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MATHEMATICAL MODELS OF EPIDEMICS

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PREFACE

The present tract is a monograph in the true sense. It is not a survey of all kinds of mathematical models of contagious diseases neither a survey of existing literature. It deals with a single mathematical model which finds its origin in the pioneering paper of KERMACK & McKENDRICK (1927). It is written from a practical point of view showing what can be done with relatively simple analytical and numerical techniques.

When introducing the subject HOPPENSTEADT (1975) advocated focussing attention on very simple models. "This will illustrate one of the main reasons for studying epidemic models: finite complicated systems often have features which are correctly described by much simpler versions. When this is the case, and these particular features are the things of interest, it is necessary only to consider a simple model." One general conclusion should be mentioned here already. Our work confirms the all-importance of the logistic function as the standard way of describing epidemic behaviour. Perhaps even the general Kermack and McKendrick is too general. In its set-up features are included which only result in minor variations of the outcome. After all, an epidemic is usually described by daily counts of new illness cases. But these may be affected by all kinds of accuracies.

The author wishes to express his thanks to members of the Department of Mathematical Physics and Biomathematics for their critical remarks which led to so many improvements. In particular we mention O. Diekmann and J. Grasman of the Amsterdam Mathematical Centre and also J.A.J. Metz of the Leiden University: all members of a national biomathematics working group. Many ideas worked out here found their origin in lively discussions with them.

CHAPTER 1

INTRODUCTION AND GENERAL SURVEY

The mathematical analysis of epidemics starts perhaps with the pioneering paper of KERMACK & McKENDRICK (1927), to be cited as K&K, in which they proposed a fairly general mathematical model of a contagious disease and formulated what later became known as their celebrated "threshold theorem". This paper was preceded by an equally remarkable paper on the "applications of mathematics to medical problems" (1926) by the second author alone. Many ideas that can be found in the recent literature can be traced back to those papers.

The subject seems to have been neglected, at least among mathematicians, for some thirty years, but a new impetus was given when in 1956 Kendall gave an address for the third Berkely symposium on mathematical statistics and probability in which he drew attention to the fundamental work of the two Scotchmen. Kendall discussed a simplified model used by K&K for illustrative purposes but the original more general model remained rather unnoticed. In 1970 the Dutch theoretical biologists, Reddingius and Metz revived the interest in this general model. Their work was a source of inspiration for a number of mathematicians of the Amsterdam Mathematical Centre and the Amsterdam University including the author of this tract.

A few terms are used here in a mathematical sense rather than in a strict medical sense. The individuals of a population of constant size are usually grouped into classes of susceptibles, ills and immunes. However, sometimes a person who recovered from a contagious disease is still counted as an ill until he looses immunity. Throughout the text we use the same notation for the fundamental variables and parameter. The relative number of susceptibles is always indicated by $x(t)$. The graph of $x(t)$ is called the epidemic curve on the epidemic function. The relative number of ills is indicated by $y(t)$. However, this variable has not much practical value since it may comprise the totality of infective ills, non-infective ills, immunes and dead individuals. Therefore we mainly use $v(t)$, the rate of new ills, as

the second fundamental variable. The rate or amount of infectivity A(t) of an ill depends on the time t passed since his infection. The total amount of infectivity, γ is defined as

$$\gamma = \int_0^\infty A(t)dt.$$

This parameter serves as a threshold with $\gamma > 1$ for an epidemic and $\gamma < 1$ for a passing disease. A possible epidemic is usually started at $t = 0$ by introducing a small fraction ε of infective ills into a population of susceptibles. In some models the start of an epidemic is put at minus infinity, i.e. the unknown remote past. Models of that kind appear to pose interesting problems.

Much attention is also paid to K&K type models with a discrete time variable. They are close analogues of the models with a continuous time variable but they have the advantage that they lend themselves readily to the use of the small programmable pocket calculators such as the HP 41 C. In fact discrete time models are closer to reality. Again they go back to the early days of mathematical epidemiology but they seem to be fairly neglected up to recent times because of lack of computer facilities.

It is not attempted to give a historical survey of the subject. There are a number of good papers and books giving full attention to history and the existing literature. A selected bibliography may be found at the end of this tract.

After this introduction a survey will be given of the subsequent sections.

2. The special model considered by Kendall is studied. An explicit analytic solution is given in the form of an integral expression. It is shown that this solution can be approximated by a logistic function. Asymptotic analysis and perturbation techniques are applied with respect to ε. It is shown that the half-way time t_m of the epidemic is proportional to $\ell n\ 1/\varepsilon$.

3. The general model as considered by K&K, Reddingius and others is described. In this model the initial fraction of infectives may have an arbitrary composition. However, Example 3.2 shows that x(t) is hardly influenced by this arbitrariness. This suggests using a model (3.15), (3.16) with the simplest possible initial condition where the initial fraction of infectives entirely consists of new ills. This model is described by a nonlinear integral equation of convolution type. It can already be found in K&K's original paper. Formula (3.20), Fig. 3.2 and Table 3.1 show the dependance of the limit value

$x(\infty)$ of susceptibles on γ and ε. Fig. 3.2 and Table 3.1 illustrate the significance of γ as a threshold parameter.

4. For a mathematical model of an epidemic it is irrelevant whether an individual is removed by death or by gaining permanent immunity. In this chapter the effect of removal is considered but this turns out to be a different and more detailed way of book-keeping. No essentially new features are added to the model of Chapter 3.

5. The main subject of this chapter is the derivation of an analytical estimate of the half-way time t_m. At first the general model of Chapter 3 is taken with the simple initial condition. Using Laplace transformation techniques the linearized model can be solved. In this way we obtain the approximate solution (5.11) with a corresponding estimate (5.12) of t_m. Next the model of Chapter 3 with a general initial condition is considered. Using a slightly different perturbation procedure we arrive at the approximate solution (5.18) and the half-way time estimate (5.20). Example 5.2 shows the effect of a more general initial condition upon $x(t)$.
To the general model of Chapter 3 we may apply the limit operation

$$t \to t_m(\varepsilon) + t, \quad \varepsilon \to 0.$$

By this a model is obtained for the double-infinite time range $(-\infty,\infty)$. It is described by the integral equation (5.21). This equation shows bifurcation with respect to γ. For $\gamma \leq 1$ there is only the trivial solution $x \equiv 1$, i.e. no disease. For $\gamma > 1$ there exists an infinity of non-trivial solutions all differing by a time shift. Example 5.3 shows that $x(t)$ is logistic for a block-function $A(t)$.

6. In this brief chapter existence and uniqueness are discussed. It is shown that a solution of the epidemic integral equation can be obtained as a Picard sequence. More details on existence and uniqueness may be found in the literature.

7. The epidemic model of Chapter 3 with the simple initial condition is considered for the important case of a block-function. This model has got already due attention in the literature. It has been found possible to simplify the analysis which led to the stream-lined treatment as given here. The approximate solution (7.25) is again of the logistic kind. It is shown that for $\varepsilon \to 0$ the solution converges to the logistic function thereby confirming the result of Example 5.3.

8. Computer facilities have given new life to discrete models. In this section the continuous K&K model is paralleled by a model with a discrete time variable. It appears that the latter model imitates the continuous in an almost ideal way. In order to avoid repeated reference to other sections a review is given of the continuous model in the form (8.1), (8.2). The corresponding discrete model is given by (8.14). Its discussion is essentially that of the continuous model (8.2). Instead of Laplace transformation the techniques of generating power series is used. The half-way time is estimated as in the continuous model. Example 8.1 is the analogue of the special K&K model of Chapter 2. However, no explicit solution is available here. In Example 8.2 the block-function case is studied. A numerical calculation shows that the logistic function is a surprisingly good approximation.

9. This chapter and the next one are dealing with the effect of the shape of A(t) upon the epidemic function x(t). In the present chapter we investigate the effect of an incubation period with for A(t) a shifted block function. We draw attention to the similarity principle as shown in Fig. 9.1. According to this principle and confirmed by analytical and numerical results the main effect of a postponed period of infectivity is a slowing-down of the time scale in proportion to the portion of the mid-point of A(t). Both the continuous and the discrete model are considered. The general conclusion is that the influence of an incubation period upon the logistic shape of the epidemic curve x(t) is almost negligible.

10. In the preceding chapter it became clear that small deviations of the epidemic curve are well compatible with large variations of A(t). This is a warning that the determination of A(t) from a given epidemic function x(t) may pose a rather ill-posed problem. If x(t) would be known as a given continuously differentiable function then A(t) can be found from its Laplace transform. The calculation is carried out explicitly for a logistic function. This gives the expected result of a block-function in agreement with Example 5.3. However, in an actual situation x(t) is only approximately known. The early beginning of an epidemic disease is often lost in obscurity and the future course is equally uncertain. A possible method is to approximate the better known middle part of x(t) by say a logistic function. A better method perhaps, is the determination of the moments of A(t) when considered as a statistical distribution function from those of $\dot{x}(t)$ and of $\dot{x}(t)/x(t)$. The main formulae are collected in (10.42). However, it turns out that only the first few moments of A(t) can be determined within a remarkable degree of

accuracy due to the heavy loss of significant digits in the higher moments. This is demonstrated in Table 10.1 for the case of a shifted block-function.

11. In this section an important new element is added to the model. It will be assumed that after recovery there is a period of immunity of finite duration. The model in its most general form is described by the equations (11.9), (11.14) and (11.17). With a somewhat crude way of book-keeping - a removed individual is counted as a non-infective ill - the model can be reduced to two integral equations (11.20) for the susceptibles $x(t)$ and the new ills $v(t)$. Again this model can be paralleled by a double-infinite model (11.29). The local stability of the possible equilibrium depends on the behaviour of the roots of equation (11.34). In Example 11.1 this is worked out for the simple K&K model of Chapter 2 with temporal immunity.

12. This chapter contains a few worked-out cases illustrating various points of the general model of the preceding chapter. Again, the logistic function appears to be a standard description of epidemic behaviour. Special attention is asked for example C with an exponentially diminishing infectivity $A(t)$ and with a constant rate of recovery. It appears that for $t \to \infty$, $x(t)$ tends to a stable equilibrium. Fig. 12.3 shows a numerical case in which this equilibrium is approached in an oscillatory way.

13. The important case of a block-function of duration p is studied here with the additional feature of an immunity period of fixed duration q. The continuous model has the two parameters $\gamma(1+\varepsilon)$ and q/p. The stability of the possible asymptotic equilibrium is pictured in the parameter space by a region of stability in Fig. 13.1 bounded by a critical line. Numerical illustrations of an unstable case show that this model may well represent a periodic epidemic disease. However, a rigorous mathematical proof of the existence of an asymptotically periodic solution is still a desideratum.

14. Again the block-function is studied but now for the discrete model. The treatment follows closely that of the continuous model. However, the analysis of stability is more complicated here because of the additional effect of the time discretization. For each pair of values (p,q) where p and q have the same meaning as in the preceding section there exists a critical value of $\gamma(1+\varepsilon)$ for which the equilibrium becomes unstable. Table 14.1 gives critical values of some lower values of p and q. The discrete model considered here lends itself very easily to the use of a programmable pocket calculator such as the HP 41 C. In Table 14.2 and 14.3 numerical illustrations are given of a stable case and of an unstable case. In Fig. 14.1 the unstable case is

illustrated graphically. It appears that x(t) shows a behaviour with almost periodic oscillations.

15. If the discretization of the time is too coarse the model discussed in the preceding chapter may give false results due to possible multiple counting of new ills. In order to eliminate this unwanted side-effect the model is improved by using probability arguments. The improved model is described by the equations (15.6). Table 15.1 shows that for a large epidemic, with $\gamma = 3.2$, the two models may give rather different results. The stability analysis is very similar to that of the preceding chapter. Table 15.2 is the counterpart of Table 14.1. The general conclusion is that the new model is much more stable than that of the previous chapter.

CHAPTER 2

THE SPECIAL K&K MODEL

> *Much insight can be obtained as to the process by which epidemics in limited populations run their peculiar courses, and end in final extinction from the consideration of the special case...* (K&K p.712)

The model is described by the equations

$$(2.1)\qquad \begin{cases} \dot{x} = -axy, \\ \dot{y} = axy - by, \\ \dot{z} = by, \end{cases}$$

for $t > 0$.

In this model K&K consider a demagraphically closed population of individuals partitioned into the following three classes:

$x(t)$ number of susceptibles at time t.
$y(t)$ number of infectives.
$z(t)$ those that are "removed", i.e., died or recovered from the disease and got immunity.

The first equation expresses the assumption that the individuals are mixing homogeneously as the molecules of a mixture of two gases. Further that the number of new infections in a small time interval xdt is proportional to the number of meetings between susceptibles and infectives. The second and third equation express the assumption that infectives have a constant chance of being "removed".

Of course we always have

$$(2.2)\qquad x + y + z = \text{constant}$$

a relation which is consistent with (2.1).

In accordance with K&K we take the initial condition

$$(2.3) \qquad x_0 = N, \quad y_0 = \varepsilon N, \quad z_0 = 0,$$

where $0 < \varepsilon < 1$.

A number of interesting conclusions can be drawn from (2.1) without actually solving the equations. We note that $x(t)$ always decreases. The initial behaviour of $y(t)$ depends on the sign of $ax_0 - b$ or, what is the same, on the value of the dimensionless parameter

$$(2.4) \qquad \lambda = \frac{b}{aN}$$

with respect to 1.

If $\lambda > 1$, $y(t)$ starts decreasing and stays so. This means that the disease dies out without epidemic behaviour. However, if $\lambda < 1$, $y(t)$ starts increasing and stays so until $ax(t) = b$. Thus an epidemic may develop in that case which is at its peak for $x = b/a$. After that $y(t)$ decreases again. In both cases $y(t) \to 0$ for $t \to \infty$. This observation implies the so-called threshold theorem.

For $\lambda > 1$ there will be no epidemic.
For $\lambda < 1$ an epidemic may develop.

In order to make the statement more precise we have to solve the system (2.1).

Let us introduce scaled dimensionless variables by means of

$$x \to Nx', \quad y \to Ny', \quad z \to Nz', \quad t \to (aN)^{-1}t' \ .$$

Then the equations (2.1) and the relation (2.2) can be replaced by

$$(2.4) \qquad \begin{cases} \dot{x} = -xy, \\ \dot{y} = xy - \lambda y, \\ x+y+z = 1+\varepsilon, \end{cases}$$

with the initial conditions

(2.5) $\quad x(0) = 1, \quad y(0) = \varepsilon.$

From (2.4) we obtain the single d.e.

(2.6) $\quad \frac{dy}{dx} = -1 + \frac{\lambda}{x}$.

Its solution satisfying the boundary condition (2.5) is

(2.7) $\quad y = 1 + \varepsilon - x + \lambda \ell n\, x.$

Although the time variable has been eliminated this equation neatly describes a possible epidemic behaviour. The right-hand side of (2.7) has a maximum for $x = \lambda$. There are two situations whether or not this maximum falls within the x-domain (0,1).

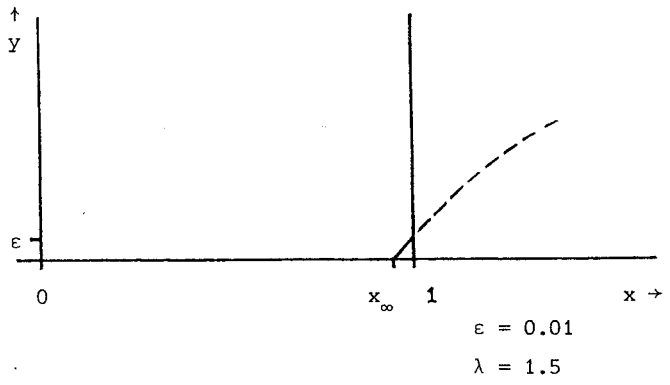
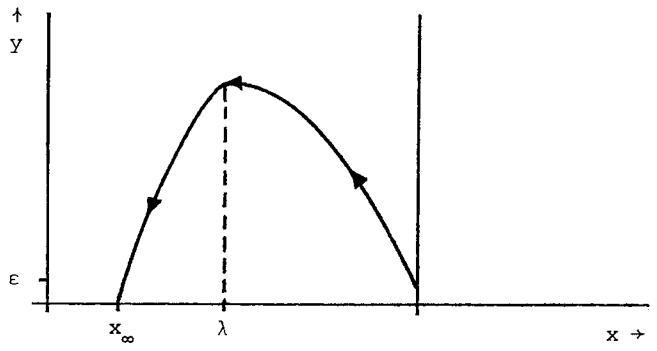


Fig. 2.1.

A typical situation for $\lambda > 1$ is sketched in Fig. 2.1. The process starts at the point $(1,\varepsilon)$ and proceeds along the curve downwards to the point $(x_\infty,0)$ where x_∞ is the lower zero of $1 + \varepsilon - x + \lambda \ell n\, x$. If $\lambda - 1 >> \varepsilon$ we have the good estimate $x_\infty = 1 - \varepsilon(\lambda-1)^{-1}$. However, a similar plot for $\lambda < 1$ confirms the observation made above for the epidemic case. A typical situation is sketched in Fig. 2.2.



ε = 0.01

λ = 0.5

Fig. 2.2.

The process starts again at $(1,\varepsilon)$ but now an epidemic is started which is at its peak - i.e., with a maximum total of ills - at the point (λ, y_{max}) where

$$y_{max} = 1 + \varepsilon - \lambda + \lambda \ell n\, \lambda. \tag{2.8}$$

When the epidemic is over at the point $(x_\infty, 0)$ there remains a fraction of individuals who though susceptible were not infected. In Table 2.1 we have collected a few values of x_∞ and y_{max} for various values of λ with $\varepsilon = 0$.

$\lambda = x_{max}$	y_{max}	x_∞
0.2	0.48	0.007
0.3	0.34	0.041
0.4	0.23	0.107
0.5	0.15	0.203
0.6	0.09	0.324
0.7	0.05	0.467
0.8	0.02	0.629
0.9	0.01	0.807

Table 2.1

In order to determine the course of the epidemic in real time we combine the d.e. $\dot{x} = -xy$ with the relation (2.6). By integration of

$$-\,dt = \frac{dx}{xy}$$

we obtain

$$t = \int_{x}^{1} \frac{d\xi}{\xi(1+\varepsilon-\xi+\lambda\,\ell n\,\xi)} \tag{2.9}$$

This expression defines a function t(x) inversion of which gives the required epidemic behaviour x(t) as a function of time. Numerical integration gives a curve which rather looks like a logistic curve as in Fig. 2.3.

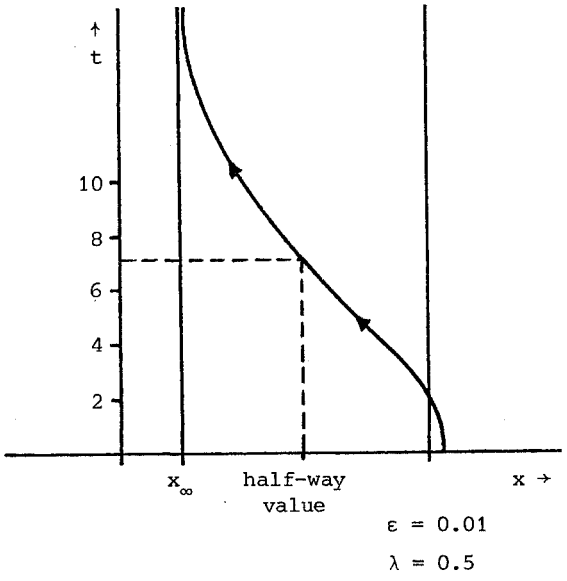


Fig. 2.3

Although the right-hand side of (2.9) cannot be expressed in elementary functions it is not difficult to study its behaviour for $\varepsilon \to 0$, in particular when x is close to 1. Then we have the appropriate equality

$$-\ell n\, x \approx 1-x.$$

Then (2.9) may be approximated by

$$t = \int_x^1 \frac{d\xi}{\xi(\varepsilon+(1-\xi)(1-\lambda))} \tag{2.10}$$

integration of which is elementary. We have

$$t = \frac{1}{1+\varepsilon-\lambda} \ell n \frac{\varepsilon+(1+x)(1-\lambda)}{\varepsilon x} . \tag{2.11}$$

Inversion gives

$$x = \frac{1+\varepsilon-\lambda}{1-\lambda+\varepsilon \exp(1+\varepsilon-\lambda)t} , \tag{2.12}$$

the well-known logistic function.

This means that the epidemic behaviour of the model (2.1) or (2.4) can be approximated by the logistic law provided x(t) stays close to x(0). Let $t = t_m$ be the time for which x has a given value x_m then t_m depends on ε in such a way that $t_m \to \infty$ for $\varepsilon \to 0$. This follows at once from (2.9) since the integral diverges when $\varepsilon = 0$. If the approximation (2.10) is used we would obtain

$$t_m = \frac{1}{1-\lambda} \ell n \frac{(1-x_m)(1-\lambda)}{\varepsilon x_m} + O(\varepsilon \ell n \varepsilon), \tag{2.13}$$

or, more roughly,

$$t_m(\varepsilon) \approx \frac{1}{1-\lambda} \ell n \frac{1}{\varepsilon} + \text{constant}. \tag{2.14}$$

We shall now study (2.9) more carefully by looking for an asymptotic expansion in ε which is also valid for moderate values of x. We may expect a leading term as given in (2.14). This prompts us to split off the approximation (2.10). Again with $1-\lambda \approx -\ell n\xi$ we proceed as follows

$$t = \int_x^1 \frac{d\xi}{\xi(\varepsilon-(1-\lambda)\ell n\xi)} + \int_x^1 \left(\frac{1}{1+\varepsilon-\xi+\lambda\ell n\xi} - \frac{1}{\varepsilon-(1-\lambda)\ell n\xi}\right) \frac{d\xi}{\xi} =$$

$$= \frac{1}{1-\lambda} \ell n \frac{\varepsilon-(1-\lambda)\ell nx}{\varepsilon} + \int_x^1 \frac{-1+\xi-\ell n\xi}{(1-\xi+\lambda\ell n\xi)(-(1-\lambda)\ell n\xi)} \frac{d\xi}{\xi} + o(1),$$

so that for $t = t_m$, $x = x_m$

$$(2.15) \qquad t_m = \frac{1}{1-\lambda} \ell n \frac{(1-\lambda)\, \ell n\, 1/x_m}{\varepsilon} + C(x) + o(1).$$

where

$$(2.16) \qquad C(x_m) = \frac{1}{1-\lambda} \int_0^{\ell n\, 1/x_m} \frac{-1+e^{-\eta}+\eta}{\eta(1-e^{-\eta}-\lambda\eta)}\, d\eta.$$

As a numerical illustration we consider the approximations (2.13), (2.14) and (2.15) for the special case $\lambda = 0.05$, $\gamma = 2$. For x_m we take the half-way value $x_m = \frac{1}{2}(1+x_\infty)$.

	$\varepsilon = 0.01$	$\varepsilon = 0.001$	$\varepsilon = 0.0001$
x_∞	0.197	0.202	0.203
$x_m = \frac{1}{2}(1+x_\infty)$	0.598	0.601	0.602
t_m (2.13)	7.03	11.61	16.21
t_m (2.14)	9.21	13.82	18.42
t_m (2.15)	7.70	12.29	16.89

Table 2.2

It may be of some use to derive an estimate of $C(x)$. If x_m is close to 1 the integrand can be approximated by

$$\frac{-1+e^{-\eta}+\eta}{\eta(1-e^{-\eta}-\lambda\eta)} = \frac{\frac{1}{2}+\ldots}{(1-\eta)-\frac{1}{2}\eta+\ldots}.$$

Integration is elementary and gives

$$(2.17) \qquad C(x_m) \approx \frac{1}{1-\lambda} \ell n \frac{1-\lambda}{1-\lambda+\frac{1}{2}\ell n\, x_m}$$

In the special case considered above with $\lambda = 0.5$ we find $C(x_m) \approx 1.42$ whereas numerical integration of (2.16) yields 1.21.

The formulae (2.13), (2.14) and (2.15) may be interpreted by saying that each halving of the original ε-fraction of ills causes a constant delay of $\ell n 2/(1-\lambda)$ in the time for which the epidemic attains its peak.

To say that an epidemic is at its height may mean different things. The first definition is the time t_m when the number y of ills is maximum. In this model y is given by (2.8) with $x = \lambda$. A second definition is the time t_m for which $x = \frac{1}{2}(x_0+x_\infty)$. In that case we may say that then the epidemic is half-way through. A third more practical definition is the time for which the rate $-\dot{x}$ of new ills, the daily number of diseases is maximum.

