 2} &= \theta^j + \epsilon a_j(t, 0, \hat{\theta}) \end{aligned}$$

where  $\epsilon$  is the coordinate on  $\mathbf{R}^{0|1}$  and  $(t^1, \dots, t^p, \theta^2, \dots, \theta^q)$  are the coordinates on  $\mathbf{R}^{p|q-1}$ , and  $\hat{\theta}$  denotes the  $\theta$ -indices  $2, \dots, q$ . The  $\alpha(t, 0, \hat{\theta})$  and  $a(t, 0, \hat{\theta})$  are the functions  $\alpha_i$  and  $a_\sigma$  where we substitute  $t$  for  $y$ , let  $\theta^1 = 0$ , and substitute  $\hat{\theta}$  for  $\eta^2, \dots, \eta^q$ . We claim that the map  $\pi$  is a diffeomorphism in a neighborhood of the origin. Indeed, the Jacobian of  $\pi$  at 0 is

$$J = \text{Ber} \begin{pmatrix} I_p & * & 0 \\ 0 & a_1(0) & 0 \\ 0 & * & I_{q-1} \end{pmatrix} = a_1^{-1}(0) \neq 0.$$

So we may think of  $(t, \epsilon, \hat{\theta})$  as coordinates on  $\mathbf{R}^{p|q}$  with  $\pi$  being a change of coordinates. Under this change of coordinates, we have

$$\frac{\partial}{\partial \epsilon} = \sum_i \frac{\partial y^i}{\partial \epsilon} \frac{\partial}{\partial y^i} + \sum_j \frac{\partial \eta^j}{\partial \epsilon} \frac{\partial}{\partial \eta^j},$$



which amounts to

$$\frac{\partial}{\partial \epsilon} = \sum_i \alpha_i(t, 0, \hat{\theta}) \frac{\partial}{\partial y^i} + \sum_j a_j(t, 0, \hat{\theta}) \frac{\partial}{\partial \eta^j}.$$

The  $\alpha_i(t, 0, \hat{\theta})$  and  $a_j(t, 0, \hat{\theta})$  terms must be expressed as functions of  $(y, \eta)$ . Notice that by a simple Taylor series expansion,  $\alpha_i(y, \eta) = \alpha_i(t^i + \epsilon \alpha_i, \epsilon a_1, \theta^{k \geq 2} + \epsilon a_k) = \alpha_i(t^i, 0, \hat{\theta}) + \epsilon \beta_i$  for some odd function  $\beta_i$ . Similarly we get  $a_j(y, \eta) = a_j(t, 0, \hat{\theta}) + \epsilon b_j$  for some even function  $b_j$ . Thus we can write

$$\frac{\partial}{\partial \epsilon} = \chi + \epsilon Z$$

for some even vector field  $Z$ . Recall that  $\eta^1 = \epsilon \hat{a}_1$  where  $\hat{a}_1$  is an even invertible section. Hence  $\epsilon = \eta^1 A$  from some invertible even section  $A$ .

Then we see that under the change of coordinates given by  $\pi$ ,

$$\frac{\partial}{\partial \epsilon} - \eta^1 \underbrace{A \cdot Z^*}_{=Z'} = \chi$$

where  $Z^*$  denotes the pullback of  $Z$  by  $\pi$  and  $Z'$  is some even vector field (since both  $A$  and  $Z$  are even). Now,

$$\begin{aligned} \chi^2 = 0 &\implies \left( \frac{\partial}{\partial \epsilon} - \eta^1 Z' \right)^2 = 0 \\ &\implies \underbrace{\left( \frac{\partial}{\partial \epsilon} \right)^2}_{=0} - \frac{\partial}{\partial \epsilon} (\eta^1 Z') - (\eta^1 Z') \frac{\partial}{\partial \epsilon} + \underbrace{(\eta^1 Z')^2}_{=0} = 0 \\ &\implies -\hat{a}_1 Z' + \eta^1 \frac{\partial}{\partial \epsilon} Z' - \eta^1 Z' \frac{\partial}{\partial \epsilon} = 0 \\ &\implies \hat{a}_1 Z' = 0 \\ &\implies Z' = 0, \end{aligned}$$

so we really have  $\frac{\partial}{\partial \epsilon} = \chi$  under the change of coordinates. ■

Now we can prove of the full local Frobenius theorem.

**Theorem 4.1.10.** (*Local Frobenius Theorem*) Let  $\mathcal{D}$  be an integrable (involutive) distribution of rank  $r|s$ . Then there exist local coordinates so that  $\mathcal{D}$  is spanned by

$$\frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^r}, \frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^s}.$$

*Proof.* Let  $\{X_1, \dots, X_r, \chi_1, \dots, \chi_s\}$  be a basis of vector fields for the distribution  $\mathcal{D}$ . By Lemma 4.1.6 we may assume that these basis elements supercommute, so then  $\mathcal{D}' = \text{span}\{X_1, \dots, X_r\}$  is a subdistribution, and by lemma 4.1.8 we get that there exist coordinates so that  $X_i = \frac{\partial}{\partial t^i}$ .

We then use the fact that  $[\chi_1, X_i] = 0$  for all  $i$ , to see that  $\chi_1$  depends only on coordinates  $(t^{r+1}, \dots, \theta^q)$  (as in the proof of Lemma 4.1.8). In fact, this is not completely accurate. If we express  $\chi_1$  as in (4.1), we see that it is only the  $\beta_{ik}$  and  $b_{lk}$  which depend only on the coordinates  $(t^{r+1}, \dots, \theta^q)$ . However, we can always kill off the first  $r$   $\partial/\partial t^i$  terms by subtracting off appropriate linear combinations of the  $\{X_1 = \partial/\partial t^1, \dots, X_r = \partial/\partial t^r\}$ .

Then since  $\chi_1^2 = 0$ , by Lemma 4.1.9 we may change only the coordinates  $(t^{r+1}, \dots, \theta^q)$  and express  $\chi_1 = \frac{\partial}{\partial \theta^1}$ . For  $\chi_2$  we apply the same idea: that  $[\chi_2, X_i] = 0$  and  $[\chi_2, \chi_1] = 0$  again shows that  $\chi_2$  depends only on coordinates  $(t^{r+1}, \dots, t^p, \theta^2, \dots, \theta^q)$ , and again applying Lemma 4.1.9 gives  $\chi_2 = \frac{\partial}{\partial \theta^2}$ . And so on with  $\chi_3, \dots, \chi_s$ . ■

We are now in a position to state and prove the global Frobenius theorem on supermanifolds.

## 4.2 The Global Frobenius Theorem on Supermanifolds

**Theorem 4.2.1.** (*Global Frobenius Theorem*) *Let  $M$  be a  $C^\infty$ -supermanifold, and let  $\mathcal{D}$  be an integrable distribution on  $M$ . Then given any point of  $M$  there is a unique maximal supermanifold corresponding to  $\mathcal{D}$  which contains that point.*

*Proof.* Let  $\mathcal{D} = \text{span}\{X_1, \dots, X_r, \chi_1, \dots, \chi_s\}$  as in the previous section (again the  $X_i$  are even and the  $\chi_j$  are odd). Then let  $\mathcal{D}_0 = \text{span}\{X_1, \dots, X_r\}$ ; this subdistribution maps odd sections to odd sections, and so descends to an integral distribution  $\widetilde{\mathcal{D}}_0$  on  $\widetilde{M}$ . Let  $x \in |M|$ . Then by the classical global Frobenius Theorem, there is a unique maximal integral manifold  $\widetilde{M}_x \subset \widetilde{M}$  of  $\widetilde{\mathcal{D}}_0$  containing  $x$ . We want to build a sheaf of commutative superalgebras on  $\widetilde{M}_x$ .

By the local Frobenius theorem, given any point  $y \in |M|$ , there exists an open coordinate neighborhood around  $y$ ,  $U_y \subset |M|$ , so that  $U_y$  is characterized by coordinates  $(t, z, \theta, \eta)$  (i.e.  $\mathcal{O}_M(U_y) = C^\infty(t, z)[\theta, \eta]$ ) where  $\mathcal{D}$  is given by the

$\mathcal{I}_M$ -span of  $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}\}$ . Now let  $U \subset |M|$  and define the following presheaf  $\mathcal{I}$ :

$$\mathcal{I}(U) = \langle \{f \in \mathcal{O}_M(U) \mid \forall y \in \widetilde{M}_x \cap U, \exists V_y \subset U \text{ so that } f|_{V_y} \in C^\infty(z)[\eta]\} \rangle.$$

We claim that  $\mathcal{I}$  is a subsheaf of  $\mathcal{O}_M$ . Again let  $U \subset |M|$  be an open subset and let  $\{U_\alpha\}$  be an open covering of  $U$  so that for  $s_\alpha \in U_\alpha$  we have  $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ . We know that there exists a unique  $s \in \mathcal{O}_M(U)$  so that  $s|_{U_\alpha} = s_\alpha$ . Let  $y \in \widetilde{M}_x \cap U$ , then  $y \in U_\alpha$  for some  $\alpha$ . Then there exists  $V_y \subset U_\alpha$  where  $s|_{V_y} \in C^\infty(z)[\eta]$  since  $s|_{V_y} = s_\alpha|_{V_y}$ . Hence  $\mathcal{I}$  is a subsheaf of  $\mathcal{O}_M$ . Moreover  $\mathcal{I}$  is an ideal sheaf by construction.

It is clear that if  $p \notin \widetilde{M}_x$ , then  $\mathcal{I}_p = \mathcal{O}_{M,p}$  since we can find some neighborhood of  $p$ ,  $W_p \cap \widetilde{M}_x = \emptyset$ , where  $\mathcal{I}(W_p) = \mathcal{O}_M(W_p)$ . Thus the support of  $\mathcal{I}$  is  $\widetilde{M}_x$ , and we have defined a quasi-coherent sheaf of ideals with support  $\widetilde{M}_x$  which defines

a unique closed subspace of  $M$ . By going to coordinate neighborhoods it is clear that this closed subspace is in fact a closed subsupermanifold which we shall now call  $M_x$ .

The maximality condition is clear. From the classical theory we have that the reduced space is maximal, and locally we can verify that we have the maximal number of odd coordinates that  $\mathcal{D}$  allows. ■

### 4.3 Lie Subalgebra, Subgroup Correspondence

From a slice of the tangent bundle of a given supermanifold, the global Frobenius theorem allows us to build a subsupermanifold. We now use this construction to make the super Lie group/algebra super Lie subgroup/subalgebra correspondence. We begin with a technical lemma we will need later.

**Lemma 4.3.1.** *Let  $M$  be a supermanifold,  $N \subset M$  a subsupermanifold, and let  $\varphi : M \longrightarrow M$  be a diffeomorphism so that  $\varphi(N) \subset N$  and  $\tilde{\varphi}(\tilde{N}) = \tilde{N}$ . Then  $\varphi(N) = N$ .*

*Proof.* Because  $\varphi$  is a diffeomorphism,  $\varphi(N)$  is a super submanifold of  $M$  with the same super dimension as  $N$ . We assume that they both have the same underlying space. Then since they have the same odd dimension and  $\varphi(N)$  sits inside  $N$ , they must be the same space. This can be checked at the coordinate neighborhood level. ■

Let  $G$  be a super Lie group with super Lie algebra  $\text{Lie}(G)$ . Let  $(H', \mathfrak{h})$  be a pair consisting of an ordinary Lie subgroup and a Lie superalgebra, so that

1.  $H' \subset \tilde{G}$  is a Lie subgroup;
2.  $\mathfrak{h} \subset \text{Lie}(G)$  is a super Lie subalgebra.

**Theorem 4.3.2.** *There is a super Lie subgroup  $H$  of  $G$  so that*

1.  $\tilde{H} = H'$ ;
2.  $\text{Lie}(H) = \mathfrak{h}$ .

*Proof.* Let  $\mathcal{D}$  be the distribution generated by  $\mathfrak{h}$  on  $G$ , i.e.  $\mathcal{D} = \langle \mathfrak{h} \rangle_{\mathcal{T}_G}$ .

Then we can use the Global Frobenius Theorem to get maximal integral super submanifolds  $G_p$  through each point  $p \in G$  which correspond to  $\mathcal{D}$ . Since  $H'$  is second countable, it is the union of a countable number of connected components. Take a collection of points  $p_i \in H'$ , each of which corresponds to exactly one of the  $G_{p_i}$  and let  $H$  be the supermanifold of the union of these maximal integral super submanifolds, i.e.  $H = \cup G_{p_i}$ . This construction makes it clear that  $\tilde{H} = H'$ .

All that is left to show is that  $H$  is in fact a super Lie group. We already have a morphism  $\mu_H : H \times H \longrightarrow G$  which comes from restricting the multiplication morphism  $\mu : G \times G \longrightarrow G$ . This gives the sheaf map

$$\mu_H^* : \mathcal{O}_G \longrightarrow \mathcal{O}_{H \times H}.$$

Let us restrict our view to some coordinate neighborhood of  $G$ , and let  $(t, z, \theta, \eta)$  represent the coordinates on  $G$  so that  $H$  is described locally by the vanishing of the coordinates  $(z, \eta)$ . This is equivalent to saying that

$\mathfrak{h}$  kills these coordinates; for  $h \in \mathfrak{h}$ ,  $h(z) = h(\eta) = 0$ . To show that  $\mu_H^*$  is actually a map from  $\mathcal{O}_H \longrightarrow \mathcal{O}_{H \times H}$  we have to show that  $\mu_H^*$  vanishes on  $(z, \eta)$ .

Let  $h \in \mathfrak{h}$ . Because  $\mathfrak{h} \subset \text{Lie}(G)$ ,  $h$  is left invariant, and hence commutes with  $\mu$  (more precisely, with  $\mu_H$ ). Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_G & \xrightarrow{\mu_H^*} & \mathcal{O}_{H \times H} \\ h \downarrow & & \downarrow \text{id}_H \otimes h \\ \mathcal{O}_G & \xrightarrow{\mu_H^*} & \mathcal{O}_{H \times H}. \end{array}$$

We already know  $\mu_H^* \circ h(z) = 0$ , from which commutativity gives that  $id_H \otimes h \circ \mu_H^*(z) = 0$ . Thus  $\mu_H^*(z)$  is killed by  $id_H \otimes h$  which implies that  $\mu_H^*(z) = 0$  since  $h$  ranges over all left invariant vector fields. Similarly,  $\mu_H^*(\eta) = 0$ . Thus we have a product structure on  $H$ .

Last we show that there is an inverse map  $\iota : H \longrightarrow H$ . Consider the morphism

$$\nu : G \times G \longrightarrow G \times G$$

given by  $\nu = (id, \mu)$  which we define as follows via  $T$ -points. Let  $T$  be any supermanifold. Then

$$\nu(T) : G(T) \times G(T) \longrightarrow G(T) \times G(T)$$

so that for  $g, h \in G(T)$ ,

$$(g, h) \mapsto (g, gh).$$

It is then clear that  $\nu(T)$  is bijective for all  $T$ , and so  $\nu$  is thus a diffeomorphism of  $G \times G$  to itself by

Yoneda's Lemma. Recall that  $H \times H \subset G \times G$  is a closed subsupermanifold. From the arguments above,  $\nu$  maps  $H \times H$  into itself. Moreover, we claim that  $\widetilde{\nu(H \times H)} = \widetilde{H \times H}$ . First note that  $\widetilde{H \times H} = \widetilde{H} \times \widetilde{H}$  and that for any ordinary manifold  $S$ , we have that  $\widetilde{\nu}(S) : \widetilde{H}(S) \times \widetilde{H}(S) \longrightarrow \widetilde{H}(S) \times \widetilde{H}(S)$  is a bijection because  $\widetilde{H}$  is a Lie group. Thus

$$\widetilde{\nu}(\widetilde{H} \times \widetilde{H}) = \widetilde{H} \times \widetilde{H}$$

and we can use the general Lemma (4.3.1) to see that  $\nu(H \times H) = H \times H$  from which it follows that the inverse map of  $G$  descends to  $H$ .

The necessary diagrams commute (associativity, inverses, etc.) because they do for  $G$  and all the maps for  $H$  are derived from those of  $G$ . We have thus

produced a super Lie subgroup  $H$  of  $G$  with the additional property that  $\text{Lie}(H) = \mathfrak{h}$ . ■

## CHAPTER 5

### Supervarieties and Superschemes

#### 5.1 Basic definitions

In this section we give the basic definitions of algebraic supergeometry. Because we are in need of a more general setting in the next two chapters we no longer assume the ground field to be  $\mathbf{R}$ .

Let  $k$  be a commutative ring.

Assume all superalgebras are associative, commutative (i.e.  $xy = (-1)^{p(x)p(y)}yx$ ) with unit and over  $k$ . We denote their category with  $(\text{salg})$ . For a superalgebra  $A$  let  $J_A$  denote the ideal generated by the odd elements i. e.  $J_A = \langle A_1 \rangle_A$ . Denote the quotient  $A/J_A$  by  $A^r$ .

