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Mathematical Modelling on Blood Flow Under Atherosclerotic Condition

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Abstract: The present study discusses the effect of stenosis on flow rate, resistance to flow and wall shear stress for different parameters. A two –layered mathematical model has been incorporated here by considering the peripheral layer as Newtonian fluid and the core layer as Bingham-plastic type non-Newtonian fluid. The numerical results are presented in graphical form.

Keywords: Stenosis, Flux, Wall Shear Stress, Bingham-Plastic Fluid, Resistance to Flow, Yield Stress

1. Introduction

The study of blood flow characteristics are of great importance due to the unusual flow of blood for unusual fluid properties. Blood is a very complex fluid, as blood is formed by the suspension of fluid particles in an aqueous solution, called plasma which is composed of 90% of water and 7% protein. In human blood, red blood cells are more in number than other cells. The aforesaid cells play an important role in carrying oxygen from the lungs to all parts of the body and the removal of carbon dioxide which is one of the waste products of cell metabolism in the body to the lungs. About 45% of the total blood volume is occupied by red cells. Of the remaining 1% are white cells or leucocytes entrusted with the function to resist the body to infection. 5% of the total blood volume constituted by Platelets and they perform a pivotal role in blood clotting. The normal blood flow is disturbed abnormally in presence of arterial diseases whose consequences cause several types of cardiovascular diseases. Blood flow characteristics mainly depend on resistance to flow. If bore of the vessel is reduced, resistance to flow is increased, and so normal blood flow is disturbed abruptly. Resistance to flow mainly depends on arterial occlusion such as stenosis and aneurysm. So if stenosis is formed blood flow is insufficient to reach every cell of the body and as a result nutrient supplement is insufficient to reach each cell. Many investigators (Majumdar et. al. [1], Sanyal and Maiti [2], Sanyal et. al. [3]) have presented two-layered mathematical model to study non-Newtonian behaviour of blood. Several

researchers (Tu and Deville [4], Lee and Fung [5], Jung et. al. [6], Krumholz et. al.[7], Peterson et. al. [8], Rathod and Tanveer [9], Shukla [10], Maiti [11]) have presented experimental results to study the various aspect of blood flow through stenosed condition. Lerche [12] has analysed a two-layered fluid model with both the fluids as Newtonian with different viscosities. But when blood flows in an artery, tendency of erythrocytes is to migrate towards the centre of the vessel, making relatively cell free in the plasma layer, so it is more appropriate to take the fluid of the core layer as non-Newtonian type with different viscosity. Many mathematician (Aroesty et. al [13], Chakraborty et. al. [14], Halder et. al. [15], Lee [16]) have shown the behaviour of blood flow by considering both the layers either Newtonian or non-Newtonian fluid. Pontrelli [17], Sankar and Hemalatha [18], Shalman et. al. [19] have proposed various types of mathematical models by considering the peripheral layer as Newtonian fluid and the core layer as non-Newtonian fluid. They have shown that stenosis is very much important in micro circulation, where peripheral layer thickness and viscosity effects dominate the blood flow characteristics. Srivastava et. al. [20] have shown the effect of overlapping stenosis on blood flow by considering two-layered fluid model. In a recent paper Singh et. al. [21] have studied the effect of magnetic field on blood flow by presenting two-dimensional of blood flow with variable viscosity.

In the present analysis a two-layered mathematical model is proposed to study the axisymmetric flow of blood by considering the peripheral layer as Newtonian fluid and the core layer as Bingham plastic type non–Newtonian fluid in presence of arterial stenosis.

Let us consider the steady flow of blood through an axially symmetric but radially non-symmetric stenosed inelastic cylindrical arterial tube.

2. Mathematical Formulation

Here a two-layered blood flow model consisting of a central core layer has been considered which is suspension of red cells in plasma of radius R_1 and a peripheral plasma layer of thickness $(R - R_1)$ in the stenotic region.

