Chapter 4

SEVERAL SECTORS; FIXED PRICES; NO SUBSTITUTION

4.1. Models without Interindustry Deliveries

4.11 In this chapter, models will be discussed in which the economy is subdivided into a number of sectors, each characterized by a group of products or by one product. We shall deal with the subject by the method of *decreasing abstraction*, that is, we start with very simple models and gradually make them more complicated. In this chapter prices will be assumed fixed, that is, invariable. The conditions under which this assumption is justified have been discussed in Sec. 1.55. Moreover, the hypothesis will be made that substitution of one commodity group for another in no way occurs, neither on the side of demand or consumption nor on that of supply or production. Finally, in this first section, no interindustry deliveries will be supposed to exist. Although this assumption may never materialize completely, it will often apply approximately, in particular if the sectors represent vertically integrated combinations of activities. It will be assumed, however, that there is one *capital good* needed for the production of all sectors.

To begin with, a *closed* economy will be studied.

Sectors will be indicated throughout this book by an upper index h running from 1 to H. In the models with one capital good only, this good will have $h = 1$.

4.2. Closed Economy; One Capital Good

4.21 The variables of the model are

v^h volume of production of commodity h ($h = 1, \ldots, H$)
c^h volume of consumption of commodity h

s savings
j investment
y income

4.22 The relations of the model are

$$j = \sum_h \frac{\kappa^h}{\theta} (v^h_{t+\theta} - v^h) \tag{4.22.1}$$

Investment is equal to the sum of the quantities of the capital good needed by each sector for its future expansion. In this equation θ represents the gestation period, taken equal for all sectors; κ^h the capital-output ratio for sector h.

$$s = j \tag{4.22.2}$$

Investment is assumed to be financed from savings.

$$s = \sigma y \tag{4.22.3}$$

Savings are a given proportion of income, where σ is the propensity to save.

$$y = \sum_h v^h \tag{4.22.4}$$

Income equals the sum total of all production.

$$v^h = c^h \qquad h = 2, \ldots, H \tag{4.22.5}$$
$$v^1 = j \tag{4.22.5'}$$

Production, in the absence of exports and of interindustry deliveries, equals consumption for goods 2, . . . , H and investment for good 1.

$$c^h = \gamma^h(y - s) + \bar{c}^h \qquad (h = 2, \ldots, H) \tag{4.22.6}$$
$$c^1 = 0 \tag{4.22.6'}$$

Consumption depends linearly on income minus savings and is zero for good 1. Here, γ^h are marginal propensities to consume and $\bar{c}^h$ are intercepts of the Engel curves. We shall assume that total consumption adds up to income minus savings and that consumption cannot be negative. This means that the constants γ^h and $\bar{c}^h$ must satisfy the following conditions:

$$\sum_h \gamma^h = 1 \tag{4.22.7}$$
$$\sum_h \bar{c}^h = 0 \tag{4.22.8}$$
$$c^h \geq 0 \tag{4.22.9}$$

As a consequence, there may be, for various intervals of the variable $y - s$, different sets of coefficients γ^h and $\bar{c}^h$.

Another consequence is that one of the equations, for example (4.22.2), is dependent on the others. The total number of independent equations therefore amounts to $2H + 3$ and equals the number of variables.

4.23 As always, the model can be used for the solution of various problems. In this section, we shall concentrate on the problem of planning development with a given value of σ, based perhaps on the considerations presented in Chap. 2. The model then serves to determine the development of the various sectors which is compatible with the given value of σ. This problem constitutes one of the simplest examples of the second stage of planning after σ has been determined in a first stage. This second stage supplies us with the relative sizes of the sectors, which, in their turn, enable us to calculate such average coefficients as the average capital-output ratio, assumed at a certain value in the macromodel used in stage 1. The second stage may thus lead to a value of the average capital-output ratio different from the one assumed before. If necessary a second round of stages in planning can then be undertaken.

4.24 The *nature of the solution* may first be illustrated by an example in which $H = 3$ and $\theta = 1$. The system of equations can be reduced to equations containing the v^h only as unknowns; in this particular case, they are, for $t = 0$

$$v_1^2 = \gamma^2(1 - \sigma)(v_1^1 + v_1^2 + v_1^3) + \bar{c}^2 \tag{4.24.1}$$

$$v_1^3 = \gamma^3(1 - \sigma)(v_1^1 + v_1^2 + v_1^3) + \bar{c}^3 \tag{4.24.2}$$

$$\sigma(v_0^1 + v_0^2 + v_0^3) = \kappa^1(v_1^1 - v_0^1) + \kappa^2(v_1^2 - v_0^2) + \kappa^3(v_1^3 - v_0^3) \tag{4.24.3}$$

4.25 For the *special case* where $\kappa^1 = \kappa^2 = \kappa^3 = \kappa$, we can write the system in the form

$$v_1^2 = \gamma^2(1 - \sigma)y_1 + \bar{c}^2$$

$$v_1^3 = \gamma^3(1 - \sigma)y_1 + \bar{c}^3$$

$$\sigma y_0 = \kappa(y_1 - y_0) \qquad \text{or} \qquad y_1 = \left(1 + \frac{\sigma}{\kappa}\right) y_0$$

Here, the third equation is identical with the corresponding relation in a Domar-Harrod model, yielding the development over time of y; the first and second equation then yield the individual v's.

4.26 The system (4.24.1) to (4.24.3) is a *generalization.* The last equation may now be written

$$\sigma y_0 = (y_1 - y_0) \sum_h \kappa^h \frac{v_1^h - v_0^h}{y_1 - y_0} \tag{4.26.1}$$

This means that we can still handle the model as a Domar-Harrod macromodel if we define

$$\kappa = \sum_h \kappa^h \frac{v_1^h - v_0^h}{y_1 - y_0} \tag{4.26.2}$$

that is, if we make κ a weighted average of the κ^h with variable weights $(v_1{}^h - v_0{}^h)/(y_1 - y_0)$. Since the unknowns are appearing in the weights, this is not a solution in the mathematical sense, but constitutes a hint for practical approximations.

4.27 The *exact solution* can be undertaken along the following lines. We solve the system (4.24.1) to (4.24.3) for the v_1 by first writing it

$$\begin{aligned} -\gamma^2(1-\sigma)v_1{}^1 + [1-\gamma^2(1-\sigma)]v_1{}^2 - \gamma^2(1-\sigma)v_1{}^3 &= \tilde{c}^2 \\ -\gamma^3(1-\sigma)v_1{}^1 - \gamma^3(1-\sigma)v_1{}^2 + [1-\gamma^3(1-\sigma)]v_1{}^3 &= \tilde{c}^3 \qquad (4.27.1) \\ \kappa^1 v_1{}^1 + \kappa^2 v_1{}^2 + \kappa^3 v_1{}^3 &= (\kappa^1+\sigma)v_0{}^1 + (\kappa^2+\sigma)v_0{}^2 + (\kappa^3+\sigma)v_0{}^3 \end{aligned}$$

From the first two equations we can derive $v_1{}^2$ and $v_1{}^3$ as linear functions of $v_1{}^1$ without any lag. Substituting in the third equation, we find, for $v_1{}^1$, a linear expression in

$$(\kappa^1+\sigma)v_0{}^1 + (\kappa^2+\sigma)v_0{}^2 + (\kappa^3+\sigma)v_0{}^3 = \Sigma(\kappa^h+\sigma)v_0{}^h$$

this being the only combination of initial values $v_0{}^h$ which we have to know in order to find all future values of all variables.

