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Mathematical Modeling

of Large Elastic-Plastic Deformations

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Abstract

The paper is devoted to development and numerical implementation for a method for investigation of stress-strain state of the solids with large elastic-plastic deformations. Calculation algorithm is based on the linearized equation of virtual work, defined to actual state. The arc-length method is used. A spatial discretization is based on the finite element method. The developed algorithm of investigation of large elastic-plastic deformations is tested on the solution of the necking of circular bar.

Keywords: large deformations, nonlinear elasticity, plasticity, finite element method

Introduction

There are many publications, where solutions of nonlinear problems of a solid mechanic are discussed, for example [10-13]. In this paper a numerical algorithm of the investigation of stress-strain state of the elastic–plastic solids with large deformations is described.

1 Kinematics

The deformation gradient tensor **F**, the left Cauchy–Green tensor $\mathbf{B}=\mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}$, the velocity gradient $\mathbf{h}=\dot{\mathbf{F}}\cdot\mathbf{F}^{-1}$ and the deformation rate $\mathbf{d}=\mathrm{sym}(\mathbf{h})$ are used for describing of kinematics of a continuum. The basic relationships are described in [1, 6-9].

2 Constitutive equations

The constitutive equations are obtained using the free energy function and yield function. The Cauchy stress tensor is defined as [1, 5]

$$\mathbf{\Sigma}=\frac{2}{J}\mathbf{B}\cdot\frac{\partial\psi}{\partial\mathbf{B}}, \tag{1}$$

where $J=\det(\mathbf{F})$ is a changing of volume, ψ is the free energy per unit volume in the reference configuration. For isotropic material the free energy is defined as

$$\psi=\psi\left(I_{1\mathbf{B}},I_{2\mathbf{B}},I_{3\mathbf{B}}\right),$$

where $I_{i\mathbf{B}}$ is the corresponding invariants of tensor $\mathbf{B}$.

After linearization (1) the rate of Cauchy stress is defined as

$$\dot{\mathbf{\Sigma}}=2\left\{\frac{1}{J}\dot{\mathbf{B}}\cdot\frac{\partial\psi}{\partial\mathbf{B}}+\frac{1}{J}\left[\mathbf{B}\cdot\frac{\partial^2\psi}{\partial\mathbf{B}^2}\right]\cdot\cdot\dot{\mathbf{B}}-\frac{1}{J}\mathbf{B}\cdot\frac{\partial\psi}{\partial\mathbf{B}}I_{1\mathbf{d}}\right\}=\mathbf{\Lambda}_{\Sigma}\cdot\cdot\mathbf{d}+\mathbf{h}\cdot\mathbf{\Sigma}+\mathbf{\Sigma}\cdot\mathbf{h}^{\mathrm{T}}-\mathbf{\Sigma}I_{1\mathbf{d}},$$

where $\mathbf{\Lambda}_{\Sigma}=\frac{4}{J}\mathbf{B}\cdot\frac{\partial^2 W}{\partial\mathbf{B}\partial\mathbf{B}}\cdot\mathbf{B}$.

Or

$$\mathbf{\Sigma}^{Tr}=\mathbf{\Lambda}_{\Sigma}:\mathbf{d}, \tag{2}$$

where $\mathbf{\Sigma}^{Tr}=\dot{\mathbf{\Sigma}}+\mathbf{h}\cdot\mathbf{\Sigma}+\mathbf{\Sigma}\cdot\mathbf{h}^{\mathrm{T}}-I_{1\mathbf{d}}\mathbf{\Sigma}$ is the Truesdell stress rate.

The theory of flow is used for describing plastic deformation [2-5]. The total deformation rate is represented as a sum of elastic and plastic parts: $\mathbf{d}=\mathbf{d}^e+\mathbf{d}^p$ [2-5]. For the plastic deformation rate must hold association flow rule:

$$\mathbf{d}^p=\dot{\gamma}\frac{\partial\Phi}{\partial\mathbf{\Sigma}}, \tag{3}$$

where $\dot{\gamma}$ is the consistency parameter, Φ is a yield function.

3 Integration algorithm of the flow rules

For the solution problem of a plastic flow the general return method is used [5, 14-16]. The solution at time ${}^{k}t$ is known. The stress at time ${}^{k+1}t={}^{k}t+\Delta\,{}^{k+1}t$ is defined from the initial conditions at time ${}^{k}t$. The trial stress is defined as

$$ {}^{k+1}\tilde{\mathbf{\Sigma}} = {}^{k}\mathbf{\Sigma} + {}^{k+1}\Delta\mathbf{\Sigma}. \qquad (4)$$

The plastic flow $\Delta^{k+1}\gamma$ can be computed from equation

$$ {}^{k+1}\mathbf{\Sigma} + \Delta^{k+1}\gamma \frac{\partial^{k+1}\Phi}{\partial^{k+1}\Sigma} \cdot\cdot {}^{k+1}\mathbf{\Sigma} = {}^{k+1}\tilde{\mathbf{\Sigma}}. \qquad (5)$$

The solution is obtained by solving a nonlinear system of equations (5) with linearized Newton scheme using (2)–(4).