The inverse of (2.10) can also be determined in a more direct way by starting from the d.e.

$$\dot{x} = -x(1+\varepsilon-x+\lambda \ell n\, x) \tag{2.18}$$

and taking x in the form of a perturbation series

$$x = 1 + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots \tag{2.19}$$

It appears to be a slightly better procedure to change x into a logarithmic variable u by making the substitution

$$x = \exp - u \tag{2.20}$$

first. Then (2.18) takes the form

$$\dot{u} = 1 + \varepsilon - e^{-u} - \lambda u. \tag{2.21}$$

Now we apply the perturbation series

$$u = \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \ldots . \tag{2.22}$$

The usual perturbation procedure leads to the following equations

$$\begin{aligned} &\dot{u}_1 - (1-\lambda)u_1 = 1, \\ &\dot{u}_2 - (1-\lambda)u_2 = -\tfrac{1}{2} u_1^2 \quad \text{etc.} \end{aligned} \tag{2.23}$$

They all have the same initial condition

$$u_1 = u_2 = \ldots = 0 \quad \text{for } t = 0$$

Integration is elementary. We find

$$u_1 = \frac{1}{1-\lambda}\,(e^{(1-\lambda)t}-1), \tag{2.24}$$

and next

$$u_2 = -\frac{e^{(1-\lambda)t}}{(1-\lambda)^2}\,(\sinh(1-\lambda)t - (1-\lambda)t). \tag{2.25}$$

This result is easily seen to be compatible with (2.12). In fact from (2.12) we obtain for $\varepsilon \to 0$

$$\frac{1}{x(t)} = 1 + \varepsilon u_1(t) + O(\varepsilon^2),$$

and from (2.20) and (2.22)

$$\frac{1}{x(t)} = \exp(\varepsilon u_1 + \varepsilon^2 u_2 + \ldots) = 1 + \varepsilon u_1(t) + O(\varepsilon^2).$$

CHAPTER 3

A GENERAL K&K MODEL

We consider a population of individuals who are subjected to a disease the course of which may be described in the following way.

Let a susceptible be infected at time $\tau = 0$ then the rate of infectivity, his capability of infecting others, is given by a non-negative function $A(\tau)$ which is a characteristic of the given disease. In an actual situation there may be numbers τ_1 and τ_2 such that $A(\tau) = 0$ for $0 \leq \tau \leq \tau_1$ and $\tau > \tau_2$ and $A(\tau) \neq 0$ for $\tau_1 < \tau < \tau_2$. The lowest number marks the incubation time and $\tau_2 - \tau_1$ measures the length of the period of infectiveness. It may be useful, however, to consider mathematical models with $\tau_1 = 0$ and with $\tau_2 = \infty$.

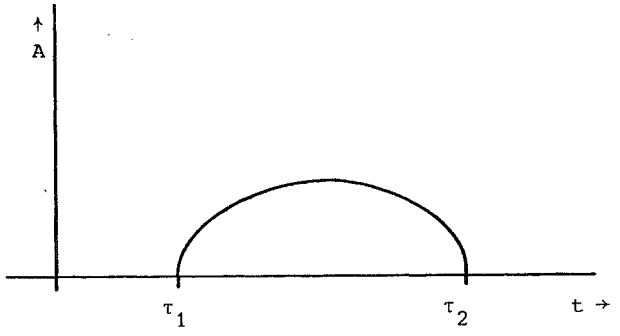


Fig. 3.1

In this model the assumption is made that the diasease gives permanent immunity. Once an individual became infected he is counted as an ill from that time onward. Thus there is no removal from the class of ills neither by recovery nor by death.

The following classes will be used

$x(t)$ number of susceptibles at time t.

$v(t,\tau)d\tau$ number of ills who are in the phase $(\tau,\tau+d\tau)$ of the disease at time t.

$y(t)$ total number of ills at time t.

We have of course

$$y(t) = \int_0^\infty v(t,\tau)d\tau \tag{3.1}$$

and next

$$x(t) + y(t) = \text{constant}. \tag{3.2}$$

The absence of removal from any class of ills implies that

$$v(t+h,\tau+h) = v(t,\tau) \tag{3.3}$$

for any possible time shift h. This means that

$$\begin{aligned} v(t,\tau) &= v(t-\tau) \quad \text{for } t > \tau, \\ v(t,\tau) &= v(0,\tau-t) \quad \text{for } t \leq \tau, \end{aligned} \tag{3.4}$$

where we have used the shorter notation

$$v(t) = v(t,0) \tag{3.5}$$

The chance of infection is proportional to the number of susceptibles and the total amount of infectivity as measured by $\int A(\tau)v(t,\tau)d\tau$. This gives the following relation for the rate of new ills $v(t)$

$$v(t) = ax(t) \int_0^\infty A(\tau)v(t,\tau)d\tau, \tag{3.6}$$

where the factor a may be time dependent if one wishes to incorporate effects

such as seasonal influence. However, we shall consider only the case of a constant factor. By a proper scaling it can be arranged that a = 1 and also that

(3.7) $\quad x(t) + y(t) = 1.$

In what follows it is assumed that this scaling has been carried out.

As in the previous chapter we consider an initial situation with

(3.8) $\quad x(0) = 1, \quad y(0) = \varepsilon.$

The ills introduced "from the outside" may be in various phases of their illness. This means that $v(0,\tau)$ is some given function of τ, say

(3.9) $\quad v(0,\tau) = \varepsilon g(\tau), \quad \int_0^\infty g(\tau)d\tau = 1.$

The number of new ills in a small time interval h can be expressed as an increase of the number of susceptibles

$$x(t) - x(t+h) = -h\dot{x}(t).$$

In this way we obtain for the rate of new ills the equation

(3.10) $\quad v(t) = -\dot{x}(t).$

The equations (3.4), (3.6), (3.9) and (3.10) can be combined into a single equation

(3.11) $\quad \dot{x}(t) = x(t)\{\int_0^t A(\tau)\dot{x}(t-\tau)d\tau - \varepsilon \int_t^\infty A(\tau)g(\tau-t)d\tau\}$

By integration the following simpler equation is obtained

(3.12) $\quad -\ell nx(t) = \int_0^t A(\tau)\{1-x(t-\tau)\}d\tau + \varepsilon A_1(t),$

where

$$(3.13) \qquad A_1(t) = \int_0^\infty \{A(\tau) - A(\tau+t)\}G(\tau)d\tau$$

with

$$G(t) = \int_0^t g(\tau)d\tau.$$

The transition from (3.11) into (3.12) may seem somewhat complicated but the reader will have no difficulty in the verification of (3.12) by direct differentiation and by noting its validity for $t = 0$.

In most actual situations the way in which an epidemic disease begins is rather obscure so that it has little meaning of paying much attention to the specific form of the initial distribution function $g(\tau)$. A convenient choice is

$$(3.14) \qquad g(\tau) = \delta(\tau),$$

Dirac's symbolic delta function. This is equivalent to the assumption that the ills introduced into the population at $t = 0$ are just infected and are at the very beginning of the course of their disease.

The assumption (3.14) gives

$$G(t) = 1, \quad A_1(t) = \int_0^t A(\tau)d\tau$$

so that the equations (3.11) and (3.12) can be written in the simpler form

$$(3.15) \qquad \dot{x}(t) = x(t)\{\int_0^t A(\tau)\dot{x}(t-\tau)d\tau - \varepsilon A(t)\},$$

and

$$(3.16) \qquad -\ell nx(t) = \int_0^t \{1+\varepsilon-x(t-\tau)\}A(\tau)d\tau.$$

Both equations can be found in K&K's original papers.

The following example shows that the special K&K model can be obtained as a special case of (3.16) if for $A(\tau)$ an exponential function is taken.

EXAMPLE 3.1.

$$A(t) = e^{-\lambda t}, \quad g(\tau) = \delta(\tau).$$

The equation (3.16) can be written as

$$-\ell nx = e^{-\lambda t} \int_0^t e^{\lambda\tau}\{1+\varepsilon-x(\tau)\}d\tau.$$

From this we obtain by differentiation

$$-\frac{\dot{x}}{x} - \lambda\ell nx = 1 + \varepsilon - x$$

or

$$\dot{x} = -x(1+\varepsilon-x+\lambda\ell nx),$$

but this equation is equivalent to (2.9).

The special case considered in the preceding example may serve as a test-case for studying the influence of $g(\tau)$ upon $x(t)$.

EXAMPLE 3.2.

$$A(t) = e^{-\lambda t}, \quad g(\tau) = \omega e^{-\omega\tau}.$$

From (3.13) we obtain

$$G(\tau) = 1 - e^{-\omega\tau}$$

and

$$A_1(t) = \frac{\omega}{\lambda(\omega+\lambda)} (1-e^{-\lambda t})$$

so that (3.12) can be written as

$$-\ell nx = e^{-\lambda t} \int_0^t e^{\lambda\tau}\{1-x(\tau)\}d\tau + \frac{\varepsilon\omega}{\lambda(\omega+\lambda)} (1-e^{-\lambda t})$$

and next as in the preceding example

$$-\frac{\dot{x}}{x} - \lambda\ell nx = 1 + \frac{\varepsilon\omega}{\omega+\lambda} - x.$$

We see that the only influence of $g(\tau)$ upon the epidemic behaviour $x(t)$ is a slightly different value of ε.

The initial amount of infectivity is given by

$$\varepsilon \int_0^\infty A(\tau)g(\tau)d\tau = \frac{\varepsilon\omega}{\omega+\lambda} .$$

This means that this special model is insensitive to changes in the parameter ω of the initial exponential distribution function $g(\tau)$ provided the initial amount of infectivity is kept constant.

We shall now consider (3.12) by taking for $A(t)$ and $g(t)$ arbitrary non-negative functions. In order to avoid unnecessary mathematical complications we assume that $A(t) \to 0$ for $t \to \infty$ and that

$$\gamma = \int_0^\infty A(t)dt \tag{3.17}$$

is finite.

Since $x(t)$ is a monotonously decreasing function of t the limit x_∞ of x for $t \to \infty$ exists. From (3.12) we obtain for $t \to \infty$

$$-\ell n x_\infty = \gamma(1-x_\infty) + \varepsilon \lim_{t\to\infty} A_1(t) .$$

From (3.13) we obtain

$$\varepsilon \lim_{t\to\infty} A_1(t) = \varepsilon \int_0^\infty A(\tau)G(\tau)d\tau,$$

which can be interpreted as the total amount of infectivity produced by the initial number of ills $\varepsilon g(\tau)$ in the course of their disease. In fact, the number $\varepsilon g(\tau)d\tau$ of the ills at $t = 0$ in the phase τ of their disease contributes in the course of time the amount of infectivity

$$\varepsilon g(\tau)d\tau \int_\tau^\infty A(t)dt.$$

For all ills at $t = 0$ the total amount is

$$\varepsilon \int_0^\infty g(\tau)\{\int_\tau^\infty A(t)dt\}d\tau = \varepsilon \int_0^\infty A(\tau)G(\tau)d\tau.$$

Thus we have

$$-\ell n x_\infty = \gamma(1-x_\infty+\varepsilon_0), \tag{3.18}$$

where

$$\gamma\varepsilon_0 = \varepsilon \int_0^\infty A(\tau)G(\tau)d\tau. \tag{3.19}$$

Normally we take $g(\tau) = \delta(\tau)$. $G(\tau) = 1$ and then ε_0 coincides with ε. From now on we shall drop the zero subscript. It is easy to show that the function $\gamma(x_\infty)$ defined as

$$\gamma = \frac{-\ell n x_\infty}{1+\varepsilon-x_\infty} \tag{3.20}$$

decreases from ∞ to 0 in the interval $0 \leq x_\infty \leq 1$. Thus for a given value of γ (3.18) defines a single root. The function $\gamma(x_\infty)$ of (3.20) is plotted below for $\varepsilon = 0.01$. In Table 3.1 we have collected a few values of x_∞ for various given values of γ and ε. Again we observe a threshold behaviour. For $\gamma \leq 1$ their is only a slightly passing disease whereas for $\gamma > 1$ a true epidemic may occur. Of course there is no clear distinction between an epidemic and a disease. However, in the limit situation $\varepsilon \to 0$ there is a quite different behaviour of $x(t)$ on either side of $\gamma = 1$.

It appears that for $\gamma << 1$, x is very close to 1. Using the approximation

$$-\ell n x_\infty \approx 1 - x_\infty$$

we find the estimation

$$x_\infty \approx 1 - \frac{\varepsilon\gamma}{1-\gamma}, \quad \gamma << 1. \tag{3.21}$$

For $\gamma >> 1$, x_∞ is rather small and we have from (3.20) by neglecting $\varepsilon - x_\infty$

$$x_\infty \approx \exp - \gamma, \quad \gamma >> 1. \tag{3.22}$$

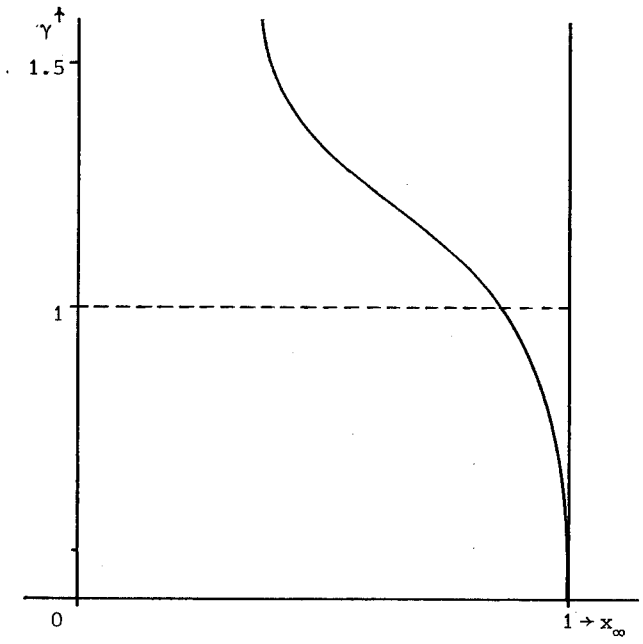


Figure 3.2

γ	$\varepsilon = 0$	$\varepsilon = 0.001$	$\varepsilon = 0.01$
0.6	1	0.998	0.985
0.7	1	0.997	0.978
0.8	1	0.996	0.963
0.9	1	0.991	0.933
0.95	1	0.984	0.905
1	1	0.956	0.865
1.01	0.980	0.945	0.856
1.02	0.961	0.933	0.846
1.05	0.906	0.890	0.814
1.1	0.824	0.815	0.757
1.2	0.686	0.682	0.646
1.3	0.577	0.574	0.550
1.4	0.489	0.487	0.469
1.5	0.417	0.416	0.401

Table 3.1. Values of x_∞ for given ε and γ.

CHAPTER 4

THE MODEL WITH REMOVAL

The book-keeping of the general model considered in the preceding chapter may be improved by taking into account the effect of removal. It will be assumed that for each class $(\tau,\tau+d\tau)$ of the ills there exists a given rate $\psi(\tau)$ of removal. This means that during a small time interval h from the number of individuals $v(t,\tau)d\tau$ a fraction $h\psi(\tau)$ is removed. Some ills may die or may recover thereby gaining permanent immunity. However, in the mathematical model both possibilities amount to the same thing. In addition to the variables $x(t)$ and $y(t)$ we have to use a variable $z(t)$ for the number of those that are removed up to time t. With a proper scaling, using fraction instead of numbers, we have of course

$$x(t) + y(t) + z(t) = 1. \tag{4.1}$$

The various phases of the disease are passed according to the so-called transport equation

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial \tau} = -\psi(\tau)v. \tag{4.2}$$

This equation may be derived as follows. During a small time interval h the individuals of the class $(\tau,\tau+d\tau)$ move to the next class $(\tau+h,\tau+h+d\tau)$. Due to the removal of the fraction $h\psi(\tau)$ we have the following equation of balance

$$v(t+h,\tau+h)d\tau - v(t,\tau)d\tau = -h\psi(\tau)v(t,\tau)d\tau.$$

By taking the derivative with respect to h the transport equation (4.2) is obtained without difficulty. The transport equation (4.2) has the solution

$$
\begin{aligned}
&v(t,\tau) = v(t-\tau)B(\tau) && \text{for } t > \tau, \\
&v(t,\tau) = \varepsilon g(\tau-t)\,\frac{B(\tau)}{B(\tau-t)}\,, && \text{for } t \le \tau,
\end{aligned}
\tag{4.3}
$$

where

$$
B(t) = \exp - \int_0^t \psi(\tau)\,d\tau, \tag{4.4}
$$

and where $v(t)$ and $g(t)$ have the same meaning (3.5), (3.9) as in the previous chapter. The solution (4.3) may be divided in a standard fashion by using the technique of characteristics. In a slightly different approach we observe that (4.2) has the general solution

$$
v(t,\tau) = f(t-\tau)B(\tau),
$$

where $f(t)$ is an arbitrary function which can be determined by using the initial conditions for $t = 0$ and for $\tau = 0$.

The special solution (3.4) is obtained from (4.3) by taking $\psi(\tau) \equiv 0$ and $B(t) \equiv 1$.

In this chapter we consider only the particular case (3.14) which describes a sudden infection at $t = 0$. Then (4.3) can be written as

$$
\begin{aligned}
&v(t,\tau) = v(t-\tau)B(\tau), && \text{for } t > \tau, \\
&v(t,\tau) = \varepsilon\delta(\tau-t)B(\tau), && \text{for } t \le \tau.
\end{aligned}
\tag{4.5}
$$

The rate of new ills is described by the relation (3.6) or, after scaling with $a = 1$,

$$
v(t) = x(t) \int_0^\infty A(\tau)v(t,\tau)\,d\tau. \tag{4.6}
$$

Substitution of (4.5) in (4.6) gives

$$
v(t) = x(t)\,\{\int_0^t A(\tau)B(\tau)v(t-\tau)\,d\tau + \varepsilon A(t)B(t)\}. \tag{4.7}
$$

To this we may add the relation (3.10) or

$$
v(t) = -\dot{x}(t). \tag{4.8}
$$

The equations (4.7) and (4.8) can be combined into a single equation as (3.15) and (3.16) the only difference being that A(t) has to be replaced by A(t)B(t)

$$A(t) \to A(t)B(t). \tag{4.9}$$

To these equations we may add relations for obtaining y(t) and z(t). From (3.1), (4.5) and (4.8) we get

$$y(t) = -\int_0^t B(\tau)\dot{x}(t-\tau)d\tau + \varepsilon B(t). \tag{4.10}$$

Finally

$$z(t) = 1 - x(t) - y(t). \tag{4.11}$$

EXAMPLE 4.1. We consider the extreme case in which an ill is a constant source of infection but also has a constant probability of removal by either death or recovery.

If we take

$$A(t) = 1, \quad \psi(\tau) = \lambda$$

we have from (4.4) B(t) = exp - λt, so that

$$A(t)B(t) = e^{-\lambda t}$$

Thus we have obtained the mathematical model of Example 3.1, i.e., K&K's special model with a different interpretation.

CHAPTER 5

SINGULAR PERTURBATION

The integral equations (3.15) and (3.16) present themselves as an invitation to applying perturbation techniques with respect to ε. GRASMAN and MATKOVSKY (1976) have worked out this case in detail. The obvious idea is to apply a perturbation series

$$x(t,\varepsilon) = 1 + \varepsilon x_1(t) + \varepsilon^2 k_2(t) + \ldots . \tag{5.1}$$

Substitution in (3.15) gives for the term of the lowest order

$$\dot{x}_1(t) = \int_0^t A(\tau)\dot{x}_1(t-\tau)d\tau - A(t). \tag{5.2}$$

The standard technique of dealing with (5.2) is Laplace transformation as defined by

$$\bar{A}(s) = \int_0^\infty e^{-st}A(t)st. \tag{5.3}$$

By this (5.2) is transformed into

$$s\bar{x}_1(s) = s\bar{A}(s)\bar{x}_1(s) - \bar{A}(s) \tag{5.4}$$

so that

$$\bar{x}_1(s) = \frac{-\bar{A}(s)}{s(1-\bar{A}(s))} . \tag{5.5}$$

From this $x_1(t)$ can be determined by some method of inverse Laplace transformation.

It is of historical interest to note that the linearized equation (5.2) and its solution (5.5) can be traced back to K&K's original paper. This is the more remarkable because they used Laplace transformation long before it

became a popular working-tool among mathematicians.

Following K&K we may concentrate on functions $A(t)$ which are a linear combination of a finite number of terms such as $t^m e^{-\lambda t}$ where m is a non-negative integer and Re $\lambda > 0$. Then $\bar{A}(s)$ is a rational function of s and the same is true for $\bar{x}_1(s)$. The original $x_1(t)$ is then also a linear combination of similar terms determined by the roots of $\bar{A}(s) - 1 = 0$. Considering the epidemic case $\gamma > 1$, the only case of interest, we note that

$$\bar{A}(0) = \gamma \tag{5.6}$$

and that there exists a positive real number σ such that

$$\bar{A}(\sigma) = \int_0^\infty e^{-\sigma t} A(t)\,dt = 1. \tag{5.7}$$

This value $s = \sigma$ is a pole of $\bar{x}_1(s)$ with the largest real part. It is a little mathematical exercise to show that this pole is simple and that there are no other poles on the line Re $s = \sigma$.

The standard theorems of the asymptotic behaviour of Laplace transforms then say that

$$x_1(t) \approx -\frac{1}{\sigma} e^{\sigma t} \operatorname{Res} \frac{1}{1-\bar{A}(s)}, \quad t \to \infty. \tag{5.8}$$

The residue of $(1-\bar{A})^{-1}$ at $s = \sigma$ will be written as $1/\mu$ where

$$\mu = \int_0^\infty e^{-\sigma t} tA(t)\,dt. \tag{5.9}$$

Thus we have

$$x_1(t) \approx -\frac{1}{\mu\sigma} e^{\sigma t}, \quad t \to \infty. \tag{5.10}$$

The first two terms of the perturbation series give the approximation

$$x(t) \approx 1 - \frac{\varepsilon}{\mu\varepsilon} e^{\sigma t}. \tag{5.11}$$

Clearly, the expansion (5.1) cannot be uniformly valid for all values of t. For $t \approx \frac{1}{\sigma} \ell n \frac{1}{\varepsilon}$ the second term of (5.1) is of the same order as the first term. Both terms when used as an approximation of $x(t,\varepsilon)$ give an estimate for the time $t = t_m$ for which the epidemic is at its height, say when

$x = \frac{1}{2}(1+x_\infty)$.

From (5.11) we obtain

$$t_m \approx \frac{1}{\sigma} \ell n \frac{1}{\varepsilon} + \frac{1}{\sigma} \ell n \frac{\mu\sigma}{2} (1-x_\infty). \tag{5.12}$$

This estimation will be compared to the corresponding estimation (2.13) in the special K&K model.

EXAMPLE 5.1. The special K&K model is obtained for $A(t) = \exp - \lambda t$. A simple calculation shows that $\sigma = 1 - \lambda$ and $\mu = 1$ so that

$$t_m \approx \frac{1}{1-\lambda} \ell n \frac{(1-x_\infty)(1-\lambda)}{2\varepsilon}$$

which is in good agreement with (2.18).

As a slightly different procedure we may apply the perturbation technique after the substitution $x = \exp - u$ as (2.20) in the special K&K model. In order to avoid unnecessary repetition we now consider the integral equation (3.12) with an arbitrary initial condition.

With (2.20) we write

$$u(t) = \int_0^t A(\tau)\{1 - e^{-u(t-\tau)}\}d\tau + \varepsilon A_1(t). \tag{5.13}$$

The perturbation series is now

$$u(t) = \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \ldots . \tag{5.14}$$

For the lowest order term we have the linear integral equation

$$u_1(t) = \int_0^t A(\tau) u_1(t-\tau) d\tau + A_1(t). \tag{5.15}$$

Its solution can be found by using Laplace transformation as before. We find

$$\bar{u}_1(s) = \frac{\bar{A}_1(s)}{1-\bar{A}(s)} \tag{5.16}$$

as a generalization of (5.5).

For $t \to \infty$ the dominant contribution to $u_1(t)$ comes from the pole $s = \sigma$, so that

$$u_1(t) \approx \frac{\bar{A}_1(\sigma)}{\mu} e^{\sigma t}, \quad t \to \infty, \tag{5.17}$$

a similar result as (5.10), and when εu_1 is used as an approximation of u

$$x(t) \approx \exp\left(-\frac{\varepsilon \bar{A}_1(\sigma)}{\mu} e^{\sigma t}\right), \tag{5.18}$$

similar to (5.11).

The numerical factor $\bar{A}_1(\sigma)$ is determined by (3.13) and depends of course on the initial condition as given by $g(\tau)$. However, we can make the following estimation

$$A_1(t) \le \int_0^\infty A(\tau)G(\tau)d\tau \le \int_0^\infty A(\tau)d\tau = \gamma,$$

and

$$\begin{aligned} A_1(t) &= \int_0^\infty A(\tau)G(\tau)d\tau - \int_t^\infty A(\tau)G(\tau-t)d\tau \\ &\ge \int_0^\infty A(\tau)G(\tau)d\tau - \int_t^\infty A(\tau)G(\tau)d\tau \\ &= \int_0^t A(\tau)G(\tau)d\tau \ge 0. \end{aligned}$$

From this we obtain

$$0 \le \bar{A}_1(\sigma) \le \frac{\gamma}{\sigma}. \tag{5.19}$$

The following example gives a more explicit result for the special case $g(\tau) = \omega \exp - \omega\tau$ considered already in Example 3.2.