The formulas so far discussed can be conveniently used for *numerical extrapolation.*

4.28 Since only one combination of initial values $\Sigma(\kappa^h+\sigma)v_0{}^h$ has to be given in order to determine the future movement, the *general mathematical solution*[1] will contain one arbitrary constant only, reflecting the fact that our system (4.24.1) to (4.24.3) can be replaced by a final equation of the first order. This is due to the facts that (1) we have one capital good only and (2) the gestation periods are all equal to one. The general mathematical solution consists in assuming

$$v_t{}^h = \bar{v}_0{}^h e^{\omega t} + \bar{v}^h \qquad h = 1, 2, 3 \qquad (4.28.1)$$

and trying this solution out. Substituting it into the system (4.27.1) we shall get two types of terms, namely, terms proportional to $e^{\omega t}$ and constant terms. In order that the solutions satisfy the equations, the following conditions must be fulfilled:

$$\text{(I)} \quad \begin{aligned} -\gamma^2(1-\sigma)\bar{v}_0{}^1 + [1-\gamma^2(1-\sigma)]\bar{v}_0{}^2 - \gamma^2(1-\sigma)\bar{v}_0{}^3 &= 0 \\ -\gamma^3(1-\sigma)\bar{v}_0{}^1 - \gamma^3(1-\sigma)\bar{v}_0{}^2 + [1-\gamma^3(1-\sigma)]\bar{v}_0{}^3 &= 0 \\ \kappa^1\bar{v}_0{}^1 + \kappa^2\bar{v}_0{}^2 + \kappa^3\bar{v}_0{}^3 &= [(\kappa^1+\sigma)\bar{v}_0{}^1 \\ &\quad + (\kappa^2+\sigma)\bar{v}_0{}^2 + (\kappa^2+\sigma)\bar{v}_0{}^3]e^{-\omega} \end{aligned}$$

$$\text{(II)} \quad \begin{aligned} -\gamma^2(1-\sigma)\bar{v}^1 + [1-\gamma^2(1-\sigma)]\bar{v}^2 - \gamma^2(1-\sigma)\bar{v}^3 &= \tilde{c}^2 \\ -\gamma^3(1-\sigma)\bar{v}^1 - \gamma^3(1-\sigma)\bar{v}^2 + [1-\gamma^3(1-\sigma)]\bar{v}^3 &= \tilde{c}^3 \\ \kappa^1\bar{v}^1 + \kappa^2\bar{v}^2 + \kappa^3\bar{v}^3 &= (\kappa^1+\sigma)\bar{v}^1 \\ &\quad + (\kappa^2+\sigma)\bar{v}^2 + (\kappa^3+\sigma)\bar{v}^3 \end{aligned}$$

The first set (I) originates from the terms in $e^{\omega t}$ and the second (II) from the constant terms. From II we find that the $\bar{v}^h$ are uniquely determined

[1] See, e.g., William J. Baumol, "Economic Dynamics," pp. 151ff., New York, 1959.

by the additive constants $\bar{c}^h$ and the coefficients of the system of equations, that is, by the structural constants.

Conditions (I) are linear homogeneous in the $\bar{v}_0{}^h$, meaning that, on the one hand, one of them can be chosen arbitrarily and that, on the other hand, their coefficients have to obey the condition that their determinant is zero.

$$\begin{vmatrix} -\gamma^2(1-\sigma) & 1-\gamma^2(1-\sigma) & -\gamma^2(1-\sigma) \\ -\gamma^3(1-\sigma) & -\gamma^3(1-\sigma) & 1-\gamma^3(1-\sigma) \\ \omega'\kappa^1-(\kappa^1+\sigma) & \omega'\kappa^2-(\kappa^2+\sigma) & \omega'\kappa^3-(\kappa^3+\sigma) \end{vmatrix} = 0 \qquad (4.28.2)$$

where ω' has been written for e^{ω}.

This condition determines ω', and since ω' occurs only in the last line, the equation is linear in ω' and has one root only. This fits with our previous finding that the general solution contains only one arbitrary constant.

This constant must be so determined that the only expression in the values $v_0{}^h$ which occurs in the system (4.27.1), namely, $\sum_h (\kappa^h + \sigma)v_0{}^h$, assumes the correct numerical value. This is not the usual situation in systems of equations of the type considered, but represents what mathematicians call a *degeneration*, as a consequence of the absence of lagged variables in the first two equations of (4.27.1). We shall deal with the usual, more general case later, when we are no longer assuming the existence of one capital good only. The fact just explained has a very interesting economic implication. Even when the values $v_0{}^1$, $v_0{}^2$, $v_0{}^3$ do not fit the demand equations (4.22.5) and hence do not fit the *pattern of balanced growth* as defined by our equations, the values for $t = 1$, that is, $v_1{}^1$, $v_1{}^2$, $v_1{}^3$, will fit already. Although in more general models we shall meet the phenomenon of a *transition or adaptation period*, during which the economy, in a few steps, approaches this pattern, this adaptation takes place in one time unit—being the gestation period—here. As mentioned in this section, such an adaptation is possible in one time unit only if we have one capital good and a gestation period of one unit.

The models to be discussed later will, step by step, be made more complicated. It seemed useful, however, to treat this simple situation first. The reader will have no difficulties in removing the restriction to three sectors only; this may be suggested as an exercise.

4.3. Models with Second-order Capital Goods

4.31 Important practical problems arise from the fact that capital goods are often *specific*, that is, can be used only for one or a few related production processes. In particular, if a switch in general development

policy is aimed at, for example, in order to develop one sector less quickly than foreseen and another sector more quickly, considerable difficulties may come up, because some capital goods cannot be used for the purposes now aimed at.

So far this problem has not been recognized in our models. In the one-sector models of Chaps. 2 and 3, all goods have been considered interchangeable. We have only spoken of *production* and assumed that any portion of this production could be *saved*, making it part of the country's capital stock. In some models discussed in this chapter, capital goods have been considered interchangeable: their supply could be used in one sector or another. As long as capital grows, no difficulties arise. If, however, production in some sectors should fall, there would be difficulties. Suppose the volume of savings should be lowered. This means that less capital goods and more consumer goods would have to be produced and that, correspondingly, less capacity in the capital goods industry would be needed. As another example, assume that production in one sector of consumer goods should be lowered and in another sector raised. In both cases, capital goods would be in surplus in some sector and short in another. Our formulas tacitly assumed that a transfer was possible. If the goods are specific, this is not correct, however, and our formulas apply only if all capacities are rising, or, at most, falling by no more than annual depreciation. It may be said in defense of our formulas that development planning, which is the purpose for which our models are built, does not require, as a rule, reductions in capacity. This is not true for adaptation periods, however; nor is it true for the sectors producing so-called inferior goods.