4 Variational formulation

The research algorithm is based on an Update Lagrange formulation. The principle of virtual work in terms of the virtual velocity is used [1-5]:

$$\int_{\Omega} \mathbf{\Sigma} : \boldsymbol{\delta}\mathbf{d} d\Omega = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\delta}\mathbf{v} d\Omega + \int_{S^{\sigma}} \mathbf{p} \cdot \boldsymbol{\delta}\mathbf{v} dS,$$

where Ω is the current volume, S^{σ} is the surface on which the force $\mathbf{p}$ is applied, $\mathbf{f}$ is the body force vector, $\mathbf{v}$ is a velocity vector. After linearization the system of linear equations is obtained, where the unknown is the increment of displacement in the current state $\boldsymbol{\Delta}^{k+1}\mathbf{u}$. For solving general system of equations the arc-length method is applied [5]. The current state is defined as ${}^{k+1}\mathbf{R} = {}^{k}\mathbf{R} + \Delta^{k+1}\mathbf{u}$. Then the trial stress is calculated by (4). And if $\Phi({}^{\mathbf{k+1}}\tilde{\mathbf{\Sigma}}) \le 0$ then the Cauchy stress ${}^{k+1}\mathbf{\Sigma} = {}^{k+1}\tilde{\mathbf{\Sigma}}$, else the radial return method with an iterative refinement of the current mode of deformation is applied [5].

5 Numerical example

As an example the potential of elastic deformation is considered:

$$\psi = \frac{\lambda + 2\mu}{8}\left(I_{1\mathbf{B}} - 3\right)^2 + \mu\left(I_{1\mathbf{B}} - 3\right) - \frac{\mu}{2}\left(I_{2\mathbf{B}} - 3\right),$$

where λ, μ are Lame parameters. The von Mises yield criterion with isotropic hardening is used:

$$\Phi = \sigma_i - \xi(\chi) \le 0,$$

where $\sigma_i = \sqrt{\frac{3}{2} dev\mathbf{\Sigma} : dev\mathbf{\Sigma}}$, $\xi(\chi) = \sigma_T + h\chi + (\sigma_{\infty} - \sigma_T)(1 - e^{-\delta\chi})$ is the hardening function. The material data for isotropic elasticity and the von Mises yield condition are given as follows: $E = 206.9$ GPa, $\nu = 0.29$, $\sigma_{\infty} = 0.715$ GPa, $\sigma_T = 0.450$ GPa, $h = 0.129$, $\delta = 16.93$.

The numerical implementation is based on the finite element method. An 8-node brick element is used [2-5, 9].

Necking of a circular bar. The necking of a circular bar is an example widely investigated in the literature; see e.g. [14] or [15]. To initialize the necking process radius in the center is reduced by 1.8 %. Fig. 1 displays the final deformed structure and the equivalent plastic strain, which concentrates in the necking zone. The results are in very good agreement with the computational reference solutions of [14] and [15].

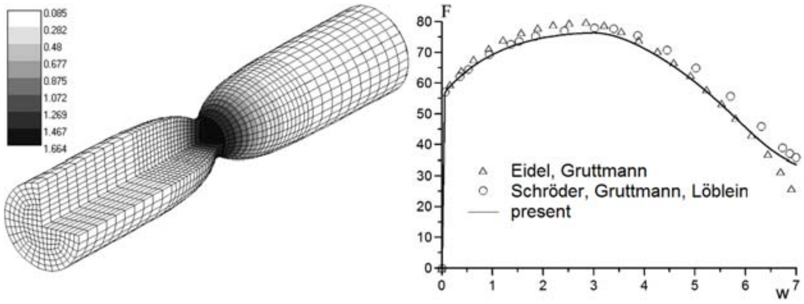

Fig. 1. Left: Equivalent plastic strain at final structure.
Right: Computational results of applied force F [kN] versus axial elongation w [mm].

Conclusion. A numerical algorithm of the investigation of stress-strain state of the solids with large elastic-plastic deformations is described. The constitutive equations are obtained using the free energy function and yield function. The general radial return method is applied. Calculation algorithm is based on the linearized equation of virtual work, defined to actual state. The arc-length method is used. The von Mises yield criterion with isotropic hardening is considered. A spatial discretization is based on the finite element method. The developed algorithm of investigation of large elastic-plastic deformations is tested on the solution of the necking of circular bar problem. The results of solutions and comparison with results obtained by other authors are presented. Solved problems demonstrate the efficiency of research methods of elastic-plastic problems.

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