The shape of constriction in the peripheral plasma layer and core layer may be taken as ([11], [20])

$$\frac{\left(R(z), R_1(z)\right)}{R_0} = (1, \alpha) - \frac{(\delta, \delta_1)}{R_0 L_0^s} \frac{s^{\frac{s}{s-1}}}{(s-1)} \left[L_0^{(s-1)}(z - d) - (z - d)^s\right], \qquad d \leq z \leq d + L_0$$

$$= (1, \alpha), \quad \text{Otherwise} \tag{1}$$

where $R(z)$ is the radius of the tube in the stenotic region, R_0 is the radius outside the stenosis, L_0 is the length of the stenosis, d indicates its location and $s(\geq 2)$ is the shape parameter, α is the ratio of the central core radius to the tube radius outside the stenotic region and (δ, δ_1) are the maximum height of the stenosis and bulging of the interface in the stenotic region located at $z = d + \frac{L_0}{s^{\frac{1}{(s-1)}}}, \frac{\delta}{R_0} \ll 1$.
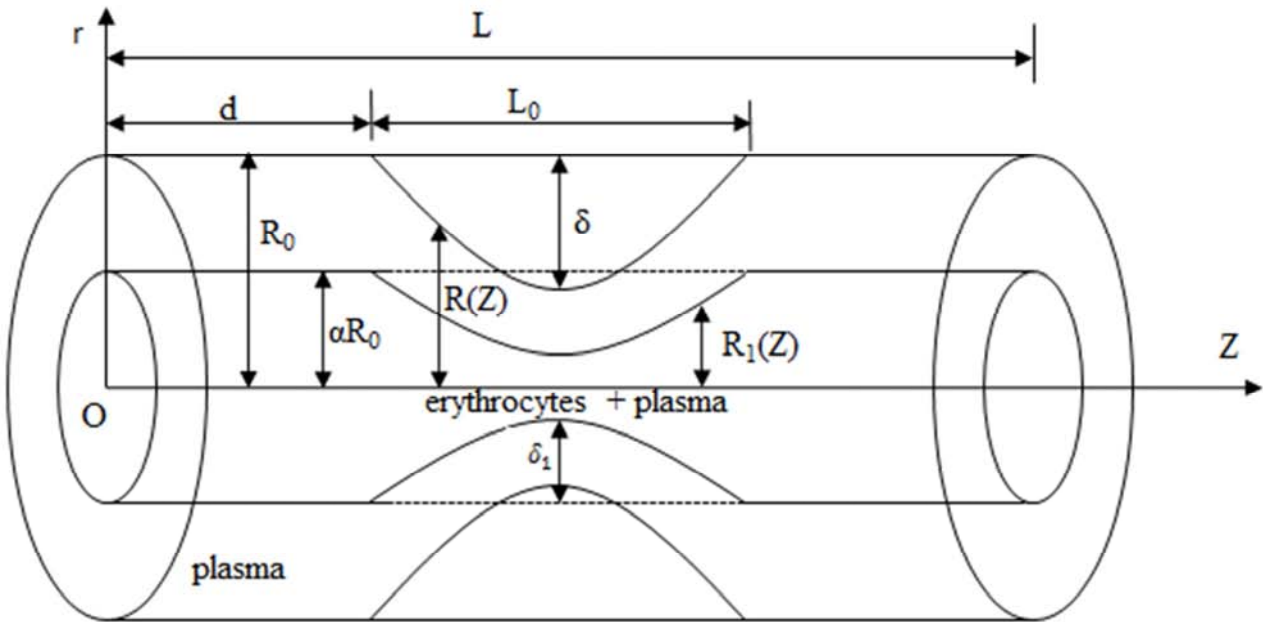

Figure 1. Geometry of stenosis.

The equation of motion for laminar, incompressible steady fully developed flow is given by

$$-\frac{dp}{dz} + \frac{1}{r} \frac{\partial}{\partial r}[r\tau] = 0 \tag{2}$$

where (r, z) are the axial co-ordinates and $\frac{dp}{dz}$ is the pressure gradient.