EXAMPLE 5.2.

$$g(\tau) = \omega e^{-\omega\tau}.$$

From (3.11) and (3.12) we obtain

$$A_1(t) = \omega \int_0^t \{ \int_0^\infty e^{-\omega(\tau-t)} A(\tau)\,d\tau - \int_0^t e^{\omega(t-\tau)} A(\tau)\,d\tau \}\,dt$$

$$= \omega \int_0^t \{ e^{\omega t} \bar{A}(\omega) - e^{\omega t} * A(t) \}\,dt$$

so that

$$\bar{A}_1(s) = \frac{\omega}{s} \{ \frac{\bar{A}(\omega)}{s-\omega} - \frac{1}{s-\omega} . \bar{A}(s) \}.$$

The final result is

$$\bar{A}_1(\sigma) = \frac{\omega}{\sigma(\omega-\sigma)} \{1 - \bar{A}(\omega)\}.$$

For $\omega \to \infty$ we obtain $\bar{A}_1(\sigma) \to 1/\sigma$ as it should be.

From (5.18) we may derive the following estimate for the time $t = t_m$ for which $x(t)$ takes its half-way value $x_m = \frac{1}{2}(1+x_\infty)$

$$(5.20) \qquad t_m \approx \frac{1}{\sigma} \ell n \frac{1}{\varepsilon} + \frac{1}{\sigma} \ell n \left(\frac{\mu}{\bar{A}_1(\sigma)} \ell n \frac{1}{x_m} \right).$$

This formula again shows that the epidemic process is little influenced by the way the epidemic is started.

The effect of $\varepsilon \to 0$ to $x(t)$ can be described as follows. The half-way time t_m increases as $\frac{1}{\sigma} \ell n \frac{1}{\varepsilon}$ so that the graph of $x(t)$ as a whole is shifted to the right while keeping the same form up to an ε-perturbation. This may prompt us to introduce a new time scale by taking t_m as the zero time.

$$t \to t + t_m .$$

If this trick is applied to the integral equation (3.12) we get

$$-\ell n x(t+t_m) = \int_0^{t+t_m} A(\tau)\{1 - x(t-\tau+t_m)\}\,d\tau + \varepsilon A_1(t+t_m) .$$

With a proper redefinition of x

$$x(t+t_m) \to x(t)$$

we obtain for $\varepsilon \to 0$, $t_m \to \infty$, the following result

$$-\ell nx(t) = \int_0^\infty A(\tau)\{1 - x(t-\tau)\}d\tau. \tag{5.21}$$

This integral equation has a number of interesting properties which are discussed in detail by DIEKMANN (1977). We note that if x(t) is a solution of (5.21) any translate x(t+h) is also a solution. It is natural to impose as an extra condition

$$x(0) = x_m, \tag{5.22}$$

where x_m can be chosen as the half-way value $\frac{1}{2}(1+x_\infty)$. This condition seems to determine a unique solution of (5.21). However, there is always the trivial solution $x(t) \equiv 1$. In the next section it will be shown that this is the only solution for $\gamma \leq 1$. For $\gamma > 1$ a non-trivial solution can be determined. This solution x(t) is a monotonously decreasing function with $x(-\infty) = 1$ and where $x_\infty = x(\infty)$ is obtained from the relation (3.18) with $\varepsilon = 0$ i.e.

$$-\ell nx_\infty = \gamma(1-x_\infty). \tag{5.23}$$

EXAMPLE 5.3. We consider (5.23) for the block-function

$$A(t) = \begin{cases} \gamma & \text{for } 0 \leq t \leq 1, \\ 0 & \text{for } t > 1. \end{cases} \tag{5.24}$$

Substitution in (5.21) gives

$$-\ell nx = \gamma - \gamma \int_{t-1}^{t} x(\tau)d\tau, \tag{5.25}$$

which by differentiation passes into the difference-differential equation

$$\frac{\dot{x}}{x} = \gamma\{x(t) - x(t-1)\}. \tag{5.26}$$

It is perhaps a bit of a surprise that this equation has the following simple solution

$$x(t) = \frac{c+c^t}{1+c^t}, \qquad 0 < c \leq 1, \tag{5.27}$$

as can be verified by direct substitution. We note that $c = x_\infty$ and that $x(0) = \frac{1}{2}(1+x_\infty)$.

The block function case will be discussed in much more detail in a later chapter. Then a more systematic derivation of the above-given solution can be found.

The integral equation (5.21) has a monotonously decreasing solution $x(t)$ with $x(-\infty) = 1$. The behaviour of $x(t)$ for $t \to -\infty$ is not known in advance but a reasonable guess is

$$x(t) \approx 1 - c \exp st. \tag{5.28}$$

If this is substituted in (5.21) we obtain as a first order approximation

$$ce^{st} = c \int_0^\infty e^{s(t-\tau)} A(\tau) d\tau$$

or

$$\bar{A}(s) = \int_0^\infty e^{-s\tau} A(\tau) d\tau.$$

This gives $s = \sigma$ in view of (5.7).

The constant c cannot be determined which is self-evident since it is connected with the time-translation property of the solution of (5.21). Thus for $\gamma > 1$ (5.28) with an arbitrary positive value of c may serve as a start for the non-trivial solution of (5.21).

CHAPTER 6

EXISTENCE AND UNIQUENESS

We consider first the model (3.12) which describes an epidemic process with an arbitrary start at t = 0. After the substitution x = exp - u we obtain the Volterra integral equation (5.13) which we repeat here for convenience

$$u(t) = \int_0^t A(\tau)\{1 - e^{-u(t-\tau)}\}d\tau + \varepsilon A_1(t), \tag{6.1}$$

where $0 \le A_1(t) \le \gamma$ with $A_1(0) = 0$.

It is a well-known fact that such an equation has a unique solution which can be obtained as the limit of the following Picard process

$$u_{n+1}(t) = \int_0^t A(\tau)\{1 - \exp - u_n(t-\tau)\}d\tau + \varepsilon A_1(t), \tag{6.2}$$

with

$$u_0(t) = \varepsilon A_1(t).$$

It is not difficult to show that the Picard sequence is monotonuously non-decreasing. We know already that the solution of (6.1) is a monotonuous non-decreasing function of time with u(0) = 0 and $u(\infty) = -\ell n x_\infty$.

Bounds of u(t) can be obtained from (6.1) as follows

$$u(t) \le \int_0^t A(\tau)u(t-\tau)d\tau + \varepsilon A_1(t) \le u(t) \int_0^t A(\tau)d\tau + \varepsilon\gamma$$

so that

$$u(t) \le \frac{\varepsilon\gamma}{1-\int_0^t A(\tau)d\tau} \tag{6.3}$$

In the non-epidemic case $\gamma < 1$ we may take $t = \infty$ in the right-hand side of (6.3) so that

$$u(t) \leq \frac{\varepsilon\gamma}{1-\gamma}, \quad \gamma < 1. \tag{6.4}$$

for all values of t. In the epidemic case $\gamma \geq 1$ the inequality (6.3) is not of much use.

The corresponding "homogeneous" integral equation

$$u(t) = \int_0^\infty A(\tau)\{1 - e^{-u(t-\tau)}\}d\tau, \tag{6.5}$$

obtained from (5.20) by the substitution $x = \exp - u$ belongs to a class of integral equations which has been studied by DIEKMANN (1977). He assumes that A(t) is a piecewise continuous non-negative function.

The equation (6.5) has always the trivial solution $u(t) \equiv 0$. This is the only solution in the non-epidemic case $\gamma < 1$. This can be proved by considering the corresponding inequality

$$u(t) \leq \int_0^\infty A(\tau)u(t-\tau)d\tau \tag{6.6}$$

from which we obtain for all T

$$\max_{t\leq T} u \leq \gamma \max_{t\leq T} u,$$

but this implies $u \equiv 0$. For $\gamma = 1$ the same argument would lead to a constant value of u(t) but the only constant value compatible with (6.5) is zero.

Diekmann proves the existence of a monotonous non-decreasing solution of (6.5) for $\gamma > 1$ satisfying $u(t) > 0$. In a further theorem he also proves uniqueness up to an arbitrary time shift. This conclusion is based upon the following property

THEOREM 6.1. *Let*

$$\int_0^\infty e^{-\varepsilon t}A(t)dt < \infty$$

for some positive ε. *Then if* u(t) *is a monotonous non-decreasing solution of* (6.5) *there exists a constant* c *such that*

(6.7) $$\lim_{t\to-\infty} e^{-\sigma t}u(t) = c.$$

where σ *is given by* (5.7).

Again a solution of (6.5) can be constructed as the limit of the Picard process

(6.8) $$u_{n+1}(t) = \int_0^\infty A(\tau)\{1 - \exp - u_n(t-\tau)\}d\tau$$

with

(6.9) $$u_0(t) = c \exp \sigma t,$$

where c is an arbitrary positive constant. The first step gives

$$u_1(t) = \int_0^\infty A(\tau)\{1 - u_0(t-\tau) - e^{-u_0(t-\tau)}\}d\tau + u_0(t).$$

It appears that $u_1(t) \le u_0(t)$ and next

$$u_0(t) - u_1(t) \le \frac{1}{2}\int_0^\infty A(\tau)u_0^2(t-\tau)d\tau$$

or

(6.10) $$u_0(t) - u_1(t) \le \frac{1}{2}\kappa c^2 \exp 2\sigma t,$$

where κ is defined by

(6.11) $$\kappa = \bar{A}(2\sigma) = \int_0^\infty e^{-2\sigma t}A(t)dt.$$

We note that $\kappa < 1$. For $n \ge 1$ we have

(6.12) $$u_n(t) - u_{n+1}(t) = \int_0^\infty A(\tau)\{e^{-u_n(t-\tau)} - e^{-u_{n-1}(t-\tau)}\}d\tau,$$

which shows that the Picard sequence is non-increasing. Thus $\lim_{n\to\infty} u_n(t)$ exists for all values of t. However, to avoid the non-trivial solution it is necessary to make some estimation. Let us make the hypothesis

(6.13) $$u_{n-1}(t) - u_n(t) \le \frac{1}{2}c^2\kappa^n \exp 2\sigma t.$$

For n = 1 this is already true in view of (6.10). For higher values of n the validity follows in a recursive way from (6.12) by turning it into the inequality

$$u_n(t) - u_{n+1}(t) \leq \int_0^\infty A(\tau)\{u_{n-1}(t-\tau) - u_n(t-\tau)\}d\tau.$$

We are now able to estimate the limit of $u_n(t)$ by means of

$$u = u_0 - \lim_{n\to\infty} \sum_{k=1}^{n} (u_{k-1} - u_k). \tag{6.14}$$

Using (6.13) we obtain without difficulty

$$u(t) \geq ce^{\sigma t} - \frac{\frac{1}{2}\kappa c^2}{1-\kappa} e^{2\sigma t}. \tag{6.15}$$

This inequality excludes the possiblity of a zero limit for the epidemic case $\gamma > 1$ since the right-hand side of (6.15) is positive in some left-infinite interval $(-\infty, t_0)$.

CHAPTER 7

THE BLOCK-FUNCTION CASE

In this chapter we consider the model (3.14), (3.16) with a constant infectivity of finite duration

$$A(t) = b\theta(a-t) \tag{7.1}$$

where $\theta(t)$ is the standard notation of the unit-step function. Substitution of (7.1) in (3.16) gives

$$-\ell nx(t) = b \int_0^{\min(a,t)} \{1 + \varepsilon - x(t-\tau)\}d\tau. \tag{7.2}$$

If the time is scaled according to $t \to$ at the relation (7.2) passes into

$$-\ell nx(t) = \gamma \int_0^{\min(1,t)} \{1 + \varepsilon - x(t-\tau)\}d\tau, \tag{7.3}$$

where $\gamma = ab$. This means that it is sufficiently general to consider the case (7.1) with $a = 1$, $b = \gamma$

$$A(t) = \gamma\theta(1-t). \tag{7.4}$$

This model was originally suggested by COOKE (1967), further developed and discussed by HOPPENSTEADT and WALTMAN (1970). WILSON (1972) showed that an explicit analytic solution could be obtained. Here the solution is obtained in a slightly different but more direct way.

Substitution of (7.4) in (3.15) gives

$$\begin{aligned} -\dot{x}(t)/x(t) + \gamma x(t) &= (1+\varepsilon)\gamma, \quad \text{for } 0 \le t < 1, \\ -\dot{x}(t)/x(t) + \gamma x(t) &= -\gamma x(t-1), \quad \text{for } t > 1. \end{aligned} \tag{7.5}$$

with $x(0) = 1$ and continuity at $t = 1$.

The essential trick of dealing with (7.5) is a substitution which is well-known in the theory of the Riccati equation. We introduce a new function $F(t)$ satisfying

$$\gamma x(t) = -\dot{F}(t)/F(t) \tag{7.6}$$

and

$$F(0) = 1.$$

Explicitly we have

$$F(t) = \exp - \gamma \int_0^t x(\tau)d\tau. \tag{7.7}$$

By this the equations (7.5) are transformed into

$$\begin{aligned} \ddot{F}(t)/\dot{F}(t) &= -(1+\varepsilon)\gamma \qquad \text{for } 0 \le t < 1, \\ \ddot{F}(t)/\dot{F}(t) &= \dot{F}(t-1)/F(t-1) \qquad \text{for } t > 1. \end{aligned} \tag{7.8}$$

These equations can be integrated. Using the initial conditions

$$F(0) = 1, \quad \dot{F}(0) = -\gamma \tag{7.9}$$

we find without difficulty

$$\dot{F}(t) = -\gamma \exp - (1+\varepsilon)\gamma t, \quad \text{for } 0 \le t < 1, \tag{7.10}$$

and

$$\dot{F}(t) = \dot{F}(1)F(t-1), \quad \text{for } t > 1. \tag{7.11}$$

Let us write for convenience

$$c = -\dot{F}(1) = \gamma \exp - (1+\varepsilon)\gamma. \tag{7.12}$$

In view of the relation (3.18) we also have

(7.13) $\quad c = \gamma x_\infty \exp - \gamma x_\infty .$

The equation (7.10) is solved by

(7.14) $\quad F(t) = \dfrac{\varepsilon+\exp-(1+\varepsilon)\gamma t}{1+\varepsilon}, \quad 0 \le t \le 1.$

The equation (7.11) can then be solved for $1 \le t \le 2$ and next for each subsequent time interval of unit length. However, this is a rather tedious procedure and it seems difficult to obtain conclusions for large values of time. A better idea is to apply Laplace transformation. Then from (7.11) we obtain

$$\int_1^\infty e^{-st}\dot{F}(t)\,dt = -c \int_1^\infty e^{-st}F(t-1)\,dt$$

and next

$$\int_0^\infty e^{-st}\dot{F}(t)\,dt = \int_0^1 e^{-st}\dot{F}(t)\,dt - ce^{-s}\int_0^\infty e^{-st}F(t)\,dt.$$

Using (7.14) and the initial condition $F(0) = 1$ we can write this as

$$(s+ce^{-s})\bar{F}(s) = 1 - \frac{\gamma-ce^{-s}}{s+(1+\varepsilon)\gamma}$$

so that

(7.15) $\quad \bar{F}(s) = \dfrac{s+\varepsilon\gamma+c\exp-s}{(s+(1+\varepsilon)\gamma)(s+c\exp-s)}$

The original $F(t)$ can be found using standard techniques. Since the right-hand side of (7.15) is a meromorphic function, $F(t)$ can be represented as the sum of the residues of $\bar{F}(s)\exp st$ at the poles of $\bar{F}(s)$. We note that $s = -(1+\varepsilon)\gamma$ is no pole since for that value of s also the numerator vanishes. Therefore the poles of $\bar{F}(s)$ are the zeros of $s + c\exp - s$. It is not difficult to show that there are two real zeros and an infinite set of complex zeros. The relation (7.13) shows that $s_1 = -\gamma x_\infty$ is a negative real zero. The other real zero s_2 is also negative. It appears that $s_2 = -\gamma$ for $\varepsilon = 0$. For $\varepsilon \neq 0$ the following approximation can be obtained

(7.16) $\quad s_2 = -\gamma - \dfrac{\varepsilon\gamma^2}{\gamma-1} + O(\varepsilon^2), \quad \gamma \neq 1.$

However, for $\gamma = 1$ we have

(7.17) $$s_{1,2} = -1 \pm \sqrt{2\varepsilon} + O(\varepsilon).$$

The complex zeros can be determined in the following way. Writing

(7.18) $$s = -p + qi$$

we obtain from $-s = c \exp -s$ the two relations

(7.19) $$p = ce^{-p}\cos q,$$
$$q = ce^{-p}\sin q,$$

or equivalently

(7.20) $$p = q/\tan q,$$

and

(7.21) $$q = (c^2e^{2p} - p^2)^{\frac{1}{2}}.$$

The relations (7.20) and (7.21) can be plotted as curves in the p,q-plane. The complex zeros are determined by their intersections (see Fig. 7.1 and Table 7.1). It is not difficult to derive the following estimate for the k-th pair of complex roots

(7.22) $$p_k \approx \ell n \frac{(k+\frac{1}{2})\pi}{c},$$
$$q_k \approx (k+\tfrac{1}{2})\pi - \frac{1}{(k+\frac{1}{2})\pi} \ell n \frac{(k+\frac{1}{2})\pi}{c}.$$

γ	c	s_1	s_2	s_3, s_4
1	0.368	-1	-1	-2.61 ± 4.15i
1.1	0.366	-0.906	-1.1	-2.61 ± 4.15i
1.2	0.361	-0.823	-1.2	-2.61 ± 4.15i
1.3	0.354	-0.750	-1.3	-2.63 ± 4.15i
1.4	0.345	-0.685	-1.4	-2.66 ± 4.15i
1.5	0.335	-0.626	-1.5	-2.69 ± 4.14i

Table 7.1. Roots of s+cexp-s = 0 for $\varepsilon = 0$.

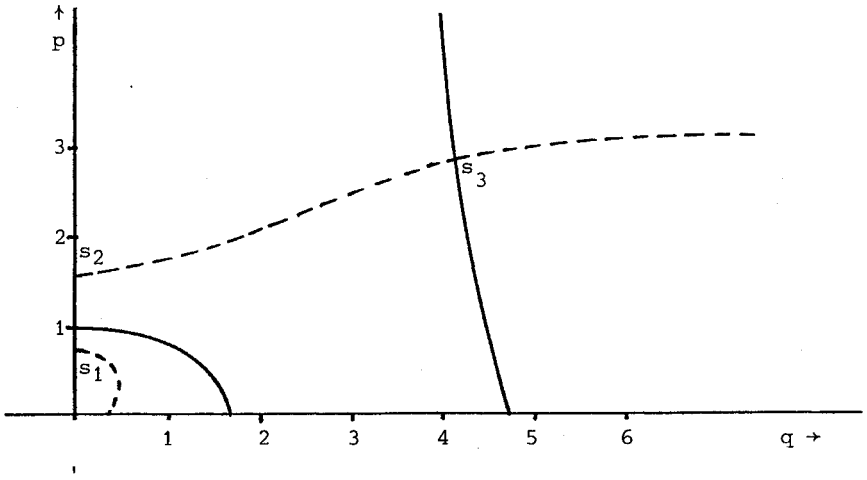


Fig. 7.1. Graphs of (7.19), (7.20) for $\gamma = 1.5$, $\varepsilon = 0$.

The expansion theorem of the Laplace transformation gives the following expansion of the original F(t)

$$F(t) = \sum_{k=1}^{\infty} \frac{\varepsilon\gamma e^{s_k t}}{(\gamma+\varepsilon\gamma+s_k)(1+s_k)} . \tag{7.23}$$

According to (7.22) this series converges for all values of t. For moderate and large values of t the first two terms, which correspond to the two real zeros, already give a good approximation. A little calculation shows that

$$F(t) = \frac{\varepsilon e^{-\gamma x_\infty t}}{(1-\gamma x_\infty)(1-x_\infty)} + e^{-\gamma t} + \dots . \tag{7.24}$$

We note that all terms in the expansion of F(t) are $O(\varepsilon)$, even uniformly in t, with the exception of the second term.

If the approximation (7.24) is used in (7.6) we obtain the following interesting result

$$x(t) \approx \frac{\varepsilon x_\infty + (1-\gamma x_\infty)(1-x_\infty)\exp-(1-x_\infty)\gamma t}{\varepsilon + (1-\gamma x_\infty)(1-x_\infty)\exp-(1-x_\infty)\gamma t} . \tag{7.25}$$

This approximation may be considered as very good for moderate and large values of t. Even for $t = 0$ we observe a small error of order ε. For $t \to \infty$

we have $x \to x_\infty$ and for $\varepsilon \to 0$ we have $x \to 1$.

The half-way time t_m for which $x = \frac{1}{2}(1+x_\infty)$ is found to be

$$(7.26) \qquad t_m \approx \frac{1}{\gamma(1-x_\infty)} \ell n \frac{(1-\gamma x_\infty)(1-x_\infty)}{\varepsilon}$$

This expression would be correct if (7.25) were correct. It is of interest to compare this formula with the predicted half-way time from the general-theory as given by (5.12). The results for a few specific cases are given in Table 7.2. The agreement is satisfactory.

If a new time scale is introduced as in Chapter 5 by

$$t \to t_m + t,$$

then for $\varepsilon \to 0$ we obtain from (7.25)

$$(7.27) \qquad \lim_{\varepsilon \to 0} x(t+t_m) = \frac{x_\infty + \exp -(1-x_\infty)\gamma t}{1 + \exp -(1-x_\infty)\gamma t} .$$

γ	x_∞	σ	μ	t_m (6.26)	t_m (5.12)
1.1	0.815	0.194	0.485	14.5	11.2
1.2	0.682	0.376	0.468	10.6	8.9
1.3	0.574	0.550	0.455	8.5	7.2
1.4	0.487	0.715	0.441	7.1	6.1
1.5	0.416	0.874	0.428	6.1	5.4

Table 7.2. of half-way times for $\varepsilon = 0.001$

In view of the remarks made in connection with (7.23) the right-hand side of (7.27) is the correct analytic solution satisfying the equations (5.25) and (5.26) of example 5.2. As in (5.27) we may write

$$(7.28) \qquad x(t) = \frac{x_\infty + x_\infty^t}{1+x_\infty^t}$$

in view of the relation (3.18) with $\varepsilon = 0$

$$(7.29) \qquad (1-x_\infty)\gamma = -\ell n x_\infty .$$

CHAPTER 8

A DISCRETE MODEL

To the continuous model considered in Chapter 3 we may associate a corresponding discrete model, i.e., a model with discrete time steps. For the ease of reference we first summarize the main points of the continuous model.

Let $x(t)$ be the fraction of susceptibles at time t and $v(t)dt$ the fraction of new ills in the period $(t,t+dt)$. Further let a fraction ε of new ills be introduced at $t = 0$. Then if $A(t)$ denotes the rate of infectivity we have the integral equation (3.15) or (3.16) viz.

$$\dot{x}(t) = x(t)\{\int_0^t A(\tau)\dot{x}(t-\tau)d\tau - \varepsilon A(t)\}, \tag{8.1}$$

and

$$-\ell n x(t) = \int_0^t \{1 + \varepsilon - x(t-\tau)\}A(\tau)d\tau. \tag{8.2}$$

The final fraction of susceptibles x_∞ at the end of the epidemic is determined by (3.20) or

$$\ell n \frac{1}{x_\infty} = \gamma(1 + \varepsilon - x_\infty), \tag{8.3}$$

where

$$\gamma = \int_0^\infty A(t)dt \tag{8.4}$$

is a measure of the total infectivity of a single ill. For $\gamma \leq 1$, the non-epidemic case, the value of x_∞ is very close to 1. For $\gamma > 1$, the epidemic case, x_∞ may be considerably smaller than 1 notwithstanding the small value of ε. As long as x stays close to 1 it can be approximated by the solution of the linearized version of (8.2)

$$u(t) = \int_0^t \{\varepsilon + u(t-\tau)\}A(\tau)d\tau \tag{8.5}$$

where

$$x(t) = \exp - u(t).$$

The solution of (8.5) can be obtained by using Laplace transformation. If $\bar{u}(s)$ denotes the Laplace transform of $u(t)$ we have

$$\bar{u}(s) = \frac{\varepsilon\bar{A}(s)}{s(1-\bar{A}(s))} . \tag{8.6}$$

This approximation can be used to predict the time t_m for which $x = \frac{1}{2}(1+x_\infty)$, the so-called half-way time. According to (5.20) we have

$$t_m \approx \frac{1}{\sigma} \ell n \frac{1}{\varepsilon} + c, \tag{8.7}$$

where σ is the positive real root of $\bar{A}(s) = 1$ and where c is a constant.

Of course the model described by (8.1) or (8.2) can be solved numerically by using some standard technique for dealing with Volterra integral equations.