In chapter 2 we have introduced the notion of *superspace* and of *superscheme*. Recall that a superspace  $X = (|X|, \mathcal{O}_X)$  is a topological space  $|X|$  together with a sheaf of superalgebras  $\mathcal{O}_X$  such that  $\mathcal{O}_{X,x}$  is a local superalgebra, i.e. it has a unique two sided maximal homogeneous ideal.

The sheaf of superalgebras  $\mathcal{O}_X$  is a sheaf of  $\mathcal{O}_{X,0}$ -modules, where  $\mathcal{O}_{X,0}(U) =_{\text{def}} \mathcal{O}_X(U)_0$ ,  $\forall U$  open in  $|X|$ .

Let  $\mathcal{O}_X^r$  denote the sheaf of algebras:

$$\mathcal{O}_X^r(U) = \mathcal{O}_X(U)/J_{\mathcal{O}_X(U)}$$



We will call  $X^r = (|X|, \mathcal{O}_X^r)$  the *reduced space* associated to the superspace  $X = (|X|, \mathcal{O}_X)$ . This is a locally ringed space in the classical sense.

Recall that given two superspaces  $X = (|X|, \mathcal{O}_X)$  and  $Y = (|Y|, \mathcal{O}_Y)$  a *morphism*  $f : X \longrightarrow Y$  of superspaces is given by a pair  $f = (|f|, f^*)$  such that

1.  $|f| : X \rightarrow Y$  is a continuous map.
2.  $f^* : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$  is a map of sheaves of superalgebras on  $|Y|$ , that is for all  $U$  open in  $|Y|$  there exists a morphism  $f_U^* : \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(|f|^{-1}(U))$ .
3. The map of local superalgebras  $f_p^* : \mathcal{O}_{Y,|f|(p)} \rightarrow \mathcal{O}_{X,p}$  is a local morphisms i.e. sends the maximal ideal of  $\mathcal{O}_{Y,|f|(p)}$  into the maximal ideal of  $\mathcal{O}_{X,p}$ .

Recall that a *superscheme*  $S$  is a superspace  $(|S|, \mathcal{O}_S)$  such that  $(|S|, \mathcal{O}_{S,0})$  is a quasi coherent sheaf of  $\mathcal{O}_{x,1}$ -modules. A *morphism* of superschemes is a morphisms of the corresponding superspaces.

For any open  $U \subset X$  we have the superscheme  $U = (|U|, \mathcal{O}_X|_U)$ , called *open subscheme* in the superscheme  $X$ .

One of the most important examples of superscheme is given by the spectrum of the even part of a given superalgebra (the topological structure) together with a certain sheaf of superalgebras on it that plays the role of the structural sheaf in the classical theory. Let's see this construction in detail.

**Definition 5.1.1.**  $\text{Spec} A$ .

Let  $A$  be an object of  $(\text{salg})$ . We have that  $\text{Spec}(A_0) = \text{Spec}(A^r)$ , since the algebras  $A^r$  and  $A_0$  differ only by nilpotent elements.

Let's consider  $\mathcal{O}_{A_0}$  the structural sheaf of  $\text{Spec}(A_0)$ . The stalk of the sheaf at the prime  $\mathfrak{p} \in \text{Spec}(A_0)$  is the localization of  $A_0$  at  $\mathfrak{p}$ . As for any superalgebra,

$A$  is a module over  $A_0$ . We have indeed a sheaf  $\mathcal{O}_A$  of  $\mathcal{O}_{A_0}$ -modules over  $\text{Spec} A_0$  with stalk  $A_{\mathbf{p}}$ , the localization of the  $A_0$ -module  $A$  over each prime  $\mathbf{p} \in \text{Spec}(A_0)$ .

$$A_{\mathbf{p}} = \left\{ \frac{f}{g} \mid f \in A, g \in A_0 - \mathbf{p} \right\}$$

The localization  $A_{\mathbf{p}}$  has a unique two-sided maximal ideal which consists of the maximal ideal in the local ring  $(A_{\mathbf{p}})_0$  and the generators of  $(A_{\mathbf{p}})_1$  as  $A_0$ -module. For more details on this construction see [13] II §5.

$\mathcal{O}_A$  is a sheaf of superalgebras and  $(\text{Spec} A_0, \mathcal{O}_A)$  is a superscheme that we will denote with  $\underline{\text{Spec}} A$ . Notice that on the open sets:

$$U_f = \{\mathbf{p} \in \text{Spec} A_0 \mid (f) \not\subset \mathbf{p}\}, \quad f \in A_0$$

we have that  $\mathcal{O}_A(U_f) = A_f = \{a/f^n \mid a \in A\}$ .

**Definition 5.1.2.** An *affine superscheme* is a superspace that is isomorphic to  $\underline{\text{Spec}} A$  for some superalgebra  $A$  in  $(\text{salg})$ . An *affine algebraic supervariety* is a superspace isomorphic to  $\underline{\text{Spec}} A$  for some *affine* superalgebra  $A$  i. e. a finitely generated superalgebra such that  $A/J_A$  has no nilpotents. We will call  $A$  the *coordinate ring* of the supervariety.

**Proposition 5.1.3.** A superspace  $S$  is a superscheme if and only if it is locally isomorphic to  $\underline{\text{Spec}} A$  for some superalgebra  $A$ , i. e. for all  $x \in |S|$ , there exists  $U_x \subset |S|$  open such that  $(U_x, \mathcal{O}_S|_{U_x}) \cong \underline{\text{Spec}} A$ . (Clearly  $A$  depends on  $U_x$ ).

*Proof.* Since  $S$  is a superscheme, by definition  $S' = (|S|, \mathcal{O}_{S,0})$  is an ordinary scheme, that is, it admits an open cover  $S' = \cup V_i$  so that  $V_i \cong \underline{\text{Spec}} A_{i,0}$  where  $A_{i,0}$  is a commutative algebra. Let  $x \in |S|$  and let  $U_i = (|V_i|, \mathcal{O}_S|_{V_i})$ , such that  $x \in |U_i|$ .

The  $U_i$  can be chosen so that there exists a  $A_{i,0}$ -module  $A_{i,1}$  such that  $\mathcal{O}_{S,1}|_{U_i} \cong \mathcal{O}_{A_{i,1}}$ , where  $\mathcal{O}_{A_{i,1}}$  denotes the sheaf induced by the  $A_{i,0}$ -module  $A_{i,1}$ . So we have that:

$$\mathcal{O}_S|_{U_i} = \mathcal{O}_{S,0}|_{U_i} \oplus \mathcal{O}_{S,1}|_{U_i} = \mathcal{O}_{A_{i,0}} \oplus \mathcal{O}_{A_{i,1}} = \mathcal{O}_{A_i}$$

Since these are sheaves of superalgebras,  $A_i$  is also a superalgebra, in fact  $A_i = \mathcal{O}_S|_{U_i}(U_i)$ . Hence  $U_i \cong \underline{\text{Spec}} A_i$ . The other direction is clear.  $\blacksquare$

Given a superscheme  $X = (|X|, \mathcal{O}_X)$ , the scheme  $(|X|, \mathcal{O}_X^r)$  is called the *reduced scheme* associated to  $X$ . Notice that the reduced scheme associated to a given superscheme may not be reduced, i.e.  $\mathcal{O}_X^r(U)$ ,  $U$  open in  $X$ , can contain nilpotents.

As in the classical setting we can define closed subsuperschemes.

**Definition 5.1.4.** A *closed subsuperscheme*  $Y$  of a given superscheme  $X$  is such that  $|Y| \subset |X|$  and  $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}$  for a quasi-coherent sheaf of ideals in  $\mathcal{O}_X$ .

Notice that if  $X = \underline{\text{Spec}} A$ , closed subschemes are in one to one correspondence with ideals in  $A$  as it happens in the ordinary case.

**Example 5.1.5.** 1. *Affine superspace*  $\mathbf{A}^{m|n}$ .

Consider the polynomial superalgebra  $k[x_1 \dots x_m, \xi_1 \dots \xi_n]$  over an algebraically closed field  $k$  where  $x_1 \dots x_m$  are even indeterminates and  $\xi_1 \dots \xi_n$  are odd indeterminates (see chapter 1). We will call  $\underline{\text{Spec}} k[x_1 \dots x_m, \xi_1 \dots \xi_n]$  the affine superspace of superdimension  $m|n$  and we denote it by  $\mathbf{A}^{m|n}$ .

$$k[\mathbf{A}^{m|n}] = k[x_1 \dots x_m, \xi_1 \dots \xi_n].$$

As a topological space  $\text{Spec}k[x_1 \dots x_m, \xi_1 \dots \xi_n]_0$  will consists of the even maximal ideals

$$(x_i - a_i, \xi_j \xi_k), \quad i = 1 \dots m, \quad j, k = 1 \dots n$$

and the even prime ideals

$$(p_1 \dots p_r, \xi_j \xi_k), \quad i = 1 \dots m, \quad j, k = 1 \dots n$$

where  $(p_1 \dots p_r)$  is a prime ideal in  $k[x_1 \dots x_m]$ .

The structural sheaf of  $\mathbf{A}^{m|n}$  will have stalk at the point  $\mathbf{p} \in \text{Spec}k[\mathbf{A}^{m|n}]_0$ :

$$k[\mathbf{A}^{m|n}]_{\mathbf{p}} = \left\{ \frac{f}{g} \mid f \in k[\mathbf{A}^{m|n}], \quad g \in k[\mathbf{A}^{m|n}]_0, \quad g \notin \mathbf{p} \right\}.$$

## 2. Supervariety over the sphere $S^2$ .

Consider the polynomial superalgebra generated over an algebraically closed field  $k$   $k[x_1, x_2, x_3, \xi_1, \xi_2, \xi_3]$ , and the ideal

$$\mathcal{I} = (x_1^2 + x_2^2 + x_3^2 - 1, x_1 \cdot \xi_1 + x_2 \cdot \xi_2 + x_3 \cdot \xi_3).$$

Let  $k[X] = k[x_1, x_2, x_3, \xi_1, \xi_2, \xi_3]/\mathcal{I}$  and  $X = \underline{\text{Spec}}k[X]$ .  $X$  is a supervariety whose reduced variety  $X^r$  is the sphere  $S^2$ . A maximal ideal in  $k[X]_0$  is given by:

$$\mathbf{m} = (x_i - a_i, \xi_i \xi_j) \quad \text{with } i, j = 1, 2, 3, \quad a_i \in k \quad \text{and} \quad a_1^2 + a_2^2 + a_3^2 = 1.$$

**Observation 5.1.6.** Let  $(\text{affine sschemes})$  denote the category of affine superschemes.

The functor

$$\begin{array}{ccc} F : (\text{salg})^o & \longrightarrow & (\text{affine sschemes}) \\ A & \longmapsto & \underline{\text{Spec}}A \end{array}$$

gives an equivalence between the category of superalgebras and the category of affine superschemes. The inverse functor is given by:

$$\begin{aligned} G : (\text{affine sschemes}) &\longrightarrow (\text{salg}) \\ \underline{\text{Spec}} A &\longmapsto \mathcal{O}_A(A_0) \cong A. \end{aligned}$$

We need to specify both  $F(\phi)$ , for  $\phi : A \longrightarrow B$  and  $G(f)$ , for  $f : \underline{\text{Spec}} A \longrightarrow \underline{\text{Spec}} B$  and show that they realize a bijection:

$$\text{Hom}_{(\text{salg})} (A, B) \cong \text{Hom}_{(\text{affine sschemes})} (\underline{\text{Spec}} B, \underline{\text{Spec}} A)$$

Let  $\phi : A \longrightarrow B$  and  $\phi_0 = \phi|_{A_0}$ . We want to build a morphism  $f = F(\phi) : \underline{\text{Spec}} B \longrightarrow \underline{\text{Spec}} A$ . We have immediately  $|f| : \text{Spec} B_0 \longrightarrow \text{Spec} A_0$  defined as  $|f|(\mathbf{p}) = \phi_0^{-1}(\mathbf{p})$ . We also have a map

$$f_U^* : \mathcal{O}_A(U) \longrightarrow \mathcal{O}_B(|f|^{-1}(U)), \quad U \text{ open in } \text{Spec} A_0,$$

defined as in the classical case. That is if  $a \in \mathcal{O}_A(U)$ ,  $f_U^*(a) : \mathbf{p} \mapsto \phi_{\mathbf{p}}(a(|f|(\mathbf{p})))$  and  $\phi_{\mathbf{p}} : A_{\phi_0^{-1}(\mathbf{p})} \longrightarrow B_{\mathbf{p}}$ ,  $\mathbf{p} \in \text{Spec} B_0$ .

Vice-versa if we have a map  $f : \underline{\text{Spec}} B \longrightarrow \underline{\text{Spec}} A$ , since global sections of the structural sheaves coincide with the rings  $B$  and  $A$  respectively, we obtain immediately a map from  $A$  to  $B$ :

$$G(f) = f_{\text{Spec} A_0}^* : \mathcal{O}_A(\text{Spec} A_0) \cong A \longrightarrow \mathcal{O}_B(\text{Spec} B_0) \cong B.$$

When we restrict the functor  $F$  to the category of affine superalgebras, it gives an equivalence of categories between affine superalgebras and affine supervarieties.

We now would like to give an example of a non affine superscheme which is of particular importance: the projective superspace.

**Example 5.1.7.** *Projective superspace*

Let  $S = k[x_0, \dots, x_m, \xi_1 \dots \xi_m]$ .  $S = \sigma - 0 \oplus S_1$  is a  $\mathbf{Z}/2\mathbf{Z}$  and  $\mathbf{Z}$  graded algebra and the two gradings are compatible. Define the topological space  $\text{Proj}S_0$  as the set of  $\mathbf{Z}$ -homogeneous non irrelevant primes in  $S_0$ , with the Zariski topology.  $\text{Proj}S_0$  is covered by open affine  $U_i$  consisting of those primes non containing  $(x_i)$ . As in the classical setting we have that

$$U_i = \text{Spec}k[x_0 \dots \hat{x}_i \dots x_m, \xi_1 \dots \xi_m]_0, \quad i = 1 \dots m.$$

So we can define the sheaves

$$\mathcal{O}_{U_i} = \mathcal{O}_{k[x_0 \dots \hat{x}_i \dots x_m, \xi_1 \dots \xi_m]}$$

corresponding to these open affine subsets. One can check that these sheaves glue to give a sheaf  $\mathcal{O}_S$  on all  $\text{Proj}S$ . So we define *projective superspace*  $\mathbf{P}^{m|n}$ , as the superscheme  $(\text{Proj}S_0, \mathcal{O}_S)$ .

The same construction can be easily repeated for a generic  $\mathbf{Z}$ -graded superalgebra.

## 5.2 The functor of points

As in  $C^\infty$  geometry, we employ the functor of points approach from algebraic geometry to better handle to nilpotent elements and to bring back geometric intuition.

**Definition 5.2.1.** For a superscheme  $X$ , the *functor of points* of  $X$  is a representable functor

$$h_X : (\text{sschemes})^o \rightarrow (\text{sets}), \quad h_X(Y) = \text{Hom}_{(\text{sschemes})}(Y, X)$$

$h_X(Y)$  are called the  $Y$ -points of the superscheme  $X$ .

In the previous chapters, we have used the same notation to denote both a supergeometric object, say a supermanifold, and its functor of points. In this chapter, however, we want to make a distinction, since we will also deal with non representable functors.

As in the ordinary setting, the functor of points of a superscheme  $h_X$  is determined by looking at its restriction to the affine superschemes  $h_X^a$ , that is looking at the functor

$$h_X^a : (\text{salg}) \longrightarrow (\text{sets}) , \quad h_X^a(A) = \text{Hom}_{(\text{sschemes})}(\underline{\text{Spec}}A, X).$$

This is proven in the same way as the ordinary case. In fact a morphism  $\phi \in \text{Hom}(Y, X)$  is determined by its restrictions to the open affine subschemes that form an open cover of  $Y$ .

When the superscheme  $X$  is affine, i.e.  $X = \underline{\text{Spec}}R$ ,  $h_X^a$  is representable. In fact by Observation 5.1.6:

$$h_X^a(A) = \text{Hom}_{(\text{sschemes})}(\underline{\text{Spec}}A, \underline{\text{Spec}}R) = \text{Hom}_{(\text{salg})}(R, A).$$

**Observation 5.2.2.** Since we have the equivalence of categories between affine superschemes and superalgebras, we can define an affine superscheme equivalently as a representable functor

$$F : (\text{salg}) \longrightarrow (\text{sets}) , \quad F(B) = \text{Hom}_{(\text{salg})}(A, B).$$

**Remark 5.2.3.** To simplify notation we drop the suffix  $a$  in  $h_X^a$ , the context will make clear whether we are considering  $h_X$  or its restriction to affine superschemes. Moreover, whenever want the restriction of  $h_X$  to affine superschemes we will not use a different functor name for  $h_X(A)$ ,  $h_X : (\text{salg}) \longrightarrow (\text{sets})$  and for  $h_X(\underline{\text{Spec}}A)$   $h_X : (\text{affine sschemes}) \longrightarrow (\text{sets})$ .

**Observation 5.2.4.** Let  $X^0$  be an affine variety over an algebraically closed field  $k$ . Consider an affine supervariety  $X$  whose reduced part coincides with  $X^0$ . Then one can immediately check that the  $k$ -points of  $X$  correspond to the points of the affine variety  $X^0$ .