The boundary conditions are
(i) τ is finite at $r = 0$ (regularity condition)
(ii) $u = 0$ at $r = R(z)$ (no- slip condition)

The relationship between shear stress and shear rate for Bingham plastic fluid is given by

$$\tau = \mu\left(-\frac{\partial u}{\partial r}\right) + \tau_0 \tag{3}$$

Integrating (1) and using the boundary condition (i) we get,

$$\tau = \frac{Pr}{2} \tag{4}$$

where,

$$P = \frac{dp}{dz}$$

The volumetric flow rate Q_1 at the core layer is given by

$$Q_1 = \int_0^{R_1(z)} 2\pi r u \, dr$$

$$= \int_0^{R_1(z)} \pi r^2 \left(-\frac{\partial u}{\partial r}\right) dr$$

$$= \pi \int_0^{R_1(z)} \frac{r^2}{\mu_c} \left(\frac{Pr}{2} - \tau_0\right) dr$$

$$= \frac{\pi}{\mu_c}\left[\frac{PR_1^4}{8} - \frac{\tau_0 R_1^3}{3}\right] \tag{5}$$

where the no-slip boundary condition is used.

The volumetric flow rate Q_2 at the plasma layer is given by

$$Q_2 = \int_{R_1(z)}^{R(z)} \pi r^2 \frac{P}{2\mu_p} dr$$

$$= \frac{\pi P}{8\mu_p}[R^4 - R_1^4] \tag{6}$$

where μ_c, μ_p are respectively the viscosities of the central core layer and plasma layer respectively.

Thus the total flux is given by

$$Q = Q_1 + Q_2$$

$$= \frac{\pi}{8\mu_p}\left[P(R^4 - (1-\mu)R_1^4 - \frac{8\tau_0\mu}{3}R_1^3\right] \tag{7}$$

where

$$\mu = \frac{\mu_p}{\mu_c}$$

From which the pressure gradient P can be written as

$$P = \frac{\frac{8\mu_p Q}{\pi} + \frac{8\tau_0\mu}{3}R_1^3}{R^4 - (1-\mu)R_1^4} \tag{8}$$

The pressure drop along the length of the tube is given by

$$p_1 - p_0 = \frac{\frac{8\mu_p QL}{\pi} + \frac{8\tau_0\mu}{3}R_0^3\left[(L-L_0)\alpha^3 + \int_d^{d+L_0}\left(\frac{R_1}{R_0}\right)^3 dz\right]}{R_0^4\left[(L-L_0) - (L-L_0)(1-\mu)\alpha^4\right]\left[\int_d^{d+L_0}\left\{\left(\frac{R}{R_0}\right)^4 - (1-\mu)\left(\frac{R_1}{R_0}\right)^4\right\}dz\right]} \tag{9}$$

Where p_1 and p_0 are respectively the pressure at $z = L$ and $z = 0$.

The resistance to flow is given by

$$\lambda = \frac{p_1 - p_0}{Q} \tag{10}$$

The resistance to flow for Newtonian flow is given by

$$\lambda_N = \frac{8\mu_p L}{\pi R_0^4} \tag{11}$$

Thus the non-dimensional resistance to flow is given by

$$\bar{\lambda} = \frac{\lambda}{\lambda_N} = \frac{1 + \frac{\tau_0\mu\pi}{3\mu_p Q}R_0^3\left[\left(1-\frac{L_0}{L}\right)\alpha^3 + \int_d^{d+L_0}\left(\frac{R_1}{R_0}\right)^3 dz\right]}{(L-L_0)[1-(1-\mu)\alpha^4]\left[\int_d^{d+L_0}\left\{\left(\frac{R}{R_0}\right)^4 - (1-\mu)\left(\frac{R_1}{R_0}\right)^4\right\}dz\right]} \tag{12}$$