However, instead of replacing a continuous model by a numerically more suited discrete model it is preferable to set up a discrete model straight from the beginning by breaking up the time in periods of equal length, say days or hours. Although we shall speak of days a "day" may mean in fact any period of time in applications.

Let x_n denote the fraction of susceptibles at the n-th day and $v_{n,k}$ the fraction of ills at the n-th day in the k-th day of their illness. The fraction $v_{n,0}$ of the new ills will be denoted by v_n as in the continuous model. If removals from the class of ills are not counted as such - from the continuous model we know that this does not affect the generality of the mathematical model - we have

$$\begin{aligned} v_{n,k} &= v_{n-k} && \text{for } n > k, \\ v_{n,k} &= \varepsilon && \text{for } n = k, \\ v_{n,k} &= 0 && \text{for } n < k, \end{aligned} \tag{8.8}$$

under the assumption that initially a fraction of new ills ε is introduced into an otherwise healthy population. Of course we have

(8.9) $\qquad v_{n+1} = x_n - x_{n+1} \quad \text{for } n \geq 0.$

The law of infection, the rate at which new ills arise is the following discrete analogue of (8.1)

(8.10) $\qquad v_{n+1} = \phi(x_n, x_{n+1}) \sum_{j=0}^{n} A_j \, v_{n-j}$

with a certain arbitrariness in the choice of ϕ. The sequence $A_0, A_1, A_2, \ldots$ is the discrete analogue of the infectivity function A(t) with

(8.11) $\qquad \gamma = \sum_{j=0}^{\infty} A_j$

on the total infectivity.

In an attempt to make the model as simple as possible we might take $\phi = x_n$, $\phi = x_{n+1}$ or some intermediate value such as the arithmetic or geometric mean. Our choice is motivated by the intention of obtaining a discrete analogue of the simpler integral equation (8.2). To that purpose we take the intermediate

(8.12) $\qquad \phi = \dfrac{x_n - x_{n+1}}{\ell n x_n - \ell n x_{n+1}} \; .$

That indeed $x_{n+1} < \phi < x_n$ follows easily from the inequality

$$\frac{x_n - x_{n+1}}{x_n} < \int_{x_{n+1}}^{x_n} \frac{d\xi}{\xi} < \frac{x_n - x_{n+1}}{x_{n+1}} .$$

Substitution of (8.12) into (8.10) gives in view of (8.9)

(8.13) $\qquad \ell n \, \dfrac{x_n}{x_{n+1}} = \sum_{j=0}^{n} A_j v_{n-j}$

or more fully

(8.13a) $\qquad \ell n \, \dfrac{x_n}{x_{n+1}} = \sum_{j=0}^{n-1} A_j x_{n-1-j} - \sum_{j=0}^{n} A_j x_{n-j} + (1+\varepsilon) A_n$

for $n \geq 1$ and

$$\ell n \, \frac{1}{x_1} = -A_0 + (1+\varepsilon) A_0 \quad \text{for } n = 0.$$

By summation with respect to n we obtain

$$(8.14) \qquad \ell n \frac{1}{x_{n+1}} = \sum_{j=0}^{n} (1 + \varepsilon - x_{n-j})A_j, \quad n \geq 0,$$

with $x_0 = 1$, as the obvious discrete analogue of (8.2).

The model (8.14) can be used as the starting-point for further investigations. For $n \to \infty$, x_n has the limit x_∞ for which the relation (8.3) holds as in the continuous case. The linear approximation of (8.14) is

$$(8.15) \qquad u_{n+1} = \sum_{j=0}^{n} (\varepsilon + u_{n-j})A_j, \qquad n \geq 0,$$

with

$$x_n = \exp - u_n,$$

obviously the discrete analogue of (8.5). This linear convolution equation can be solved by introducing the generating functions

$$(8.16) \qquad U(s) = \sum_{n=0}^{\infty} u_n s^n$$

and

$$(8.17) \qquad A(s) = \sum_{n=0}^{\infty} A_n s^{n+1}.$$

It is not difficult to show that the set of equations (8.14) can be transformed into the single relation

$$U(s) = \left(\frac{\varepsilon}{1-s} + U(s)\right)A(s)$$

so that

$$(8.18) \qquad U(s) = \frac{\varepsilon A(s)}{(1-s)(1-A(s))},$$

the "discrete" analogue of (8.6). Only for simple infectivity models it can be hoped to find relatively simple expressions for A(s) and for u.

EXAMPLE 8.1. We consider the case $A_n = a^{\lambda n}$ with $\lambda < 1$. We note that $\gamma = a/(1-\lambda)$. Thus if we assume that $a + \lambda > 1$ we may have an epidemic. From (8.17) and (8.18) we obtain

$$U(s) = \frac{\varepsilon a s}{(1-s)(1-(a+\lambda)s)}$$

from which

$$u_n = \frac{\varepsilon a}{a+\lambda-1}\{(a+\lambda)^n - 1\},$$

and finally

$$x_n \approx \exp - \frac{\varepsilon a}{a+\lambda-1}(a+\lambda)^n.$$

Of course this approximation is only good in so far as x_n is close to 1. The approximation may be used to predict the day m for which the epidemic is half-way through. It is not difficult to show that

$$m \approx \frac{1}{\ell n(a+\lambda)} \ell n \frac{1}{\varepsilon} + c,$$

a similar result as (8.7).

In the general case where A(s) is not a simple function of s we may still derive an estimate of u_n for moderate and large values of n. Let us assume for simplicity that A(s) is a quotient of two polynomials then the same is true for U(s) and we may write

$$U(s) = \varepsilon \sum_{j=1}^{p} \frac{c_j}{1-s/\sigma_j}, \tag{8.19}$$

where $\sigma_1, \sigma_2, \ldots, \sigma_p$ are the poles of U(s). From (8.16) it follows that

$$u_n = \varepsilon \sum_{j=1}^{p} c_j \sigma_j^{-n} \tag{8.20}$$

so that the behaviour of u_n for $n \to \infty$ is determined by the poles with the least absolute value. In the epidemic case $\gamma = A(1) > 1$ there exists a positive real pole σ for which $A(\sigma) = 1$ with the property that there are no other poles with $|s| < \sigma$. However, it is possible that there are complex roots with $|s| = \sigma$. Leaving this possibility aside we may say that

$$u_n \approx \varepsilon c \sigma^{-n}, \quad A(\sigma) = 1, \tag{8.21}$$

or

(8.22) $$x_n \approx \exp - \varepsilon c \sigma^{-n}.$$

The predicted half-way time then follows as in Example 8.1 as

(8.23) $$m \approx \frac{1}{\sigma} \ell n \frac{1}{\varepsilon} + c,$$

where c is some constant.

We shall end this chapter with a numerical illustration for the block-function case.

EXAMPLE 8.2. We take

$$A_j = a, \qquad \text{for } 0 \le j \le p-1$$

and $A_j = 0$ otherwise. Thus we consider an illness with a constant rate of infectivity during the first days of illness. We consider the epidemic case only assuming

$$\gamma = ap > 0.$$

From (8.14) we obtain the model

$$x_{n+1} = \exp - a \sum_{j=0}^{\min(n,p-1)} (1 + \varepsilon - x_{n-j}).$$

For numerical purposes this can be rewritten as

$$x_{n+1} = x_n \exp - a(1+\varepsilon-x_n) \qquad \text{for } n < p,$$

$$x_{n+1} = x_n \exp - a(x_{n-p} - x_n) \qquad \text{for } n \ge p.$$

Starting with $x_0 = 1$ the subsequent x_n can easily be calculated by means of a programmable pocket calculator. The following table shows some values for the typical case $a = 0.2$, $p = 7$ (i.e. $\gamma = 1.4$) with $\varepsilon = 0.01$ and $\varepsilon = 0.001$. In Example 5.2 and in Chapter 7 (cf. formula 7.28) it has been shown that for $\varepsilon \to 0$ the continuous model of the block-function case yields the logistic law

$$x(t) = \frac{c+c^t}{1+c^t}$$

where $c = x_\infty$ and $t = 0$ marks the half-way value $\frac{1}{2}(1+x_\infty)$. It is reasonable to expect that, also in the discrete-time model considered here, the logistic curve is a good approximation for small values of ε. Table 8.1 shows a surprisingly good fit of the logistic curve in the numerical case $a = 0.2$, $p = 7$ with $\varepsilon = 0.001$.

The first mathematical model with a discrete time scale was given in the renowned paper of Kermack and McKendrick. However, they regarded it merely as a step towards the continuous model.

In recent years discrete-time models appear to come more into prominence. Obviously this has to do with their suitability of numerical calculations. It is not difficult to set up programs for a programmable pocket calculator such as the HP 41 C. In fact, we have used this instrument for most of our numerical illustrations.

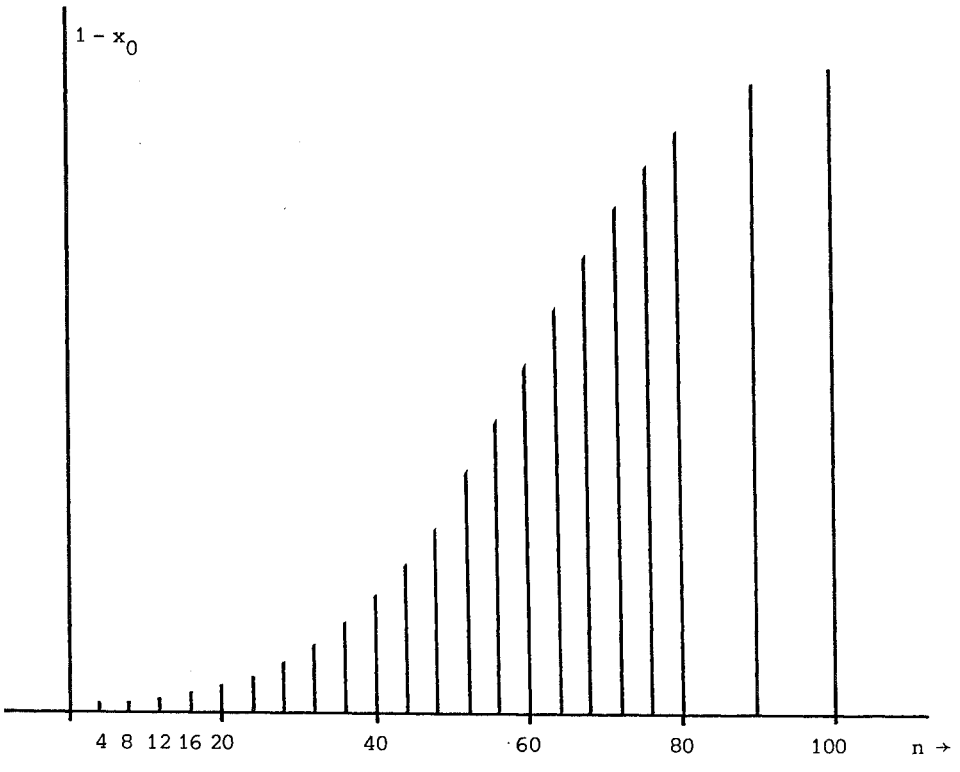


Fig. 8.1. Discrete epidemic model of Table 8.1

n	$1000(1-x_n)$ $\varepsilon = 0.01$	$100(1-x_n)$ $\varepsilon = 0.001$	logistic fit $\varepsilon = 0.001$	t
4	11	1		
8	30	3		
12	56	6		
16	89	10		
20	131	15		
24	180	23	25	-4.10
28	234	33	36	-3.60
32	290	47	50	-3.10
36	342	66	68	-2.60
40	389	91	93	-2.10
44	427	121	123	-1.60
48	457	158	160	-1.10
50	469	179	180	-0.85
52	479	201	202	-0.60
54	488	223	224	-0.35
56	495	246	247	-0.10
58	501	270	270	0.15
60	507	293	293	0.40
62	511	315	316	0.65
64	515	337	337	0.90
66	518	357	357	1.15
68	520	376	376	1.40
70	522	394	393	1.65
72	524	410	409	1.90
74	525	424	423	2.15
76	526	437	436	2.40
78	527	448	447	2.65
80	528	458	457	2.90
90	530	489	489	4.15
100	531	503	503	5.40
∞	531	513	513	∞

Table 8.1.

CHAPTER 9

THE EFFECT OF AN INCUBATION PERIOD

Again we consider the mathematical model (3.15), (3.16) as in Chapter 7 but now with the infectivity function

$$A(t) = \begin{cases} 0 & \text{for } t < \alpha, \\ \gamma/\beta & \text{for } \alpha \le t \le \alpha+\beta, \\ 0 & \text{for } t > \alpha+\beta. \end{cases} \tag{9.1}$$

The constants α and β denote the length of the incubation period and of the period of infectivity. The constant γ has the same meaning as before of the total infectivity.

The special case $\alpha = 0$ has been considered at length in Chapter 7. There we discovered the important fact that $x(t)$ is very close to the logistic function

$$x(t) = \frac{x_\infty + x_\infty^{(t-t_m)\gamma/\beta}}{1+x_\infty^{(t-t_m)\gamma/\beta}} \tag{9.2}$$

where t_m is the half-way time, i.e.

$$x(t_m) = \tfrac{1}{2}(1+x_\infty).$$

We pose ourselves the following question. What happens to the epidemic curve $x(t)$ when we take increasing values of the incubation time α while keeping β, γ and ε constant.

The mathematical model (3.16) is given by

$$-\ell n x(t) = \frac{\gamma}{\beta} \int_{\alpha}^{\min(\alpha+\beta,t)} \{1 + \varepsilon - x(t-\tau)\}d\tau, \qquad t > \alpha, \tag{9.3}$$

with

$$x(t) = 1 \quad \text{for } 0 \leq t \leq \alpha.$$

The similarity principle says that if $x(t)$ is the solution of (9.3) with the parameters α, β, γ then $x(t/\lambda)$ is the solution of (9.3) with the parameters $\lambda\alpha$, $\lambda\beta$ and γ (cf. Fig. 9.1).

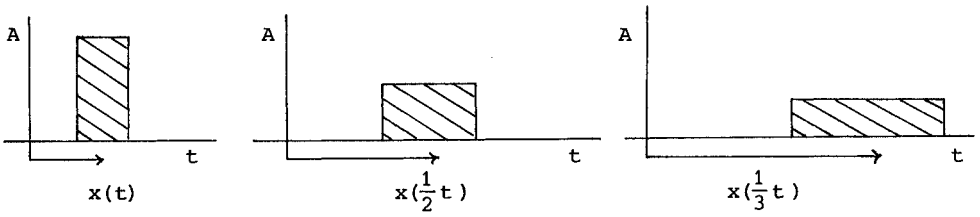


Fig. 9.1. The similarity principle

As figure 9.1 shows the similarity principle corresponds for increasing values of λ to an increase of both incubation time and period of infectivity function. However, we want to increase the incubation time without changing the shape of the infectivity function. Still in both cases γ is kept constant and it is this constant that dominates the ultimate behaviour of $x(t)$. We recall the well-known relation

$$-\ell n x_\infty = \gamma(1 + \varepsilon - x_\infty). \tag{9.4}$$

Therefore the first impression of the effect of an increasing value of α is that of a slowing down of the time scale by a factor $\alpha + \frac{1}{2}\beta$, the time interval between the start of the incubation period and the mid-point of the period of infectivity, i.e.

$$t \to \frac{\beta}{2\alpha+\beta}\, t. \tag{9.5}$$

Although it is virtually impossible to solve the non-linear integral equation (9.3) by analytic means a study of its more tractable linear approximation may help us to gain more insight.

Proceeding as sketched in Chapter 8 we have for $x(t) \approx 1$ the linear

approximation $x(t) \approx \exp - u(t)$ with $u(t)$ as determined by its Laplace transform (see form. 8.6)

$$(9.6) \qquad \bar{u}(s) = \frac{\varepsilon \bar{A}(s)}{s(1-\bar{A}(s))}$$

or explicitly using (9.1)

$$(9.7) \qquad \bar{u}(s) = \frac{\varepsilon\gamma(e^{-\alpha s} - e^{-(\alpha+\beta)s})}{s(\beta s - \gamma e^{-\alpha s} + \gamma e^{-(\alpha+\beta)s})} \quad .$$

Let us suppose that γ slightly exceeds 1, in fact the most important case, then there is a single small positive pole σ of (9.7) which dominates the behaviour of $u(t)$ for $t \to \infty$. From

$$(9.8) \qquad \beta\sigma = \gamma e^{-\alpha\sigma} - \gamma e^{-(\alpha+\beta)\sigma}$$

we obtain the first-order solution

$$(9.9) \qquad \sigma \approx \frac{\gamma-1}{\gamma(\alpha+\frac{1}{2}\beta)}$$

For $t \to \infty$ we have

$$(9.10) \qquad u(t) \approx \frac{\varepsilon e^{\sigma t}}{-\sigma \bar{A}'(\sigma)}$$

A little calculation shows that $\sigma\bar{A}'(\sigma) = -\gamma + 1$ so that

$$(9.11) \qquad u(t) \approx \frac{\varepsilon}{\gamma-1} e^{\sigma t}$$

or better

$$(9.12) \qquad x(t) \approx \{\exp \frac{-\varepsilon}{\gamma-1} \exp \frac{\gamma-1}{\gamma} \frac{t}{\alpha+\frac{1}{2}\beta}\} \quad .$$

This result confirms the conjecture of the time transformation by the factor $\alpha + \frac{1}{2}\beta$. It becomes clear that at least for moderate values of t within the validity of approximation (9.12) this is the only effect of the parameters α and β.

The next and more difficult question is the effect upon the shape of

the epidemic curve. Let us consider the extreme case $\beta \to 0$. This is equivalent to the assumption $\alpha \to \infty$ together with an appropriate transformation of t. In this way we hope to obtain the effect of an extremely long incubation period in proportion to the infectivity period. For $\alpha \to 0$, (9.3) becomes

$$-\ell nx(t) = \gamma\{1 + \varepsilon - x(t-\alpha)\}, \tag{9.13}$$

a recurrent relation for which we write

$$x_{n+1} = \exp - \gamma(1 + \varepsilon - x_n) \tag{9.14}$$

with

$$x_0 = 1. \tag{9.15}$$

An explicit analytic expression for x_n is out of the question, but a numerical calculation on a programmable pocket calculator is an easy matter. In Table 9.1 we have considered the case $\gamma = 1.4$, $\varepsilon = 0.001$. The table gives $1000\ (1-x_n)$ "total number of ills" for various values of n "days". We have tried to approximate x_n by a logistic function

$$\frac{x_\infty + x_\infty^{\mu t}}{1 + x_\infty^{\mu t}} \tag{9.16}$$

where $n = t + t_m$. The results for $\mu = 0.51$ and $t_m = 14.5$ are also given in Table 9.1.

The agreement is surprisingly very good. We are tempted to draw the conclusion that the influence of the incubation parameter α upon the logistic shape of the epidemic curve is almost negligible.

The influence of an incubation period may also be studied by using the discrete model (8.14). With the notation of Example 8.2 we may take

$$A_j = \begin{cases} 0 & \text{for } 0 \le j < p_0, \\ a & \text{for } p_0 \le j \le p_0 + p - 1, \\ 0 & \text{for } j > p_0 + p - 1. \end{cases} \tag{9.17}$$

This gives

n	1000 $(1-k_n)$	logistic fit
2	3	5
4	10	11
6	22	22
8	45	43
10	85	83
11	114	111
12	148	146
13	189	188
14	233	233
15	280	280
16	325	325
17	366	367
18	402	402
19	431	430
20	454	453
22	484	482
24	499	498
26	506	506
28	510	509
30	512	511
∞	513	513

Table 9.1

$$(9.18)\qquad \begin{cases} x_0 = x_1 = \ldots = x_{p_0} = 1, \\ x_{n+1} = \exp - a \sum_{j=p_0}^{\min(n,p_0+p-1)} (1 + \varepsilon - x_{n-j}). \end{cases}$$

Using the linear approximation as sketched in Section 8 we have formula (8.18) with

$$A(s) = a s^{p_0+1} \frac{1-s^p}{1-s} .$$

The dominant pole σ of (8.18) satisfies $A(\sigma) = 1$. A little calculation yields the good approximation

$$(9.19)\qquad \sigma \approx 1 - \frac{\gamma-1}{\gamma(p_0+\frac{1}{2}p+\frac{1}{2})}$$

obviously the analogue of (9.9).

Again we find the same influence of p_0 upon the time scale which is slowed down in proportion to the factor $p_0 + \frac{1}{2}p + \frac{1}{2}$.

In order to test the logistic fit we have calculated the cases $\gamma = 1.4$, $\varepsilon = 0.01$ with $p = 7$ and $p_0 = 1,2,3,4,5,6$. The results are given in Table 9.2.

n	p_0=1	p_0=2	p_0=3	p_0=4	p_0=5	p_0=6
4	6	4	2	0	0	0
8	21	14	10	8	6	4
12	37	26	20	16	14	12
16	58	42	32	25	20	16
20	88	60	45	36	30	25
24	116	81	61	48	39	33
28	152	107	80	63	51	42
32	193	136	101	79	64	53
36	237	168	125	97	78	64
40	281	203	151	117	94	77
44	324	239	180	139	111	92
48	364	276	210	163	131	107
52	399	312	241	189	151	124
56	429	347	273	216	173	142
60	453	378	304	243	196	161
64	473	406	334	271	220	181
68	488	431	363	298	245	202
72	499	451	389	325	270	224
76	508	468	412	351	294	246
80	514	482	432	374	318	268
84	519	493	450	396	341	291
88	522	502	465	416	363	312
92	525	508	477	434	383	333
96	527	514	488	449	402	353
100	528	518	496	462	419	372

Table 9.2 of 1000 $(1-x_n)$

In all those cases a good logistic fit is possible with

$$1 - x_n = \frac{0.531}{1+0.531^{\alpha(n-n_m)}} \quad . \tag{9.20}$$

The corresponding "best possible" values of α and n_m, the half-way time, are given in Table 9.3 together with the values obtained from (9.20) which differ in more than a single last digit from those in Table 9.2.

p_0	1	2	3	4	5	6
n_m	38.6	46.9	55.1	63.2	71.3	79.5
α	0.130	0.109	0.094	0.0825	0.0725	0.066
n						
20	94	72	59	50	46	41
24	123	91	72	61	54	48
28	157	113	88	73	64	55
32	195	140	107	87	75	64
36	237	170	129	104	88	74
40		203	154	122	102	86
44			181	143	118	98
48				165	136	112
52				190	155	128
56					176	145
60					198	163
64						182

Table 9.3

It appears that the middle and later phases of the epidemic function can still be closely approximated by a logistic function. The incubation period does not affect the logistic shape of the epidemic curve. The earlier part of the epidemic function seems to lag somewhat behind the logistic approximation but this effect can partly be explained by the rather unnatural initial condition that initially the ε-fraction of ills is supposed to be just infected.

As a final example we consider the case

$$p_0 = 6, \quad p = 7, \quad \gamma = 1.4, \quad \varepsilon = 0.001.$$

The results are given in Table 9.4. The logistic fit (9.20) with $x_\infty = 0.513$ $n_m = 148.4$ and $\alpha = 0.0545$ appears to be particularly good, even for the earlier phase of the epidemic.

n	1000 $(1-x_n)$	logistic fit
50	13	14
60	19	20
70	28	28
80	40	39
90	56	55
100	77	75
110	103	102
120	136	135
130	175	174
140	218	218
150	264	264
160	310	310
170	353	352
180	391	390
190	422	420
200	447	445
210	466	464
220	480	478
230	490	488
240	497	495
250	502	501
∞	513	513

Table 9.4

CHAPTER 10

THE INVERSE PROBLEM

Let us imagine that a population is suffering from an epidemic disease of unknown nature. Is it possible to determine the infectivity function A(t) from a given epidemic course? Taking the continuous model (3.15) or (3.16) as our starting point we observe that the problem is nothing more than the inversion of a convolution integral. The appropriate technique is Laplace transformation £ with the convention (5.3). Thus from (3.16) we obtain at once

$$(10.1) \qquad -£\,\ell nx = (\frac{1+\varepsilon}{s} - £x)\bar{A}(s).$$

We may expect that $\bar{A}(s)$ hardly changes when $\varepsilon \to 0$. If we take $\varepsilon = 0$ and if x(t) is known for all values of t the Laplace transform of A(t) is known and there remains the rather standard procedure of carrying out the inverse transformation.

However, there are a number of difficulties. The behaviour of A(t) for small values of t is determined by the behaviour of $\bar{A}(s)$ for large values of s. But this in turn requires an accurate knowledge of the behaviour of x(t) for small values of t. In the preceding chapter we have seen that for a shifted block-function (9.1) the epidemic curve is rather insensitive to the extent of the shift. In all cases we found that x(t) is very close to a logistic function with the exception of the initial phase where x is close to 1 and t is small. Unfortunately the initial phase of an epidemic is almost lost in obscurity and, moreover, a deterministic continuous mathematical model is no good representation of an actual epidemic disease as long as a few individuals are involved.