**Examples 5.2.5.** 1. *Affine superspace revisited.*

Let  $A \in (\text{salg})$  and let  $V = V_0 \oplus V_1$  be a free supermodule (over  $k$ ). Let  $(\text{smod})$  denote the category of  $k$ -modules. Define

$$V(A) = (A \otimes V)_0 = A_0 \otimes V_0 \oplus A_1 \otimes V_1.$$

In general this functor is not representable. However, if  $V$  is finite dimensional we have:

$$(A \otimes V)_0 \cong \text{Hom}_{(\text{smod})}(V^*, A) \cong \text{Hom}_{(\text{salg})}(\text{Sym}(V^*), A)$$

where  $\text{Sym}(V^*)$  denotes the symmetric algebra over the dual space  $V^*$ . Recall that  $V^*$  is the set of linear maps  $V \rightarrow k$  not necessarily preserving the parity, and  $\text{Sym}(V^*) = \text{Sym}(V_0^*) \otimes \wedge V_1^*$ , where  $\wedge V_1^*$  denotes the exterior algebra over the ordinary space  $V_1$ .

Let's fix a basis for  $V$  and let  $\dim V = p|q$ . The functor  $V$  is represented by:

$$k[V] = k[x_1 \dots x_p, \xi_1 \dots \xi_q]$$

where  $x_i$  and  $\xi_j$  are respectively even and odd indeterminates.

Hence the functor  $V$  is the functor of points of the affine supervariety  $\mathbf{A}^{m|n}$  introduced in Example 5.1.5.

We also want to remark that the functor  $D_V$  defined as:

$$D_V(A) =_{\text{def}} \text{Hom}_{(\text{smod})}(V, A)$$



is representable for any  $V$  (not necessarily finite dimensional), and it is represented by the superalgebra  $\text{Sym}(V)$ . Clearly  $V = D_V$  when  $V$  is finite dimensional.

## 2. Supermatrices.

Let  $A \in (\text{salg})$ . Define  $M_{m|n}(A)$  as the set of endomorphisms of the  $A$ -supermodule  $A^{m|n}$ . Choosing coordinates we can write:

$$M_{m|n}(A) = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \right\}$$

where  $a$  and  $b$  are  $m \times m$ ,  $n \times n$  blocks of even elements and  $\forall, \alpha, \beta$   $m \times n$ ,  $n \times m$  blocks of odd elements.

This is the functor of points of an affine supervariety represented by the commutative superalgebra:  $k[M(m|n)] = k[x_{ij}, \xi_{kl}]$  where  $x_{ij}$ 's and  $\xi_{kl}$ 's are respectively even and odd variables with  $1 \leq i, j \leq m$  or  $m+1 \leq i, j \leq m+n$ ,  $1 \leq k \leq m$ ,  $m+1 \leq l \leq m+n$  or  $m+1 \leq k \leq m+n$ ,  $1 \leq l \leq m$ .

Notice that  $M_{m|n} \cong h_{\mathbf{A}^{m^2+n^2|2mn}}$ .

## 5.3 A representability criterion

We now want to single out among all the functors  $F : (\text{salg}) \longrightarrow (\text{sets})$  those that are the functor of points of superschemes.

We first need the definition of local functor and open subfunctor.

Let  $A$  be a superalgebra. Given  $f \in A_0$ , let  $A_f$  denote:

$$A_f =_{\text{def}} \{a/f^n | a \in A\}.$$

The sets  $U_f = \text{Spec}(A_f)_0$  are open sets in the topological space  $X = \text{Spec}A_0$ . In fact recall that since  $A_0$  is an ordinary commutative algebra, by definition the open sets in the Zariski topology of  $\text{Spec}A_0$  are:

$$U_I = \{\mathbf{p} \in \text{Spec}A_0 \mid I \not\subset \mathbf{p}\}$$

for all the ideals  $I$  in  $A_0$ .

We now want to define the notion of open subfunctor of a functor  $F$ . If we assume  $F$  is the functor of points of superscheme  $X$  an open subfunctor could simply be defined as the functor of points of an open superscheme  $U \subset X$ . However because we are precisely interested in a characterization of those  $F$  that come from superschemes, we have to carefully extend this notion.

**Definition 5.3.1.** Let  $U$  be a subfunctor of a functor  $F : (\text{salg}) \longrightarrow (\text{sets})$  (this means that we have a natural transformation  $U \longrightarrow F$  such that  $U(A) \longrightarrow F(A)$  is injective for all  $A$ ). We say that  $U$  is an *open subfunctor* of  $F$  if for all  $A \in (\text{salg})$  given a natural transformations  $f : h_{\underline{\text{Spec}}A} \longrightarrow F$ , the subfunctor  $f^{-1}(U)$  is equal to  $h_V$ , for some open  $V$  in  $\underline{\text{Spec}}A$  where

$$f^{-1}(U)(R) =_{\text{def}} f_R^{-1}(U(R)), \quad f_R : h_{\underline{\text{Spec}}A}(R) \longrightarrow F(R).$$

We say  $U$  is an *open affine subfunctor* of  $F$  if it is open and representable.

**Observation 5.3.2.** Let  $X = (|X|, \mathcal{O}_X)$  be a superscheme and  $U \subset X$  open affine in  $X$ . Then  $h_U$  is an open affine subfunctor of  $h_X$ .

By Yoneda's lemma  $f : h_{\underline{\text{Spec}}A} \longrightarrow h_X$  corresponds to a map  $f' : \underline{\text{Spec}}A \longrightarrow X$ . Let  $V = f'^{-1}(U)$  open in  $\underline{\text{Spec}}A$ . We claim

$$f_R^{-1}(h_U(R)) = h_V(R).$$

Let  $\phi \in h_{\underline{\text{Spec}}A}(R)$ , then  $f_R(\phi) = f' \cdot \phi \in h_X(R)$ .

Hence if  $f_R(\phi) \in h_U(R)$  immediately:

$$\phi : \underline{\text{Spec}}R \longrightarrow V = f^{-1}(U) \longrightarrow \underline{\text{Spec}}A.$$

So  $f_R(\phi) \in h_U(R)$  if and only if  $\phi \in h_V(R)$ .

We want to define the notion of an open cover of a functor.

**Definition 5.3.3.** Let  $F : (\text{salg}) \longrightarrow (\text{sets})$  be a functor.  $F$  is covered by the open subfunctors  $(U_i)_{i \in I}$ , if and only if for any affine superscheme  $\underline{\text{Spec}}A$  and map  $f : h_{\underline{\text{Spec}}A} \longrightarrow F$  we have that the fibered product  $h_{\underline{\text{Spec}}A} \times_F U_i \cong h_{V_i}$  and  $(V_i)_{i \in I}$  is an open cover of  $\underline{\text{Spec}}A$ . (For the definition of fibered product see the Appendix 7).

Notice that by the very definition of open subfunctor the functor  $h_{\underline{\text{Spec}}A} \times_F U_i$  is always representable. In fact it is equal to  $f^{-1}(U)$  which is by definition the functor of points of an open and affine  $V_i$  in  $\underline{\text{Spec}}A$ .

Before going to our main result we need the notion of local functor.

**Definition 5.3.4.** A functor

$$F : (\text{salg}) \longrightarrow (\text{sets})$$

is called *local* or *sheaf in the Zariski topology*, if for each  $A \in (\text{salg})$ , there exists  $f_i \in A_0$ ,  $i \in I$ ,  $(f_i, i \in I) = (1)$ , such that for every collection of  $\alpha_i \in F(A_{f_i})$  which map to the same element in  $F(A_{f_i f_j})$ , then there exists a unique  $\alpha \in F(A)$  mapping to each  $\alpha_i$ .

**Proposition 5.3.5.** *The functor of points  $h_X$  of a superscheme  $X$  is local.*

*Proof.* We briefly sketch the proof since it is the same as in the ordinary case. Let the notation as in the previous definition. Consider a collection of maps  $\alpha_i \in h_X(A_{f_i})$  which map to the same element in  $h_X(A_{f_i f_j})$ . Each  $\alpha_i$  consists of two maps:  $|\alpha_i| : \text{Spec} A_{f_i 0} \longrightarrow |X|$  and a family of  $\alpha_{i,U}^* : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{A_{f_i}}(|\alpha|^{-1}(U))$ . The fact the  $|\alpha_i|$  glue together is clear. The gluing of the  $\alpha_i^*$ 's to give  $\alpha : \underline{\text{Spec}} A \longrightarrow X$  depends on the fact that  $\mathcal{O}_X, \mathcal{O}_{A_f}$  are sheaves. ■

We are ready to state the result that characterizes among all the functors from (salg) to (sets) those which are the functors of points of superschemes.

**Theorem 5.3.6.** *A functor*

$$F : (\text{salg}) \longrightarrow (\text{sets})$$

*is the functor of points of a superscheme  $X$ , i. e.  $F = h_X$  if and only if*

1.  *$F$  is local.*
2.  *$F$  admits a cover by affine open subfunctors.*

*Proof.* Again the proof of this result is similar to that in the ordinary case. We include a sketch of it for lack of an appropriate reference. We first observe that if  $h_X$  is the functor of points of a superscheme, by 5.3.5 it is local and by 5.3.2 it admits a cover by open affine subfunctors.

Let's now assume to have  $F$  satisfying the properties (1) and (2) of 5.3.6. We need to construct a superscheme  $X = (|X|, \mathcal{O}_X)$  such that  $h_X = F$ . The construction of the topological space  $|X|$  is the same as in the ordinary case. Let's sketch it.

Let  $\{h_{X_\alpha}\}_{\alpha \in A}$  be the affine open subfunctors that cover  $F$ . Define  $h_{X_{\alpha\beta}} = h_{X_\alpha} \times_F h_{X_\beta}$ . ( $X_{\alpha\beta}$  will correspond to the intersection of the two open affine  $X_\alpha$  and  $X_\beta$  in the superscheme  $X$ ). Notice that  $h_{X_\alpha} \times_F h_{X_\beta}$  is representable.

We have the commutative diagram:

$$\begin{array}{ccc} h_{X_{\alpha\beta}} = h_{X_\alpha} \times_F h_{X_\beta} & \xrightarrow{j_{\beta,\alpha}} & h_{X_\beta} \\ \downarrow j_{\alpha,\beta} & & \downarrow i_\beta \\ h_{X_\alpha} & \xrightarrow{i_\alpha} & F \end{array}$$

As a set we define:

$$|X| =_{\text{def}} \coprod_{\alpha} |X_\alpha| / \sim$$

where  $\sim$  is the following relation:

$$\forall x_\alpha \in |X_\alpha|, x_\beta \in |X_\beta|, x_\alpha \sim x_\beta \iff \exists x_{\alpha\beta} \in |X_{\alpha\beta}|, j_{\alpha,\beta}(x_{\alpha\beta}) = x_\alpha, j_{\beta,\alpha}(x_{\alpha\beta}) = x_\beta.$$

This is an equivalence relation.  $|X|$  is a topological space and  $\pi_\alpha : |X_\alpha| \hookrightarrow |X|$  is an injective map.

We now need to define a sheaf of superalgebras  $\mathcal{O}_X$ , by using the sheaves in the open affine  $X_\alpha$  and “gluing”.

Let  $U$  be open in  $|X|$  and let  $U_\alpha = \pi_\alpha^{-1}(U)$ . Define:

$$\mathcal{O}_X(U) =_{\text{def}} \{(f_\alpha) \in \prod_{\alpha \in I} \mathcal{O}_{X_\alpha}(U_\alpha) \mid j_{\beta,\gamma}^*(f_\beta) = j_{\gamma,\beta}^*(f_\gamma), \forall \beta, \gamma \in I\}.$$

The condition  $j_{\beta,\gamma}^*(f_\beta) = j_{\gamma,\beta}^*(f_\gamma)$  simply states that to be an element of  $\mathcal{O}_X(U)$ , the collection  $\{f_\alpha\}$  must be such that  $f_\beta$  and  $f_\gamma$  agree on the intersection of  $X_\beta$  and  $X_\gamma$  for any  $\beta$  and  $\gamma$ .

One can check that  $\mathcal{O}_X$  is a sheaf of superalgebras.

We now need to show  $h_X \cong F$ . We are looking for a functorial bijection  $h_X(A) = \text{Hom}_{(\text{sschemes})}(\underline{\text{Spec}} A, X) = F(A)$ , for all  $A \in (\text{salg})$ . It is here that we use the hypothesis of  $F$  being local.

To simplify the notation let  $T = \underline{\text{Spec}} A$ . We also write  $h_X(T)$  instead of  $h_X(A)$ . So we want to show  $h_X(T) \cong F(T)$ .

We first construct a natural transformation  $\rho_T : F(T) \longrightarrow h_X(T)$ .

Let  $t \in F(T) = \text{Hom}(h_T, F)$ , by Yoneda's lemma. Consider the diagram:

$$\begin{array}{ccc} h_{T_\alpha} =_{\text{def}} h_{X_\alpha} \times_F h_T & \longrightarrow & h_T \\ \downarrow t_\alpha & & \downarrow t \\ h_{X_\alpha} & \xrightarrow{i_\alpha} & F. \end{array}$$

Notice that  $\{T_\alpha\}$  form an open affine cover of  $T$ . Since by Yoneda's lemma:  $\text{Hom}(h_{T_\alpha}, h_{X_\alpha}) \cong \text{Hom}(T_\alpha, X_\alpha)$  we obtain a map:  $t_\alpha : T_\alpha \longrightarrow X_\alpha \subset X$ . One can check that the maps  $t_\alpha$  glue together to give a map  $t' : T \longrightarrow X$ , hence  $t' \in h_X(T)$ . So we define  $\rho_T(t) = t'$ .

Next we construct another natural transformation  $\sigma_T : h_X(T) \longrightarrow F(T)$ , which turns out to be the inverse of  $\rho$ .

Assume we have  $f \in h_X(T)$  i.e.  $f : T \longrightarrow X$ . Let  $T_\alpha = f^{-1}(X_\alpha)$ . We immediately obtain maps  $g_\alpha : T_\alpha \longrightarrow X_\alpha \subset F$ . By Yoneda's lemma,  $g_\alpha$  corresponds to a map  $g_\alpha : h_{T_\alpha} \longrightarrow h_{X_\alpha}$ . Since  $F$  is local, the maps  $i_\alpha \cdot g_\alpha$  glue together to give a map  $g : h_T \longrightarrow F$ , i. e. an element  $g \in F(T)$ . Define  $\sigma(f) = g$ .

One can directly check that  $\rho$  and  $\sigma$  are inverse to each other and that the given correspondence is functorial. ■

This theorem has an important corollary.

**Corollary 5.3.7.** *Fibered products exist in the category of superschemes. The fibered product  $X \times_Z Y$ , for superschemes  $X, Y, Z$  with morphisms  $f : X \longrightarrow Z$ ,  $g : Y \longrightarrow Z$  is the superscheme whose functor of points is  $h_X \times_{h_Z} h_Y$ .*

*Proof.* The proof follows the classical proof, and full details can be found for example in [9] I §1, 5.1. For completeness we will briefly sketch the argument. Let

$F = X \times_Z Y$ . We want to show  $F$  is representable. One can check that  $F$  is local. We then want to show it can be covered by open affine subfunctors. Let  $\{Z_i\}$  be a cover by affine open subschemes of  $Z$ . Define  $X_i = X \times_Z Z_i = f^{-1}(Z_i)$  and  $Y_i = Y \times_Z Z_i = g^{-1}(Z_i)$ . Let  $X_{i\alpha}$  and  $Y_{j\beta}$  open affine covers of  $X_i$  and  $Y_j$  respectively. One can check  $X_{i\alpha} \times_{Z_i} Y_{j\beta}$  form an affine open cover of  $F$ . Hence  $F$  is representable.  $\blacksquare$

**Remark 5.3.8.** One could also prove directly the existence of fibered product in the category of superschemes. This is done exactly as in the classical case, see for example Theorem 3.3 in chapter II of [13].

**Remark 5.3.9.** Theorem 5.3.6 can be stated also in the  $C^\infty$  category:

*Let  $F$  be a functor  $F : (\text{smfld}) \longrightarrow (\text{sets})$ , such that when restricted to the category of manifolds is representable.*

*Then the functor  $F$  is representable if and only if:*

- 1.  $F$  is local, i. e. it has the sheaf property.*
- 2.  $F$  is covered by open supermanifold functors.*

where an open supermanifold functors is a subfunctor  $U$  of  $F$  such that for all  $f : h_X \longrightarrow F$ ,  $f^{-1}(U) = h_V$  where  $V$  is a submanifold of  $X$  (here  $h_X$  denotes the functor of points of the supermanifold  $X$ ).

The proof of this result in the  $C^\infty$  category is essentially the same as the one seen in the algebraic category.

## 5.4 The Grassmannian superscheme

In this section we want to discuss the grassmannian of the  $r|s$ -dimensional superspaces inside a super vector space of dimension  $m|n$ ,  $r < m$ ,  $s < n$ . We will show that it is a superscheme using the Theorem 5.3.6. This is a particularly important example since it is the first non trivial example of a non affine superscheme.