The wall shear stress is given by

$$\tau_s = \frac{\left[\frac{4\mu_p Q}{\pi} + \frac{4\tau_0\mu}{3}R_1^3\right]R}{R^4 - (1-\mu)R_1^4} \tag{13}$$

The wall shear stress for Newtonian fluid in the absence of stenosis is given by

$$\tau_N = \frac{4\mu_p Q}{\pi R_0^4} \tag{14}$$

Thus the wall shear stress in dimensionless form can be written as

$$\bar{\tau} = \frac{\tau_s}{\tau_N} \tag{15}$$

The wall shear stress at the throat of the stenosis is given by

$$\tau_{wm} = \frac{\left[\frac{4\mu\mu_c Q}{\pi R_0^3} + \frac{4}{3}\tau_0\mu\left(\alpha - \frac{\delta}{R_0}\right)^3\right]\left(1 - \frac{\delta}{R_0}\right)}{\left[\left(1-\frac{\delta}{R_0}\right)^4 - (1-\mu)\left(\alpha - \frac{\delta}{R_0}\right)^4\right]} \tag{16}$$

Thus the non-dimensional wall shear stress at the throat of the stenosis can be written as

$$\overline{\tau_m} = \frac{\tau_{wm}}{\tau_N} \tag{17}$$

3. Results and Discussions

To interpret the present analysis, the results are shown graphically with the help of MATLAB- 7.6.

Figure 2-3 represent the variation of flux Q for different values of α and μ. It is found that the flux Q decreases with the increase of α but increases with the increase of μ as stenosis developed.

Figure 4-8 show the fluctuation of non-dimensional resistance to flow $\bar{\lambda}$ for various parameters with respect to stenosis size. It is observed that $\bar{\lambda}$ increases with the increase of α, L_0, but opposite phenomenon occurs when s and μ increase. It also increases with the increase of yield stress.

Figure 9-10 represent the variation of wall non-dimensional wall shear stress $\bar{\tau}$ with the variation of stenosis size, α and μ. It is clear from figures that $\bar{\tau}$ increases as α increase, but the reverse effect occur when μ increases.

Figure 11-12 depict the fluctuation of non-dimensional wall shear stress at the throat of the stenosis $\overline{\tau_m}$ for the variation of α and μ with respect to stenosis size. The same phenomenon occurs as in $\bar{\tau}$.
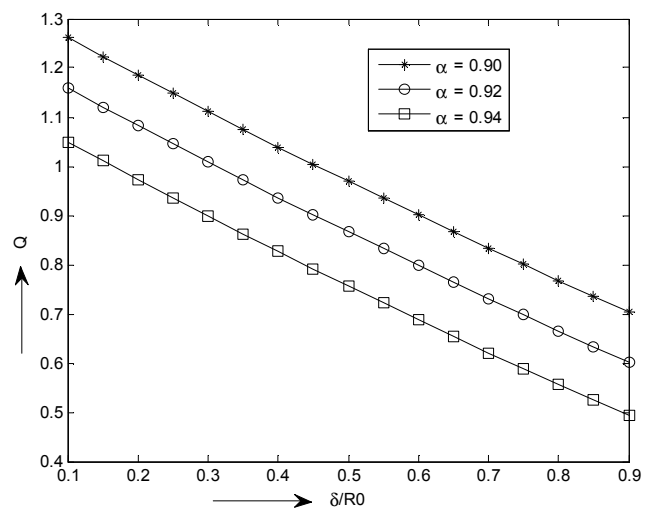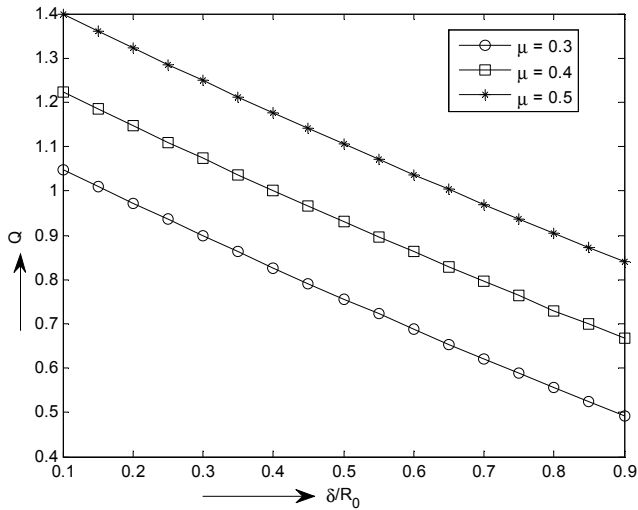