These considerations may lead us to consider the double-infinite model (5.21) which we use here in its differentiated form

$$(10.2) \qquad \dot{x}/x = \int_0^\infty A(\tau)\dot{x}(t-\tau)\,d\tau.$$

To this we apply the double-sided Laplace transformation which gives

$$\bar{A}(s) = \frac{\int_{-\infty}^{\infty} e^{-st} \dot{x}/x \, dt}{\int_{-\infty}^{\infty} e^{-st} \dot{x} \, dt} \tag{10.3}$$

In (10.2) and (10.3) the time scale can be shifted in an arbitrary way. Usually we shall take $t = 0$ for the half-way value

$$x(0) = \tfrac{1}{2}(1 + x_\infty). \tag{10.4}$$

Both $-\dot{x}$ and $-\dot{x}/x$ have the appearance of a Gaussian distribution function. Their behaviour for $t \to \pm\infty$ follows from

$$\begin{aligned} x(t) &\approx x_\infty + c_1 e^{-\sigma_+ t} && t \to \infty, \\ x(t) &\approx 1 - c_2 e^{\sigma_- t} && t \to -\infty, \end{aligned} \tag{10.5}$$

where c_1, c_2 are arbitrary positive constants and where σ_+, σ_- in the epidemic case $\gamma > 1$ are determined by

$$\bar{A}(\sigma_-) = 1, \quad \bar{A}(-\sigma_+) = \frac{1}{x_\infty}. \tag{10.6}$$

Thus numerator and denominator of (10.3) exist at least in the strip $-\sigma_+ < \operatorname{Re} s < \sigma_-$.

Let us assume that $x(t)$ can be approximated by a logistic curve with the equation

$$x(t) = \frac{c+e^{-\lambda t}}{1+e^{-\lambda t}}, \qquad -\infty < t < \infty. \tag{10.7}$$

where $t = 0$ marks the half-way time $x(0) = \frac{1}{2}(1+c)$. It may be somewhat a surprise that (10.3) can be used to obtain $A(t)$ using purely analytical methods - we know the answer already but we pretend not knowing it.

The Laplace transform of $-\dot{x}(t)$ is calculated as follows

$$-\int_{-\infty}^{\infty} e^{-st} \dot{x}(t)\,dt = s \int_{-\infty}^{\infty} e^{-st}(1-x)\,dt = s(1-c) \int_{-\infty}^{\infty} \frac{e^{-st}}{1+e^{-\lambda t}}\,dt =$$

$$= (1-c)\,\frac{\pi s/\lambda}{\sin \pi s/\lambda}$$

Similarly

$$-\int_{-\infty}^{\infty} e^{-st}\,\dot{x}/x\,dt = (1-c^{s/\lambda})\frac{\pi}{\sin \pi s/\lambda}$$

Then (10.3) gives

$$\bar{A}(s) = \frac{\lambda}{1-c}\,\frac{1-\exp-\frac{s}{\lambda}\ell n\frac{1}{c}}{s} \tag{10.8}$$

Inversion gives the - expected - result.

$$\begin{aligned} A(t) &= \frac{\lambda}{1-c} \qquad \text{for } 0 \le t < \frac{1}{\lambda}\,\ell n\,\frac{1}{c}\,, \\ A(t) &= 0 \qquad\quad \text{for } t > \frac{1}{\lambda}\,\ell n\,\frac{1}{c}\,. \end{aligned} \tag{10.9}$$

We repeat that the double-infinite integral equation (10.2) or equivalently

$$\ell n\,\frac{1}{x} = \int_{0}^{\infty} \{1 - x(t-\tau)\}A(\tau)\,d\tau \tag{10.10}$$

for the block function

$$A(t) = \begin{cases} \gamma & \text{for } 0 \le t < 1 \\ 0 & \text{for } t > 1 \end{cases} \tag{10.11}$$

has the logistic solution

$$x(t) = \frac{c+c^{t}}{1+c^{t}}\,, \tag{10.12}$$

where

$$\ell n\,\frac{1}{c} = \gamma(1-c)\,. \tag{10.13}$$

It may be asked whether a small deviation δx from (10.12) necessarily implies a small deviation δA of (10.11). In view of the experiences gained before we do not expect this to be the case. However, we shall investigate what follows from (10.10) by using the well-known perturbation technique

$$x = x_0 + \delta x, \quad A = A_0 + \delta A$$

where $x_0(t)$ and $A_0(t)$ are given by (10.12) and (10.11). The first-order perturbation term gives

$$(10.14) \qquad \frac{\delta x}{x_0} = \gamma \int_0^1 \delta x(t-\tau)d\tau - \gamma \int_0^\infty \{1 - x_0(t-\tau)\}\delta A(\tau)d\tau.$$

Substitution of (10.12) and (10.11) gives

$$(10.15) \qquad \gamma(1-c) \int_0^\infty \frac{\delta A(\tau)}{1+c^{t-\tau}}\, d\tau = \gamma \int_{t-1}^t \delta x(\tau)d\tau - \frac{\delta x}{x_0}.$$

Making the substitution

$$(10.16) \qquad \delta A(t) = \phi(c^t)$$

we can write this as

$$(10.17) \qquad \int_0^1 \frac{\phi(u)}{u+v}\, du = \psi(v), \qquad 0 < v < \infty,$$

where

$$(10.18) \qquad v = c^t$$

and where

$$(10.19) \qquad \psi(v) = \gamma \int_{t-1}^t \delta x(\tau)d\tau - \frac{\delta x}{x_0}$$

is a small function.

The inversion of the integral equation (10.17) poses interesting mathematical problems. However, without entering into analytical subtleties it is clear at first sight that it is well possible for δA being locally large whereas δx is always small. If e.g.

$$\delta A = 2^{2n} v^n (1-v)^n$$

with

$$\delta A(v = \tfrac{1}{2}) = 1$$

we have

$$0 < \psi(v) \leq 2^{2n}B(n,n+1) \approx \sqrt{\pi/n}.$$

In view of the difficulties of determining A(t) from (10.1) or (10.3) by analytical techniques we offer the following alternative more global method in which the statistical moments of A(t) can be determined from those of $-\dot{x}/x$ and $-\dot{x}$.

We recall a few well-known facts from probability theory first. Let f(t) be a probability distribution, i.e. a non-negative function satisfying

$$(10.20) \qquad \int_{-\infty}^{\infty} f(t)dt = 1$$

then the k-th moment $m_k(f)$ is defined by

$$(10.21) \qquad m_k(f) = \int_{-\infty}^{\infty} t^k f(t)dt, \qquad k = 0,1,2,\ldots .$$

Of course $m_0(f) = 1$. The first moment gives the mean. It is often better to relate the moments to the mean by making an appropriate shift. The corresponding so-called reduced moments $\mu_k(f)$ are defined as

$$(10.22) \qquad \mu_k(f) = \int_{-\infty}^{\infty} (t-m_1(f))^k f(t), \qquad k = 0,1,2,\ldots .$$

Of course we have $\mu_0 = 1$ and $\mu_1 = 0$. The second reduced moment defines the variance or squared standard deviation. The standard deviation itself is usually written as $\sigma(f)$ and gives a measure of the width of f(t). From (10.21) and (10.22) we obtain the following much-used relation

$$(10.23) \qquad \sigma^2(f) = m_2(f) - m^2(f).$$

The third reduced moment measures the asymmetry of f(t) and is called the skewness. The fourth moment measures the shape of the hill at the mean and is sometimes called the peakedness.

EXAMPLE 10.1.

(a) $\qquad f(t) = \lambda e^{-\lambda t}, \qquad t \geq 0,$

with f(t) = 0 for t < 0. We have

$$m_1 = \frac{1}{\lambda} \qquad \sigma^2 = \frac{1}{\lambda^2} \qquad \mu_3 = \frac{2}{\lambda^3} \qquad \mu_4 = \frac{9}{\lambda^4}$$

(b) $f(t) = 0 \quad$ for $t < \alpha$

$f(t) = 1/\beta \quad$ for $\alpha \leq t \leq \beta$

$f(t) = 0 \quad$ for $t > \beta$.

We have

$$m_1 = \alpha + \tfrac{1}{2}\beta, \qquad \sigma^2 = \frac{1}{12}\beta^2, \qquad \mu_3 = 0, \qquad \mu_4 = \frac{1}{80}\beta^4.$$

(c) $$f(t) = \lambda e^{-\lambda t}(1+e^{-\lambda t})^2.$$

We have

$$m_1 = 0, \qquad \sigma^2 = \frac{\pi^2}{3\lambda^2}, \qquad \mu_3 = 0, \qquad \mu_4 = \frac{7\pi^4}{15\lambda^4}.$$

The successive moments of a function given by an analytic expression can sometimes be obtained from a so-called moment generating function such as the double-sided Laplace transformation

$$(10.24) \qquad \int_{-\infty}^{\infty} e^{-st} f(t)\,dt = \sum_{k=0}^{\infty} \frac{1}{k!} m_k(f)(-s)^k.$$

The reduced moments can be obtained in a similar way as

$$(10.25) \qquad e^{m_1 s} \int_{-\infty}^{\infty} e^{-st} f(t)\,dt = \sum_{k=0}^{\infty} \frac{1}{k!} \mu_k(f)(-s)^k.$$

After this little excursion we return to the inverse problem. Our starting-point is the double-infinite model (5.21) which gave us the relation (10.3) between Laplace transforms. We suppose that $c = x_\infty$ is known and that at least the central portions of $-\dot{x}/x$ and $-\dot{x}$, i.e. the region where the epidemic is half-way, are known so that it is possible to determine the first few moments within a reasonable degree of accuracy. We recall the well-known relation (5.23) or

$$(10.26) \qquad \ell n \frac{1}{c} = \gamma(1-c).$$

This relation gives the first important datum of A(t), its surface, the total amount of infectivity

The relations (10.24) and (10.25) show that it is a simple matter to convert (10.3) into a relation of moment generating functions. First we have

to perform a suitable norming. We introduce as normed probability distributions

$$(10.27)\qquad f(t) = \frac{-\dot{x}/x}{\ell n\ 1/c}\ , \qquad g(t) = \frac{-\dot{x}}{1-c}\ , \qquad a(t) = \frac{A}{\gamma}\ .$$

The relation (10.3) can now be rewritten as

$$(10.28)\qquad \bar{a}(s)\bar{g}(s) = \bar{f}(s)$$

or explicitly

$$(1-sm_1(a)+\ldots)(1-sm_1(g)+\ldots) = 1 - sm_1(f) + \ldots\ .$$

Comparing equal powers of s we find

$$(10.29)\qquad m_1(a) = m_1(f) - m_1(g)$$

and similar relations for the higher moments, but for them it is better to rewrite (10.28) as a relation between generating functions of reduced moments. Using (10.29) we have

$$(10.30)\qquad e^{sm_1(a)}\bar{a}(s)\cdot e^{sm_1(g)}\bar{g}(s) = e^{sm_1(f)}\bar{f}(s)$$

or explicitly

$$\sum_{i=0}^{\infty}\frac{(-s)^i}{i!}\mu_i(a)\cdot\sum_{j=0}^{\infty}\frac{(-s)^j}{j!}\mu_j(g) = \sum_{k=0}^{\infty}\frac{(-s)^k}{k!}\mu_k(f),$$

so that generally

$$(10.31)\qquad \sum_{j=0}^{\infty}\binom{k}{j}\mu_{k-j}(g)\mu_j(a) = \mu_k(f).$$

The first few relations are

$$\begin{aligned}&\mu_2(a) + \mu_2(g) = \mu_2(f),\\ &\mu_3(a) + \mu_3(g) = \mu_3(f),\\ &\mu_4(a) + 6\mu_2(g)\mu_2(a) + \mu_4(g) = \mu_4(f).\end{aligned}\qquad (10.32)$$

Next we consider the discrete model as given by (8.13), i.e. the discrete counterpart of (10.2)

$$\ell n \frac{x_n}{x_{n+1}} = \sum_{j=0}^{n} A_j v_{n-j} , \qquad n \geq 0. \tag{10.33}$$

with $x_0 = 1$ and where

$$v_n = x_n - x_{n-1} , \qquad n \geq 1, \tag{10.34}$$

with $v_0 = \varepsilon$.

For easy reference we repeat the "integrated version" (8.14) of (10.33)

$$\ell n \frac{1}{x_{n+1}} = \sum_{j=0}^{n} (1+\varepsilon-x_{n-j})A_j \tag{10.35}$$

from which for $n \to \infty$ we obtain the well-known relation

$$\ell n \frac{1}{c} = \gamma(1+\varepsilon-c), \qquad c = x_\infty, \tag{10.36}$$

with

$$\gamma = \sum_{n=0}^{\infty} A_n. \tag{10.37}$$

Proceeding as in the continuous case we define normed discrete distribution functions by

$$f_n = \frac{\ell n\ x_n/x_{n+1}}{\ell n\ 1/c} , \qquad g_n = \frac{v_n}{1+\varepsilon-c} , \qquad a_n = \frac{A_n}{\gamma} \tag{10.38}$$

As before the convolution equation (10.33) can be transformed into an elementary relation between moment generating functions

$$a(w)g(w) = f(w), \tag{10.39}$$

where

$$f(w) = \sum_{n=0}^{\infty} e^{nw} f_n, \quad g(w) = \sum_{n=0}^{\infty} e^{nw} g_n, \quad a(w) = \sum_{n=0}^{\infty} e^{nw} a_n. \tag{10.40}$$

The statistical moments are obtained by expansion with respect to w

(10.41) $$f(w) = 1 + m_1(f)w + \tfrac{1}{2}m_2(f)w^2 + \ldots .$$

As before we obtain the relations (10.29) and (10.31). For practical purposes the following three relations are the most important

(10.42)
$$\gamma = \frac{\ell n\ 1/c}{1+\varepsilon-c}$$
$$m_1(a) = m_1(f) - m_1(g), \qquad \sigma^2(a) = \sigma^2(f) - \sigma^2(g).$$

The moments of the continuous or discrete infectivity function are small with respect to those of f and g. The distribution functions of f and g are very similar so that numerical calculations may involve the loss of significant digits. The following example shows this in an almost dramatic way.

We consider the discrete epidemic model (9.18) with the shifted block function

$$A_n = \begin{cases} 0 & 0 \le n \le 5 \\ 0.2 & 6 \le n \le 12 \\ 0 & n \ge 13 \end{cases}$$

and with $\varepsilon = 0.01$.

This model is a particular case of (9.17), $p = 7$, $p_0 = 6$, $a = 0.2$. The course of the epidemic function given in Table 9.2.

We expect to find

$$m_1(a) = 9, \qquad \sigma^2(a) = 4$$

The following table obtained using a HP41 pocket calculator shows the build-up of the mean and variance of the various distributions.

n	$1000(1-x_n)$	$m_1(f)$	$m_1(g)$	$m_1(f)-m_1(g)$	$\sigma^2(f)$	$\sigma^2(g)$	$\sigma^2(f)-\sigma^2(g)$
120	448	58.9	58.2	0.7	1553	1339	174
140	491	72.6	68.6	4.0	1531	1409	123
160	513	80.9	74.6	6.3	1499	1447	52
180	523	85.4	77.8	7.6	1512	1500	12
200	528	87.7	79.4	8.3	1549	1549	0
220	530	88.8	80.1	8.7	1586	1588	-2
240	531	89.3	80.5	8.8	1614	1613	1
260	531	89.6	80.7	8.9	1632	1628	4
280	531	89.7	80.7	9.0	1642	1637	5

Table 10.1

This example shows that the later phases of the epidemic process when the epidemic is almost over may contribute significantly to the determination of the mean and variance of the infectivity distribution. Even the variance seems difficult to obtain within a reasonable precision. The determination of any higher moment is an almost impossible task.

CHAPTER 11

A MODEL WITH TEMPORAL IMMUNITY

The models considered in the Chapters 3, 4 and 8 will now be extended to include the effects of temporal immunity. Again we consider a population of susceptibles, ills and immunes. Effects of birth and natural death are ignored. Immigration and emigration are excluded.

The individuals are divided into the following classes:

$x(t)$ the relative number of susceptibles at time t with initially $x(0) = 1$.

$y(t)$ the total number of ills at time t with $y(0) = \varepsilon$ ($\varepsilon << 1$).

$z(t)$ those that are removed either by death or by gaining permanent immunity with $z(0) = 0$.

We always have

$$x(t) + y(t) + z(t) = 1 + \varepsilon. \tag{11.1}$$

The class of ills is subdivided into classes

$v(t,\tau)d\tau$ the relative number of ills who at time t are in the phase $(\tau,\tau+d\tau)$ of the disease.

We have of course

$$y(t) = \int_0^\infty v(t,\tau)d\tau. \tag{11.2}$$

As in Section 3 we define

$$v(t) = v(t,0) \tag{11.3}$$

as the rate of new ills. This means that during a small time interval h from

the total number of individuals a fraction hv(t) gets an infection that results in a disease.

As in Chapter 3 we assume that a possible epidemic process is started by the presence at t = 0 of a small fraction $\varepsilon g(\tau)$ of ills being in various phases of the disease. Repeating (3.9) we have

$$v(0,\tau) = \varepsilon g(\tau), \qquad \int_0^\infty g(\tau)d\tau = 1. \tag{11.4}$$

The philosophy behind this is that at t = 0 to a healthy but susceptible population a very small number of ills is introduced from the outside. In our book-keeping immunes are not counted.

It will be assumed that for each class of ills there exist a rate of removal $\psi(\tau)$ and a rate of loss of immunity $\chi(\tau)$. This means that during a small time interval h from the $v(t,\tau)d\tau$ individuals of the class $(\tau,\tau+d\tau)$ a fraction $h\psi(\tau)$ is removed by death or otherwise and a fraction $h\chi(\tau)$ looses immunity and is added to the reservior of susceptibles. The life-story of a particular susceptible may be described as follows. At some time he is infected and passes the various phases of the disease. For a certain period he is infective and may infect others. After recovery he is still counted as an ill but as an ill who cannot transmit the disease to others and who is immune to infection. At some time, however, he may loose immunity and is counted as a susceptible thereafter.

In the course of time the ills pass through the various illness classes according to the transport equation

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial \tau} = -\psi(\tau)v - \chi(\tau)v, \tag{11.5}$$

which is obtained from (4.2) by replacing ψ by $\psi + \chi$.

Its solution is as (4.3)

$$\begin{aligned} v(t,\tau) &= v(t-\tau)B(\tau) \qquad \text{for } t > \tau, \\ v(t,\tau) &= \varepsilon g(\tau-t)\,\frac{B(\tau)}{B(\tau-t)} \qquad \text{for } t \le \tau, \end{aligned} \tag{11.6}$$

where now B(t) is defined by

$$B(t) = \exp -\left(\int_0^t \psi(\tau)d\tau + \int_0^t \chi(\tau)d\tau\right). \tag{11.7}$$

The process of infection, ideal mixing of all individuals, is described by (4.6) or

$$v(t) = x(t) \int_0^\infty A(\tau)v(t,\tau)d\tau, \tag{11.8}$$

where A(t) has the usual meaning of the infectivity function.

Substitution of (11.6) in (11.8) gives us (4.7)

$$v(t) = x(t)\{\int_0^t C(\tau)v(t-\tau)d\tau + \varepsilon C_1(t)\}, \tag{11.9}$$

where

$$C(t) = A(t)B(t), \tag{11.10}$$

and

$$C_1(t) = \int_0^\infty C(\tau+t)\,\frac{g(t)}{B(t)}\,d\tau. \tag{11.11}$$

In working models it is normally sufficient to consider only the special case

$$g(t) = \delta(t), \tag{11.12}$$

where $\delta(t)$ is Dirac's delta-function, the mathematical form of the assumption that initially all ills are at the very beginning of their disease. In that case we have simply $C_1(t) = C(t)$.

So far the model is essentially as described in the Chapters 3 and 4. However, the change in the class of susceptibles is no longer described by the simple relation (3.10) but instead we have

$$\dot{x}(t) = -v(t) + \int_0^\infty \chi(\tau)v(t,\tau)d\tau. \tag{11.13}$$

Substitution of (11.6) gives as (11.9) the result

$$\dot{x}(t) = -v(t) + \int_0^t D(\tau)v(t-\tau)d\tau + \varepsilon D_1(t), \tag{11.14}$$

where

$$D(t) = \chi(t)B(t) \tag{11.15}$$

and

$$D_1(t) = \int_0 D(\tau+t)\,\frac{g(t)}{B(t)}\,d\tau. \tag{11.16}$$

In the special case (11.12) $D_1(t)$ coincides with $D(t)$ of course.

If (11.6) is substituted in (11.2) we find for the total number of ills

$$y(t) = \int_0^t B(\tau)v(t-\tau)d\tau + \varepsilon B_1(t), \tag{11.17}$$

where

$$B_1(t) = \int_0^\infty B(\tau+t)\,\frac{g(\tau)}{B(\tau)}\,d\tau. \tag{11.18}$$

Again, for (11.12) $B_1 \equiv B$.

Together, the equations (11.9), (11.14) and (11.17) give a complete and fairly general description of the course of an epidemic.

It has been remarked in Chapter 4 that a different and somewhat cruder way of book-keeping may result in simplifications. Let us assume that removed individuals are still counted as ills. Then the z-class remains empty and (11.1) takes the simpler form

$$y(t) = 1 + \varepsilon - x(t). \tag{11.19}$$

Then a simpler set of equations is obtained by combining this relation with (11.9) and (11.17) as follows

$$\boxed{\begin{aligned} v(t) &= x(t)\{\int_0^t C(\tau)v(t-\tau)d\tau + \varepsilon C_1(t)\}, \\ 1 + \varepsilon - x(t) &= \int_0^t B(\tau)v(t-\tau)d\tau + \varepsilon B_1(t). \end{aligned}} \tag{11.20}$$

In this set $B(t)$ is defined by (11.7) with $\psi \equiv 0$, thus

$$B(t) = \exp - \int_0^t \chi(\tau)d\tau,$$

(11.21)

$$C(t) = A(t)\exp - \int_0^t \chi(\tau)d\tau.$$

The model (11.20) will be our starting-point for further investigations. We restrict the discussion to that of the case (11.12) so that $B_1 \equiv B$ and $C_1 \equiv C$. In order to avoid mathematical complications we may assume that $A(t)$ and $\chi(t)$ are non-negative step-wise continuous functions, that $A(t) \to 0$ and $B(t) \to 0$ for $t \to \infty$ and that the total infectivity

$$\gamma = \int_0^\infty A(t)dt$$

is finite.

Let us suppose that for $t \to \infty$, $x(t)$ and $v(t)$ have limits x_∞ and v_∞ then from (11.23) we obtain at once

$$v_\infty = x_\infty v_\infty \int_0^\infty C(t)dt,$$

(11.22)

$$x_\infty = 1 + \varepsilon - v_\infty \int_0^\infty B(t)dt.$$

These relations are satisfied for $v_\infty = 0$, $x_\infty = 1 + \varepsilon$. This would mean that the disease dies out eventually. If $v_\infty \neq 0$ we should have

$$x_\infty = 1 \Big/ \int_0^\infty C(t)dt,$$

(11.23)

$$v_\infty = (1+\varepsilon-x_\infty) \Big/ \int_0^\infty B(t)dt.$$

However, it is possible that $x(t)$ and $v(t)$ have no limits but that they exhibit some sort of oscillating behaviour. In that case the model may describe certain epidemics of periodic recurrency.

The initial-value problem (11.20) can be parallelled by a double-infinite model in which the epidemic is thought to have been started in the infinite remote past. In that case the transport equation (11.5) has the solution

$$v(t,\tau) = v(t-\tau)B(\tau), \qquad \tau \geq 0, \tag{11.24}$$

valid for all values of t.

Substitution of (11.8) gives

$$v(t) = x(t) \int_0^\infty A(\tau)B(\tau)v(t-\tau)d\tau. \tag{11.25}$$

The equation (11.14) has to be replaced by

$$\dot{x}(t) = -v(t) + \int_0^\infty \chi(\tau)B(\tau)v(t-\tau)d\tau. \tag{11.26}$$

Instead of (11.17) we have

$$y(t) = \int_0^\infty B(\tau)v(t-\tau)d\tau. \tag{11.27}$$

If there is no removal, i.e. if $z \equiv 0$, we may use the following analogue of (11.1)

$$x(t) + y(t) = 1. \tag{11.28}$$

Then the set (11.20) takes the following form

$$\boxed{\begin{aligned} v(t) &= x(t) \int_0^\infty C(\tau)v(t-\tau)d\tau, \\ x(t) &= 1 - \int_0^\infty B(\tau)v(t-\tau)d\tau. \end{aligned}} \tag{11.29}$$

If this set has a solution $x(t), v(t)$ then any time translate $x(t+h), v(t+h)$ is also a solution.

The trivial solution $x(t) \equiv 1$, $v(t) \equiv 0$ corresponds to the non-epidemic case which is here a total absence of disease. If

$$\int_0^\infty C(t)dt < 1$$

this is clearly the only solution of (11.29).