Consider the functor  $Gr : (\text{salg}) \longrightarrow (\text{sets})$ , where for any superalgebra  $A$ ,  $Gr(A)$  is the set of projective  $A$ -submodules of rank  $r|s$  of  $A^{m|n}$  (for the definition of the rank of a projective  $A$ -module see the Appendix 7).

Equivalently  $Gr(A)$  can also be defined as:

$$Gr(A) = \{\alpha : A^{n|m} \longrightarrow L \mid \alpha \text{ surjective, } L \text{ projective } A\text{-module of rank } r|s\} \\ (\text{modulo equivalence}).$$

We need also to specify  $Gr$  on morphisms  $\psi : A \longrightarrow B$ .

Given a morphism  $\psi : A \rightarrow B$  of superalgebras, we can give to  $B$  the structure of right  $A$ -module by setting

$$a \cdot b = \psi(a)b, \quad a \in A, b \in B.$$

Also, given an  $A$ -module  $L$ , we can construct the  $B$ -module  $L \otimes_A B$ . So given  $\psi$  and the element of  $Gr(A)$ ,  $f : A^{m|n} \rightarrow L$ , we have an element of  $Gr(B)$ ,

$$Gr(\psi)(f) : B^{m|n} = A^{m|n} \otimes_A B \rightarrow L \otimes_A B.$$

We want to show that  $Gr$  is the functor of points of a superscheme.

We will start by showing it admits a cover of open affine subfunctors. Consider the multiindex  $I = (i_1, \dots, i_r | \mu_1, \dots, \mu_s)$  and the map  $\phi_I : A^{r|s} \longrightarrow A^{m|n}$  where  $\phi_I(x_1, \dots, x_r | \xi_1, \dots, \xi_s)$  is the  $m|n$ -uple with  $x_1, \dots, x_r$  occupying the position



$i_1, \dots, i_r$  and  $\xi_1, \dots, \xi_s$  occupying the position  $\mu_1, \dots, \mu_s$  and the other positions are occupied by zero. For example, let  $m = n = 2$  and  $r = s = 1$ . Then  $\phi_{1|2}(x, \xi) = (x, 0|0, \xi)$ .

Now define the subfunctors  $v_I$  of  $Gr$  as follows. The  $v_I(A)$  are the maps  $\alpha : A^{m|n} \longrightarrow L$  such that  $\alpha \cdot \phi_I$  is invertible.

We want to show that the  $v_I$  are open affine subfunctors of  $Gr$ . The condition that  $v_I$  is an open subfunctor is equivalent to asking that  $f^{-1}(v_I)$  is open for any map  $f : \text{Spec}A \longrightarrow Gr$ .

By Yoneda's lemma, a map  $f : \text{Spec}A \longrightarrow Gr$  corresponds to a point  $f$  in  $Gr(A)$ . So we are asking if there exists an open subscheme  $V_I$  in  $\text{Spec}A$ , such that

$$h_{V_I}(B) = \{\psi : A \longrightarrow B \mid Gr(\psi)(f) \in v_I(B)\} \subset h_{\underline{\text{Spec}A}}(B)$$

To show  $V_I$  is open, consider the matrix

$$Z = (f(e_{i_1}) \dots f(e_{i_r}), f(e_{\iota_1}) \dots f(e_{\iota_s}))$$

and define  $b_A(f)$  the product of the determinants of the two even diagonal blocks of  $Z$ .

If  $b_A(f)$  is invertible, then any map  $\psi : A \longrightarrow B$  is forced to send  $b_A(f)$  into an invertible element in  $B$ , hence all maps  $Gr(\psi)(f)$  are in  $v_I(B)$ . Hence  $V_I = \underline{\text{Spec}A}$ .

If  $b_A(f)$  is zero, then no map can send  $b_A(f)$  into an invertible element, so  $V_I$  is empty.

The only non trivial case is when  $b_A(f)$  is non-zero and not invertible. In this case since  $b_A(f)$  is sent to an invertible element in  $B$  by  $\psi$  we have a one to one

correspondence between such maps  $\psi$  and  $\psi' : A[b_A(f)^{-1}] \longrightarrow B$ . So we have obtained that  $h_{V_I} \cong \underline{\text{Spec}} A[b_A(f)^{-1}]$  which is open in  $\underline{\text{Spec}} A$ .

It remains to show that these subfunctors cover  $Gr$ .

Given  $f \in Gr(A)$ , that is a function from  $h_{\underline{\text{Spec}} A} \longrightarrow Gr$ , we have that since  $f$  is surjective, there exists at least an index  $I$  so that  $b_A(f)$  is invertible, hence  $f \in v_I(A)$  for this  $I$ . The above argument shows that we obtain a cover of any  $\underline{\text{Spec}} A$  by taking  $v_I \times_{Gr} h_{\underline{\text{Spec}} A}$ .

Finally we want to show that  $Gr$  is local. This is immediate once we identify  $Gr(A)$  with coherent sheaves with locally constant of rank  $r|s$ .

$$Gr(A) \cong \{\mathcal{F} \subset \mathcal{O}_A^{m|n} / \mathcal{F} \text{ is a subsheaf, of locally constant rank } r|s\}$$

where  $\mathcal{O}_A^{m|n} = k^{m|n} \otimes \mathcal{O}_A$ .

By its very definition this functor is local.

This identification is possible since by the Appendix 7.2 we prove that a projective module  $M$  is locally free and the correspondence between coherent sheaves and finitely generated modules in the supersetting.

So we have shown that  $Gr$  is the functor of points of a superscheme that we will call the *supergrassmannian* of  $r|s$  subspaces into a  $m|n$  dimensional space.

## 5.5 The infinitesimal theory

In this section we discuss the infinitesimal theory of superschemes. We define the notion of tangent space to a superscheme and to a supervariety at a point

of the underlying topological space. We then use these definitions in explicit computations.

Let  $k$  be a field.

**Definition 5.5.1.** Let  $X = (|X|, \mathcal{O}_X)$  be a superscheme (supervariety). We say that  $X$  is *algebraic* if it admits an open affine finite cover  $\{X_i\}_{i \in I}$  such that  $\mathcal{O}_X(X_i)$  is a finitely generated superalgebra for each  $X_i$ .

Unless otherwise specified all superschemes are assumed to be algebraic.

Given a superscheme  $X = (|X|, \mathcal{O}_X)$  each point of  $x$  in the topological space  $|X|$  belongs to an open affine subsuperscheme  $\underline{\text{Spec}} A$ ,  $x \cong \mathbf{p} \in \text{Spec} A_0$ , so that  $\mathcal{O}_{X,x} \cong A_{\mathbf{p}}$ . Recall that  $A_{\mathbf{p}}$  is the localization of the  $A_0$ -module  $A$  into the prime ideal  $\mathbf{p} \subset A_0$  and that

$$A_{\mathbf{p}} = \left\{ \frac{f}{g} \mid g \in A_0 - \mathbf{p} \right\}.$$

The local ring  $A_{\mathbf{p}}$  contains the maximal ideal  $\mathbf{p}_x$  generated by the maximal ideal in the local ring  $(A_{\mathbf{p}})_0$  and the generators of  $(A_{\mathbf{p}})_1$  as  $A_0$ -module.

We want to define the notion of a rational point of a scheme. We will then define the tangent space to a scheme in a rational point.

**Definition 5.5.2.** Let  $X = (|X|, \mathcal{O}_X)$  be a superscheme. A point  $x \in |X|$  is said to be *rational* if  $\mathcal{O}_{X,x}/\mathbf{p}_x \cong k$ .

**Remark 5.5.3.** As in the commutative case we have that if  $k$  is algebraically closed all closed points of  $|X|$  are rational. This is because the field  $\mathcal{O}_{X,x}/\mathbf{p}_x$  is a finite algebraic extension of  $k$ . (see [1] 7.9 for more details).

(Recall that a point  $x \in |X|$  is *closed* if it corresponds to a maximal ideal in  $\text{Spec} A_0$ , where  $(\text{Spec} A_0, \mathcal{O}_A) \subset X$  is any affine open neighbourhood of  $x$ ).

It is important not to confuse the points of the underlying topological space  $|X|$  with the elements obtained via the functor of points,  $h_X(A)$  for a generic  $A \in (\text{salg})$ . These are called  $A$ -points of the superscheme  $X$ . The next observation clarifies the relationship between the points of  $X$  and  $h_X$  the functor of points of  $X$ .

**Observation 5.5.4.** There is a bijection between the rational points of a superscheme  $X$  and the set of its  $k$ -points  $h_X(k)$ . In fact, an element  $(|f|, f^*) \in h_X(k)$ ,  $|f| : \text{Spec} k \longrightarrow |X|$ ,  $f^* : \mathcal{O}_{X,x} \longrightarrow k$ , determines immediately a point  $x = |f|(0)$ , which is rational.

**Definition 5.5.5.** Let  $A$  be a superalgebra and  $M$  an  $A$ -module. Let  $D : A \longrightarrow M$  be an additive map with the property  $D(a) = 0$ ,  $\forall a \in k$ . We say that  $D$  is an *super derivation* if:

$$D(fg) = D(f)g + (-1)^{p(D)p(f)} fD(g), \quad f, g \in A$$

where  $p$  as always denotes the parity.

**Definition 5.5.6.** Let  $X = (|X|, \mathcal{O}_X)$  be a superscheme and  $x$  a rational point in  $|X|$ . We define *tangent space of  $X$  at  $x$* :

$$T_x X = \text{Der}(\mathcal{O}_{X,x}, k)$$

where  $k$  is viewed as  $\mathcal{O}_{X,x}$ -module via the identification  $k \cong \mathcal{O}_{X,x} / \mathfrak{m}_{X,x}$ , where  $\mathfrak{m}_{X,x}$  is the maximal ideal in  $\mathcal{O}_{X,x}$ .

The next proposition gives an equivalent definition for the tangent space.

**Proposition 5.5.7.** *Let  $X$  be a superscheme, then:*

$$T_x X = \text{Der}(\mathcal{O}_{X,x}, k) \cong \underline{\text{Hom}}_{(\text{smod})}(\mathbf{m}_{X,x}/\mathbf{m}_{X,x}^2, k).$$

Note that  $\mathbf{m}_{X,x}/\mathbf{m}_{X,x}^2$  is a  $\mathcal{O}_{X,x}$ -supermodule which is annihilated by  $\mathbf{m}_{X,x}$ , hence it is a  $k = \mathcal{O}_{X,x}/\mathbf{m}_{X,x}$ -supermodule i.e. a super vector space.

*Proof.* Let  $D \in \text{Der}(\mathcal{O}_{X,x}, k)$ . Since  $D$  is zero on  $k$  and  $\mathcal{O}_{X,x} = k \oplus \mathbf{m}_{X,x}$  we have that  $D$  is determined by its restriction to  $\mathbf{m}_{X,x}$ ,  $D|_{\mathbf{m}_{X,x}}$ . Moreover since  $\mathbf{m}_{X,x}$  acts as zero on  $k \cong \mathcal{O}_{X,x}/\mathbf{m}_{X,x}$  one can check that

$$\begin{array}{ccc} \psi : \text{Der}(\mathcal{O}_{X,x}, k) & \longrightarrow & \underline{\text{Hom}}_{(\text{smod})}(\mathbf{m}_{X,x}/\mathbf{m}_{X,x}^2, k) \\ D & \longmapsto & D|_{\mathbf{m}_{X,x}} \end{array}$$

is well defined.

Now we construct the inverse. Let  $\alpha : \mathbf{m}_{X,x} \longrightarrow k$ ,  $\alpha(\mathbf{m}_{X,x}^2) = 0$ . Define

$$D_\alpha : \mathcal{O}_{X,x} = k \oplus \mathbf{m}_{X,x} \longrightarrow k, \quad D_\alpha(a, f) = \alpha(f).$$

This is a well defined superderivation.

Moreover one can check that the map  $\alpha \mapsto D_\alpha$  is  $\psi^{-1}$ . ■

The next proposition provides a characterization of the tangent space, that is useful for explicit calculations.

**Proposition 5.5.8.** *Let  $X = (|X|, \mathcal{O}_X)$  be a supervariety  $x \in |X|$  a rational closed point. Let  $U$  be an affine neighbourhood of  $x$ ,  $\mathbf{m}_x \subset k[U]$  the maximal ideal corresponding to  $x$ . Then*

$$T_x X \cong \underline{\text{Hom}}_{(\text{smod})}(\mathbf{m}_{X,x}/\mathbf{m}_{X,x}^2, k) \cong \underline{\text{Hom}}_{(\text{smod})}(\mathbf{m}_x/\mathbf{m}_x^2, k).$$

*Proof.* The proof is the same as in the ordinary case and is based on the fact that localization commutes with exact sequences. ■

Let's compute explicitly the tangent space in an example.

**Example 5.5.9.** Consider the affine supervariety represented by the coordinate ring:

$$\mathbf{C}[x, y, \xi, \eta]/(x\xi + y\eta).$$

Since  $\mathbf{C}$  is algebraically closed, all closed points are rational. Consider the closed point  $P = (1, 1, 0, 0) \cong \mathbf{m}_P = (x - 1, y - 1, \xi, \eta) \subset \mathbf{C}[x, y, \xi, \eta]/(x\xi + y\eta)$  (we identify  $(x_0, y_0, 0, 0)$  with maximal ideals in the ring of the supervariety, as we do in the commutative case). By Proposition 6.3.5, the tangent space at  $P$  is given by all the functions  $\alpha : \mathbf{m}_P \longrightarrow k$ ,  $\alpha(\mathbf{m}_P^2) = 0$ .

A generic  $f \in \mathbf{m}_P$  lifts to the family of  $f = f_1 + f_2(x\xi + y\eta) \in \mathbf{C}[\mathbf{A}^{1|1}] = \mathbf{C}[x, y, \xi, \eta]$  with  $f_1(1, 1, 0, 0) = 0$  and where  $f_2$  is any function in  $\mathbf{C}[\mathbf{A}^{1|1}] = \mathbf{C}[x, y, \xi, \eta]$ . Thus  $f$  can be formally expanded in power series around  $P$  (see [23] for more details).

$$f = \frac{\partial f_1}{\partial x}(P)(x - 1) + \frac{\partial f_1}{\partial y}(P)(y - 1) + \left(\frac{\partial f_1}{\partial \xi}(P) + f_2(P)\right)\xi +$$

$$\left(\frac{\partial f_1}{\partial \eta}(P) + f_2(P)\right)\eta + \text{higher order terms.}$$

Define:

$$X = \frac{\partial f_1}{\partial x}(P), \quad Y = \frac{\partial f_1}{\partial y}(P), \quad \Xi = \frac{\partial f_1}{\partial \xi}(P), \quad E = \frac{\partial f_1}{\partial \eta}(P).$$

These are coordinates for the supervector space  $\mathbf{M}_P/\mathbf{M}_P^2$ ,  $\mathbf{M}_P = (x - 1, y - 1, \xi, \eta) \subset \mathbf{C}[x, y, \xi, \eta]$ . A basis for the dual space  $(\mathbf{M}_P/\mathbf{M}_P^2)^*$  consists of sending the coefficient of one of the  $x - 1, y - 1, \xi, \eta$  to a non zero element and the others

to zero. This gives relations that allow us to eliminate the parameter  $f_2(P)$ . We get equations:

$$\Xi + f_2(P) = 0, \quad E + f_2(P) = 0.$$

Eliminating the parameter we get the equation for the tangent space:

$$\Xi + E = 0.$$

So we have described the tangent space  $(\mathbf{m}_P/\mathbf{m}_P^2)^*$  as a subspace of  $(\mathbf{M}_P/\mathbf{M}_P^2)^*$ , the tangent space to the affine superspace  $\mathbf{A}^{m|n}$ .

There is yet another way to compute the tangent space, in the case  $X$  is an affine supervariety. Before we examine this construction, we must understand first the notion of differential of a function and of a morphism.

**Definition 5.5.10.** Let  $X = (|X|, \mathcal{O}_X)$  be a superscheme,  $x$  a rational point.

Consider the projections:

$$\pi : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\mathbf{m}_{X,x} \cong k, \quad p : \mathbf{m}_{X,x} \longrightarrow \mathbf{m}_{X,x}/\mathbf{m}_{X,x}^2$$

Let  $f \in \mathcal{O}_{X,x}$ , we define *value of  $f$  at  $x$* :

$$f(x) =_{\text{def}} \pi(f).$$

We also define *differential of  $f$  at  $x$* :

$$(df)_x =_{\text{def}} p(f - f(x)).$$

We now want to define value and differential of a section in a point.