Two attitudes are possible in the face of this problem. Surplus capital goods may be left idle, or they may be used for further production, as long as they are available. The former attitude requires a distinction, in our models, between capacity to produce and actual production, a distinction not made so far. We shall present a simple example of such a treatment (see Sec. 4.35). The latter attitude implies that the adaptation process is retarded as a consequence of specificity of capital goods. In this section, an example of this coincidence will be discussed (see Sec. 4.36).

Before dealing with these examples, we shall discuss another implication of specificity of capital goods.

4.32 As soon as capital goods are assumed to be specific, it becomes necessary to distinguish between capital goods to produce consumer goods and capital goods to produce capital goods. In a well-known terminology, the former may be called *first-order capital goods*, leaving another category which could then be called *second-order capital goods*.

There may well be scope even to introduce higher orders too. It is interesting to note that a strict adherence to the concept of specificity would force us to make one of two assumptions: either the highest-order capital goods can be produced without the aid of further capital goods, that is, with the aid of labor only, or there is an infinite number of *orders*. Both are unrealistic, the latter assumption even inconsistent with the existence of capital goods. A more realistic assumption is that for some order specificity no longer applies. The simplest assumption evidently is that this applies already to second-order capital goods. Thus, for example, we may assume that there are two types of capital goods, looms and lathes. Looms are specific and can only produce textiles (representing consumer goods) and lathes are not: they can be used for producing looms or for producing more lathes. In this train of thought there are two sectors producing capital goods, one producing looms, the other producing lathes, but their capacities can be switched. Statistically perhaps the nearest approximation to lathes is the product group of machine tools.

4.33 We shall now analyze the consequences of the introduction of second-order capital goods for the general process of development as brought out by some of the simplest models. It is not necessary to distinguish between different sorts of consumer goods, and in this respect our models will be similar to those presented in Chaps. 2 and 3. In the first model, without gestation periods, the *variables* are

v^1 production of first-order capital goods (looms)
v^2 production of second-order capital goods (lathes)
v^3 production of consumer goods (textiles)
v total production, equal to income

The *relations* are

$$v^1 = \kappa^3 \dot{v}^3 \tag{4.33.1}$$

The production of first-order capital goods is proportional to the rate of increase in consumer-goods production; κ^3 is the capital-output ratio in the consumer-goods industry.

$$v^2 = \kappa^1 \dot{v}^1 + \kappa^2 \dot{v}^2 \tag{4.33.2}$$

The production of second-order capital goods is the sum total of the production for loom making and lathe making, both depending on the rate of increase in the production of these two items, the coefficients being the capital-output ratios for producing looms (κ^1) and lathes (κ^2), respectively.

$$v^3 = (1 - \sigma)v \tag{4.33.3}$$

The production of consumer goods equals their demand, which equals income minus savings.

$$v = v^1 + v^2 + v^3 \tag{4.33.4}$$

Income is equal to the sum total of all production figures.

4.34 The general solution of the equations just presented will yield us the rate of increase in production that can be obtained from a given rate of savings σ. The solution appears to be simple: by substituting (4.33.3) into (4.33.1) and (4.33.1) into (4.33.2) we obtain

$$v^2 = \kappa^1\kappa^3(1 - \sigma)\ddot{v} + \kappa^2\dot{v}^2 \tag{4.34.1}$$

Substituting (4.33.3) into (4.33.1) and both into (4.33.4) yields

$$v = \kappa^3(1 - \sigma)\dot{v} + v^2 + (1 - \sigma)v \tag{4.34.2}$$

or

$$v^2 = \sigma v - \kappa^3(1 - \sigma)\dot{v} \tag{4.34.3}$$

It follows that

$$\dot{v}^2 = \sigma\dot{v} - \kappa^3(1 - \sigma)\ddot{v} \tag{4.34.4}$$

which can be substituted into (4.34.1):

$$\sigma v - [(1 - \sigma)\kappa^3 + \sigma\kappa^2]\dot{v} + \kappa^3(\kappa^2 - \kappa^1)(1 - \sigma)\ddot{v} = 0 \tag{4.34.5}$$

As a possible solution[1] we try $v = v_0e^{\omega t}$, implying that $\dot{v} = \omega v_0e^{\omega t}$ and $\ddot{v} = \omega^2 v_0 e^{\omega t}$. Substituting in (4.34.5), we find that ω must satisfy

$$\sigma - [(1 - \sigma)\kappa^3 + \sigma\kappa^2]\omega + \kappa^3(\kappa^2 - \kappa^1)(1 - \sigma)\omega^2 = 0 \tag{4.34.6}$$

This equation has two roots which are rather complicated expressions in the coefficients κ^1, κ^2, κ^3, and σ. For some special cases it is easy, however, to determine them. Thus, for $\kappa^2 = \kappa^1$, the last term in (4.34.6) vanishes, and there is only one root $\omega = \sigma/\kappa$, if $\kappa = (1 - \sigma)\kappa^3 + \sigma\kappa^2$, that is, a weighted average of κ^3 and κ^2 with weights $(1 - \sigma)$ and σ. This is the result Domar-Harrod models yield.

For $\kappa^3 = \kappa^2 = \kappa$ and $\kappa^2 - \kappa^1 = \Delta\kappa$ we have

$$\sigma - \kappa\omega + \kappa\,\Delta\kappa(1 - \sigma)\omega^2 = 0 \tag{4.34.7}$$

For small values of σ we can solve this equation explicitly and find two roots:

$$\omega_1 = \frac{\sigma}{\kappa} \qquad \omega_2 = \frac{1}{\Delta\kappa(1 - \sigma)} - \frac{\sigma}{\kappa} \tag{4.34.8}$$

The first root corresponds to a Domar-Harrod process with the value κ as the relevant capital-output ratio.

The second root may assume high numerical values, pointing to the possibility of a high rate of growth. Further analysis shows, however, that the corresponding process of quick growth cannot last long under

[1] See, e.g., Kells, *loc. cit.*

the assumption made with regard to σ, namely, the assumption of a constant σ. This constitutes a difference in comparison with the simplest Domar-Harrod model discussed before, where σ can take any constant value.

The question may then be asked what other behavior is compatible with the technical relations in the present model. One of the simplest patterns conceivable is a pattern with a constant proportion of new investment devoted to the production of looms.[1] This assumption implies the replacement of Eq. (4.33.3) by another one running

$$\dot{v}^1 = \varphi \dot{v}^2 \tag{4.33.3'}$$

with

$$0 \leq \varphi \leq \infty \tag{4.34.9}$$

It can be shown that development now satisfies the equation

$$v_t = v_0{}^1 + v_0{}^3 - \varphi v_0{}^2 \frac{\kappa^3\kappa^1\varphi + \kappa^3\kappa^2 + 1}{\kappa^3(\kappa^1\varphi + \kappa^2)} + \frac{v_0{}^1 - \varphi v_0{}^2}{\kappa^3} t + \left[\frac{\varphi}{\kappa^3(\kappa^1\varphi + \kappa^2)} + 1 + \varphi\right] v_0{}^2 e^{\frac{t}{\kappa^1\varphi + \kappa^2}} \tag{4.34.10}$$

The rate of development ω is no longer a constant now; it is highest for $\varphi = 0$ and asymptotically approaches $\omega_\infty = 1/\kappa^2$.