In the epidemic case the set (11.29) may have the equilibrium solution $x = \bar{x}$, $v = \bar{v}$ where

$$\bar{x} = \left(\int_0^\infty C(t)dt\right)^{-1}, \qquad \bar{v} = (1-\bar{x})\left(\int_0^\infty B(t)dt\right)^{-1}, \tag{11.30}$$

i.e. (11.23) with $\varepsilon = 0$.

In order to investigate the stability of this stationary state we perform a small perturbation

$$(11.31) \qquad x = \bar{x} + C_1 e^{st}, \qquad v = \bar{v} + C_2 e^{st}$$

where C_1 and C_2 are small constants and where s is a complex parameter.

Substitution of (11.31) and (11.29) followed by linearization with respect to C_1 and C_2 gives

$$(11.32) \qquad \begin{aligned} &\bar{v}C_1 + \bar{x}(\bar{x}\bar{C}(s)-1)C_2 = 0, \\ &C_1 + \bar{B}(s)C_2 = 0, \end{aligned}$$

where $\bar{B}(s)$ and $\bar{C}(s)$ are the Laplace transforms of B(t) and C(t)

$$(11.33) \qquad \bar{B}(s) = \int_0^\infty e^{-st}B(t)dt, \qquad \bar{C}(s) = \int_0^\infty e^{-st}C(t)dt.$$

The compatibility equation obtained from (11.32) by elimination of C_1 and C_2 is

$$(11.34) \qquad \bar{D}(s) \equiv \bar{v}\bar{B}(s) + \bar{x} - \bar{x}^2\bar{C}(s) = 0.$$

The equilibrium $x = \bar{x}$, $v = \bar{v}$ is stable when all roots of (11.34) have a negative real part, i.e. when they are situated in the left-hand half-plane Re $s < 0$. If, however, at least one root has a positive real part the equilibrium is unstable.

EXAMPLE 11.1. We consider a disease with the infectivity function $A(t) = \exp - \lambda t$ such that after a fixed period $t = q$ all ills (ill in the sense of book-keeping) become anew susceptibles, i.e. $B(t) = \theta(q-t)$. With the absence of removal the model is that of (11.29) with

$$C(t) = e^{-\lambda t}\theta(q-t).$$

According to (11.30) we have

$$\bar{x} = \lambda/(1-e^{-q\lambda}), \qquad \bar{v} = (1-\bar{x})/q.$$

The eigenvalue equation (11.34) can be written in the form

$$(1-e^{-qs})\left(1+\frac{\lambda}{s}-\frac{\bar{x}^2}{\bar{v}}e^{-q\lambda}\right)+\frac{\bar{x}}{\bar{v}}s=0.$$

It is not difficult to see that for some combinations of q and λ there are roots in the right-hand half-plane which implies instability. If $q \to \infty$ the eigenvalue equation is approximated by

$$(1-e^{-qs})\frac{\lambda}{s}+\frac{\bar{x}}{\bar{v}}s=0.$$

It can be shown by substitution that there is a pair of roots with

$$s=\pm\frac{2\pi i}{q}+\frac{4\pi^2\bar{x}}{\lambda(1-\bar{x})}\frac{1}{q^2}+\ldots .$$

When in a diagram we plot those λ, q combinations for which there are purely imaginary roots but no roots in the right-hand half-plane we obtain a critical line which separates regions of stability and of instability. From the eigenvalue equation with $qs = i\omega$ we obtain by taking the real and the imaginary part the following set of equations

$$C_1(1-\cos\omega)+C_2\frac{\sin\omega}{\omega}=0,$$

$$C_1\sin\omega-C_2\frac{1-\cos\omega}{\omega}+C_3\omega=0,$$

where

$$C_1=1-\frac{\bar{x}^2}{\bar{v}}e^{-q\lambda},\qquad C_2=q\lambda,\qquad C_3=\frac{\bar{x}}{(1-\bar{x})}.$$

For the purpose of numerical calculation we bring them into the following form

$$\omega\tan\tfrac{1}{2}\omega=-\frac{C_2}{C_1},\qquad \omega^2=\frac{2C_2}{C_3}$$

$$C_1=1-\frac{C_2C_3}{1-\exp -C_2}.$$

Starting from a given value of ω values of C_2, C_3, C_1, λ and q can be computed in succession.

A few combinations of (λ, q) values are given in Table 11.1 below. The corresponding diagram with the critical line is given in Fig. 11.1

ω	λ	q
π	1	∞
3.2	0.96	114.7
3.4	0.82	32.0
3.6	0.70	21.9
3.8	0.61	18.3
4.0	0.52	16.8
4.2	0.45	16.0
4.4	0.38	15.7
4.6	0.32	15.8
4.8	0.27	16.0
5.0	0.22	16.6
5.2	0.17	17.4
5.4	0.13	18.6
5.6	0.09	20.7
5.8	0.06	24.4
6.0	0.02	33.4
2π	0	∞

Table 11.1

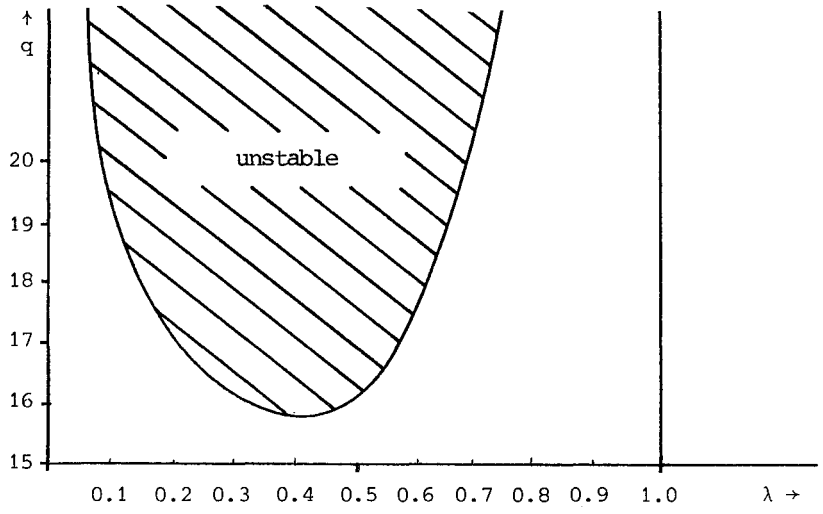


Fig. 11.1

CHAPTER 12

SOME ANALYTICAL CASES

This chapter is merely an illustration of the models of the previous chapter by considering a few special cases for which an analytical solution may be obtained.

A. Constant infectivity, constant probability of recovery, no death nor permanent immunity.

We consider the case

$$A(t) = \alpha, \qquad \psi(t) = 0, \qquad \chi(t) = \lambda. \tag{12.1}$$

The model is described by (11.20). Taking the initial condition (11.12) we have

$$B = B_1 = e^{-\lambda t}, \qquad C = C_1 = \alpha e^{-\lambda t}. \tag{12.2}$$

Then the equations (11.20) can be rewritten as

$$\begin{aligned} ve^{\lambda t} &= \alpha x\left(\int_0^t e^{\lambda\tau} v(\tau)\,d\tau + \varepsilon\right), \\ (1+\varepsilon-x)e^{\lambda t} &= \int_0^t e^{\lambda\tau} v(\tau)\,d\tau + \varepsilon. \end{aligned} \tag{12.3}$$

A suitable combination gives

$$v = \alpha x(1+\varepsilon-x). \tag{12.4}$$

Differentiation of the second equation of (12.3) gives the differential equation

$$\dot{x} = -v + \lambda(1+\varepsilon-x). \tag{12.5}$$

Substitution of (12.4) gives

$$\dot{x} = (\lambda-\alpha x)(1+\varepsilon-x), \tag{12.6}$$

the well-known differential equation of the logistic function. Its solution satisfying $x(0) = 1$ is

$$x(t) = \frac{(1+\varepsilon)(\alpha-\lambda)+\varepsilon\lambda\exp\{(1+\varepsilon)(\alpha-\lambda)t\}}{\alpha-\lambda+\varepsilon\alpha\exp\{(1+\varepsilon)(\alpha-\lambda)t\}} . \tag{12.7}$$

We observe that for $\lambda > \alpha$ we have $x(t) \to 1+\varepsilon$ so that no epidemic develops. However, for $\lambda \leq \alpha$ an epidemic may occur with $x(t) \to \lambda/\alpha$.

This is in agreement with what would follow in the general theory by considering the effective total infectivity

$$\gamma = \int_0 C(t)dt = \alpha/\lambda.$$

From (11.23) we have in the epidemic case

$$x_\infty = \frac{\lambda}{\alpha}, \qquad v_\infty = (1+\varepsilon-\frac{\lambda}{\alpha}). \tag{12.8}$$

Since $v_\infty \neq 0$ we have the situation that the disease is permanently present among the population. The daily number of recoveries and of new ills are eventually in equilibrium.

In Fig. 12.1 we have given an illustration of $x(t)$, $v(t)$ for the numerical case $\alpha = .1$, $\lambda = 0.5$, $\varepsilon = 0.01$.

B. Constant infectivity, constant probability of recovery, constant probability of removal.

We consider the case

$$A(t) = \alpha, \quad \psi(t) = \beta, \quad \chi(t) = \lambda, \qquad (\beta \neq 0). \tag{12.9}$$

The model is described by (11.9) and (11.14). With the initial condition (11.12) we have

$$B = B_1 = e^{-(\lambda+\beta)t}, \qquad C = C_1 = \alpha e^{-(\lambda+\beta)t}. \tag{12.10}$$

The equations (11.9) and (11.14) can be rewritten as

$$
\begin{aligned}
ve^{(\lambda+\beta)t} &= \alpha x\{\int_0^t e^{(\lambda+\beta)\tau} v(\tau)\,d\tau + \varepsilon\}, \\
(\dot{x}+v)e^{(\lambda+\beta)t} &= \lambda\{\int_0^t e^{(\lambda+\beta)\tau} v(\tau)\,d\tau + \varepsilon\}.
\end{aligned}
\tag{12.11}
$$

A suitable combination gives the differential equation

$$\dot{x} = \frac{(\lambda-\alpha x)v}{\alpha x} . \tag{12.12}$$

A second differential equation can be obtained from the first equation of (12.11) by differentiation after having divided by x. We find

$$\dot{v} = v\frac{\dot{x}}{x} + v(\alpha x-\lambda-\beta). \tag{12.13}$$

Together, (12.12) and (12.13) form an autonomous system which can be treated in the usual way by considering the equilibrium and the v,x-diagram.

The equilibrium condition $\dot{x} = \dot{v} = 0$ gives $v = 0$ as the only possibility. This means that always $v \to 0$ for $t \to \infty$. However, what happens for x for $t \to \infty$ will be a matter of further investigation.

From (12.12) and (12.13) we obtain by elimination of the time variable the differential equation

$$\frac{dv}{dx} = \frac{v}{x} - \frac{\alpha x(\alpha x-\lambda-\beta)}{\alpha x-\lambda} , \tag{12.14}$$

which can be integrated.

Its solution with the initial condition

$$v = \alpha\varepsilon, \qquad x = 1 \qquad (\text{for } t = 0)$$

is

$$v = \alpha x(1+\varepsilon-x) - \beta x\ell n\frac{\alpha-\lambda}{\alpha x-\lambda} , \tag{12.15}$$

obviously a generalization of (12.4).

The limit value x_∞ of x is a root of the equation

(12.16) $\alpha(1+\varepsilon-x) = \beta \ell n \frac{\alpha-\lambda}{\alpha x-\lambda}$

It is not difficult to see that the equation has a single root. A graphical argument may be obtained from the combination of the graphs of $y = \alpha(1+\varepsilon-x)$ $y = \beta \ell n \frac{\alpha-\lambda}{\alpha x-\lambda}$ in a single diagram.

In Fig. 12.1 and 12.2 we have given a numerical illustration with

$$\alpha = 1, \quad \beta = 0.1, \quad \lambda = 0.5, \quad \varepsilon = 0.01.$$

In Fig. 12.1 x(t) and v(t) are plotted against time. In Fig. 12.2 x and v are plotted against each other both for $\beta = 0.1$ as for $\beta = 0$.

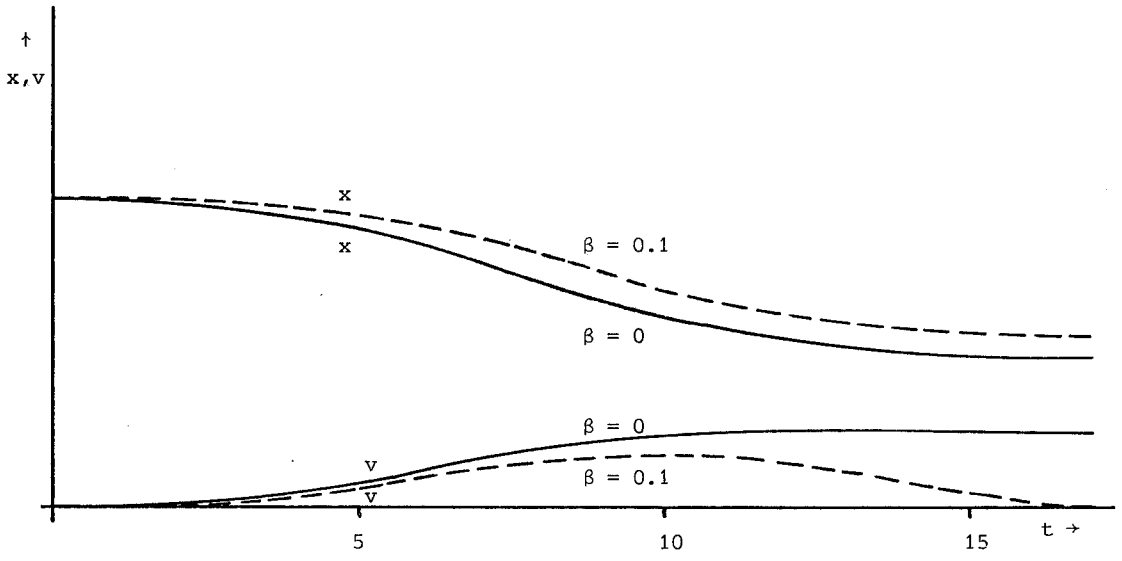


Fig. 12.1

Graphs of x(t) and v(t) for $\alpha = 1$, $\beta = 0$ and 0.1, $\lambda = 0.5$, $\varepsilon = 0.01$

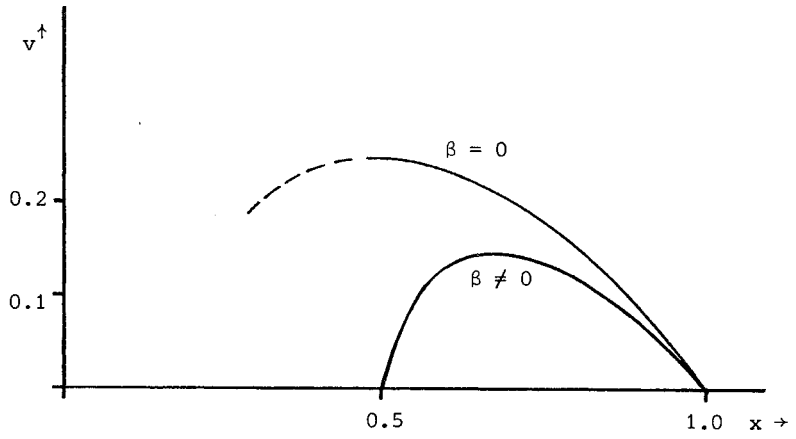


Fig. 12.2

C. Exponentially diminishing infectivity, constant probability of recovery, no removal.

We consider the following generalization of model A.

(12.17) $A(t) = \alpha \exp - \lambda_2 t, \quad \psi(t) = 0, \quad \chi(t) = \lambda_1.$

According to (11.21) we have

(12.18) $B(t) = \exp - \lambda_1 t, \qquad C(t) = \alpha \exp - (\lambda_1+\lambda_2)t.$

The equations (11.20) give

$$ve^{(\lambda_1+\lambda_2)t} = \alpha x\left(\int_0^t e^{(\lambda_1+\lambda_2)\tau} v(\tau)d\tau + \varepsilon\right),$$

(12.19)

$$(1+\varepsilon-x)e^{\lambda_1 t} = \int_0^t e^{\lambda_1 \tau} v(\tau)d\tau + \varepsilon.$$

For $\lambda_1 = \lambda$, $\lambda_2 = 0$ the set (12.3) of model A is obtained.

We consider the epidemic case $\alpha > \lambda_1 + \lambda_2$ only. If x(t) and v(t) have limits they are given by

(12.20) $$x_\infty = \frac{\lambda_1+\lambda_2}{\alpha}, \qquad v_\infty = \lambda_1(1+\varepsilon-x_\infty).$$

By differentiation we can bring (12.19) in the form of a pair of autonomous differential equations.

$$\begin{aligned} \dot{x} &= \lambda_1(1+\varepsilon) - \lambda_1 x - v, \\ \dot{v} &= v\dot{x}/x - (\lambda_1+\lambda_2)v + \alpha x v. \end{aligned} \tag{12.21}$$

There is a single equilibrium state $x = x_\infty$, $v = v_\infty$. Its nature can be investigated in the usual way by local linearization. The local Jacobian matrix is

$$\begin{pmatrix} -\lambda_1 & -1 \\ \lambda_2 v_\infty/x_\infty & -v_\infty/x_\infty \end{pmatrix} \tag{12.22}$$

to which is associated the eigenvalue equation

$$s^2 + (\lambda_1 + v_\infty/x_\infty)s + \alpha v_\infty = 0. \tag{12.23}$$

It is clear that both eigenvalues are either real and negative or complex with a negative real part. Hence the equilibrium is always stable.

In Fig. 12.3 we have given a numerical illustration for

$$\alpha = 0.8, \qquad \lambda_1 = 0.2, \qquad \lambda_2 = 0.1, \qquad \varepsilon = 0.01.$$

This gives

$$x_\infty = 0.38, \qquad v_\infty = 0.13.$$

The eigenvalue equation (12.23) has the complex roots

$$s = -0.27 \pm 0.17i.$$

The graph of $x(t)$ shows that the equilibrium value x_∞ is reached in an oscillatory way.

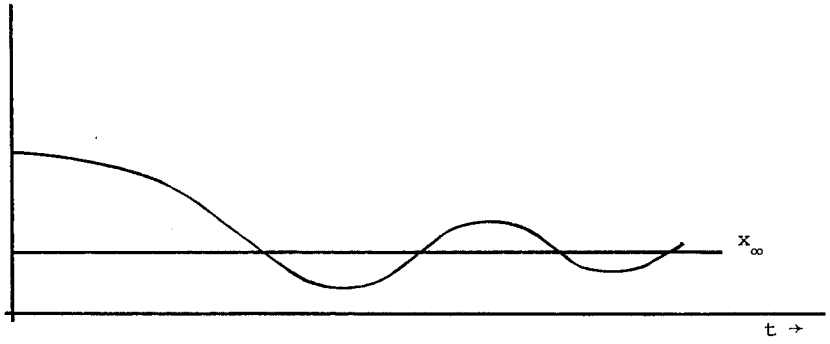


Fig. 12.3

CHAPTER 13

THE BLOCK-FUNCTION CASE WITH TEMPORAL IMMUNITY

In this chapter we consider the model studied in Chapter 7 with temporal immunity added. Formally the model is described by

$$A(t) = \frac{\gamma}{p}\,\theta(p-t), \qquad \psi(t) = 0 \qquad \text{and}$$

(13.1)

$$\chi(t) = \begin{cases} 0 & \text{for } t < q, \\ \infty & \text{for } t > q, \end{cases}$$

with $q \geq p$.

These conditions imply

$$B(t) = \theta(q-t), \qquad C(t) = A(t) = \frac{\gamma}{p}\,\theta(p-t). \tag{13.2}$$

Of course the length of the infective period p could be taken as the unit of time as in Chapter 7. However, we shall not do this for reasons of symmetry. We should realize that apart from ε the only essential parameters of the model are γ and the ratio q/p.

With the simplifying condition (11.12) we assume that the initial ε-fraction of ills are all at the very beginning of the disease. According to (13.1) there is no incubation period but it is not difficult to incorporate incubation in the model if one wishes. In order to keep the mathematics as simple as possible we took the simple model without incubation.

The relevant equations are (11.20) now written as follows

$$v(t) = x(t)\{\int_0^t A(\tau)v(t-\tau)d\tau + \varepsilon A(t)\},$$

(13.3)

$$1 + \varepsilon - x(t) = \int_0^t B(\tau)v(t-\tau)d\tau + \varepsilon B(t),$$

where A(t) and B(t) are given by (13.2). The equilibrium state is given by (11.23) as

$$(13.4) \qquad x_\infty = \frac{1}{\gamma}\,, \qquad v_\infty = \frac{1}{q}(1+\varepsilon-\frac{1}{\gamma})$$

in the epidemic case $\gamma > 1$, the only case considered here.

It is possible that $x(t) \to x_\infty$ and $v(t) \to v_\infty$ so that the equilibrium is a stable one but it seems plausible that x(t) and v(t) stay oscillating in a more or less defined way in case the equilibrium is an unstable one.

The stability of (13.4) is investigated in the usual way by making the local substitution (cf. eq. 11.31)

$$(13.5) \qquad x = x_\infty + C_1 e^{st}, \qquad v = v_\infty + C_2 e^{st},$$

where C_1 and C_2 are small constants and where s is a complex parameter.

After linearization with respect to C_1 and C_2 we obtain the linear equations (cf. eq. 11.32)

$$(13.6) \qquad \begin{aligned} \gamma v_\infty C_1 + \frac{1-ps-e^{-ps}}{ps} C_2 &= 0, \\ C_1 + \frac{1-e^{-qs}}{qs} C_2 &= 0. \end{aligned}$$

The compatibility equation (cf. eq. 11.34) is

$$(13.7) \qquad \bar{D}(s) \equiv \{\gamma(1+\varepsilon) - 1\} \frac{1-e^{-qs}}{qs} - \frac{1-ps-e^{-ps}}{ps} = 0.$$

The behaviour of its roots can be illustrated in a diagram where $\gamma(1+\varepsilon)$ and q/p are plotted against each other. Regions of stability and instability are separated by critical lines corresponding to the presence of purely imaginary roots $s = i\omega$ of $\bar{D}(s) = 0$.

From $\bar{D}(i\omega) = 0$ we obtain by taking the real and the imaginary part the equations

$$(13.8) \qquad \gamma(1+\varepsilon) - 1 = \frac{q}{p} \frac{p\omega - \sin p\omega}{-\sin q\omega} = \frac{q}{p} \frac{1-\cos p\omega}{1-\cos q\omega}\,.$$

The critical line for the case where $\bar{D}(s) = 0$ has no roots with a positive real part and when the first pair of complex conjugate roots are about to cross the imaginary axis is determined from

$$\frac{q}{p} = \frac{2}{p\omega}\left(\pi - \operatorname{arctg}\frac{1-\cos p\omega}{\omega-\sin p\omega}\right), \qquad \omega > 0,$$

$$\gamma(1+\varepsilon) = 1 + \frac{q}{p}\frac{1-\cos p\omega}{1-\cos q\omega}. \tag{13.9}$$

For a given value of ω values of q/p and $\gamma(1+\varepsilon)$ are easily obtained. A few results are collected in Table 13.1. The corresponding critical line is sketched in Fig. 13.1.

For small values of ω the relations (13.9) can be approximated by

$$\frac{q}{p} = \frac{2}{\omega}\left(\frac{1}{2}\pi + \frac{1}{3}\right),$$

$$\gamma(1+\varepsilon) = 1 + \frac{q\omega^2}{4p}. \tag{13.10}$$

Elimination of ω gives

$$\gamma(1+\varepsilon) \approx \frac{9\pi^2 pq}{4(3q-2p)^2}, \tag{13.11}$$

an approximation which appears to be surprisingly good for say $q > 3p$, as Table 13.1 shows.

ω	$\gamma(1+\varepsilon)$	q/p	$\gamma(1+\varepsilon)$ appr.
0	1	∞	1
0.1	1.08	32.1	1.08
0.2	1.16	16.4	1.16
0.3	1.25	11.1	1.25
0.4	1.34	8.52	1.34
0.5	1.44	6.95	1.43
1	1.98	3.81	1.95
1.5	2.65	2.76	2.55
2	3.51	2.23	3.25
2.5	4.63	1.91	4.05
3	6.15	1.69	
4	11.8	1.40	
5	31.4	1.21	
2π	∞	1	

Table 13.1

Critical values of $\gamma(1+\varepsilon)$ and q/p for the equilibrium x_∞, v_∞.