Figure 2. Variation of flux Q with the variation of α.

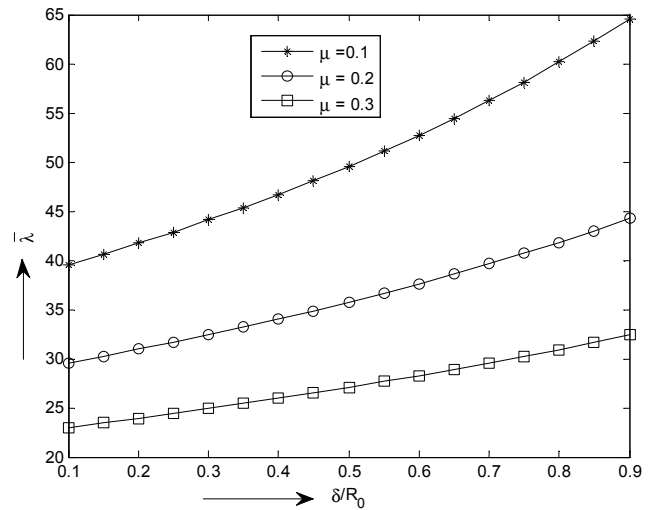***Figure 3.*** *Variation of flux Q with the variation of μ.*

Figure 6. *Variation of non-dimensional resistance to flow $\bar\lambda$ for different values of μ.*
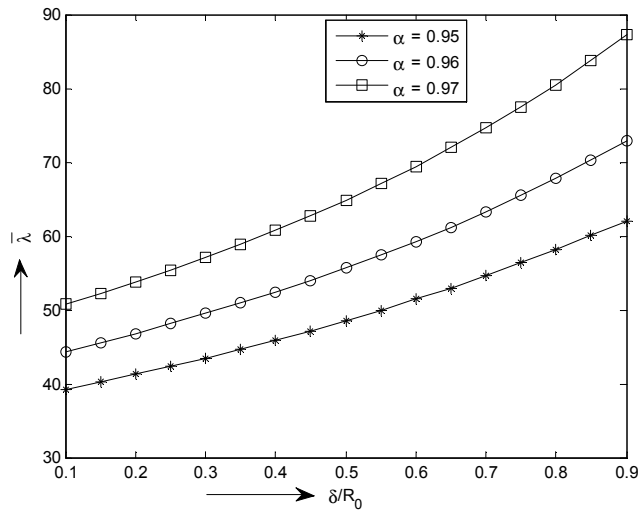

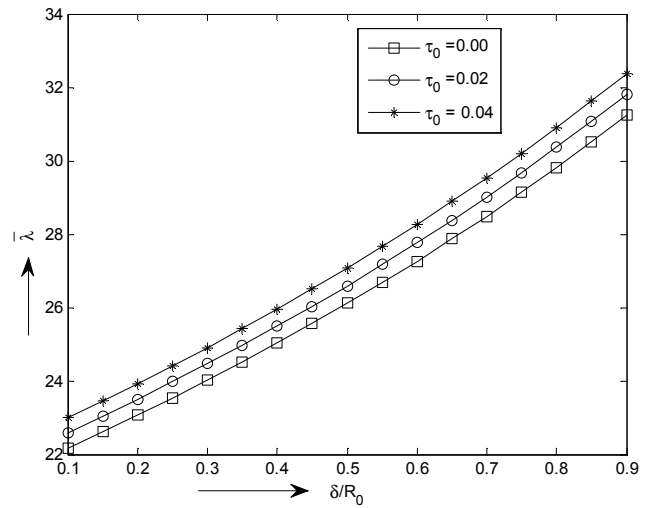***Figure 4.*** *Variation of non-dimensional resistance to flow $\bar\lambda$ for different values of α.*