If  $U$  is an open neighbourhood of  $x$  and  $f \in \mathcal{O}_X(U)$  we define *value of  $f$  at  $x$*  to be:

$$f(x) =_{\text{def}} \pi(\phi(f)), \quad \phi : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x}.$$

We define *differential of  $f$  at  $x$*

$$(df)_x =_{\text{def}} (d\phi(f))_x.$$

For example, if  $P = (x_1^0 \dots x_m^0, 0 \dots 0)$  is a closed rational point of the affine superspace  $\mathbf{A}^{m|n}$  with coordinate ring  $k[x_1 \dots x_m, \xi_1 \dots \xi_n]$  a basis of  $\mathbf{m}_P/\mathbf{m}_P^2$  is  $\{x - x_i^0, \xi_j\}_{\{i=1\dots m, j=1\dots n\}}$ . Hence:

$$(dx_i)_P = x - x_i^0, \quad (d\xi_j)_P = \xi_j, \quad i = 1 \dots m, \quad j = 1 \dots n$$

Let  $(|\alpha|, \alpha^*) : X \longrightarrow Y$  be a morphism of superschemes and  $x$  a rational point,  $|\alpha|(x)$  also rational,  $\alpha$  induces a morphism  $d\alpha_x : T_x X \longrightarrow T_{|\alpha|(x)} Y$ , by:

$$d\alpha_x(D)f = D(\alpha_x^*(f)), \quad D \in T_x X = \text{Der}(\mathcal{O}_{X,x}, k), \quad \alpha_x^* : \mathcal{O}_{Y,|\alpha|(x)} \longrightarrow \mathcal{O}_{X,x}$$

It is simple to check that if  $(|\alpha|, \alpha^*)$  is an immersion i.e. it identifies  $X$  with a subscheme of  $Y$ , then  $d\alpha_x$  is injective. Hence if  $X$  is a subsupervariety of  $\mathbf{A}^{m|n}$  it makes sense to ask for equations that determine the tangent superspace to  $X$  as a linear subsuperspace of  $T_x \mathbf{A}^{m|n} \cong k^{m|n}$ .

**Proposition 5.5.11.** *Let  $X$  be a subvariety of  $\mathbf{A}^{m|n}$  and let  $x$  be a rational closed point of  $X$ . Then*

$$T_x X = \{v \in k^{m|n} | (df)_x(v) = 0, \forall f \in I\}$$

where  $I$  is the ideal defining  $X$  in  $k[\mathbf{A}^{m|n}]$ .

*Proof.* The immersion  $\alpha : X \subset \mathbf{A}^{m|n}$  corresponds to a surjective morphism  $\phi : k[\mathbf{A}^{m|n}] \longrightarrow k[X]$ , hence  $k[X] \cong k[\mathbf{A}^{m|n}]/I$ . Let  $\mathbf{m}_x$  and  $\mathbf{M}_x$  denote respectively



the maximal ideal associated to  $x$  in  $X$  and  $\text{Spec}k[\mathbf{A}^{m|n}]$  respectively.  $\phi$  induces a surjective linear map  $\psi$  between superspaces:

$$\psi : \mathbf{M}_x / \mathbf{M}_x^2 \longrightarrow \mathbf{m}_x / \mathbf{m}_x^2.$$

Let's recall the following simple fact of linear algebra. If  $a : V_1 \longrightarrow V_2$  is a surjective linear map between finite dimensional vector spaces  $V_1, V_2$  and  $b : V_2^* \subset V_1^*$  is the injective linear map induced by  $a$  on the dual vector spaces then  $s \in \text{Im}(b)$  if and only if  $s|_{\ker(a)} = 0$ . We apply this to the differential

$$(d\alpha)_x : T_x(X) = (\mathbf{m}_x / \mathbf{m}_x^2)^* \longrightarrow T_{\alpha(x)}(\mathbf{A}^{m|n}) = (\mathbf{M}_x / \mathbf{M}_x^2)^*$$

and we see that

$$T_x(X) = \{v \in T_{\alpha(x)}(\mathbf{A}^{m|n}) | v(\ker(\psi)) = 0\}.$$

Observe that  $\ker(\psi) = \{(df)_x | f \in I\}$ . By identifying  $\mathbf{A}^{m|n} = k^{m|n}$  with its double dual  $(k^{m|n})^{**}$  we obtain the result. ■

**Remark 5.5.12.** In the notation of the previous proposition, if  $I = (f_1 \dots f_r)$  one can check that:

$$T_x X = \{v \in k^{m|n} | (df_i)_x(v) = 0, \forall i = 1 \dots r\}.$$

Let's revisit Example 5.5.9 and see how the calculation is made using Proposition 5.5.11.

**Example 5.5.13.** Consider again the supervariety represented by:

$$\mathbf{C}[x, y, \xi, \eta] / (x\xi + y\eta).$$

We want to compute the tangent space at  $P = (1, 1, 0, 0) = (x_0, y_0, \xi_0, \eta_0)$ .

$$\begin{aligned} d(x\xi + y\eta)_P &= x_0(d\xi)_P + \xi_0(dx)_P + y_0(d\eta)_P + \eta_0(dy)_P \\ &= (d\xi)_P + (d\eta)_P \cong (0, 0, 1, 1). \end{aligned}$$

Hence by 5.5.11 the tangent space is the subspace of  $k^{2|2}$  given by the equation:

$$\xi + \eta = 0.$$

## CHAPTER 6

### Algebraic Supergroups and Lie Superalgebras

In this section we introduce the notion of supergroup scheme, and of its Lie superalgebra.

Let  $k$  be a noetherian ring.

All superschemes are assumed to be algebraic.

#### 6.1 Supergroup functors and algebraic supergroups

A supergroup scheme is a superscheme whose functor of points is group valued. In order to study supergroup schemes we need first to understand the weaker notion of supergroup functor.

**Definition 6.1.1.** A *supergroup functor* is a group valued functor:

$$G : (\text{salg}) \longrightarrow (\text{sets})$$

**Remark 6.1.2.** Saying that  $G$  is group valued is equivalent to have the following natural transformations:

1. Multiplication  $\mu : G \times G \longrightarrow G$ , such that  $\mu \circ (\mu \times \text{id}) = (\mu \times \text{id}) \circ \mu$ , i. e.

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times \text{id}} & G \times G \\ \text{id} \times \mu \downarrow & & \downarrow \mu \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

2. Unit  $e : e_k \longrightarrow G$ , where  $e_k : (\text{salg}) \longrightarrow (\text{sets})$ ,  $e_k(A) = 1_A$ , such that  $\mu \circ (\text{id} \otimes e) = \mu \circ (e \times \text{id})$ , i. e.

$$\begin{array}{ccccc} G \times e_k & \xrightarrow{\text{id} \times e} & G \times G & \xleftarrow{e \times \text{id}} & e_k \times G \\ & \searrow & \downarrow \mu & \swarrow & \\ & & G & & \end{array}$$

3. Inverse  $i : G \longrightarrow G$ , such that  $\mu \circ (\text{id} \times i) = e \circ \text{id}$ , i. e.

$$\begin{array}{ccc} G & \xrightarrow{(\text{id}, i)} & G \times G \\ \downarrow & & \downarrow \mu \\ e_k & \xrightarrow{e} & G \end{array}$$

If  $G$  is the functor of points of a superscheme  $X$ , we say that  $X$  is a *supergroup scheme*. An *affine supergroup scheme* is a supergroup scheme which is an affine superscheme. To make the terminology easier we will drop the word “scheme” when speaking of supergroup schemes.

**Observation 6.1.3.** The functor of points of an affine supergroup  $G$  is a representable functor. It is represented by the superalgebra  $k[G]$ . This superalgebra has a Hopf superalgebra structure, so we identify the category of affine supergroups with the category of commutative Hopf superalgebras.

**Observation 6.1.4.** If  $k$  is a field, we may interpret the unit  $e$  of a supergroup  $G = (|G|, \mathcal{O}_G)$  as a rational point of  $G$  that we will denote with  $1_G$ . In fact  $e : e_k \longrightarrow G$ ,  $e = (|e|, e^*)$ . Define  $1_G = |e|(|e_k|)$ . This is a rational point, in fact  $\mathcal{O}_{G, 1_G}/m_{1_G} \cong k$ . Moreover by the very definition of  $e$ ,  $1_G$  has the property of a unit for the group  $|G|$ .

**Example 6.1.5.** 1. *Supermatrices*  $M_{m|n}$ . It is immediate to verify that the supermatrices discussed in chapter 1 are an affine supergroup where  $\mu$  is interpreted as the usual matrix addition.

2. *The general linear supergroup*  $GL_{m|n}$ .

Let  $A \in (\text{salg})$ . Define  $GL_{m|n}(A)$  as  $GL(A^{m|n})$  (see chapter 1) the set of automorphisms of the  $A$ -supermodule  $A^{m|n}$ . Choosing coordinates we can write

$$GL_{m|n}(A) = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \right\}$$

where  $a$  and  $b$  are  $m \times m$ ,  $n \times n$  blocks of even elements and  $\forall, \alpha, \beta$   $m \times n$ ,  $n \times m$  blocks of odd elements and  $a$  and  $b$  are invertible matrices.

It is not hard to see that this is the functor of points of an affine supergroup  $GL(m|n)$  represented by the Hopf superalgebra

$$k[GL(m|n)] = k[x_{ij}, \xi_{kl}][T]/(T\text{Ber} - 1),$$

where  $x_{ij}$ 's and  $\xi_{kl}$ 's are respectively even and odd variables with  $1 \leq i, j \leq m$  or  $m+1 \leq i, j \leq m+n$ ,  $1 \leq k \leq m$ ,  $m+1 \leq l \leq m+n$  or  $m+1 \leq k \leq m+n$ ,  $1 \leq l \leq m$  and  $\text{Ber}$  denotes the Berezinian.

In general if  $V$  is a super vector space we define the functor  $GL(V)$  as  $GL(V)(A) = GL(V(A))$ , the invertible transformations of  $V(A)$  preserving parity.

3. *The special linear group*  $SL_{m|n}$ .

For a superalgebra  $A$ , let's define  $SL_{m|n}(A)$  to be the subset of  $GL_{m|n}(A)$  consisting of matrices with Berezinian equal to 1. This is the functor of points of an affine supergroup and it is represented by the Hopf superalgebra:

$$k[SL(m|n)] = k[x_{ij}, \xi_{kl}]/(\text{Ber} - 1).$$

Similarly one can construct the functor of points and the representing Hopf superalgebras for all the classical algebraic supergroups (see [7, 23]).

## 6.2 Lie superalgebras

Assume 2 and 3 are not zero divisors in  $k$ .

In this section we define functorially the notion of Lie superalgebra.

Our definition is only apparently different from the one we have introduced in chapter 1, which is the one mostly used in the literature.

Let  $\mathcal{O}_k : (\text{salg}) \longrightarrow (\text{sets})$  be the functor represented by  $k[x]$ .  $\mathcal{O}_k$  corresponds to an ordinary algebraic variety, namely the affine line. For a superalgebra  $A$  we have that  $\mathcal{O}_k(A) = A_0$ .

**Definition 6.2.1.** Let  $\mathfrak{g}$  be a free  $k$ -module. We say that the group valued functor

$$L_{\mathfrak{g}} : (\text{salg}) \longrightarrow (\text{sets}), \quad L_{\mathfrak{g}}(A) = (A \otimes \mathfrak{g})_0$$

is a *Lie superalgebra* if there is a  $\mathcal{O}_k$ -linear natural transformation

$$[\ , \ ] : L_{\mathfrak{g}} \times L_{\mathfrak{g}} \longrightarrow L_{\mathfrak{g}}$$

that satisfies commutative diagrams corresponding to the antisymmetric property and the Jacobi identity. For each superalgebra  $A$ , the bracket  $[\ , \ ]_A$  defines a Lie algebra structure on the  $A$ -module  $L_{\mathfrak{g}}(A)$ , hence the functor  $L_{\mathfrak{g}}$  is *Lie algebra* valued. We will drop the suffix  $A$  from the bracket and the natural transformations to ease the notation.

**Remark 6.2.2.** In general, a Lie superalgebra is not representable. However if  $\mathfrak{g}$  is finite dimensional,

$$L_{\mathfrak{g}}(A) = (A \otimes \mathfrak{g})_0 = \text{Hom}_{(\text{smod})}(\mathfrak{g}^*, A) = \text{Hom}_{(\text{salg})}(\text{Sym}(\mathfrak{g}^*), A)$$

where  $(\text{smod})$  denotes the category of supermodules and  $\text{Sym}(\mathfrak{g}^*)$  the symmetric algebra over  $\mathfrak{g}^*$ . In this special case  $L_{\mathfrak{g}}$  is representable and it is an affine superscheme represented by the superalgebra  $\text{Sym}(\mathfrak{g}^*)$ .

We want to see that the usual notion of Lie superalgebra, as defined by Kac (see [14]) among many others, is equivalent to this functorial definition.

Recall that in chapter 1 we gave the following definition of Lie superalgebra.

Let  $k$  be a field,  $\text{char}(k) \neq 2, 3$ .

**Definition 6.2.3.** Let  $\mathfrak{g}$  be a super vector space. We say that a bilinear map  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$  is a *superbracket* if  $\forall x, y, z \in \mathfrak{g}$ :

- a)  $[x, y] = (-1)^{p(x)p(y)}[y, x]$
- b)  $[x, [y, z]] + (-1)^{p(x)p(y)+p(x)p(z)}[y, [z, x]] + (-1)^{p(x)p(z)+p(y)p(z)}[z, [x, y]] = 0$ .

The super vector space  $\mathfrak{g}$  satisfying the above properties is commonly defined as Lie superalgebra in the literature.

**Observation 6.2.4.** The definition 6.2.1 and 6.2.3 are equivalent. In other words given a Lie algebra valued functor  $L_{\mathfrak{g}} : (\text{salg}) \longrightarrow (\text{sets})$  we can build a super vector space  $\mathfrak{g}$  with a superbracket and vice-versa. Let's see these constructions in detail.

If we have a Lie superalgebra  $L_{\mathfrak{g}}$  there is always, by definition, a super vector space  $\mathfrak{g}$  associated to it. The superbracket on  $\mathfrak{g}$  is given following the *even rules*.

Let's see in detail what it amounts to in this case (for a complete treatment of even rules see pg 57 [7]). Given  $v, w \in \mathfrak{g}$ , since the Lie bracket on  $L_{\mathfrak{g}}(A)$  is  $A$ -linear we can define the element  $\{v, w\} \in \mathfrak{g}$  as:

$$[a \otimes v, b \otimes w] = (-1)^{p(b)p(v)} ab \otimes \{v, w\} \in (A \otimes \mathfrak{g})_0 \in L_{\mathfrak{g}}(A).$$

Clearly the bracket  $\{v, w\} \in \mathfrak{g}$  does not depend on  $a, b \in A$ . It is straightforward to verify that it is a superbracket. Let's see, for example, the antisymmetry property. Observe first that if  $a \otimes v \in (A \otimes \mathfrak{g})_0$  must be  $p(v) = p(a)$ , since  $(\mathfrak{g} \otimes A)_0 = A_0 \otimes \mathfrak{g}_0 \oplus A_1 \otimes \mathfrak{g}_1$ . So we can write:

$$[a \otimes v, b \otimes w] = (-1)^{p(b)p(v)} \{v, w\} \otimes ab = (-1)^{p(w)p(v)} ab \otimes \{v, w\}.$$

On the other hand:

$$[b \otimes w, a \otimes v] = (-1)^{p(a)p(w)} ba \otimes \{w, v\} = (-1)^{p(a)p(w)+p(a)p(b)} ab \otimes \{w, v\} =$$

$$(-1)^{2p(w)p(v)} ab \otimes \{w, v\} = \{w, v\}.$$

By comparing the two expressions we get the antisymmetry of the superbracket. For the super Jacobi identity the calculation is the same.

A similar calculation also shows that given a supervector space with a superbracket one obtains a Lie superalgebra.

Hence a Lie superalgebra  $L_{\mathfrak{g}}$  according to Definition 6.2.3 is equivalent to a supervector space  $\mathfrak{g}$  with a superbracket. With an abuse of language we will refer to both  $\mathfrak{g}$  and  $L_{\mathfrak{g}}$  as "Lie superalgebra".

**Remark 6.2.5.** Given a supervector space  $\mathfrak{g}$  one may also define a Lie superalgebra to be the representable functor  $D_{\mathfrak{g}} : (\text{salg}) \longrightarrow (\text{sets})$  so that

$$D_{\mathfrak{g}}(A) = \text{Hom}_{(\text{smod})} (\mathfrak{g}^*, A) = \text{Hom}_{(\text{salg})} (\text{Sym}(\mathfrak{g}^*), A)$$



with a  $\mathcal{O}_k$  linear natural transformation  $[\cdot, \cdot] : D_{\mathfrak{g}} \times D_{\mathfrak{g}} \longrightarrow D_{\mathfrak{g}}$  satisfying the commutative diagrams corresponding to antisymmetry and Jacobi identity. When  $\mathfrak{g}$  is finite dimensional this definition coincides with the previous one, however we have preferred the one given in 6.2.3 since its immediate equivalence with the definition mostly used in the literature.

The purpose of the next two sections is to naturally associate a Lie superalgebra  $\text{Lie}(G)$  to a supergroup  $G$ .

### 6.3 $\text{Lie}(G)$ as tangent superspace to a supergroup scheme

For the rest of this chapter, let  $k$  be a field,  $\text{char}(k) \neq 2, 3$ .

Let  $G$  be a supergroup functor.