It may be left to the reader to qualify this statement.

4.35 A very simple model will now be discussed in which the capacity to produce enters as a variable alongside production. Let there be two sectors, indicated by superscripts 1 and 2, producing consumer goods and capital goods, respectively. The variables of the model are

k^1 capital-goods stock in sector 1
k^2 capital-goods stock in sector 2
v^1 production of consumer goods
v^2 production of capital goods

The relations are

$$k^h \geq \kappa^h v^h \qquad h = 1, 2 \tag{4.35.1}$$

Production can at most be equal to capital-goods stock divided by capital-output ratio.

$$v^1 = (1 - \sigma)(v^1 + v^2) \tag{4.35.2}$$

Production of consumer goods is equal to demand, being equal to income minus savings.

$$v_t{}^2 = \sum_h (k^h_{t+1} - k_t{}^h) \tag{4.35.3}$$

[1] See also Sec. 5.1.

Production of capital goods is equal to the sum total of the rates of increase, after the gestation period, of capital-goods stocks in both industries.

The model is underdetermined under some circumstances. But it is clear that the initial values of k^1 and k^2, namely, $k_0{}^1$ and $k_0{}^2$, can be such that full utilization of these capacities (that is, $k^1 = \kappa^1 v^1$ and $k^2 = \kappa^2 v^2$) is incompatible with a given value of σ requiring that

$$\frac{v^1}{v^2} = \frac{1 - \sigma}{\sigma}$$

In such a case full utilization can exist in only one sector—in sector 1, if

$$\frac{k^2/\kappa^2}{k^1/\kappa^1} > \frac{\sigma}{1 - \sigma}$$

and in sector 2 if

$$\frac{k^2/\kappa^2}{k^1/\kappa^1} < \frac{\sigma}{1 - \sigma}$$

4.36 The discrepancies just described will present themselves when a sudden change in σ is introduced. Thus, assuming that σ has been 0.1 for some time, we have

$$v^1 : v^2 = \frac{1 - \sigma}{\sigma} = 9$$

In order to fix the ideas let us take $\kappa^1 = 2$ and $\kappa^2 = 4$. If $\sigma = 0.1$ for a succession of years already, the economy has been able to adapt itself to it, and hence we shall have

$$k^1 : k^2 = \kappa^1 v^1 : \kappa^2 v^2 = 4\tfrac{1}{2}$$

and according to (4.35.3)

$$\frac{k^1_{t+1} - k_t{}^1}{k^2_{t+1} - k_t{}^2} = 4\tfrac{1}{2}$$

also. If at a certain moment t_0 total capital amounts to 11 units, we have $k^1_{t_0} = 9$, $k^2_{t_0} = 2$.

Suppose now the government wants, in the future, σ to be 0.2 and hence $(1 - \sigma)/\sigma = 4$, implying that in future equilibrium $k^1 : k^2 = 2$. If capital goods in the two sectors were interchangeable, the stock of 11 could be redistributed between the two sectors so as to obey this condition, that is, $k^1 = \tfrac{2}{3} \times 11 = 7\tfrac{1}{3}$ and $k^2 = 3\tfrac{2}{3}$. If, however, this interchangeability does not exist, k^1 will have to remain 9, and all the economy can do is, for some time to come, to use all production v^2 to increase k^2—meaning that σ is only changed by steps. Since for $t = t_0$, $v^2 = \tfrac{2}{4} = \tfrac{1}{2}$, during that year k^2 can be brought only to the level of $2\tfrac{1}{2}$, and for the

subsequent years we have the following figures if all v^2 is used to increase k^2:

t	k^2	v^2 (rounded)	k^1/k^2	v^1/v^2	σ_t
$t_0 + 1$	2.5	0.63	3.6	7.2	0.122
$t_0 + 2$	3.13	0.78	2.88	5.76	0.148
$t_0 + 3$	3.91	0.98	2.30	4.60	0.178
$t_0 + 4$	4.89				

Evidently at time $t_0 + 3$, we have not yet reached the proportion $k^1:k^2 = 2:1$, but at time $t_0 + 4$, we have surpassed the goal, meaning that during period $t_0 + 3$ we could have devoted part of v^2 to increasing k^1, the correct policy being determined by the equations

$$k^1_{t_0+4} - k^1_{t_0+3} + k^2_{t_0+4} - k^2_{t_0+3} = 0.98$$
$$k^1_{t_0+3} = 9 \qquad k^2_{t_0+3} = 3.91$$
$$k^1_{t_0+4} = 2k^2_{t_0+4}$$

The result is $k^1_{t_0+4} = 9.26$, and $k^2_{t_0+4} = 4.63$.

The example illustrates that it takes 4 time units now to reach a path compatible with the new policy. As a consequence of the specificity of capital goods, we have an adaptation period even in this simple model with only one gestation period.

At the same time it will be clear that for small changes in σ there is no need for an adaptation period. Let us calculate what is the maximum change in σ for which no adaptation period is necessary. Evidently this change must be such that by devoting the whole of v^2 to one type of capital goods we can, in period t_0, just reach the composition of capital corresponding to the new value of σ. Since $v^2 = \frac{1}{2}$, the extreme situations are

(1) $\quad k^1_{t_0+1} = 9 \qquad k^2_{t_0+1} = 2.5 \qquad$ and $\qquad \dfrac{k^1}{k^2} = 3.6$

(2) $\quad k^1_{t_0+1} = 9.5 \qquad k^2_{t_0+1} = 2 \qquad$ and $\qquad \dfrac{k^1}{k^2} = 4.75$

The corresponding values of $(1 - \sigma)/\sigma = 2k^1/k^2$ are

(1) $\quad \dfrac{1 - \sigma}{\sigma} = 7.2 \qquad$ with $\qquad \sigma = 0.122$

(2) $\quad \dfrac{1 - \sigma}{\sigma} = 9.5 \qquad$ with $\qquad \sigma = 0.095$

Therefore, for the new σ' obeying $0.122 \geq \sigma' \geq 0.095$, there is no need for an adaptation period.

4.37 A final remark on methodology seems in order. As the reader will have observed, the problems discussed can most elegantly be

handled by giving a central role to the variables k^1 and k^2, or generally all the components of the capital stock. This is the way a number of problems of this type—and considerably more complicated—have been dealt with in mathematico-economic publications.[1]

4.4. Open Economy; One Capital Good

4.41 The variables of the model are

v^h volume of production of commodity h ($h = 1, \ldots, H$)
c^h volume of consumption
e^h volume of exports (negative values of e^h representing imports)
s savings
j investment
y income

4.42 The relations of the model are Eqs. (.1), (.2), (.3), (.4), and (.6) of model 4.22, plus a number of relations indicating the development over time of exports, for example,

$$e^h = e_0{}^h + \bar{e}_1{}^h t \tag{4.42.1}$$

These equations express, in a very simple way, the limitations upon an expansion of exports, often felt to be very real in underdeveloped countries.