Figure 7. *Variation of non-dimensional resistance to flow $\bar\lambda$ for different values of yield stress τ_0.*
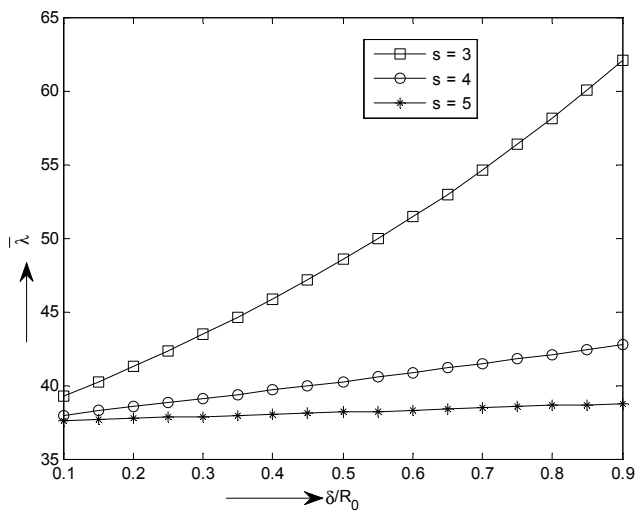

Figure 5. *Variation of non-dimensional resistance to flow $\bar\lambda$ for different values of shape parameter s.*
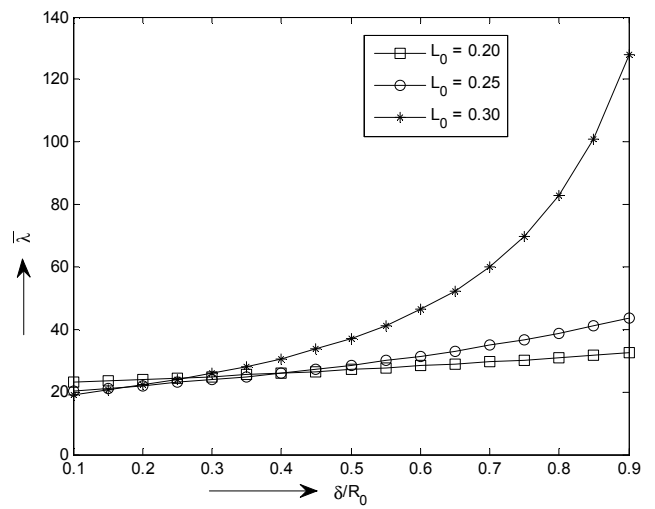