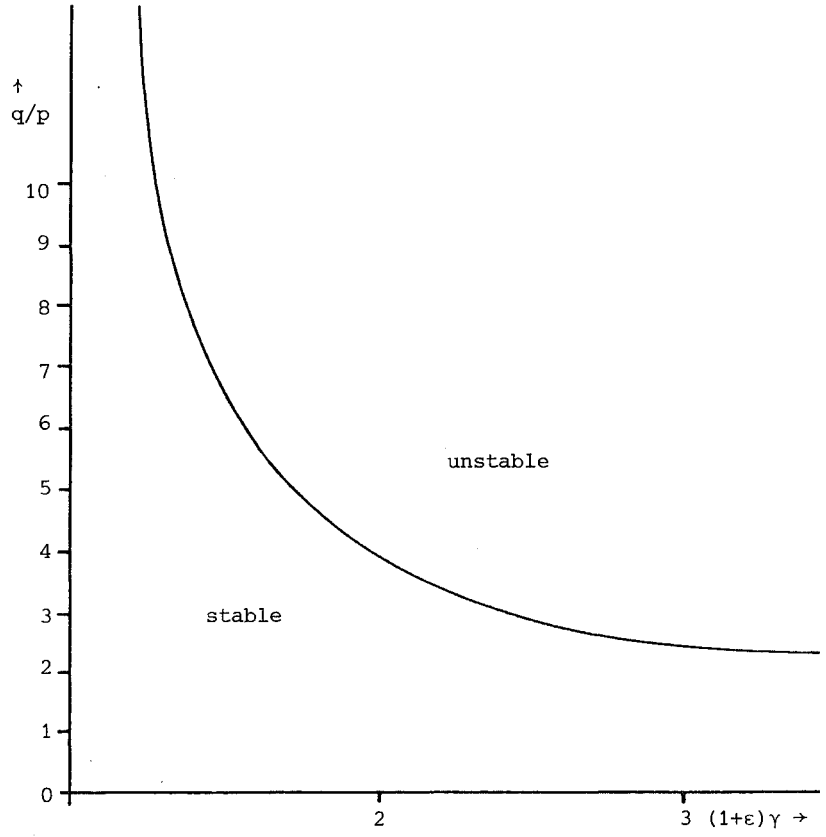


Fig. 13.1

Critical line for equilibrium

The system (13.3) has been studied numerically by L.J. Bonder in the way described below. For simplicity the unit of time is taken such that $p = \gamma$.

For the initial period $0 \leq t \leq p$ the equations (13.3) can be written as

$$v(t) = x(t)\{\int_0^t v(\tau)d\tau + \varepsilon\},$$

$$x(t) = 1 - \int_0^t v(\tau)d\tau. \tag{13.12}$$

This system can be reduced to first-order differential equations for which there is the following elementary solution

$$x(t) = \frac{1+\varepsilon}{1+\varepsilon\exp(1+\varepsilon)t}\,,$$

(13.13)

$$v(t) = -\dot{x}(t).$$

This solution may serve as a start for the integration of (13.3) for $t > p$. Then (13.3) can be brought into the following form

$$\frac{d}{dt}\,x = -v + f,$$

(13.14)

$$\frac{d}{dt}\,\frac{v}{x} = v - g,$$

where

(13.15) $\quad f(t) = v(t-q), \quad g(t) = v(t-p).$

For each subsequent interval $np < t < (n+1)p$, $n = 1,2,\ldots$ the set (13.14) can be solved numerically in a straightforward manner with f and g as given functions determined by the solution in a previous interval.

From Bonder's results we reproduce a few pictures. In Fig. 13.2 we show x(t) and v(t) for the unstable case $\gamma = p = 2$, $q = 8$, $\varepsilon = 0.005$ with a corresponding x,v-diagram in Fig. 13.3. In Figs. 13.4 and 13.5 similarly the case $\gamma = p = 2$, $q = 10$, $\varepsilon = 0.001$ is shown. Finally in Figs. 13.6 and 13.7 we consider the rather extreme case $\gamma = p = 5$, $q = 15$, $\varepsilon = 0.001$.

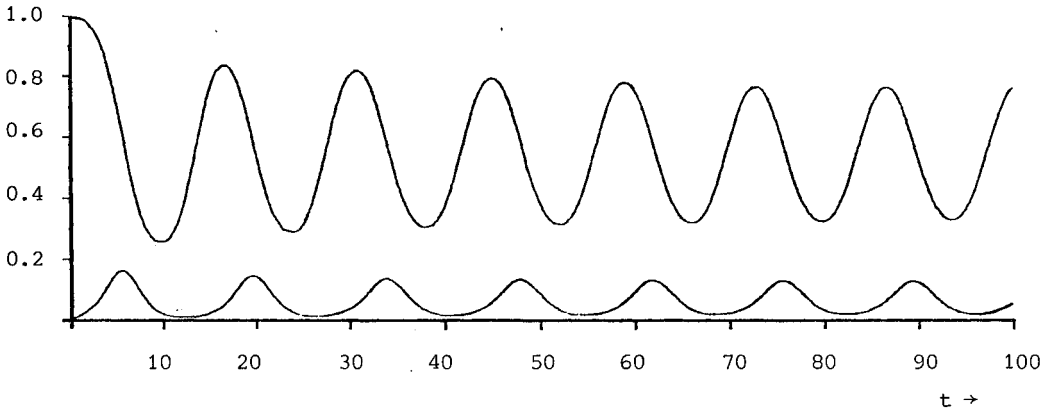


Fig. 13.2. γ=2, q/p=4, ε=0.005

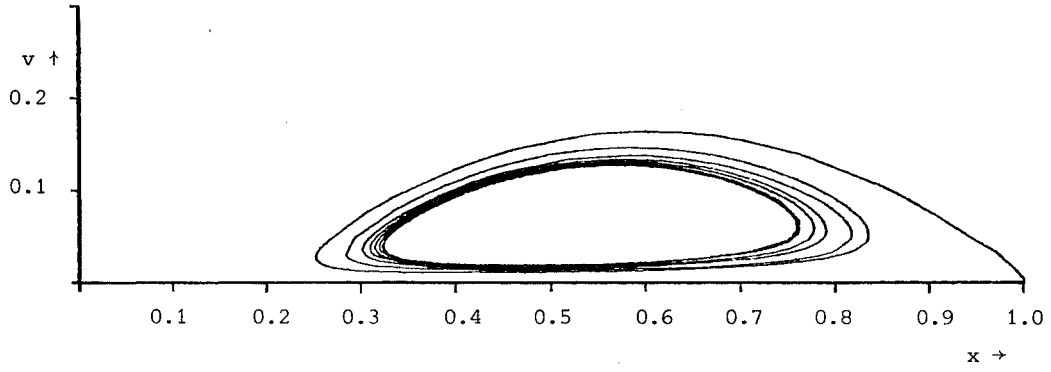


Fig. 13.3 $\gamma=2$, $q/p=4$, $\varepsilon=0.005$

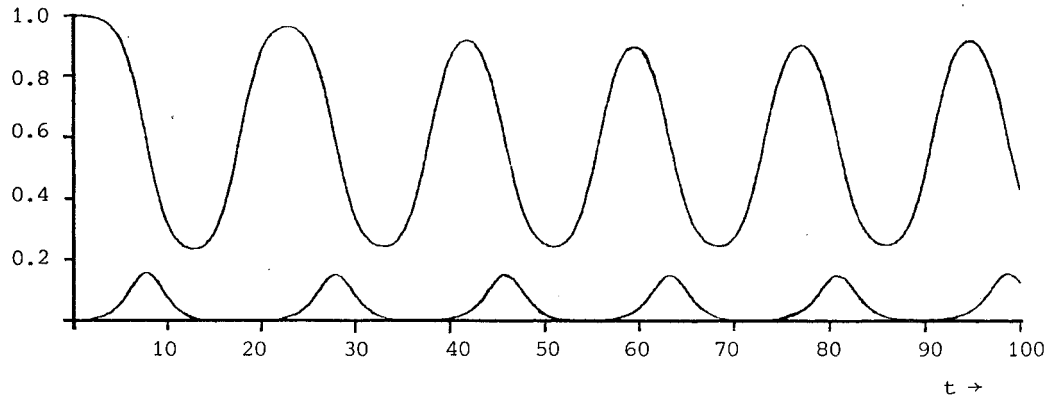


Fig. 13.4. $\gamma=2$, $q/p=5$, $\varepsilon=0.001$

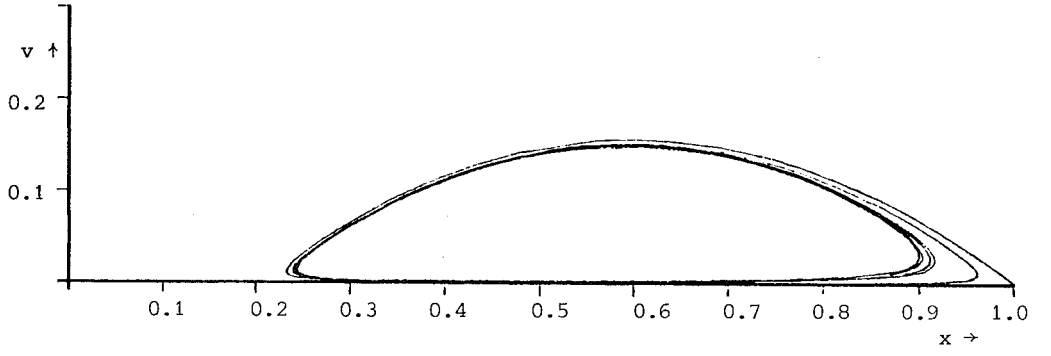


Fig. 13.5. $\gamma=2$, $q/p=5$, $\varepsilon=0.001$

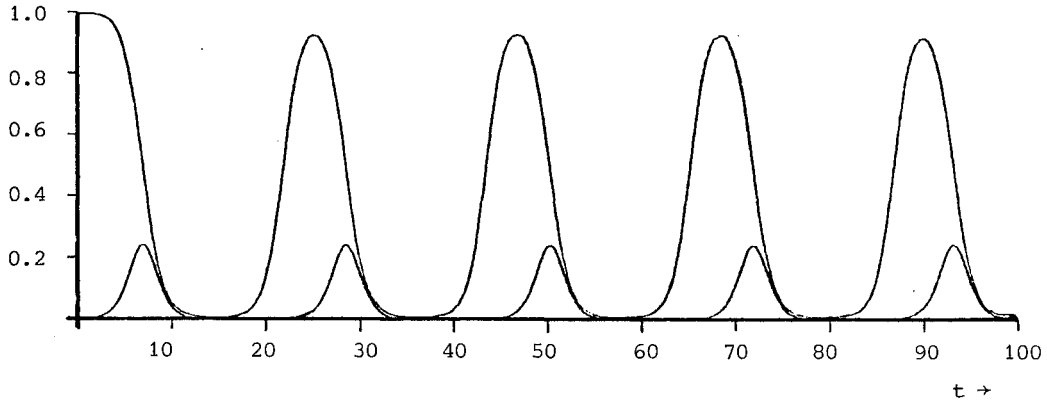


Fig. 13.6. $\gamma=5$, $q/p=2$, $\varepsilon=0.001$

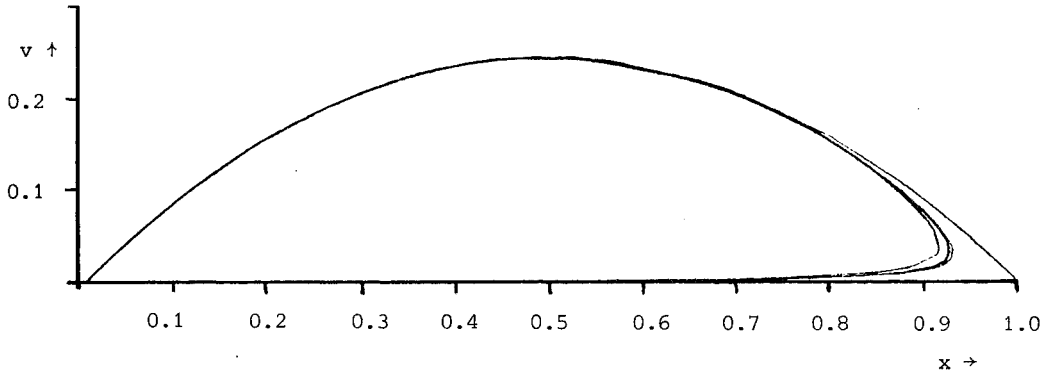


Fig. 13.7. $\gamma=5$, $q/p=2$, $\varepsilon=0.001$

The results obtained above strongly suggest that the solutions $x(t)$, $v(t)$ of (13.3) in the unstable regions tend to periodic functions. Similarly the double-infinite model

$$
\begin{aligned}
v(t) &= \frac{\gamma}{p}\, x(t) \int_0^p v(t-\tau)\,d\tau, \\
x(t) &= 1 - \int_0^q v(t-\tau)\,d\tau,
\end{aligned}
\tag{13.16}
$$

should have a periodic solution, or equivalently

$$(13.17) \qquad v(t) = \frac{\gamma}{p}\int_0^p v(t-\tau)d\tau\{1 - \int_0^q v(t-\tau)d\tau\}$$

has a periodic solution for certain combinations of γ and q/p.

We note that (13.17) is some continuous generalization of the familiar class model

$$(13.18) \qquad v_{n+1} = \gamma v_n(1-v_n), \qquad 0 < \gamma < 4$$

where the stable solution may bifurcate into periodic cycli. We shall return to this question in the final chapter.

CHAPTER 14

A DISCRETE MODEL WITH TEMPORAL IMMUNITY

The continuous model considered in Chapter 11 can be paralleled by a discrete model as in Chapter 8. For simplicity we neglect the possible effect of removal. To facilitate reference to the continuous model we repeat the main formulae. With the initial condition that at $t = 0$ there is a fraction ε of ills which are just infected. We have the equations (11.20) in the form

$$
\begin{aligned}
v(t) &= x(t)\{\int_0^t A(\tau)B(\tau)v(t-\tau)d\tau + \varepsilon A(t)B(t)\}, \\
1 + \varepsilon - x(t) &= \int_0^t B(\tau)v(t-\tau)d\tau + \varepsilon B(t),
\end{aligned}
\tag{14.1}
$$

with

$$
B(t) = \exp - \int_0^t \chi(\tau)d\tau. \tag{14.2}
$$

as the cumulative loss of immunity.

The various phases of the disease are passed according to (11.6) with (11.12) i.e.

$$
\begin{aligned}
v(t,\tau) &= v(t-\tau)B(\tau) \qquad \text{for } t > \tau, \\
v(t,\tau) &= \varepsilon\delta(\tau-t)B(t) \qquad \text{for } t \le \tau.
\end{aligned}
\tag{14.3}
$$

Let x_n denote the fraction of susceptibles and $v_{n,k}$ the fraction of ills who are at "day" n in the k-th "day" of their illness. The fraction $v_{n,0}$ of the new ills is denoted by v_n. The loss of immunity is described by a set of non-negative numbers $\chi_0, \chi_1, \chi_2, \ldots$ such that with $\chi_0 = 0$.

$$
v_{n+1,k+1} = v_{n,k} \exp - \chi_{k+1}, \qquad k \ge 0. \tag{14.4}
$$

The discrete analogue of (14.3) is then

$$
\begin{aligned}
v_{n,k} &= v_{n-k}B_k && \text{for } n > k,\\
v_{n,k} &= \varepsilon B_k && \text{for } n = k,\\
v_{n,k} &= 0 && \text{for } n < k,
\end{aligned}
\tag{14.5}
$$

where

$$B_k = \exp - (\chi_0+\chi_1+\chi_2+\ldots+\chi_k), \tag{14.6}$$

implying $B_0 = 1$.

The process of infection is as in (8.10). With the simplest choice $\phi(x_n,x_{n+1}) \equiv x_n$ we obtain the equation

$$v_{n+1} = x_n \sum_{k=0}^{n} A_k v_{n,k}, \tag{14.7}$$

where the sequence $A_0,A_1,A_2,\ldots$ describes the rate of infectivity of the various days of the disease.

A second equation is obtained by observing that the total number of ills and susceptibles stays constant

$$x_n + \sum_{k=0}^{n} v_{n,k} = 1 + \varepsilon. \tag{14.8}$$

Substitution of (14.5) in (14.7) in (14.8) gives

$$
\begin{aligned}
v_{n+1} &= x_n \sum_{k=0}^{n} A_k B_k v_{n-k}, && n \geq 0,\\
x_n &= 1 + - \sum_{k=0}^{n} B_k v_{n-k}, && n \geq 0,
\end{aligned}
\tag{14.9}
$$

with $x_0 = 1$ and $v_0 = \varepsilon$.

The equations (14.9) are the discrete analogue of (14.1). They can be combined into a single recursive equation

$$v_{n+1} = (1 + \varepsilon - \sum_{k=0}^{n} B_k v_{n-k}) \sum_{k=0}^{n} A_k B_k v_{n-k}. \tag{14.10}$$

For further discussion we consider the block-function model as the discrete analogue of the model considered in Chapter 13.

We take

$$A_0 = A_1 = \ldots = A_{p-1} = \gamma/p,$$
$$A_p = A_{p+1} = \ldots = 0, \tag{14.11}$$

and

$$B_0 = B_1 = \ldots = B_{q-1} = 1,$$
$$B_q = B_{q+1} = \ldots = 0, \tag{14.12}$$

with $q \geq p$.

Then the equation (14.9) can be written as

$$pv_{n+1} = \gamma x_n \sum_{k=0}^{\min(n,p-1)} v_{n-k},$$
$$x_n = 1 + \varepsilon - \sum_{k=0}^{\min(n,q-1)} v_{n-k}. \tag{14.13}$$

If x_∞ and v_∞ exist as the limits of x_n and v_n for $n \to \infty$ we have of course for the epidemic case

$$\gamma x_\infty = 1, \qquad \gamma v_\infty = \sigma/q, \tag{14.14}$$

where we have written

$$\sigma = (1+\varepsilon)\gamma - 1. \tag{14.15}$$

In order to investigate the stability of the stationary state x_∞, v_∞ we proceed as in Chapter 8 by studying the local linear approximation

$$x_n = x_\infty + \zeta_n, \qquad v_n = v_\infty + \eta_n. \tag{14.16}$$

If this is substituted into (14.13) we find after neglecting the quadratic terms

$$\zeta_n = -\sum_{k=0}^{q-1} \eta_{n-k},$$
$$\eta_{n+1} = \frac{\sigma}{q}\zeta_n + \frac{1}{p}\sum_{k=0}^{p-1} \zeta_{n-k}. \tag{14.17}$$

Both ζ_n and η_n can be expressed as linear sums with powers of the kind z_j^n, $j = 0,1,2,\ldots$. The eigenvalues z_j are found by substituting

$$\zeta_n = C_1 z^n, \qquad \eta_n = C_2 z^n$$

followed by elimination of C_1 and C_2. Substitution gives

$$\begin{aligned} &C_1 + \frac{1-z^{-q}}{1-z^{-1}} C_2 = 0, \\ &\frac{\sigma}{q} C_1 + (\frac{1}{p} \frac{1-z^{-p}}{1-z^{-1}} - z) C_2 = 0. \end{aligned} \tag{14.18}$$

Elimination of C_1 and C_2 gives the eigenvalue equation

$$\frac{\sigma}{q} (1-z^{-q}) = \frac{1}{p}(1-z^{-p}) + 1 - z, \tag{14.19}$$

an algebraic equation of degree q after removal of the trivial root $z = 1$.

Stability of the equilibrium x_∞, v_∞ requires that all non-trivial roots of (14.19) are within the unit circle.

Let us suppose that there is at least one root $z = \exp i\theta$ on the unit circle and no roots outside. Substitution of $z = \exp i\theta$ in (14.19) gives the two real equations

$$\begin{aligned} &\frac{p\sigma}{2} (1 - \cos q\theta) = 1 - \cos p\theta + p(1 - \cos\theta), \\ &\frac{p\sigma}{q} \sin q\theta = \sin p\theta - p\sin\theta, \end{aligned} \tag{14.20}$$

or equivalently for $p \neq 1$

$$\begin{aligned} &\tan \frac{1}{2} q\theta = \frac{p(1-\cos\theta)+(1-\cos p\theta)}{\sin p\theta - p\sin\theta}, \\ &\sigma = \frac{q}{p} \frac{\sin p\theta - p\sin\theta}{\sin q\theta} . \end{aligned} \tag{14.21}$$

Before considering the general case we turn to a few special cases.

EXAMPLE 14.1.

$$p = 1, \qquad q \geq 1.$$

For $q = 1$ there is the single eigenvalue $z = 1 - \sigma$. Stability requires $\sigma < 2$. For $q = 2$ there are two eigenvalues satisfying

$$2z^2 + (\sigma-2)z + \sigma = 0.$$

Stability requires again $\sigma < 2$. For $q > 2$ there are q eigenvalues satisfying

$$qz^q + (\sigma-q)z^{q-1} + \sigma(z^{q-2} + z^{q-3} + \ldots + 1) = 0.$$

If $\sigma = 0$ there is a single root $z = 1$ and a $(q-1)$-fold root $z = 0$. If σ increases a little, the single root moves to the position $z = 1 - \sigma/q$ and the multiple root fans out into the $q - 1$ values of $(\sigma/q)^{1/q-1}$. As σ further increases the roots are moving towards the boundary $|z| = 1$ until at a certain critical value σ_0 the first (pair of) root(s) becomes unimodular.

The possible values of σ_0 follow from (14.20) as

$$\sigma = q(1 - \cos\frac{j\pi}{q}), \qquad j = 1,3,5,\ldots$$

for

$$\theta = j\pi/q.$$

For σ_0 we have to take the lowest value

$$\sigma_0 = q(1 - \cos\frac{\pi}{q}).$$

This gives the following little table

q	1	2	3	4	5	6	7
σ_0	2	2	1.5	1.17	0.95	0.80	0.69

EXAMPLE 14.2.

$$p = 2, \qquad q \geq 2.$$

The eigenvalue equation is

$$2qz^q + (2\sigma-q)(z^{q-1} + z^{q-2}) + 2\sigma(z^{q-3} + \ldots + 1) = 0.$$

For $q = 2$ there are two eigenvalues satisfying $2z^2 + (\sigma-1)z + (\sigma-1) = 0$. Stability requires $\sigma < 3$. For $\sigma = 3$ the roots are $\exp \pm \frac{2\pi i}{3}$.

For $q > 2$ the critical value σ_0 is obtained from (14.21) by specialization. A little trigonometry shows that

$$\sigma = \frac{-q\sin\theta(1-\cos\theta)}{\sin q\theta} ,$$

with

$$\pm\theta = \frac{\pi}{q} + \frac{2}{q} \arctan \frac{\sin\theta}{2+\cos\theta} .$$

EXAMPLE 14.3.

$$q = p.$$

The relations (14.20) imply

$$\sigma_0 = p + 1 \quad \text{for} \quad \theta = \pm \frac{2\pi}{p+1} .$$

EXAMPLE 14.4.

$$q = p + 1.$$

In this rather curious special case the resulting critical σ-values are integers. With some elementary trigonometry the relations (14.20) or (14.21) lead to

$$\sigma = q,$$

$$(q-1)\sin \frac{q+1}{2} \theta = \sin \frac{q-1}{2} \theta.$$

In the general case the critical values σ_0 and the corresponding critical roots $\pm\theta_0$ ($\theta_0 > 0$) can be derived numerically from (14.21).

If q/p is large the following asymptotic estimates can be obtained from (14.21)

$$\sigma_0 = \frac{\pi^2(p+1)}{4q} + \ldots \tag{14.22}$$

for

$$\theta_0 = \frac{\pi}{q} + \frac{2\pi(p-1)}{3q^2} + \ldots \quad . \tag{14.23}$$

A few numerical results are collected in Table 14.1 which gives both σ_0 and θ_0 for a number of p,q-values.

q \ p	1	2	3	4	5	6	7
1	2 π						
2	2 $\frac{1}{2}\pi$	3 π					
3	1.5 $\frac{1}{3}\pi$	3 1.32	4 $\frac{1}{2}\pi$				
4	1.17 $\frac{1}{4}\pi$	2.31 0.94	4 1.12	5 $\frac{2}{5}\pi$			
5	0.95 $\frac{1}{5}\pi$	1.82 0.72	3.16 0.84	5 0.97	6 $\frac{1}{3}\pi$		
6	0.80 $\frac{1}{6}\pi$	1.48 0.59	2.50 0.67	4.03 0.76	6 0.85	7 $\frac{2}{7}\pi$	
7	0.69 $\frac{1}{7}\pi$	1.24 0.50	2.04 0.55	3.21 0.62	4.92 0.69	7 0.75	8 $\frac{1}{4}\pi$

Table 14.1

Critical values of $(1+\varepsilon)\gamma - 1$ and θ.

In Table 14.2 an illustration is given of the stable case p = 3, q = 6, γ = 3.2 with ε = 0.001. The equilibrium x_∞ = 0.313, v_∞ = 0.115 is approached in an oscillating way. The oscillation period is ab. 9.7 time steps. The corresponding critical value of γ (3.50 for ε = 0) would give a period of $2\pi/0.67$ = 9,4 according to Table 14.1. Both values appear to be in good agreement.