In addition we have a relation replacing (.5) of model 4.22:

$$v^h = c^h + e^h \qquad h = 2, \ldots, H \tag{4.42.2}$$
$$v^1 = j + e^1 \tag{4.42.2'}$$

4.43 In order to have a clear understanding of the logical implications of the number of such relations we may first derive a side condition from Eqs. (4.22.4), (4.42.2), (4.42.2′), (4.22.2), and (4.22.6), having in mind (4.22.7) and (4.22.8).

$$y = \sum v^h = \sum_2^H c^h + \sum_1^H e^h + j = y - s + s + \sum e^h$$
$$\therefore \sum e^h = 0 \tag{4.43.1}$$

This side condition brings out the well-known identity of internal financial equilibrium and balance-of-payments equilibrium. It implies that one of the e^h follows from the others.

As long as there is only one import item, Eq. (4.43.1) expresses the fact that these imports are determined by exports. If there are more import items, we are confronted with the difficulty that equations of the

[1] See R. Dorfman, P. A. Samuelson, and R. M. Solow, "Linear Programming and Economic Analysis," New York, 1958.

type (4.42.1) do not make much sense for import items. Condition (4.43.1) will then not be sufficient to determine each import item, and the model becomes underdetermined. A realistic way out will sometimes be the assumption that some of the sectors are *empty*, meaning, as we have seen, that no production takes place. In such a sector, imports have to be equal to consumption (in this simple model). In terms of supplementary relations it means that for empty sectors we have

$$v^h = 0 \tag{4.43.2}$$

4.44 Imports of goods which are also produced at home represent essentially a case of substitution and will not be considered in this chapter. We therefore assume that the number of empty sectors is equal to the number H^I of import items, minus one. This makes the number of relations equal to

1	(4.22.1)
1	(4.22.2)
1	(4.22.3)
1	(4.22.4)
H	(4.22.6)
H^E (number of export items)	(4.42.1)
	(4.42.2)
H	and (4.42.2′)
$H^I - 1$	(4.43.2)
Total $2H + H^E + H^I + 3$	

The variables add up to this same total and the model is, as a rule, determinate.

The *nature of the solution* of the planning problem with given value of σ may again be shown by taking the special case $H = 3$, $\theta = 1$. We assume that sector 1 is importing (but not empty) and that sectors 2 and 3 are exporting. Again, we may eliminate all variables except v^h, as in Sec. 4.27; the result is

$$\begin{aligned}
&-\gamma^2(1-\sigma)v_1{}^1 + [1-\gamma^2(1-\sigma)]v_1{}^2 - \gamma^2(1-\sigma)v_1{}^3 \\
&\qquad\qquad = \bar{c}^2 + e_0{}^2 + \bar{e}_1{}^2 t \\
&-\gamma^3(1-\sigma)v_1{}^1 - \gamma^3(1-\sigma)v_1{}^2 + [1-\gamma^3(1-\sigma)]v_1{}^3 \\
&\qquad\qquad = \bar{c}^3 + e_0{}^3 + \bar{e}_1{}^3 t \\
&\kappa^1 v_1{}^1 + \kappa^2 v_1{}^2 + \kappa^3 v_1{}^3 = (\kappa^1+\sigma)v_0{}^1 + (\kappa^2+\sigma)v_0{}^2 + (\kappa^3+\sigma)v_0{}^3
\end{aligned} \tag{4.44.1}$$

The difference with system (4.27.1) clearly is that the equations are now, in the mathematician's language, nonhomogeneous difference equations.[1] The solutions found in Sec. 4.28 have to be supplemented now by

[1] See, e.g., Baumol, *op. cit.*, p. 175.

a particular solution of the nonhomogeneous equations. Such a particular solution may be given the form

$$v_t^h = \varphi_0^h + \bar{\varphi}^h t \tag{4.44.2}$$

and values for φ_0^h and $\bar{\varphi}^h$ can be found without difficulty by substituting (4.44.2) into (4.44.1). They are uniquely determined by the structural constants of the model. The general solutions will be [see Eq. (4.28.1)]

$$v_t^h = \bar{v}_0^h e^{\omega t} + \varphi_0^h + \bar{\varphi}^h t \tag{4.44.3}$$

since the constants $\bar{v}^h$ shown in (4.28.1) are now absorbed in φ_0^h. The movements of e^2 and e^3 will therefore, generally speaking, influence those of v^h and hence those of y. The influence on y will be rather weak since v^1, v^2, and v^3 are influenced in opposite directions as a consequence of $\Sigma e^h = 0$. In special *degenerated* cases the influence on y may even be absent; this is true for $\kappa^1 = \kappa^2 = \kappa^3$ again, for then the third equation is one in y, independent of the movements of the individual v^h, as in Sec. 4.25.

4.5. Input-Output Models with Uniform Lags; One Capital Good

4.51 We are now going to introduce the phenomenon of interindustry deliveries. These constitute one important form of complementarity, a somewhat more general phenomenon of great importance to planning. In fact, the assumption that for the production of any commodity certain inputs of other different commodities are necessary is another way of stating that these various inputs are complementary. To the extent that we base our planning figures on such relationships, we do take account of such complementarities. It evidently depends on the details of the model used whether all relevant complementarities are taken account of. Thus it may happen that the necessary complementarity between the number of motor cars and the width of the roads is neglected.

In the examples to be discussed we shall, to begin with, maintain the assumption that there is only one capital good. We shall demonstrate the new features with the help of models for a closed economy; the reader will be able to apply them to open economies along the lines of Sec. 4.4.

4.52 The variables of the model are

- v^h volume of production of commodity h
- $v^{hh'}$ quantity of good h supplied to sector h', for current production of good h'
- c^h consumption of good h
- s savings
- j investment
- y income

4.53 The relations are those of Sec. 4.22, duly amended, supplemented with one set of new relations about interindustry deliveries. For the reader's convenience they are all given.

$$j = \sum_h \frac{\kappa^h}{\theta} (v^h_{t+\theta} - v^h) \tag{4.53.1}$$

$$s = j \tag{4.53.2}$$

$$s = \sigma y \tag{4.53.3}$$

$$y = \sum_h v^h - \sum_h \sum_{h'} v^{hh'} \tag{4.53.4}$$

This equation, the income definition, has been amended, since inputs used in the production of the various sectors must be deducted from gross product in order to obtain net product or income.

$$v^h = c^h + \sum_{h'} v^{hh'} \qquad h = 2, \ldots, H \tag{4.53.5}$$

$$v^1 = j$$

These equations, the balance equations, have also become more complicated now. Total production v^h now is composed of production for final consumption plus production for interindustry deliveries.

$$c^h = \gamma^h(y - s) + \bar{c}^h \qquad h = 2, \ldots, H \tag{4.53.6}$$

$$c^1 = 0 \tag{4.53.6'}$$

$$v^{hh'} = \varphi^{hh'} v^{h'} \tag{4.53.7}$$

This is the new set of technical equations expressing the assumption that inputs are proportional to outputs, with the technical coefficients $\varphi^{hh'}$ considered given. We shall consider somewhat more general relationships later.