Figure 8. *Variation of non-dimensional resistance to flow $\bar\lambda$ for different values of stenosis length L_0.*
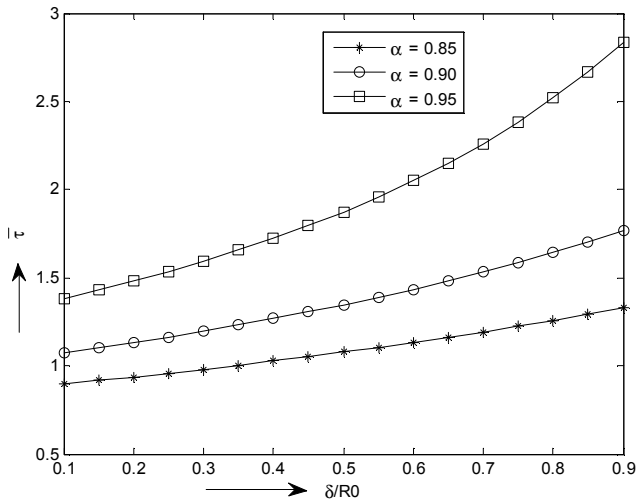
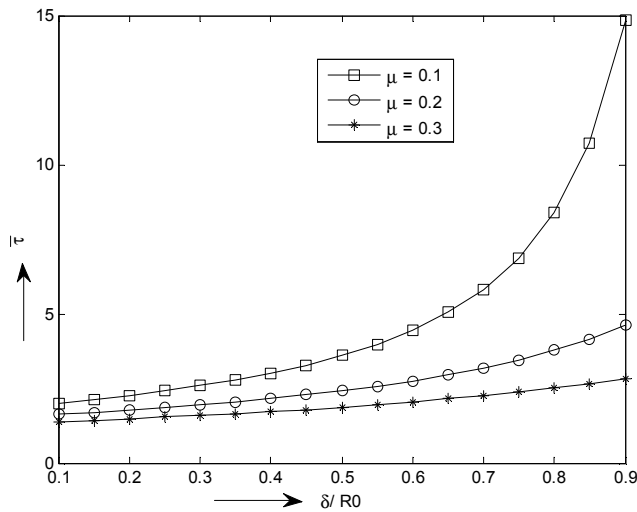Figure 9. Variation of non-dimensional wall shear stress $\overline{\tau}$ for different values of α.

Figure 10. Variation of non-dimensional wall shear stress $\overline{\tau}$ for different values of μ.
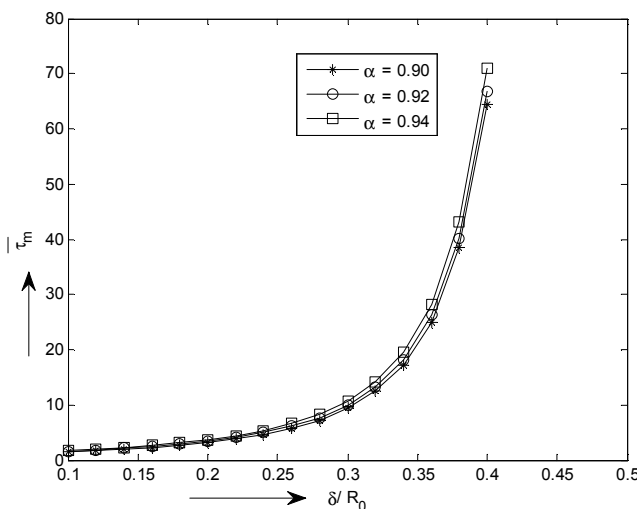

Figure 11. Fluctuation of non-dimensional wall shear stress at the throat of the stenosis $\overline{\tau_m}$ for different values of α.
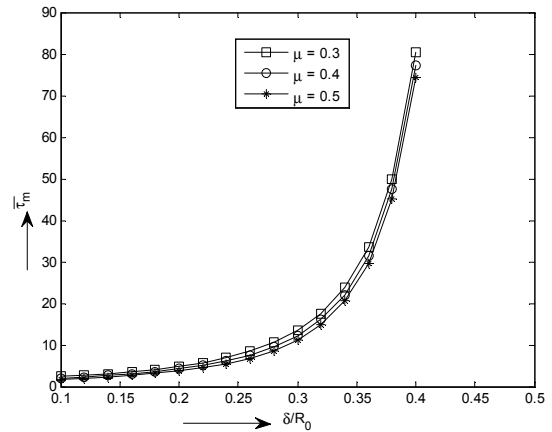

Figure 12. Fluctuation of non-dimensional wall shear stress at the throat of the stenosis $\overline{\tau_m}$ for different values of μ.

4. Conclusion

The notion of the present analysis is to study the effect of severe stenosis on resistance to flow and wall shear stress. It is observed that resistance to flow decreases as stenosis shape parameter and viscosity increases, but the reverse effect occurs when stenosis size increases. The present study is able to predict the main characteristics of the physiological flows and may have played an important role in biomedical investigations.

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