Let  $A$  be a commutative superalgebra and let  $A(\epsilon) =_{\text{def}} A[\epsilon]/(\epsilon^2)$  be the algebra of dual numbers ( $\epsilon$  here is taken as an *even* indeterminate). We have that  $A(\epsilon) = A \oplus \epsilon A$  and there are two natural morphisms:

$$\begin{aligned} i : A &\rightarrow A(\epsilon), & i(1) &= 1 \\ p : A(\epsilon) &\rightarrow A, & p(1) &= 1, \quad p(\epsilon) = 0, \quad p \cdot i = \text{id}_A \end{aligned}$$

**Definition 6.3.1.** Consider the homomorphism  $G(p) : G(A(\epsilon)) \longrightarrow G(A)$ . For each  $G$  there is a supergroup functor,

$$\text{Lie}(G) : (\text{salg}) \longrightarrow (\text{sets}), \quad \text{Lie}(G)(A) =_{\text{def}} \ker(G(p)).$$

If  $G$  is a supergroup scheme, we denote  $\text{Lie}(h_G)$  by  $\text{Lie}(G)$ .

**Example 6.3.2.** 1. *The super general linear algebra.*

We want to determine the functor  $\text{Lie}(\text{GL}_{m|n})$ . Consider the map:

$$\text{GL}_{m|n}(p) : \quad \text{GL}_{m|n}(A(\epsilon)) \quad \longrightarrow \quad \text{GL}_{m|n}(A)$$

$$\begin{pmatrix} p + \epsilon p' & q + \epsilon q' \\ r + \epsilon r' & s + \epsilon s' \end{pmatrix} \mapsto \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

with  $p, p', s, s'$  having entries in  $A_0$  and  $q, q', r, r'$  having entries in  $A_1$ ; the blocks  $p$  and  $s$  are invertible matrices. One can see immediately that

$$\text{Lie}(\text{GL}_{m|n})(A) = \ker(\text{GL}_{m|n}(p)) = \left\{ \begin{pmatrix} I_m + \epsilon p' & \epsilon q' \\ \epsilon r' & I_n + \epsilon s' \end{pmatrix} \right\}$$

where  $I_n$  is a  $n \times n$  identity matrix. The functor  $\text{Lie}(\text{GL}_{m|n})$  is clearly group valued and can be identified with the (additive) group functor  $M_{m|n}$  defined as:

$$M_{m|n}(A) = \text{Hom}_{(\text{smod})} (M(m|n)^*, A) = \text{Hom}_{(\text{salg})} (\text{Sym}(M(m|n)^*), A)$$

where  $M(m|n)$  is the supervector space

$$M(m|n) = \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right\} \cong k^{m^2+n^2|2mn}$$

where  $P, Q, R, S$  are respectively  $m \times m, m \times n, n \times m, n \times n$  matrices with entries in  $k$ . An element  $X \in M(m|n)$  is even if  $Q = R = 0$ , odd if  $P = S = 0$ .

Notice that  $M(m|n)$  is a Lie superalgebra with superbracket:

$$[X, Y] = XY - (-1)^{p(X)p(Y)} YX$$

So  $\text{Lie}(\text{GL}_{m|n})$  is a Lie superalgebra. In the next section we will see that in general we can give a Lie superalgebra structure to  $\text{Lie}(G)$  for any group scheme  $G$ .

2. *The special linear superalgebra.*

A similar computation shows that

$$\mathrm{Lie}(\mathrm{SL}_{m|n})(A) = \left\{ W = \begin{pmatrix} I_m + \epsilon p' & \epsilon q' \\ \epsilon r' & I_n + \epsilon s' \end{pmatrix} \mid \mathrm{Ber}(W) = 1 \right\}.$$

The condition on the Berezinian is equivalent to:

$$\det(1 - \epsilon s') \det(1 + \epsilon p') = 1$$

which gives:

$$\mathrm{tr}(p') - \mathrm{tr}(s') = 0.$$

Hence

$$\mathrm{Lie}(\mathrm{SL}_{m|n})(A) = \{X \in M_{m|n}(A) \mid \mathrm{Tr}(X) = 0\}.$$

Similar calculations can be done also for the other classical supergroups.

Let's now assume that  $G$  is a supergroup scheme.

We now want to show that  $\mathrm{Lie}(G) : (\mathrm{salg}) \longrightarrow (\mathrm{sets})$  is a representable functor and its representing superscheme is identified with the tangent space at the identity of the supergroup  $G$ .

**Definition 6.3.3.** Let  $X = (|X|, \mathcal{O}_X)$  be a superscheme,  $x \in |X|$ . We define *the first neighbourhood of  $X$ ,  $X_x$* , to be the superscheme  $\underline{\mathrm{Spec}} \mathcal{O}_{X,x}/m_{X,x}^2$ . The topological space  $|X_x|$  consists of the one point  $m_{X,x}$  which is the maximal ideal in  $\mathcal{O}_{X,x}$ .

**Observation 6.3.4.** There exists a natural map  $f : X_x \longrightarrow X$ . In fact we can write immediately

$$\begin{array}{ccc} |f| : \mathrm{Spec} \mathcal{O}_{X,x}/m_{X,x}^2 = \{m_{X,x}\} & \longrightarrow & X \\ m_{X,x} & \longmapsto & x \end{array}$$

$$f_U^* : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X_x}(|f|^{-1}(U)) = \mathcal{O}_{X,x}/m_{X,x}^2$$

where  $f_U^*$  is the composition of natural map from  $\mathcal{O}_X(U)$  to the direct limit  $\mathcal{O}_{X,x}$  and the projection  $\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/m_{X,x}^2$ .

We now want to make some observation on the identity element of a supergroup  $G$ . By definition we have that the identity is a map  $e : \underline{\text{Spec}} k \longrightarrow G$ . This corresponds to a map of the functor of points:  $h_{\underline{\text{Spec}} k} \longrightarrow h_G$  assigning to the only map  $1_A \in h_{\underline{\text{Spec}} k}(A)$  a map that we will denote  $1_{G(A)} \in h_G(A) = \text{Hom}(\underline{\text{Spec}} A, G)$ . The topological space map  $|1_{G(A)}|$  sends all the maximal ideals in  $\text{Spec} A$  to  $1_G \in |G|$ . The sheaf map  $\mathcal{O}_G \longrightarrow k$  is the evaluation at  $1_G$  that is  $\mathcal{O}_G(U) \longrightarrow \mathcal{O}_{G,1_G} \longrightarrow \mathcal{O}_{G,1_G}/m_{G,1_G} \cong k$  (the identity is a rational point). Hence it is immediate to verify that  $1_{G(A)}$  factors through  $G_1$  (the first neighbourhood at the identity  $1_G$ ), i. e.  $1_{G(A)} : \underline{\text{Spec}} A \longrightarrow G_1 \longrightarrow G$ . This fact will be crucial in the proof of the next theorem.

**Theorem 6.3.5.** *Let  $G$  be an algebraic supergroup. Then*

$$\text{Lie}(G)(A) = \text{Hom}_{(\text{smod})} (m_{G,1_G}/m_{G,1_G}^2, A) = (A \otimes T_{1_G}(G))_0$$

where  $T_{1_G}(G)$  denotes the tangent space in the rational point  $1_G$ .

*Proof.* Let  $d : m_{1_G}/m_{1_G}^2 \longrightarrow A$  be a linear map. Let  $d'$  be the map:

$$\mathcal{O}_{G,1_G}/m_{1_G}^2 \cong k \oplus m_{1_G}/m_{1_G}^2 \longrightarrow A(\epsilon)$$

$$(s, t) \longmapsto s + d(t)\epsilon.$$

So we have  $d' \in h_{G_1}(A(\epsilon))$  since  $G_1$  is a superscheme represented by  $\mathcal{O}_{G,1_G}/m_{1_G}^2$ .

This shows that we have a correspondence between  $h_{G_1}(A)$  and the elements of  $\text{Hom}_{(\text{smod})} (m_{1_G}/m_{1_G}^2, A)$ . Let  $\phi : G_1 \longrightarrow G$  be the map described in 6.3.3. By

Yoneda's lemma  $\phi$  induces  $\phi_{A(\epsilon)} : h_{G_1}(A(\epsilon)) \longrightarrow h_G(A(\epsilon))$ , hence we have a map

$$\begin{aligned} \psi : \text{Hom}_{(\text{smod})} (m_{1_G}/m_{1_G}^2, A) &\longrightarrow h_G(A(\epsilon)) \\ d &\mapsto \phi_{A(\epsilon)}(d'). \end{aligned}$$

The following commutative diagram shows that  $\psi(d) \in \ker(h_G(p)) = \text{Lie}(G)(A)$ .

$$\begin{array}{ccc} h_{G_1}(A(\epsilon)) & \xrightarrow{h_{G_1}(p)} & h_{G_1}(A) \\ d'' & \mapsto & 1_{G_1(A)} \\ \downarrow & & \downarrow \\ h_G(A(\epsilon)) & \xrightarrow{h_G(p)} & h_G(A) \\ \psi(d) & \mapsto & 1_{G(A)}. \end{array}$$

We now want to build an inverse for  $\psi$ . Let  $z \in \ker(h_G(p))$  i.e.  $h_G(p)z = 1_{G(A)}$ , where:

$$h_G(p) : h_G(A(\epsilon)) = \text{Hom}(\underline{\text{Spec}}A(\epsilon), G) \longrightarrow h_G(A) = \text{Hom}(\underline{\text{Spec}}A, G)$$

Here  $z$  factors via  $G_1$  and this is because  $1_{G(A)}$  splits via  $G_1$  (recall  $p : A(\epsilon) \longrightarrow A$  and it induces  $p^\# : \underline{\text{Spec}}A \longrightarrow \underline{\text{Spec}}A(\epsilon)$ ). Since  $z$  factors via  $G_1$ , that is  $z : \underline{\text{Spec}}A(\epsilon) \longrightarrow G_1 \longrightarrow G$ , this provides immediately a map  $\underline{\text{Spec}}A(\epsilon) \longrightarrow G_1$  corresponding to an element in  $\text{Hom}_{(\text{smod})} (m_{1_G}/m_{1_G}^2, A)$ .  $\blacksquare$

## 6.4 The Lie superalgebra of a supergroup scheme

We now want to show that, for any supergroup scheme  $G$ ,  $\text{Lie}(G)$  is a Lie superalgebra.

To ease the notation, throughout this section  $G$  will denote the functor of points of a supergroup scheme.

**Observation 6.4.1.**  $\text{Lie}(G)$  has a structure of  $\mathcal{O}_k$ -module. In fact let  $u_a : A(\epsilon) \longrightarrow A(\epsilon)$  be the endomorphism,  $u_a(1) = 1$ ,  $u_a(\epsilon) = a\epsilon$ , for  $a \in A_0$ .  $\text{Lie}(G)$  admits a  $\mathcal{O}_k$ -module structure, i.e. there is a natural transformation  $\mathcal{O}_k \times \text{Lie}(G) \longrightarrow \text{Lie}(G)$ , such that for any superalgebra  $A$

$$(a, x) \mapsto ax =_{\text{def}} \text{Lie}(G)(u_a)x, \quad a \in \mathcal{O}_k(A), \quad x \in \text{Lie}(G)(A).$$

For subgroups of  $\text{GL}(m|n)(A)$ ,  $ax$  corresponds to the multiplication of the matrix  $x$  by the even scalar  $a$ .

We now want to define a natural transformation  $[\cdot, \cdot] : \text{Lie}(G) \times \text{Lie}(G) \longrightarrow \text{Lie}(G)$  which has the properties of a superbracket.

Let  $\text{GL}(\text{Lie}(G))(A)$  be the (multiplicative) group of linear automorphisms and  $\text{End}(\text{Lie}(G))(A)$  be the (additive) group of linear endomorphisms of  $\text{Lie}(G)(A)$ . The natural  $\mathcal{O}_k$ -module structure of  $\text{Lie}(G)$  gives two group functors (one multiplicative the other additive)

$$\text{GL}(\text{Lie}(G)) : (\text{salg}) \longrightarrow (\text{sets}), \quad \text{End}(\text{Lie}(G)) : (\text{salg}) \longrightarrow (\text{sets}).$$

One can check

$$\text{Lie}(\text{GL}(\text{Lie}(G))) = \text{End}(\text{Lie}(G)).$$

**Definition 6.4.2.** The *adjoint action*  $\text{Ad}$  of  $G$  on  $\text{Lie}(G)$  is defined as the natural transformation

$$\text{Ad} : G \longrightarrow \text{GL}(\text{Lie}(G))$$

$$\text{Ad}(g)(x) = G(i)(g)xG(i)(g)^{-1}, \quad g \in G(A), \quad x \in \text{Lie}(G)(A).$$

The *adjoint action*  $\text{ad}$  of  $\text{Lie}(G)$  on  $\text{Lie}(G)$  is defined as

$$\text{ad} =_{\text{def}} \text{Lie}(\text{Ad}) : \text{Lie}(G) \longrightarrow \text{Lie}(\text{GL}(\text{Lie}(G))) = \text{End}(\text{Lie}(G)).$$

On  $\text{Lie}(G)$  we are ready to define a *bracket*:

$$[x, y] =_{\text{def}} \text{ad}(x)y, \quad x, y \in \text{Lie}(G)(A).$$

**Observation 6.4.3.** One can check that

$$\text{Ad}(g) = \text{Lie}(c(g))$$

where  $c(g) : G(A) \longrightarrow G(A)$ ,  $c(g)(x) = gxg^{-1}$ .

Our goal is now to prove that  $[\cdot, \cdot]$  is a Lie bracket.

In the next example we work out the bracket for  $\text{GL}_{m|n}$ . This example will be crucial for the next propositions.

**Example 6.4.4.** We want to see that in the case of  $\text{GL}_{m|n}$ , the Lie bracket  $[\cdot, \cdot]$  coincides with the bracket defined in Example 6.3.2. We have:

$$\text{Ad} : \text{GL}(A) \longrightarrow \text{GL}(\text{Lie}(\text{GL}_{m|n}))(A) = \text{GL}(\text{M}_{m|n}(A))$$

$$g \longmapsto \text{Ad}(g),$$

Since  $G(i) : \text{GL}_{m|n}(A) \longrightarrow \text{GL}_{m|n}(A(\epsilon))$  is an inclusion, if we identify  $\text{GL}_{m|n}(A)$  with its image we can write:

$$\text{Ad}(g)x = gxg^{-1}, \quad x \in \text{M}_{m|n}(A).$$

By definition we have:  $\text{Lie}(\text{GL}(\text{M}_{m|n}))(A) = \{1 + \epsilon\beta \mid \beta \in \text{GL}(\text{M}_{m|n})(A)\}$  So we have, for  $a, b \in \text{M}_{m|n}(A) \cong \text{Lie}(\text{GL}_{m|n})(A) = \{1 + \epsilon a \mid a \in \text{M}_{m|n}(A)\}$ :

$$\text{ad}(1 + \epsilon a)b = (1 + \epsilon a)b(1 - \epsilon a) = b + (ab - ba)\epsilon = b + \epsilon[a, b].$$

Hence  $\text{ad}(1 + \epsilon a) = \text{id} + \epsilon\beta(a)$ , with  $\beta(a) = [a, \cdot]$ .

It is important to observe that in  $G(A(\epsilon))$  it is customary to write the product of two elements  $x$  and  $y$  as  $xy$ . However as elements of  $Lie(G)(A)$ , their product is written as  $x + y$  (hence the unit is 0 and the inverse of  $x$  is  $-x$ ). In order to be able to switch between these two way of writing it is useful to introduce the notation  $e^{\epsilon x}$ .

**Definition 6.4.5.** Let  $\phi : A(\epsilon) \longrightarrow B$  be a superalgebra morphism such that  $\phi(\epsilon) = \alpha$ . We define  $e^{\alpha x} = G(\phi)(x)$ .

The following properties are immediate:

1.  $e^{\epsilon x} = x$ ,
2.  $e^{\alpha(x+y)} = e^{\alpha x} e^{\alpha y}$ ,
3.  $e^{\alpha(ax)} = e^{a\alpha x}$ ,
4. If  $f : G \longrightarrow H$ ,  $f(e^{\alpha x}) = e^{\alpha Lie(f)x}$ .