4.54 The main application of the model will be for planning the sector production figures, once a value for σ has been chosen. As in Sec. 4.2, we may again easily obtain equations for v^h by the elimination of all other variables. These equations are somewhat more complicated but not essentially different from those obtained in Sec. 4.2. They may be summarized as follows:

$$v^h = \gamma^h(1 - \sigma) \sum_{h'} \left(1 - \sum_{h''} \varphi^{h''h'}\right) v^{h'} + \bar{c}^h + \sum_{h'} \varphi^{hh'} v^{h'} \tag{4.54.1}$$

$$\sigma \sum_h \left(1 - \sum_{h'} \varphi^{h'h}\right) v^h = \sum_{h'} \frac{\kappa^{h'}}{\theta} (v^h_{t+\theta} - v_t{}^h) \tag{4.54.2}$$

Since their mathematical shape is the same as before, all that was said about the nature of the solution applies.

4.6. Input-Output Models with Uniform Lags; Several Capital Goods

4.61 We now remove the restriction that there is only one capital good. Instead we recognize that almost all commodities can, in certain circumstances, play the role of capital goods. This is true in particular if we go into the details of inventory accumulation as a form of investment. This new approach, then, does not make the distinction between capital goods and other goods, but the distinction between two types of operation: current production and investment or the extension of production capacity. Accordingly, there are two input-output matrices of coefficients. We still stick to the assumption of uniform lags, that is, that the gestation periods for investments in all sectors are equal.

4.62 The list of variables now becomes

v^h production of good h
$v^{hh'}$ quantity of good h used in current production of h'
$w^{hh'}$ quantity of good h used to extend capacity of sector h'
c^h consumption of good h
s savings
y income

4.63 The list of relations will be

$$w^{hh'} = \frac{\kappa^{hh'}}{\theta}(v_{t+\theta}^{h'} - v_t^{h'}) \tag{4.63.1}$$

The new setup makes it necessary to introduce the new coefficients $\kappa^{hh'}$, to be called *partial capital coefficients* for investment of good h into sector h'.

$$s = \sum_h \sum_{h'} w^{hh'} \tag{4.63.2}$$

$$s = \sigma y \tag{4.63.3}$$

$$y = \sum_h v^h - \sum_h \sum_{h'} v^{hh'} = \sum_h v^h - \sum_{h'} \sum_h v^{h'h} \tag{4.63.4}$$

It is useful to state that the inputs $w^{hh'}$ for investment must not be deducted here. They represent dispositions of income spending rather than costs to current production as the $v^{hh'}$ are.

$$v^h = c^h + \sum_{h'} v^{hh'} + \sum_{h'} w^{hh'} \tag{4.63.5}$$

Production now is directed to three different purposes and, for open economies, even to four: consumption, current interindustry deliveries, such deliveries for investment purposes, and, in an open economy, exports (or, negatively, imports).

$$c^h = \gamma^h(y - s) + \bar{c}^h \tag{4.63.6}$$

This equation now also applies to $h = 1$.

$$v^{hh'} = \varphi^{hh'}v^{h'} \tag{4.63.7}$$

We can write the income definition in a shorter form,

$$y = \sum_h \varphi^{0h}v^h = \sum_h y^h \tag{4.63.4'}$$

by the introduction of the symbol

$$\varphi^{0h} = 1 - \sum_{h'} \varphi^{h'h} \tag{4.63.4''}$$

Here, y^h is income derived from sector h.

It should be kept in mind that the $H + H^2$ terms of y are grouped here according to receiving sectors and not to supplying sectors. This latter arrangement could be presented by a definition equation

$$y = \sum_h \left(v^h - \sum_{h'} v^{hh'}\right) = \sum_h \bar{v}^h \tag{4.63.4'''}$$

where $\bar{v}^h$ is the net quantity available for consumption, investment, and exports. Depending on the problems we have to solve, we shall use one or the other grouping in what follows.

4.64 Although the model can be used, as always, for different purposes, we shall illustrate its use by considering the problem of planning production of the sectors for a succession of years, once the savings ratio σ has been chosen. As in our previous examples, it is easy to eliminate all other variables and we are left with the system (for a closed economy)

$$v^h = \gamma^h(1 - \sigma) \sum_{h'} \varphi^{0h'}v^{h'} + \bar{c}^h + \sum_{h'} \varphi^{hh'}v^{h'} + \sum_{h'} \frac{\kappa^{hh'}}{\theta} (v^{h'}_{t+\theta} - v_t^{h'}) \tag{4.64.1}$$

The difference with system (4.27.1) is that the differences $v^{h'}_{t+\theta} - v_t^{h'}$ now occur in each equation; as a consequence the logical structure of the system becomes more complicated, leading to the planning problem of the *adaptation period*, already mentioned in Sec. 4.28. We solve Eq. (4.64.1) for $v^h_{t+\theta}$ and, for the sake of convenience, take $\theta = 1$. The solution may be written

$$v^h_{t+1} = \sum_{h'} \chi^{hh'}v_t^{h'} + \bar{\chi}^h \tag{4.64.2}$$

where the χ's are functions of the coefficients and additive structural constants appearing in (4.64.1).

Before continuing our mathematical argument we shall find it useful to explain in economic terms what these equations mean: how is it that, given the production volumes in period t, those of the subsequent period $t + 1$ are determined? The explanation is that the given volumes of production require given current inputs and hence fix current income. Income determines savings and quantities demanded for consumption of each commodity. Thus there remain quantities to be invested. With given—and different—proportions, in which, for each sector, investment inputs have to be combined, there will be, as a rule, only one way of using up all quantities of goods available for investment. It is for this reason that the future production pattern follows from the present one. Evidently a determinacy develops that is caused by the completely rigid production functions. Because of various substitution possibilities, to be discussed in later chapters, reality is not this rigid. Yet the model is a useful exercising ground and does bring out a number of problems of real life connected with complementarities.

4.65 In order to solve the planning problem under these circumstances, we shall, as usual, try a solution of the shape

$$v^h = \bar{v}_0{}^h e^{\omega t} + \bar{v}^h = \bar{v}_0{}^h \omega'^t + \bar{v}^h \tag{4.65.1}$$

where, in a somewhat loose way, ω may be called the rate of growth of the system and, for small ω, $\omega' \sim 1 + \omega$. Upon substituting these expressions into (4.64.2), we shall again find two types of terms, namely, constant terms and terms depending on t. In order that the assumed solution (4.65.1) satisfies, relation (4.64.2) must be satisfied identically by both types of terms separately. Similar to conditions (I) and (II) in Sec. 4.28, we get two sets of conditions here too; corresponding to (II) we have, for the constant terms,

$$\bar{v}^h = \sum_{h'} \chi^{hh'} \bar{v}^h + \bar{\chi}^h \tag{4.65.2}$$

From these equations the $\bar{v}^h$ can be determined as functions of the structural constants.