0.	1.000	0.001	50.	0.367	0.152	100.	0.301	0.132
1.	0.999	0.001	51.	0.289	0.154	101.	0.274	0.124
2.	0.997	0.002	52.	0.231	0.135	102.	0.263	0.113
3.	0.992	0.005	53.	0.210	0.109	103.	0.272	0.104
4.	0.984	0.000	54.	0.229	0.089	104.	0.297	0.099
5.	0.968	0.016	55.	0.281	0.081	105.	0.328	0.100
6.	0.940	0.029	56.	0.349	0.084	106.	0.354	0.106
7.	0.887	0.054	57.	0.409	0.095	107.	0.363	0.115
8.	0.796	0.094	58.	0.431	0.113	108.	0.352	0.125
9.	0.650	0.150	59.	0.405	0.134	109.	0.326	0.130
10.	0.453	0.206	60.	0.347	0.148	110.	0.296	0.129
11.	0.251	0.217	61.	0.282	0.146	111.	0.275	0.121
12.	0.127	0.154	62.	0.237	0.129	112.	0.270	0.112
13.	0.103	0.078	63.	0.225	0.107	113.	0.281	0.104
14.	0.147	0.049	64.	0.246	0.091	114.	0.305	0.101
15.	0.253	0.044	65.	0.294	0.086	115.	0.332	0.103
16.	0.413	0.046	66.	0.353	0.089	116.	0.351	0.109
17.	0.568	0.061	67.	0.398	0.100	117.	0.355	0.117
18.	0.630	0.092	68.	0.410	0.117	118.	0.342	0.125
19.	0.574	0.134	69.	0.383	0.134	119.	0.318	0.128
20.	0.447	0.176	70.	0.331	0.144	120.	0.293	0.125
21.	0.299	0.192	71.	0.277	0.139	121.	0.278	0.118
22.	0.185	0.160	72.	0.243	0.123	122.	0.277	0.110
23.	0.142	0.104	73.	0.238	0.105	123.	0.290	0.105
24.	0.165	0.069	74.	0.262	0.094	124.	0.311	0.103
25.	0.240	0.059	75.	0.306	0.090	125.	0.334	0.106
26.	0.357	0.060	76.	0.355	0.094	126.	0.348	0.111
27.	0.477	0.071	77.	0.389	0.105	127.	0.348	0.119
28.	0.541	0.097	78.	0.392	0.120	128.	0.333	0.124
29.	0.514	0.131	79.	0.364	0.134	129.	0.312	0.126
30.	0.419	0.164	80.	0.318	0.139	130.	0.292	0.123
31.	0.303	0.175	81.	0.274	0.133	131.	0.281	0.116
32.	0.210	0.152	82.	0.250	0.119	132.	0.283	0.110
33.	0.171	0.110	83.	0.251	0.104	133.	0.297	0.105
34.	0.188	0.080	84.	0.275	0.095	134.	0.316	0.105
35.	0.251	0.069	85.	0.316	0.094	135.	0.335	0.108
36.	0.346	0.069	86.	0.356	0.099	136.	0.344	0.113
37.	0.441	0.080	87.	0.380	0.109	137.	0.341	0.120
38.	0.490	0.103	88.	0.377	0.122	138.	0.326	0.124
39.	0.468	0.132	89.	0.348	0.133	139.	0.307	0.124
40.	0.391	0.157	90.	0.308	0.136	140.	0.292	0.121
41.	0.297	0.163	91.	0.273	0.129	141.	0.285	0.115
42.	0.223	0.143	92.	0.256	0.116	142.	0.289	0.109
43.	0.193	0.110	93.	0.262	0.104	143.	0.302	0.106
44.	0.210	0.086	94.	0.287	0.097	144.	0.320	0.106
45.	0.266	0.076	95.	0.323	0.097	145.	0.334	0.110
46.	0.346	0.077	96.	0.356	0.103	146.	0.340	0.115
47.	0.421	0.088	97.	0.372	0.113	147.	0.334	0.120
48.	0.456	0.108	98.	0.364	0.124	148.	0.321	0.123
49.	0.433	0.133	99.	0.336	0.132	149.	0.304	0.122
						150.	0.292	0.119

Table 14.2

Values of n, x_n, v_n for p=3, q=6, ε=0.001, γ=3.2 stable case.

0.	1.000	0.001	50.	0.148	0.045	100.	0.534	0.058
1.	0.999	0.001	51.	0.224	0.036	101.	0.588	0.082
2.	0.997	0.002	52.	0.344	0.036	102.	0.560	0.116
3.	0.992	0.005	53.	0.485	0.043	103.	0.464	0.152
4.	0.984	0.008	54.	0.596	0.060	104.	0.335	0.173
5.	0.968	0.016	55.	0.626	0.088	105.	0.218	0.158
6.	0.939	0.029	56.	0.567	0.127	106.	0.150	0.112
7.	0.886	0.054	57.	0.446	0.166	107.	0.137	0.071
8.	0.794	0.093	58.	0.301	0.181	108.	0.169	0.050
9.	0.647	0.149	59.	0.185	0.152	109.	0.243	0.042
10.	0.447	0.204	60.	0.129	0.098	110.	0.353	0.042
11.	0.242	0.213	61.	0.129	0.059	111.	0.475	0.051
12.	0.111	0.146	62.	0.174	0.043	112.	0.565	0.068
13.	0.074	0.067	63.	0.264	0.037	113.	0.580	0.097
14.	0.094	0.034	64.	0.391	0.039	114.	0.517	0.134
15.	0.163	0.025	65.	0.522	0.050	115.	0.402	0.165
16.	0.290	0.022	66.	0.604	0.070	116.	0.275	0.170
17.	0.470	0.025	67.	0.599	0.103	117.	0.180	0.137
18.	0.647	0.036	68.	0.517	0.142	118.	0.140	0.090
19.	0.737	0.057	69.	0.386	0.174	119.	0.149	0.059
20.	0.712	0.092	70.	0.251	0.172	120.	0.200	0.046
21.	0.605	0.140	71.	0.159	0.131	121.	0.292	0.042
22.	0.443	0.186	72.	0.128	0.081	122.	0.411	0.046
23.	0.267	0.198	73.	0.146	0.052	123.	0.523	0.058
24.	0.142	0.149	74.	0.207	0.041	124.	0.579	0.081
25.	0.097	0.081	75.	0.311	0.039	125.	0.555	0.114
26.	0.109	0.044	76.	0.441	0.044	126.	0.464	0.150
27.	0.169	0.032	77.	0.555	0.058	127.	0.338	0.171
28.	0.281	0.028	78.	0.602	0.083	128.	0.223	0.157
29.	0.436	0.031	79.	0.564	0.119	129.	0.155	0.114
30.	0.591	0.043	80.	0.460	0.157	130.	0.140	0.073
31.	0.676	0.065	81.	0.325	0.176	131.	0.170	0.051
32.	0.657	0.100	82.	0.207	0.157	132.	0.241	0.043
33.	0.556	0.146	83.	0.143	0.108	133.	0.348	0.043
34.	0.404	0.184	84.	0.134	0.067	134.	0.468	0.051
35.	0.247	0.185	85.	0.170	0.047	135.	0.557	0.069
36.	0.143	0.136	86.	0.249	0.048	136.	0.574	0.097
37.	0.109	0.077	87.	0.364	0.041	137.	0.514	0.133
38.	0.127	0.046	88.	0.491	0.050	138.	0.402	0.163
39.	0.192	0.035	89.	0.579	0.069	139.	0.277	0.169
40.	0.306	0.032	90.	0.588	0.099	140.	0.183	0.137
41.	0.453	0.037	91.	0.519	0.136	141.	0.142	0.091
42.	0.587	0.051	92.	0.398	0.168	142.	0.151	0.060
43.	0.648	0.075	93.	0.267	0.171	143.	0.201	0.046
44.	0.612	0.112	94.	0.173	0.136	144.	0.291	0.043
45.	0.503	0.155	95.	0.135	0.087	145.	0.408	0.046
46.	0.354	0.184	96.	0.147	0.057	146.	0.518	0.059
47.	0.216	0.171	97.	0.202	0.044	147.	0.573	0.082
48.	0.135	0.117	98.	0.297	0.041	148.	0.550	0.114
49.	0.118	0.068	99.	0.421	0.045	149.	0.461	0.150

Table 14.3

Values of n, x_n, v_n for p=3, q=7, ε=0.001, γ=3.2 unstable case.

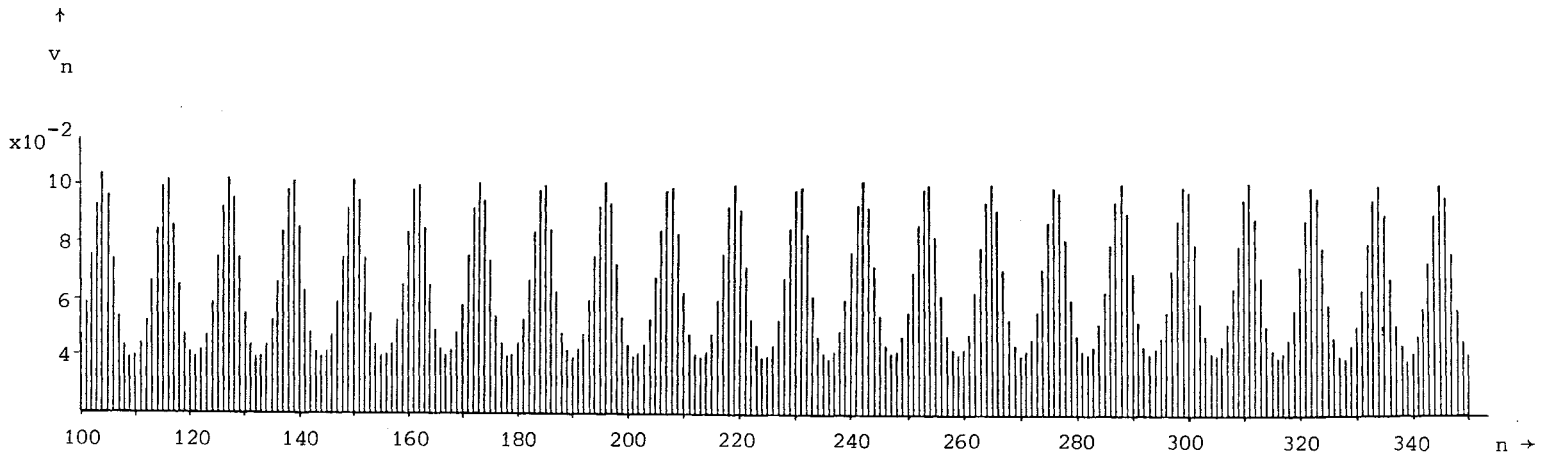


Figure 14.1

Diagram of an oscillating epidemic. Values of v_n for n = 100,...,350 with data of Table 14.3.

In Table 14.3 an illustration is given of the unstable case $p = 3$, $q = 7$, $\gamma = 3.2$ with $\varepsilon = 0.001$. The equilibrium $x_\infty = 0.312$, $v_\infty = 0.098$ is surrounded by oscillations with a period of ab. 11.6 time steps. For $n \to \infty$ they seem to tend to a stable oscillatory behaviour. The corresponding critical value of γ (3.04 for $\varepsilon = 0$) would give a period of $2\pi/0.55 = 11.4$ according to Table 14.1

CHAPTER 15

LARGE EPIDEMICS

The discrete model (14.9) considered in the previous chapter has a serious shortcoming. It may happen that after one or more time steps x_n and v_n leave their natural interval $(0,1+\varepsilon)$ and that e.g. v_n becomes negative. This rather unwanted behaviour is clearly illustrated in the special case of the block-function model (14.13) with $p = q = 1$. Then we have

$$\begin{aligned} v_{n+1} &= \gamma x_n v_n \\ x_n &= 1 + \varepsilon - v_n, \end{aligned} \tag{15.1}$$

which is equivalent to the well-known chaos model

$$z_{n+1} = a z_n (1-z_n), \tag{15.2}$$

studied and popularized by May, Oster and many others. Indeed, it suffices to take

$$v_n = (1+\varepsilon) z_n, \qquad a = (1+\varepsilon)\gamma \tag{15.3}$$

and (15.2) is readily obtained from (15.1).

Let us recall a few well-known facts. The model has a stable equilibrium $z_n = 0$ for $0 \le a \le 1$ and a stable equilibrium $z_n = 1 - a^{-1}$ for $1 < a \le 3$. However for $3 < a \le 4$ the sequence is either asymptotically periodic or chaotic. This behaviour is more or less what could be expected from an epidemic model. The value $a = 1$ appears to be a threshold value so that a threshold theorem is present. Again there is possibility of a periodic epidemic. But a is restricted to the interval $(0,4)$. For $a > 1$ the right-hand side of (15.2) may exceed the value 1 so that the next z_n becomes negative.

Computer experiments show that also for higher values of p and q

iteration of (14.13) comes to an unwanted stop when γ is sufficiently large.

That the model (14.9) or better (14.13) may become defective as γ increases can be explained as follows.

The equation (14.7) says that the number of new ills at the step n+1 is obtained by the addition of all possible "meetings" between a susceptible and an ill. A certain fraction of all meetings will result in new illness cases at the next step. However, in the case of a large infectivity constant γ, i.e. a strong epidemic, a susceptible individual may have more "effective meetings" will ills. This results in a multiple counting of a single illness case. Thus it may happen that the total number of ills in this false book-keeping Σv_{n-k} exceeds $1 + \varepsilon$ with a corresponding negative value of the number of susceptibles.

The model can be improved by using probability arguments (cf. K.L. COOKE (1975)). Let p_k be the probability of an effective contact between a specific susceptible and a specific ill from the illness class k. If there are N individuals the total number of ills in the class k is $Nv_{n,k}$ in the notation of the previous section. By "effective contact" we understand a meeting that results in infection and a new case of illness.

The probability that a certain susceptible escapes dangerous meetings for a time step is of course $(1-p_k)^{Nv_{n,k}}$ for a single illness class and

$$\prod_{k=0}^{n} (1-p_k)^{Nv_{n,k}}$$

for all infectives together. We write this probability as

$$\exp - \sum_{k=0}^{n} A_k v_{n,k},$$

where $A_k = -N \ln(1-p_k)$.

The probability of an infection is the complementary probability

$$1 - \exp - \sum_{k=0}^{n} A_k v_{n,k}.$$

Since there are Nx_n susceptibles the average number of new illness cases is

$$Nx_n(1 - \exp - \sum_{k=0}^{n} A_k v_{n,k}).$$

This gives the following improved version of (14.7)

$$v_{n+1} = x_n(1 - \exp - \prod_{k=0}^{n} A_k v_{n,k}). \tag{15.4}$$

The improved model consists of (15.4) together with the second relation of (14.9) which catalogues all individuals

$$x_n = 1 + \varepsilon - \prod_{k=0}^{n} v_{n,k}. \tag{15.5}$$

Substitution of (14.5) then gives the following basic model

$$\begin{aligned} v_{n+1} &= x_n(1 - \exp - \prod_{k=0}^{n} A_k B_k v_{n-k}), \qquad n \geq 0, \\ x_n &= 1 + \varepsilon - \prod_{k=0}^{n} B_k v_{n-k}, \qquad n \geq 0, \end{aligned} \tag{15.6}$$

with $x_0 = 1$ and $v_0 = \varepsilon$.

As in Chapter 14 we consider further the block-function model only. With (14.11) and (14.12) we obtain the following pair of equations

$$\begin{aligned} v_{n+1} &= x_n(1 - \exp - \frac{\gamma}{p} \sum_{k=0}^{p-1} v_{n-k}), \\ x_n &= 1 + \varepsilon - \sum_{k=0}^{q-1} v_{n-k}, \end{aligned} \tag{15.7}$$

with the convention that $v_j = 0$ for a negative index and with $v_0 = \varepsilon$.

If γ/p is small the first relation of (15.7) can be approximated by the first relation of the earlier model (14.13) by using the estimate $1 - \exp - z \approx z$.

In Table 15.1 both models are compared for the case $p = 3$, $q = 6$, $\gamma = 3.2$ and $\varepsilon = 0.001$. Both models show that v_n converges towards its equilibrium value in an oscillating way. Qualitatively there is not much difference but on the seond model the oscillations appear to be more damped than in the first model. The following examples will show that this is a general characteristic.

The equilibrium solution x_∞, v_∞ of (15.7) is determined by

$$\begin{aligned} v_\infty &= x_\infty(1 - \exp - \gamma v_\infty), \\ x_\infty &= 1 + \varepsilon - q v_\infty. \end{aligned} \tag{15.8}$$

By elimination of x_∞ for w defined by

$p = 3,\ q = 6,\ \varepsilon = 0.001,\ \gamma = 3.2$

block-function model (14.13)		block-function model (15.7)
v_n	n	v_n
0.044	15	0.060
0.046	16	0.062
0.061	17	0.073
0.092	18	0.093
0.134	19	0.117
0.176	20	0.137
0.192	21	0.142
0.160	22	0.130
0.104	23	0.110
0.069	24	0.091
0.059	25	0.082
...	...	...
0.086	65	0.103
0.089	66	0.103
0.100	67	0.104
0.117	68	0.105
0.134	69	0.106
0.144	70	0.107
0.139	71	0.107
0.123	72	0.107
0.105	73	0.106
0.094	74	0.105
0.090	75	0.104
...	...	...

Table 15.1

(15.9) $\quad w = \gamma v_\infty$

the following transcendental equation is obtained

(15.10) $\quad w(1 - e^{-w})^{-1} = \gamma(1+\varepsilon) - qw.$

In Fig. 15.1 we have plotted the curves $w(1 - \exp - w)^{-1}$ and the line $\gamma(1+\varepsilon) - qw$. The equilibrium is determined by their intersection. The curve $w(1 - \exp - w)^{-1}$ lies between its tangent $1 + \frac{1}{2} w$ at $w = 0$ and the line $1 + w$. This leads for γv_∞ to the following inequality

(15.11) $\quad \frac{\gamma(1+\varepsilon)-1}{q + 1/2} \leq w = \gamma v_\infty \leq \frac{\gamma(1+\varepsilon)-1}{q}.$

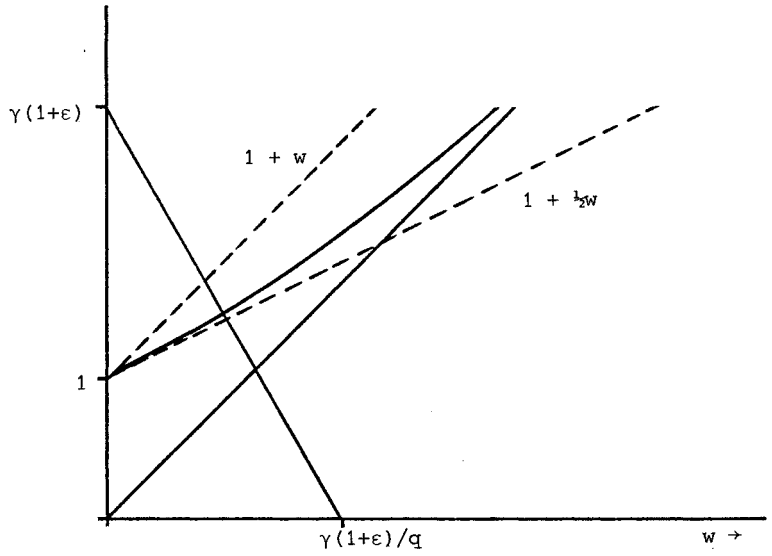


Fig. 15.1

Next we investigate the stability of the equilibrium in the same way as in the previous section.

Local linearization with

$$x_n = x_\infty + \zeta_n, \qquad v_n = v_\infty + \eta_n \tag{15.12}$$

gives

$$\begin{aligned} \eta_{n+1} &= \frac{\sigma}{q}\,\zeta_n + \frac{\rho}{p}\sum_{k=0}^{q-1}\eta_{n-k}, \\ \zeta_n &= -\sum_{k=0}^{q-1}\eta_{n-k}, \end{aligned} \tag{15.13}$$

where

$$\sigma = q(1 - e^{-\gamma v_\infty}), \qquad \rho = \gamma x_\infty e^{-\gamma v_\infty}, \tag{15.14}$$

or expressed as functions of $w = \gamma v_\infty$

$$\sigma = q(1-e^{-w}), \qquad \rho = \frac{we^{-w}}{1-e^{-w}}. \tag{15.15}$$

Substitution of

$$\zeta_n = qC_1 z^n, \qquad \eta_n = C_2 z^n \tag{15.16}$$

gives

$$\begin{aligned} &C_1 + \frac{1}{q}\,\frac{1-z^{-q}}{1-z^{-1}}\,C_2 = 0, \\ &\sigma C_1 + \left(\frac{\rho}{p}\,\frac{1-z^{-p}}{1-z^{-1}} - z\right)C_2 = 0. \end{aligned} \tag{15.17}$$

The eigenvalue equation is

$$\frac{\sigma}{q}\,\frac{1-z^{-q}}{1-z^{-1}} = \frac{\rho}{p}\,\frac{1-z^{-p}}{1-z^{-1}} - z. \tag{15.18}$$

The condition of local stability requires that all roots of (15.18) are within the unit circle.

EXAMPLE 15.1.

$$p = 1, \qquad q = 1.$$

The eigenvalue equation has the single root $z = \rho - \sigma$. From (15.14) it follows that $z > -1$. From (15.11) we obtain the inequality

$$\rho - \sigma = (\gamma(1+\varepsilon) - 1 - w)e^{-w} - 1 \le \frac{1}{3}\,\gamma(1+\varepsilon)\exp - \frac{2}{3}\,\gamma(1+\varepsilon) - 1 < 0.$$

This shows that the equilibrium is always stable.

EXAMPLE 15.2.

$$p = 1, \qquad q = 2.$$

The eigenvalue equation can be written as

$$z^2 + \left(\frac{1}{2}\,\sigma - \rho\right)z + \frac{1}{2}\,\sigma = 0.$$

If the roots z_1, z_2 are complex we have $|z_1| = |z_2| = \sqrt{\sigma/2} < 1$. If the roots are real we have the supplementary stability condition $\rho - \sigma < 1$. Again using the expressions (15.15) and the inequality (15.11) it will be seen that

this condition is always satisfied. Thus the equilibrium is stable for all values of γ.

EXAMPLE 15.3.

$$p = 1, \qquad q = 3.$$

The eigenvalue equation can be written as

$$\frac{1}{3}\sigma(1+z+z^{-1}) = \rho z - z^2.$$

Let us assume that for a critical value of $\gamma(1+\varepsilon)$ the equilibrium is beginning to become unstable. This means that the eigenvalue equation has a root of the form $z = \exp i\theta$. Substitution gives

$$\frac{1}{3}\sigma(1+2\cos\theta) = \rho\cos\theta - \cos 2\theta,$$

$$0 = \rho\sin\theta - \sin 2\theta.$$

The second equation gives the roots $\theta = 0$, $\theta = \pi$, $\theta = \arccos \rho/2$. An elementary calculation shows that the roots $\theta = 0$, $\theta = \pi$ cannot be used. Substitution of $\cos \rho/2$ in the first relation gives

$$\sigma(1+\rho) = 3.$$

With the relation (15.15) this results, rather surprisingly, in the value $w = 1$. Then from (15.10) we obtain the critical value

$$\gamma(1+\varepsilon) = \frac{4e-3}{e-1} = 4.582.$$

The conclusion is instability for $\gamma(1+\varepsilon) > 4.582$.

In the general case the lowest value of $\gamma(1+\varepsilon)$ for which the equilibrium becomes unstable can be determined as in Example 15.3 by substituting $z = \exp i\theta$ in the eigenvalue equation. Splitting of (15.18) into its real and imaginary parts gives the two relations

$$\frac{p\sigma}{q}(1-\cos q\theta) = \rho(1-\cos p\theta) + p(1-\cos\theta),$$
$$\frac{p\sigma}{q}\sin q\theta = \rho\sin p\theta - p\sin\theta. \tag{15.19}$$

They are just a little more general than the corresponding equations (14.20) of the model studied in the previous section.

They can be replaced by the equivalent set

$$\tan\frac{1}{2}q\theta = \frac{\rho(1-\cos\theta)+\rho(1-\cos p\theta)}{\rho\sin p\theta-p\sin\theta},$$
$$\sigma = \frac{q}{p}\frac{\rho\sin p\theta-p\sin\theta}{\sin q\theta}, \tag{15.20}$$

which is better suited for numerical calculations. In the following table a few values of p, q, $\gamma(1+\varepsilon)$ and θ are collected for which is the model becomes unstable.

q \ p	1	2	3	4	5	6	7
1	∞						
2	∞	∞					
3	3.58 1.28	∞	∞				
4	2.02 0.90	6.04 1.11	∞	∞			
5	1.42 0.70	3.29 0.82	8.97 0.97	∞	∞		
6	1.10 0.57	2.28 0.65	4.74 0.75	12.40 0.85	∞	∞	
7	0.91 0.48	1.75 0.54	3.22 0.60	6.37 0.68	16.54 0.76	∞	∞

Table 15.2

Critical values of $(1+\varepsilon)\gamma - 1$ and θ

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