**Observation 6.4.6.** Observe that Property 4 above and the Example 6.4.4 give us:

$$\text{Ad}(e^{\epsilon x})y = y + \epsilon[x, y] = (\text{id} + \epsilon \text{ad}(x))y$$

**Lemma 6.4.7.** Let the notation be as above and let  $\epsilon, \epsilon'$  be two elements with square zero.

$$e^{\epsilon x} e^{\epsilon' y} e^{-\epsilon x} e^{-\epsilon' y} = e^{\epsilon \epsilon' [x, y]}$$

*Proof.* By the Property 4 and Observation 6.4.6 we have that:

$$e^{\epsilon x} e^{\epsilon' y} e^{-\epsilon x} = e^{\epsilon' \text{Ad}(e^{\epsilon x})y} = e^{\epsilon' (y + \epsilon[x, y])}.$$

So we have

$$e^{\epsilon x} e^{\epsilon' y} e^{-\epsilon x} = e^{\epsilon' y} e^{\epsilon \epsilon' [x, y]} = e^{\epsilon \epsilon' [x, y]} e^{\epsilon' y}$$



which gives the result. ■

**Proposition 6.4.8.** *The bracket  $[\cdot, \cdot]$  is antisymmetric.*

*Proof.* From the previous proposition we have that:

$$e^{\epsilon\epsilon'[x,y]} = e^{\epsilon\epsilon'[-y,x]}$$

from which we get the result. ■

**Proposition 6.4.9.** *Let  $\rho : G \longrightarrow \mathrm{GL}(V)$  be a morphism of supergroup functors. Then  $\mathrm{Lie}(\rho) : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(V)$  is a Lie superalgebras morphism.*

*Proof.* Using the notation introduced previously we have that by Observation 6.4.6 and Property 4:

$$\rho(e^{\epsilon x}) = e^{\epsilon \mathrm{Lie}(\rho)x} = \mathrm{id} + \epsilon \mathrm{Lie}(\rho)x$$

Using Proposition 6.4.7 we have:

$$\rho(e^{\epsilon\epsilon'[x,y]}) = \rho(e^{\epsilon x})\rho(e^{\epsilon' y})\rho(e^{-\epsilon x})\rho(e^{\epsilon' y}).$$

Hence using Property 4:

$$\mathrm{id} + \epsilon\epsilon' \mathrm{Lie}(\rho)[x, y] = (\mathrm{id} + \epsilon \mathrm{Lie}(\rho)x)(\mathrm{id} + \epsilon' \mathrm{Lie}(\rho)y)(\mathrm{id} - \epsilon \mathrm{Lie}(\rho)x)(\mathrm{id} - \epsilon' \mathrm{Lie}(\rho)y),$$

which immediately gives:

$$\mathrm{Lie}(\rho)[x, y] = [\mathrm{Lie}(\rho)(x), \mathrm{Lie}(\rho)(y)].$$

■

**Proposition 6.4.10.** *The bracket  $[\cdot, \cdot]$  satisfies the Jacobi identity.*

*Proof.* In the previous proposition take  $\rho = \text{Ad}$ . Then we have:

$$[\text{ad}(x), \text{ad}(y)] = \text{ad}([x, y]), \quad \forall x, y \in \text{Lie}(G)(A)$$

which gives us immediately the Jacobi identity. ■

**Corollary 6.4.11.** *The natural transformation  $[\cdot, \cdot] : \text{Lie}(G) \times \text{Lie}(G) \longrightarrow \text{Lie}(G)$  defined as*

$$[x, y] =_{\text{def}} \text{ad}(x)y, \quad x, y \in \text{Lie}(G)(A).$$

*is a Lie bracket for all  $A$ .*

*Proof.* Immediate from previous propositions. ■

## 6.5 Affine algebraic supergroups. Linear representations.

We now want to restrict our attention to the case of the supergroup scheme  $G$  to be an affine algebraic group.

Let's recall few facts from chapter 1. Let  $A$  and  $B$  be superalgebras. A morphism of algebras  $f : A \longrightarrow B$  (not necessarily in  $\text{Hom}_{(\text{salg})}(A, B)$ ) is called *even* if it preserves parity, *odd* if it “reverses” the parity i.e. sends even elements in odd elements. Clearly any morphism of algebras can be written as sum of an even and an odd one. Recall that  $\text{Hom}_{(\text{salg})}(A, B)$  consists only of the even maps. The set of all morphisms between  $A$  and  $B$  is called *inner Hom*, it is denoted with  $\underline{\text{Hom}}_{(\text{salg})}(A, B)$  and it can be made an object of  $(\text{salg})$ . Its even part is  $\text{Hom}_{(\text{salg})}(A, B)$ .

**Definition 6.5.1.** Let  $G$  be an affine algebraic supergroup,  $k[G]$  its Hopf superalgebra. We define the additive map  $D : k[G] \longrightarrow k[G]$  a *left invariant superderivation* if it satisfies the following properties.

1.  $D$  is  $k$ -linear i. e.  $D(a) = 0$ , for all  $a \in k$ ,
2.  $D$  satisfies the Leibniz identity,  $D(fg) = D(f)g + (-1)^{p(D)p(f)} fD(g)$ ,
3.  $\Delta \circ D = (id \otimes D) \circ \Delta$ , where  $\Delta$  denotes the comultiplication in  $k[G]$ .

**Observation 6.5.2.** The set  $L(G)$  of left invariant derivations of  $k[G]$  is a Lie superalgebra with bracket:

$$[D_1, D_2] =_{\text{def}} D_1 D_2 - (-1)^{p(D_1)p(D_2)} D_2 D_1.$$

**Theorem 6.5.3.** *Let  $G$  be an affine supergroup scheme. Then we have natural bijections among the sets:*

- a)  $L(G)$  *left invariant derivations in*  $\text{Der}(k[G], k[G])$ ,
- b)  $\text{Der}(k[G], k)$ ,
- c)  $\text{Lie}(G)$ .

*Proof.* . Let's examine the correspondence between (a) and (b). We want to construct a map  $\phi : \text{Der}(k[G], k) \longrightarrow L(G)$ . Let  $d \in \text{Der}(k[G], k)$ . Define  $\phi(d) = (id \otimes d)\Delta$ . Then  $\phi(d) \in \text{Der}(k[G], k[G])$ , moreover it is left invariant as one can readily check. Vice-versa if  $D \in L(G)$  define  $\psi(D) = D \circ \epsilon$  ( $\epsilon$  is the counit in the Hopf algebra  $k[G]$ ). One can check that  $\psi$  is the inverse of  $\phi$ . We now want a correspondence between (b) and (c). By Theorem 6.3.5 we have that  $\text{Lie}(G) = \text{Hom}_{(\text{smod})}(T_{1_G}(G)^*, A) = \text{Der}(\mathcal{O}_{G,1_G}, k)$ . Observe that as in the commutative case:

$$\text{Der}(\mathcal{O}_{G,1_G}, k) = \text{Der}(k[G], k),$$

that is, the derivation on the localization of the ring  $k[G]$  is determined by the derivation on the ring itself. ■

We want to show that, as in the classical case, every affine algebraic supergroup  $G$  can be embedded into some  $\mathrm{GL}(m|n)$ .

**Definition 6.5.4.** Let  $f : X \longrightarrow Y$  be a superscheme morphism. We say that  $f$  is a *closed immersion* if the topological map  $|f| : |X| \longrightarrow |Y|$  is a homeomorphism of the topological space  $|X|$  onto its image in  $|Y|$  and the sheaf map  $f^* : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$  is a surjective morphism of sheaves of superalgebras.

This means that we may identify  $X$  with a closed subscheme of  $Y$ , so its sheaf is identified with  $\mathcal{O}_Y/\mathcal{I}$ , for some quasicoherent sheaf of ideals  $\mathcal{I}$ . If both  $X$  and  $Y$  are affine superschemes we have immediately that  $f$  is a closed immersion if and only if  $k[X] \cong k[Y]/I$  for some ideal  $I$ .

We now need to introduce the notion of linear representation of a supergroup.

**Definition 6.5.5.** Let  $V$  be a super vector space and  $G$  an algebraic supergroup (not necessarily affine). We define *linear representation* a natural transformation  $\rho$

$$\rho : h_G \longrightarrow \mathrm{End}(V),$$

where  $\mathrm{End}(V)$  is the functor

$$\mathrm{End}(V) : (\mathrm{salg}) \longrightarrow (\mathrm{sets}), \quad \mathrm{End}(V)(A) = \mathrm{End}(A \otimes V).$$

Here  $\mathrm{End}(A \otimes V)$  denotes the endomorphisms of the  $A$ -module  $A \otimes V$  preserving the parity. We will also say that  $G$  *acts* on  $V$ .

Assume now that  $G$  is an affine algebraic supergroup,  $k[G]$  the Hopf superalgebra representing it,  $\Delta$  and  $\epsilon$  its comultiplication and counit respectively.

**Definition 6.5.6.** Let  $V$  be a supervector space. We say that  $V$  is a *right*  $G$ -comodule if there exists a linear map:

$$\Delta_V : V \longrightarrow V \otimes k[G]$$

called a *comodule map* with the properties:

- 1)  $(\Delta_V \otimes \text{id}_G)\Delta_V = (\text{id}_V \otimes \Delta)\Delta_V$
- 2)  $(\text{id}_V \otimes \epsilon)\Delta_V = \text{id}_V$ .

where  $\text{id}_G : k[G] \longrightarrow k[G]$  is the identity map.

One can also define a left  $G$ -comodule in the obvious way.

**Observation 6.5.7.** The two notions of  $G$  acting on  $V$  and  $V$  being a (right)  $G$ -comodule are essentially equivalent. In fact, given a representation  $\rho : G \longrightarrow \text{End}(V)$ , it defines immediately a comodule map:

$$\Delta_V(v) = \rho_{k[G]}(\text{id}_G)v, \quad \text{id}_G \in h_G(k[G]) = \text{Hom}_{(\text{salg})}(k[G], k[G])$$

where we are using the natural identification (for  $A = k[G]$ )

$$\text{End}(V)(A) \cong \text{Hom}_{(\text{smod})}(V, V \otimes A).$$

Vice-versa if we have a comodule map  $\Delta_V$  we can define a representation in the following way:

$$\rho_A : h_G(A) \longrightarrow \text{End}(V)(A) \cong \text{Hom}_{(\text{smod})}(V, V \otimes A)$$

$$g \longmapsto v \mapsto (\text{id} \otimes g)(\Delta_V(v))$$

where  $g \in h_G(A) = \text{Hom}_{(\text{salg})}(k[G], A)$ .

Let's see this correspondence in a special, but important case.

**Example 6.5.8.** Let's consider the natural action of  $\mathrm{GL}_{m|n}$  on  $k^{m|n}$ :

$$\rho_A : \mathrm{GL}_{m|n}(A) \longrightarrow \mathrm{End}(k^{m|n})(A)$$

$$g = (g_{ij}) \quad \mapsto \quad e_j \mapsto \sum e_i \otimes g_{ij}$$

where  $\{e_j\}$  is the canonical homogeneous basis for the framed supervector space  $k^{m|n}$ . We identify the morphism  $g \in \mathrm{GL}_{m|n}(A) = \mathrm{Hom}_{(\mathrm{salg})}(k[G], A)$  with the matrix with entries  $g_{ij} = g(x_{ij})$ , where  $x_{ij}$ 's are the generators of  $k[\mathrm{GL}(m|n)]$ .

This corresponds to the comodule map

$$\Delta_{k^{m|n}} : k^{m|n} \longrightarrow k^{m|n} \otimes k[\mathrm{GL}(m|n)]$$

$$e_j \quad \mapsto \quad \sum e_i \otimes x_{ij}$$

where  $x_{ij}$  are the generators of the algebra  $k[\mathrm{GL}(m|n)]$ .

Vice-versa, given the comodule map as above:  $e_j \mapsto \sum e_i \otimes x_{ij}$  it corresponds to the representation:

$$\begin{aligned} \rho_A : \mathrm{GL}_{m|n}(A) &\longrightarrow \mathrm{End}(k^{m|n})(A) \\ g = (g_{ij}) &\mapsto e_j \mapsto (\mathrm{id} \otimes g)(\sum e_i \otimes x_{ij}) = \sum e_i \otimes g_{ij}. \end{aligned}$$

**Definition 6.5.9.** Let  $G$  act on the superspace  $V$ , via a representation  $\rho$  corresponding to the comodule map  $\Delta_V$ . We say that the subspace  $W \subset V$  is  $G$ -stable if  $\Delta_V(W) \subset W \otimes V$ . Equivalently  $W$  is  $G$ -stable if  $\rho_A(g)(W \otimes A) \subset W \otimes A$ .

**Definition 6.5.10.** The *right regular representation* of the affine algebraic group  $G$  is the representation of  $G$  in the (infinite dimensional) super vector space  $k[G]$  corresponding to the comodule map:

$$\Delta : k[G] \longrightarrow k[G] \otimes k[G].$$

**Proposition 6.5.11.** *Let  $\rho$  be a linear representation of an affine algebraic supergroup  $G$ . Then each finite dimensional supersubspace of  $V$  generates a finite dimensional stable subspace of  $V$ .*

*Proof.* It is the same as in the commutative case. Let's sketch it. It is enough to prove for one element  $x \in V$ . Let  $\Delta_V : V \longrightarrow V \otimes k[G]$  be the comodule structure associated to the representation  $\rho$ . Let

$$\Delta_V(x) = \sum_i x_i \otimes a_i$$

where  $\{a_i\}$  is a basis for  $k[G]$ .

We claim that  $\text{span}_k\{x_i\}$  is a  $G$ -stable subspace.

By definition of comodule we have:

$$(\Delta_V \otimes \text{id}_G)(\Delta_V(x)) = (\text{id}_V \otimes \Delta)(\Delta_V(x)),$$

that is

$$\sum_j \Delta_V(x_j) \otimes a_j = \sum_i x_i \otimes \Delta(a_i) = \sum_{i,j} x_i \otimes b_{ij} \otimes a_j.$$

Hence

$$\Delta_V(x_j) = \sum_i x_i \otimes b_{ij}.$$

The finite dimensional stable subspace is given by the span of the  $x_i$ 's. ■

**Theorem 6.5.12.** *Let  $G$  be an affine supergroup variety. Then there exists a closed embedding:*

$$G \subset \text{GL}(m|n)$$

*for suitable  $m$  and  $n$ .*

*Proof.* We need to find a surjective superalgebra morphism  $k[GL(m|n)] \longrightarrow k[G]$  for suitable  $m$  and  $n$ . Let  $k[G] = k[f_1 \dots f_n]$ , where  $f_i$  are homogeneous and chosen so that  $W = \text{span}\{f_1 \dots f_n\}$  is  $G$ -stable, according to the right regular representation. This choice is possible because of Proposition 6.5.11. We have:

$$\Delta_{k[G]}(f_i) = \sum_j f_j \otimes a_{ij}.$$

Define the morphisms:

$$\begin{array}{ccc} k[GL(m|n)] & \longrightarrow & k[G] \\ x_{ij} & \mapsto & a_{ij} \end{array}$$

where  $x_{ij}$  are the generators for  $k[GL(m|n)]$ . This is the required surjective algebra morphism. In fact, since  $k[G]$  is both a right and left  $G$ -comodule we have:

$$f_i = (\epsilon \otimes \text{id})\Delta(f_i) = (\epsilon \otimes \text{id})\left(\sum_j f_j \otimes a_{ij}\right) = \sum_j \epsilon(f_j) \otimes a_{ij}$$

which proves the surjectivity. ■

**Corollary 6.5.13.**  *$G$  is an affine supergroup scheme if and only if it is a closed subgroup of  $GL(m|n)$ .*



# CHAPTER 7

## Appendix

### 7.1 Categories

We want to make a brief summary of formal properties and definitions relative to categories. For more details one can see for example [16].

**Definition 7.1.1.** A *category*  $\mathcal{C}$  consists of a collection of objects, denoted by  $Ob(\mathcal{C})$ , and sets of *morphisms* between objects. For all pairs  $A, B \in Ob(\mathcal{C})$ , we denote the set of morphisms from  $A$  to  $B$  by  $Hom_{\mathcal{C}}(A, B)$  so that for all  $A, B, C \in \mathcal{C}$ , there exists an association

$$Hom_{\mathcal{C}}(B, C) \times Hom_{\mathcal{C}}(A, B) \longrightarrow Hom_{\mathcal{C}}(A, C)$$

called the “composition law”  $((f, g) \rightarrow f \circ g)$  which satisfies the properties

- (i) the law “ $\circ$ ” is associative,
- (ii) for all  $A, B \in Ob(\mathcal{C})$ , there exists  $id_A \in Hom_{\mathcal{C}}(A, A)$  so that we get  $f \circ id_A = f$  for all  $f \in Hom_{\mathcal{C}}(A, B)$  and  $id_A \circ g = g$  for all  $g \in Hom_{\mathcal{C}}(B, A)$ ,
- (iii)  $Hom_{\mathcal{C}}(A, B)$  and  $Hom_{\mathcal{C}}(A', B')$  are disjoint unless  $A = A'$ ,  $B = B'$  in which case they are equal.

Once the category is understood, it is conventional to write  $A \in \mathcal{C}$  instead of  $A \in Ob(\mathcal{C})$  for objects. We may also suppress the “ $\mathcal{C}$ ” from  $Hom_{\mathcal{C}}$  and just write  $Hom$  whenever there is no danger of confusion.

Essentially a category is a collection of objects which share some basic structure, along with maps between objects which preserve that structure.

**Example 7.1.2.** Let  $\mathcal{G}$  denote the category of groups. Any object  $G \in \mathcal{G}$  is a group, and for any two groups  $G, H \in \text{Ob}(\mathcal{G})$ , the set  $\text{Hom}_{\mathcal{G}}(G, H)$  is the set of group homomorphisms from  $G$  to  $H$ .

**Definition 7.1.3.** A category  $\mathcal{C}'$  is a *subcategory* of category  $\mathcal{C}$  if  $\text{Ob}(\mathcal{C}') \subset \text{Ob}(\mathcal{C})$  and if for all  $A, B \in \mathcal{C}'$ ,  $\text{Hom}_{\mathcal{C}'}(A, B) \subset \text{Hom}_{\mathcal{C}}(A, B)$  so that the composition law “ $\circ$ ” on  $\mathcal{C}'$  is induced by that on  $\mathcal{C}$ .