The conditions corresponding to (I) again are linear homogeneous in the $\bar{v}_0{}^h$, meaning that one factor for the whole set remains arbitrary and that, simultaneously, the determinant of the coefficients must vanish. This now takes the form

$$\begin{vmatrix} \chi^{11} - \omega' & \chi^{12} & \cdots & \chi^{1H} \\ \chi^{21} & \chi^{22} - \omega' & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \chi^{H1} & \chi^{H2} & \cdots & \chi^{HH} - \omega' \end{vmatrix} = 0 \tag{4.65.3}$$

Contrary to the previous situation this is now an equation of the Hth degree, which, as a rule, will have H roots. There is not one value ω', therefore, but there are H values ω'_h satisfying (4.65.3), and the general solution for v_t^h runs

$$v_t^h = \sum_{h'} \bar{v}_{h'} \bar{v}_{0h'}^h \omega_{h'}'^t + \bar{v}^h \qquad \bar{v}_{0h'}^1 = 1 \tag{4.65.4}$$

Here the roots ω' have been given a number h'; the general solution of v_t^h consists of the sum of the solutions corresponding to each ω', each solution containing a general multiplication factor $\bar{v}_{h'}$, equal for all v_t^h, and a relative value of $\bar{v}_{0h'}^h$, which has been chosen 1 for v_t^1 in all cases h'. The $\bar{v}_{0h'}^h$ are functions of the χ and $\omega'_{h'}$, but the $\bar{v}_{h'}$ are arbitrary and must be used to let the solutions fit the initial values of the variables, which must be given in order to make the problem a definite one.

4.66 So far what we have said about the solution of our problem applies just as well to the usual *analytical economic problem* of finding the future development of an economy with given initial values of the variables. In the case now considered, with gestation periods of one unit everywhere, these initial values will be v_0^h ($h = 1, \ldots, H$). It is clear that, with these values given, all future values of v_t^h can be calculated (numerically) from Eq. (4.64.2).

Generally speaking, however, the movements so found, and represented by Eq. (4.65.4), will show various components, of which only one is a smooth trend, the others being fluctuations of different periods and damping degrees. These movements will usually be considered undesirable by the planner. An additional complication is that our equations do not give a reliable picture of short-term reactions of free economic decisions and therefore the short-term fluctuations corresponding with the ω''s will be inaccurate forecasts.[1] The *planning problem therefore is not identical with the forecasting problem, but rather consists in producing, by the additional use of instruments of economic policy, a smooth development of the economy,* that is, a development along the paths

$$v_t^h = \bar{v}_1 \bar{v}_{01}^h \omega_1'^t + \bar{v}^h \qquad \bar{v}_{01}^1 = 1 \tag{4.66.1}$$

where ω'_1 represents the real root of (4.65.3) > 1.

The nature of the problems involved may be made clear by an example; we assume that $H = 3$ again. The initial values given to us by history are v_0^1, v_0^2, v_0^3. If, to begin with $t = 0$, we based our planning and our development policy on Eq. (4.64.2), that is, the system discussed in

[1] This is largely due to the fact that the acceleration principle, which is the core of long-term development, is a very inaccurate presentation of short-term reactions of a free economy.

Sec. 4.63 and summarized in Eq. (4.64.1), we would not necessarily obtain a *smooth* movement: the ratios between v_0^1, v_0^2, and v_0^3 may prevent this. For the movement to be smooth it would have to obey (4.66.1), which contains *one* arbitrary constant $\bar{v}_1$ only, and this cannot, as a rule, be chosen so as to satisfy the three conditions

$$\begin{aligned} v_0^1 &= \bar{v}_1 + \bar{v}^1 \\ v_0^2 &= \bar{v}_1\bar{v}_{01}^2 + \bar{v}^2 \\ v_0^3 &= \bar{v}_1\bar{v}_{01}^3 + \bar{v}^3 \end{aligned} \tag{4.66.2}$$

Only if our initial values happened to satisfy these relations would the problem of adaptation not exist.

We must therefore, if we wish to obey Eq. (4.66.1) later, deviate from Eq. (4.64.2) for some time, to be called the adaptation period; the questions to be answered are: how long and how must we deviate? The answers depend on the instruments available to the authorities to let the variables deviate. Taking up Eqs. (4.63.1) to (4.63.7) one by one, we find that Eqs. (.1), (.4), (.5), and (.7) because of their nature cannot be violated by additional government instruments. It may be possible to deviate from Eq. (.2), however, by importing foreign capital, from Eq. (.3) by imposing taxes used for forced savings, and from Eq. (.6) by specific taxes on some goods or by rationing.

4.67 We shall discuss three different examples with regard to the number of instruments. In example (1) we assume that the number of additional instruments is $H - 1$, that is, in the present context 2. To fix the ideas, we indicate them by Δs, being an additional term at the right-hand side of (4.63.3), to be called forced savings, and Δc^1, to be added to the equation for c^1, belonging to (4.63.6). As a consequence, Eq. (4.64.2) will have one additional term each.

$$\begin{aligned} v_{t+1}^1 &= \chi^{11}v_t^1 + \chi^{12}v_t^2 + \chi^{13}v_t^3 + \bar{\chi}^1 + \Delta\bar{\chi}^1(\Delta s, \Delta c^1) \\ v_{t+1}^2 &= \chi^{21}v_t^1 + \chi^{22}v_t^2 + \chi^{23}v_t^3 + \bar{\chi}^2 + \Delta\bar{\chi}^2(\Delta s, \Delta c^1) \\ v_{t+1}^3 &= \chi^{31}v_t^1 + \chi^{32}v_t^2 + \chi^{33}v_t^3 + \bar{\chi}^3 + \Delta\bar{\chi}^3(\Delta s, \Delta c^1) \end{aligned} \tag{4.67.1}$$

The values of $\Delta\bar{\chi}^h$ ($h = 1, 2, 3$) follow from those of Δs and Δc^1. With these additional terms we can see to it that for $t = 0$, the values of v_1^1, v_1^2, and v_1^3 obey conditions comparable to (4.66.2), namely

$$\begin{aligned} v_1^1 &= \bar{v}_1\omega_1' + \bar{v}^1 \\ v_1^2 &= \bar{v}_1\bar{v}_{01}^2\omega_1' + \bar{v}^2 \\ v_1^3 &= \bar{v}_1\bar{v}_{01}^3\omega_1' + \bar{v}^3 \end{aligned} \tag{4.67.2}$$

The three variables Δs, Δc^1, and $\bar{v}_1$ are at the planner's disposal to satisfy (4.67.2), and by applying Δs and Δc^1, the government can bring about the economy in the configuration, warranting a smooth develop-

ment without an adaptation period. The difference with our example in Sec. 4.28 is that no special measures were even required then.

In example (2) we assume that the number of additional instruments is $>H - 1$, namely 3, for example, Δs, Δc^1, and similarly Δc^2. Not only can the adaptation problem be solved then also without delay, but there is even 1 degree of freedom left, meaning that the *level* $\bar{v}_1$, from which the smooth development starts, can be chosen (within limits, of course).

In example (3), on the contrary, we assume that the number of instruments is $<H - 1$, namely 1, say Δc^1 only. Clearly the problem cannot now be solved without delay; but it can be solved for $t = 1$, that is, for v_2^1, v_2^2, and v_2^3. By applying in two consecutive time periods 0 and 1, values Δc_0^1 and Δc_1^1, respectively, we can influence the values v_1^1, v_2^1, and v_3^1. Writing down (4.67.1) for $t = 1$ and $\Delta s = 0$, we then have

$$v_2^1 = \chi^{11}v_1^1(\Delta c_0^1) + \chi^{12}v_1^2(\Delta c_0^1) + \chi^{13}v_1^3(\Delta c_0^1) + \bar{\chi}^1 + \Delta\bar{\chi}^1(\Delta c_1^1)$$

etc., symbolizing that we have 2 degrees of freedom. These can in fact be used to satisfy conditions for v_2^h similar to (4.67.2).

In a general way we can conclude from our examples that with a number of instruments equal to n, the delay will be equal to the next lower integer of the fraction $(H - 1)/n$.

4.7. Input-Output Models with Different Lags

4.71 To assume uniform gestation periods for the various sectors is far from realistic. All economies show wide variations in such periods. Many important irrigation and power projects, as well as mining projects, require periods of several or many years for their completion. Therefore it seems appropriate to introduce this feature into our models too. The alternative treatment is to break up a long process into a series of shorter ones and to introduce such commodities as semifinished or even three-tenths-finished irrigation dams, which can only be outputs from or inputs into one single process. It does not make much difference which alternative is chosen; we shall introduce the former rather than the latter. Our formulas can be easily adapted since we have already used the symbol θ, which may now be replaced by θ^h.

4.72 The set of θ^h appearing in any model may be called, together with the $w^{hh'}$, the time structure of investment. As a rule, the time structure will have to be replaced by some simpler pattern, though not by the extremely simple pattern of only one lag. Therefore, it will be very helpful to approximate it by a set of multiples of one unit lag, preferably one year.

Even so, the structure may be very different in different cases. To let the reader understand how to tackle the resulting problems the best

method seems to us to be the treatment of an example with some remarks about possibilities to generalize the results. We shall treat the simplest conceivable example, consisting of a model of two sectors only, showing gestation periods of $\theta^1 = 1$ and $\theta^2 = 2$ years.

4.73 The equations in v^h, comparable to (4.64.2), will now be

$$\begin{aligned} v^1_{t+1} &= \chi^{11}v_t{}^1 + \chi^{12}v_t{}^2 + \bar{\chi}^1 \\ v^2_{t+2} &= \chi^{21}v_t{}^1 + \chi^{22}v_t{}^2 + \bar{\chi}^2 \end{aligned} \tag{4.73.1}$$

From these equations it is clear that the initial values which we must know in order to determine the future course of the variables, say from $t = 1$ onward, are the three combinations

$$\chi^{11}v_0{}^1 + \chi^{12}v_0{}^2,\ \chi^{21}v^1_{-1} + \chi^{22}v^2_{-1},\ \chi^{21}v_0{}^1 + \chi^{22}v_0{}^2$$

Correspondingly, if we apply the standard method of general solution, we find a third-degree equation for ω', comparable to Eq. (4.65.3),

$$\begin{vmatrix} \chi^{11} - \omega' & \chi^{12} \\ \chi^{21} & \chi^{22} - \omega'^2 \end{vmatrix} = 0 \tag{4.73.2}$$

The general mathematical solution will be of the shape

$$v_t{}^h = \sum_1^3{}^{h'}\ \bar{v}_{h'}\bar{v}^h_{0h'}\omega'^t_{h'} + \bar{v}^h \tag{4.73.3}$$

with the $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$ arbitrary and $\bar{v}^1_{01} = \bar{v}^1_{02} = \bar{v}^1_{03} = 1$ [see Eq. (4.65.4)]. Again, only one of these components will be desirable, and accordingly, additional instruments of economic policy may be required, and an adaptation period may be unavoidable.

4.74 The procedure can be easily generalized for more complicated cases. The degree of the equation for ω' will no longer be H, as in the case of a uniform lag of one time unit, but $\sum_h \theta^h$, where all θ^h's are assumed to be integers. This implies that the number of components and the number of initial values will be $\sum_h \theta^h$ too. With a number of additional instruments equal to n, there may be an adaptation period if the next lower integer to $\Sigma\theta^h/n$ is >0. The period will be equal to that integer.

It might be expressly stated that in the case of a uniform lag of two years, all the phenomena discussed will be at play also. The point is that with a two-year period the variables may show fluctuations which are among the undesirable components.

4.8. Input-Output Models with Variable Coefficients

4.81 The basic assumption of input-output technique in its present form is the constancy of the coefficients. This means that coefficients

do not change over time and that they do not depend on the variables of the system. In this section some possibilities will be discussed to remove constancy over time. There are a few important examples of changes over time. Perhaps the most important example is the well-known continual fall in labor-input coefficients. Energy inputs are another example, and there are more of them. The fall in labor inputs is very marked in young industries, where it has been found that there may be a more or less general type of *learning curve* independent of the type of industry considered.

The removal of the other form of nonconstancy, that is, the introduction of coefficients depending on the variables of the system, surpasses the limits set for the present chapter. Some aspects of it will be discussed in the next chapter.

4.82 Coefficients changing over time do not constitute any difficulty to the operation called *numerical extrapolation*, illustrated by Sec. 4.28 and equally applicable to (4.44.1) and to (4.54.1) and (4.54.2). There is no difficulty indeed in changing the coefficients in each successive calculation. For practical planning purposes this method is sufficient, however, only if the model is a very simple one, requiring no adaptation by the use of additional instruments of economic policy.

4.83 A second method of dealing with the problem consists in writing the coefficients as explicit functions of time and linearizing all equations. This procedure will transform Eqs. (4.64.2) into equations of the following structure:

$$v_{t+1}^{h} = \sum_{h'} (\chi_1^{hh'} v_t^{h'} + \chi_2^{hh'} t) + \bar{\bar{\chi}}^h \tag{4.83.1}$$

where $\chi_1^{hh'}$, $\chi_2^{hh'}$, and $\bar{\bar{\chi}}^h$ depend on the old coefficients $\chi^{hh'}$ and $\bar{\chi}^h$, on the initial values of the $v^{h'}$, and on the rates of change of the old coefficients. For the details of transformations of this kind the reader may be referred to textbooks on econometrics.

The result is a system of equations of the shape of (4.44.1), the solution of which was discussed in Sec. 4.44 [see Eq. (4.44.3)]. This solution can be used only for short intervals, the length of which depends on the rate of change of the coefficients.

4.84 A third method that may sometimes be useful is the method known in physics as the method of invertible changes. It applies to slow changes only in the coefficients. It consists in using the solutions of the case with constant coefficients and substituting for these coefficients the functions of time they are supposed to be. Thus, for the problem dealt with in Sec. 4.6 (input-output models with uniform lags and several

capital goods), it means maintaining the general form (4.65.4) or (4.66.1) of the solutions, but substituting for the ω'_h the functions of time they become, once the $\chi^{hh'}$ are no longer constants but functions of time. To be sure, this can be done explicitly only in a small number of cases. For some problems implicit functions or numerical approximations of functions may also be useful, however.