**Example 7.1.4.** The category  $\mathcal{A}$  of *abelian groups* is a subcategory of the category of groups  $\mathcal{G}$ .

**Definition 7.1.5.** Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two categories. Then a *covariant* [*contravariant*] *functor*  $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$  consists of

- (1) a map  $F : \text{Ob}(\mathcal{C}_1) \longrightarrow \text{Ob}(\mathcal{C}_2)$  and
- (2) a map (denoted by the same  $F$ )  $F : \text{Hom}_{\mathcal{C}_1}(A, B) \longrightarrow \text{Hom}_{\mathcal{C}_2}(F(A), F(B))$  [ $F : \text{Hom}_{\mathcal{C}_1}(A, B) \longrightarrow \text{Hom}_{\mathcal{C}_2}(F(B), F(A))$ ] so that

- (i)  $F(\text{id}_A) = \text{id}_{F(A)}$  and
- (ii)  $F(f \circ g) = F(f) \circ F(g)$  [ $F(f \circ g) = F(g) \circ F(f)$ ]

for all  $A, B \in \text{Ob}(\mathcal{C}_1)$ .

When we say “functor” we mean covariant functor. A contravariant functor  $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$  is the same as a covariant functor from  $\mathcal{C}_1^o \longrightarrow \mathcal{C}_2$  where  $\mathcal{C}_1^o$  denotes the *opposite* category i. e. the category where all morphism arrows are reversed.

**Definition 7.1.6.** Let  $F_1, F_2$  be two functors from  $\mathcal{C}_1$  to  $\mathcal{C}_2$ . We say that there is a natural transformation of functors  $\varphi : F_1 \longrightarrow F_2$  if for all  $A \in \mathcal{C}_1$  there is a set of morphisms  $\varphi_A : F_1(A) \longrightarrow F_2(A)$  so that for any  $f \in \text{Hom}_{\mathcal{C}_1}(A, B)$  ( $B \in \mathcal{C}_1$ ),

the following diagram commutes:

$$\begin{array}{ccc}
F_1(A) & \xrightarrow{\varphi_A} & F_2(A) \\
F_1(f) \downarrow & & \downarrow F_2(f) \\
F_1(B) & \xrightarrow{\varphi_B} & F_2(B).
\end{array} \tag{7.1}$$

We say that the family of functions  $\varphi_A$  is *functorial* in  $A$ .

The notion of equivalence of categories is important since it allows to identify two categories which are apparently different.

**Definition 7.1.7.** We say that two categories  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are *equivalent* if there exists two functors  $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$  and  $G : \mathcal{C}_2 \longrightarrow \mathcal{C}_1$  such that  $FG \cong id_{\mathcal{C}_2}$ ,  $GF \cong id_{\mathcal{C}_1}$  (where  $id_{\mathcal{C}}$  denotes the identity functor of a given category, defined in the obvious way).

Next we want to formally define what it means for a functor to be *representable*.

**Definition 7.1.8.** Let  $F$  be a functor from the category  $\mathcal{C}$  to the category of sets  $\mathcal{S}$ . We say that  $F$  is *representable by*  $X \in \mathcal{C}$  if for all  $A \in \mathcal{C}$ ,

$$\begin{aligned}
F(A) &= \text{Hom}_{\mathcal{C}}(X, A), \\
F(f) : F(A) &\longrightarrow F(B), \quad F(f)(\alpha) =_{\text{def}} f \cdot \alpha
\end{aligned}$$

for all  $f : A \longrightarrow B$ .

We end our small exposition of categories by the constructing the fibered product which is be important in chapter 3.

**Definition 7.1.9.** Given functors  $A, B, C$ , from a category  $\mathcal{C}$  to the category of (sets) , and given natural transformations  $f : A \longrightarrow C$ ,  $g : B \longrightarrow C$  the *fibered*

product  $A \times_C B$  is the universal object making the following diagram commute:

$$\begin{array}{ccc} A \times_C B & \longrightarrow & B \\ \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C. \end{array}$$

One can see that:

$$(A \times_C B)(R) = A(R) \times_{C(R)} B(R) = \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

One can see that if  $g$  is injective, that is  $g_R : B(R) \subset C(R)$  we have that  $(A \times_C B)(R) = f^{-1}(B(R))$ .

The language of categories allows us to make (and prove) some sweeping generalizations about geometric objects without too much “forceful” computation. In particular, it also allows us to generalize the notion of a “point” to a  $T$ -point; this allows us to make more intuitive calculations with supergeometric objects. The main categories we discuss in this exposition are the categories of  $C^\infty$ -supermanifolds, super Lie groups (a subcategory of  $C^\infty$ -supermanifolds), superschemes, and super algebraic groups.

## 7.2 SuperNakayama’s Lemma and Projective Modules

Let  $A$  be a commutative superalgebra.

**Definition 7.2.1.** A projective  $A$ -module  $M$  is a direct summand of  $A^{m|n}$ . In other words it is a projective module in the classical sense respecting the grading:  $M_0 \subset A_0^{m|n}$ ,  $M_1 \subset A_1^{m|n}$ .

**Observation 7.2.2.** As in the classical setting being projective is equivalent to the exactness of the functor  $\text{Hom}(M, \_)$ .

We want to show that a projective  $A$ -module has the property of being locally free, that is its localization  $L_p$  into primes  $p$  of  $A_0$  is free as  $A_p$ -module. This result allows to define the *rank* of a projective module as it happens in the ordinary case.

We start with a generalization of the Nakayama's lemma.

**Lemma 7.2.3.** (*super Nakayama's Lemma*) *Let  $A$  be a local supercommutative ring with maximal homogeneous ideal  $\mathfrak{m}$ . Let  $E$  be a finitely generated module for the ungraded ring  $A$ .*

(i) *If  $\mathfrak{m}E = E$ , then  $E = 0$ ; more generally, if  $H$  is a submodule of  $E$  such that  $E = \mathfrak{m}E + H$ , then  $E = H$ .*

(ii) *Let  $(v_i)_{1 \leq i \leq p}$  be a basis for the  $k$ -vector space  $E/\mathfrak{m}E$  where  $k = A/\mathfrak{m}$ . Let  $e_i \in E$  be above  $v_i$ . Then the  $e_i$  generate  $E$ . If  $E$  is a supermodule for the super ring  $A$ , and  $v_i$  are homogeneous elements of the super vector space  $E/\mathfrak{m}E$ , we can choose the  $e_i$  to be homogeneous too (and hence of the same parity as the  $v_i$ ).*

(iii) *Suppose  $E$  is projective, i.e. there is a  $A$ -module  $F$  such that  $E \oplus F = A^N$  where  $A^N$  is the free module for the ungraded ring  $A$  of rank  $N$ . Then  $E$  (and hence  $F$ ) is free, and the  $e_i$  above form a basis for  $E$ .*

*Proof.* The proofs are easy extensions of the ones in the commutative case. We begin the proof of (i) with the following observation: if  $B$  is a commutative local ring with  $\mathfrak{n}$  a maximal ideal, then a square matrix  $R$  over  $B$  is invertible if and only if it is invertible modulo  $\mathfrak{n}$  over the field  $B/\mathfrak{n}$ . In fact if this is so,  $\det(R) \notin \mathfrak{n}$  and so is a unit of  $B$ . This said, let  $u_i$ , ( $1 < i < N$ ) generate  $E$ . If  $E = \mathfrak{m}E$ , we

can find  $m_{ij} \in \mathfrak{m}$  so that  $u_i = \sum_j m_{ij} e_j$  for all  $i$ . Hence, if  $L$  is the matrix with entries  $\delta_{ij} - m_{ij}$ , then

$$L \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = 0.$$

It is now enough to prove that  $L$  has a left inverse. Then multiplying the above from the left by  $P$ , we get  $u_i = 0$  for all  $i$  and so  $E = 0$ . It is even true that  $L$  is invertible. To prove this, let us consider  $B = A/J$  where  $J$  is the ideal generated by  $A_1$ . Since  $J \subset \mathfrak{m}$  we have

$$A \longrightarrow B = A/J \longrightarrow k = A/\mathfrak{m}.$$

Let  $L_B$  (resp.  $L_k$ ) be the reduction of  $L$  modulo  $J$  (respectively modulo  $\mathfrak{m}$ ). Then  $B$  is local, and its maximal ideal is  $\mathfrak{m}/J$  where  $L_k$  is the reduction of  $L_B$  mod  $\mathfrak{m}/J$ . But  $B$  is commutative and  $L_k = I$ , and so  $L_B$  is invertible. But then  $L$  is invertible. If more generally we have  $E = H + \mathfrak{m}E$ , then  $E/H = \mathfrak{m}(E/H)$  and so  $E/H = 0$ , which is to say that  $E = H$ .

To prove (ii), let  $H$  be the submodule of  $E$  generated by the  $e_i$ . Then  $E = \mathfrak{m}E + H$  and so  $E = H$ .

We now prove (iii). Clearly  $F$  is also finitely generated. We have  $k^N = A^N/\mathfrak{m}^N = E/\mathfrak{m}E \oplus F/\mathfrak{m}F$ . Let  $(w_j)$  be a basis of  $F/\mathfrak{m}F$  and let  $f_j$  be elements of  $F$  above  $w_j$ . Then by (ii), the  $e_i, f_j$  form a basis of  $A^N$  while the  $e_i$  (respectively the  $f_j$ ) generate  $E$  (resp.  $F$ ). Now there are exactly  $N$  of the  $e_i, f_j$ , and so if  $X$  denotes the  $N \times N$  matrix with columns  $e_1, \dots, f_1, \dots$ , then for some  $N \times N$  matrix  $Y$  over  $A$  we have  $XY = I$ . Hence  $X_B Y_B = I$  where the suffix “ $B$ ” denotes reduction modulo  $B$ . However,  $B$  is commutative and so  $Y_B X_B = I$ . Thus  $X$  has a left inverse over  $A$ , which must be  $Y$  so that  $YX = I$ . If there is a linear

relation among the  $e_i$  and the  $f_j$ , and if  $x$  is the column vector whose components are the coefficients of this relation, then  $Xx = 0$ ; but then  $x = YXx = 0$ . In particular  $E$  is a free module with basis  $(e_i)$ . ■

We now wish to give a characterization of projective modules.

**Theorem 7.2.4.** *Let  $M$  be a finitely generated  $A$ -module,  $A$  finitely generated over  $A_0$  where  $A_0$  noetherian. Then*

- i)  $M$  is projective if and only if  $M_p$  is free for all  $p$  primes in  $A_0$  and*
- ii)  $M$  is projective if and only if  $M[f_i^{-1}]$  free for all  $f_i$ 's such that  $(f_1 \dots f_r) = A_0$ .*

*Proof.* (i) If  $M$  is projective, by part (iii) of Nakayama's Lemma, we have that  $M_p$  is free since it is a module over the supercommutative ring  $A_p$ .

Now assume that  $M_p$  is free for all primes  $p \in A_0$ . Recall that

$$\mathrm{Hom}_{A[U^{-1}]}(M[U^{-1}], N[U^{-1}]) = \mathrm{Hom}_A(M, N)[U^{-1}]$$

for  $U$  a multiplicatively closed set in  $A_0$ . Recall also that given  $A_0$ -modules  $N, N', N''$ , we have that  $0 \longrightarrow N' \longrightarrow N \longrightarrow N''$  is exact if and only if  $0 \longrightarrow N'_p \longrightarrow N_p \longrightarrow N''_p$  is exact for all prime  $p$  in  $A_0$ . So given an exact sequence  $0 \longrightarrow N' \longrightarrow N \longrightarrow N''$ , since  $M_p$  is free, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}(M_p, N'_p) \longrightarrow \mathrm{Hom}(M_p, N_p) \longrightarrow \mathrm{Hom}(M_p, N''_p) \longrightarrow 0$$

for all the primes  $p$ . Hence by the previous observation,

$$0 \longrightarrow \mathrm{Hom}(M, N')_p \longrightarrow \mathrm{Hom}(M, N)_p \longrightarrow \mathrm{Hom}(M, N'')_p \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{Hom}(M, N') \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M, N'') \longrightarrow 0.$$

Hence  $M$  is projective.

(ii) That  $M_p$  is free for all primes  $p$  is equivalent to  $M[f_i^{-1}]$  being free for  $(f_1 \dots f_r) = A_0$  is a standard fact of commutative algebra and can be found in [10] p. 623 for example. ■

**Remark 7.2.5.** As in the ordinary setting we have a correspondence between projective  $A$ -modules and locally free sheaves on  $\text{Spec} A_0$ . In this correspondence, given a projective  $A$ -module  $M$ , we view  $M$  as an  $A_0$ -module and build a sheaf of modules  $\mathcal{O}_M$  on  $A_0$ . The global sections of this sheaf are isomorphic to  $M$  itself, and locally, i.e. on the open sets  $U_{f_i} = \{p \in \text{Spec} A_0 \mid (f_i) \not\subset p\}$ ,  $f_i \in A_0$ ,

$$\mathcal{O}_M(U_{f_i}) = M[f_i^{-1}].$$

More details on this construction can be found, for example, in [13] chapter 2.



## REFERENCES

- [1] M. F. Atiyah, I. G. McDonald *Introduction to commutative algebra*, Perseus books , 1969.
- [2] F. A. Berezin. *Introduction to superanalysis*. D. Reidel Publishing Company, Holland, 1987.
- [3] Yu. A. Bahturin, A. A. Mikhalev, V. M. Petrogradsky, M. V. Zaicev *Infinite dimensional Lie Superalgebras* De Gruyter Expositions in Mathematics, 7, Walter de Gruyter, 1992.
- [4] C. Bartocci, U. Bruzzo, D. Hernandez-Ruiperez. *The Geometry of Supermanifolds*. MAIA 71, Kluwer Academic Publishers, 1991.
- [5] A. Borel, *Linear algebraic groups*. Springer Verlag, New York, 1991.
- [6] D. Eisenbud and J. Harris, *The geometry of schemes*. Springer Verlag, New York, 2000.
- [7] P. Deligne, J. W. Morgan. Notes on supersymmetry following Bernstein. *Quantum fields and strings; a course for mathematicians, Vol. 1* (Princeton, NJ, 1996/1997), 41-96, Amer. Math. Soc., Providence, RI, 1999.
- [8] B. DeWitt. *Supermanifolds*. Cambridge University Press, London, 1984.
- [9] M. Demazure, P. Gabriel, *Groupes Algébriques, Tome 1*. Mason&Cie, éditeur. North-Holland Publishing Company, The Netherlands, 1970.
- [10] D. Eisenbud, *Commutative algebra with a view towards algebraic geometry*, GTM Springer Verlag, 1991.
- [11] Daniel Z. Freedman, P. van Nieuwenhuizen , S. Ferrara *Progress Toward A Theory Of Supergravity*, Phys.Rev.D 13, 3214-3218, 1976.
- [12] A. Grothendieck, M. Demazure, M. Artin [et al.]. *Schémas en groupes; séminaire de géométrie algébrique du Bois Marie*, 1962/64, SGA 3. Springer-Verlag, Berlin-New York, 1970.
- [13] R. Hartshorne. *Algebraic geometry*. Graduate Text In Mathematics. Springer-Verlag, New York, 1977.
- [14] V. G. Kac, *Lie superalgebras* Adv. in Math. 26 (1977) 8-26.

- [15] B. Kostant. Graded manifolds, graded Lie theory, and prequantization. *Differential geometrical methods in mathematical physics* (Proc. Sympos., Univ. Bonn, Bonn, (1975), pp. 177–306. *Lecture Notes in Math.*, Vol. 570, Springer, Berlin, 1977.
- [16] S. Lang. *Algebra*, Addison Wesley, 1984.
- [17] D. A. Leites. *Introduction to the theory of supermanifolds*. Russian Math. Surveys **35**: 1 (1980), 1-64.
- [18] Y. I. Manin. *Gauge field theory and complex geometry*; translated by N. Koblitz and J.R. King. Springer-Verlag, Berlin-New York, 1988.
- [19] D. Mumford. *The red book of varieties and schemes*. Springer-Verlag, Berlin-New York, 1988.
- [20] A. Salam, J. Strathdee. *Super-gauge transformations*, Nucl. Phys. B, **76**, (1974), 477-482.
- [21] G. M. Tuynman. *Supermanifolds and supergroups*. MAIA 570, Kluwer Academic Publishers, 2004.
- [22] V. S. Varadarajan. *Lie groups, Lie algebras, and their representations*. Graduate Text in Mathematics. Springer-Verlag, New York, 1984.
- [23] V. S. Varadarajan. *Supersymmetry for mathematicians: an introduction*. Courant Lecture Notes. Courant Lecture Notes Series, New York, 2004.
- [24] W. C. Waterhouse. *Introduction to affine group schemes*. Graduate Texts in Mathematics, 66. Springer-Verlag, New York-Berlin, 1979.
- [25] J. Wess, B. Zumino. *Supergauge transformations in four dimensions*. Nucl. Phys. B, **70**, (1974), 39-50.