CHAPTER 1

Mathematical Models of Hysteresis

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Contents

INTRODUCTION

This is a survey of past and ongoing research on mathematical modeling of hysteresis phenomena; it aims to illustrate some aspects of the mathematics of hysteresis and of its applications, also in connection with partial differential equations (PDEs). Following Krasnosel'skiĭ [1], first we introduce the concept of *hysteresis operator* and study some examples, then we establish connections with applications, finally we deal with related differential equations.

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We concentrate our attention upon three phenomena of relevant physical and engineering interest: hysteresis in continuum mechanics, in ferromagnetism, and in filtration through porous media. The first two topics have been dealt with in some of the monographs on hysteresis that have appeared in the last years [1–9]. These works regard hysteresis as a unifying mathematical concept for a number of phenomena; a large physical and technical literature however does not refer to this (more recent) mathematical approach.

We review some general features of elasto-plasticity and micromagnetism, and introduce a mathematical model of hysteresis in filtration through porous media. Apparently the latter topic has received little attention so far; its analysis is still open, and we outline a recent approach. The outcome of most of these models consists of PDEs that include hysteresis nonlinearities. In order to illustrate the main analytic techniques that are known for equations of that type, here we study some parabolic and hyperbolic PDEs that contain hysteresis operators.

Here is our plan. In Section 1.1 we define hysteresis as rate-independent memory, introduce the notion of hysteresis operator, illustrate some of its properties, and also define the more general class of hysteresis relations.

In Section 1.2 we briefly deal with the classic Duhem model; because of its simplicity, this seems to be a good example to start with, although it exhibits some drawbacks.

Section 1.3 is devoted to elasto-plasticity. First we define two basic hysteresis operators, the *stop* and the *play*; by means of series and parallel combinations, we then construct the larger class of Prandtl–Ishlinskiĭ models and the corresponding hysteresis operators. This setting is quite peculiar, since a number of results can be derived in the framework of the theory of Convex Analysis, that we review in the Appendix.

In Section 1.4 we deal with discontinuous hysteresis; we introduce relay operators, the Preisach model, and their vector extensions. We also provide a weak formulation, which turns out to be especially convenient in the analysis of related problems at the PDEs, cf. Section 1.10.

In Section 1.5 we outline the classic theory of ferromagnetism known as micromagnetism, and review the *Landau–Lifshitz equation*. This model accounts for a relaxation dynamic and thus is rate-dependent, but nevertheless is at the basis of ferromagnetic hysteresis; this apparent contradiction is explained by the shortness of the time-scale of relaxation. By adding a friction term and letting the relaxation time vanish, we derive a purely rate-independent mesoscopic model of ferromagnetism.

In Section 1.6 we discuss some general properties of hysteresis models with internal variables, which are represented by a population of hysteretic elements; this class includes the Prandtl–Ishlinskiĭ and Preisach models.

In Section 1.7 we illustrate how feedback and nonmonotonicity can generate hysteresis, and discuss how hysteresis may arise in space-distributed systems.

The next three sections introduce the reader to the analysis of PDEs with hysteresis. We begin in Section 1.8 by studying a simple semilinear transport equation with continuous hysteresis. In Section 1.9 we then deal with a quasilinear parabolic equation with continuous hysteresis, and in Section 1.10 with a quasilinear hyperbolic equation of second order with discontinuous hysteresis. The latter is also the weak formulation of a *free boundary problem*. For each of these equations we prove existence of a solution and, for the two former ones, also its uniqueness.

In Section 1.11 we deal with hysteresis in porous-media filtration; we introduce two rate-dependent modifications of a hysteresis model, and discuss a related boundary- and initial-value problem.

In Section 1.12 we draw conclusions, and point out the novelties that have been included in this survey, as well as some open problems; we also indicate some very recent advances in the study of hysteresis.

In these developments, notions and results of convex calculus and variational inequalities are often used; we review some elements of that theory in the Appendix.

These notes are far from having any ambition of completeness, and rather aim to introduce the reader to a research in full development. References to the literature may be found especially in the final parts of Sections 1.3, 1.5 and 1.11 and of the Appendix. An effort has been made to allow for independent reading of different sections, and to make a large part of this work accessible to a nonmathematical audience. Of course, to make an effort does not mean to attain a result... actually, this has not even been tried in Sections 1.8, 1.9 and 1.10 that are devoted to nonlinear PDEs with hysteresis.

1.1 HYSTERESIS OPERATORS

In this section we characterize hysteresis as *rate-independent memory*; we then define hysteresis operators, along the lines of Krasnosel'skiĭ and coworkers (see e.g. [1]), and discuss some of their properties.

1.1.1 HYSTERESIS AND HYSTERESIS LOOPS

Hysteresis occurs in several phenomena in physics, chemistry, biology, engineering and so on. In physics for instance we encounter it in plasticity, friction, ferromagnetism, ferroelectricity, superconductivity, adsorption and desorption, and in the recently studied materials with shape memory. More generally, hysteresis arises in phase transitions. Many other examples are known and wait for mathematical investigation.

Let us consider a simple setting, namely a system whose state is characterized by two scalar variables, u and w, which we assume to depend continuously on time, that we denote by t. Here we assume a purely phenomenologic point of view: we regard the system as a (deterministic) black box, neglect its internal constitution, and assume that the evolution of w is determined by that of u. In the terminology of system theory, u is then referred to as input and w as output. In the final part of this section we shall also consider a more general approach.

Let us outline a classic measurement apparatus, cf. e.g. [10]. By applying an electric current through a conducting solenoid wound around a ring-shaped ferromagnetic material, one can determine a coaxial magnetic field, $\vec{H}$, in the ferromagnet and control its intensity, u. This magnetic field determines a magnetic induction field, $\vec{B}$, in the ferromagnet. By winding a secondary coil around the ring and connecting it to a fluxometer, one can then measure the intensity w of $\vec{B}$. In first approximation, one can assume that the fields $\vec{H}$ and $\vec{B}$ are uniform within the ferromagnet, and regard them just as functions of time.

In Fig. 1.1 we sketch the evolution of the pair (u, w) in a simple test. If u increases from u_1 to u_2, the pair (u, w) moves along a monotone curve ABC; conversely, if u decreases from u_2 to u_1, then (u, w) moves along a different monotone curve CDA. Moreover, if u inverts its motion when $u_1 < u(t) < u_2$, then (u, w) moves into the interior of the hysteresis region, $\mathcal{L}$, namely the part of the (u, w)-plane that is bounded by the major loop $ABCDA$. (The maximum of $-uw$ along the hysteresis loop might be regarded as a crude measure of the thickness of this hysteresis loop.) This qualitative behavior must be quantitatively represented by means of specific models. Here we assume that the pair (u, w) moves along continuous curves, and speak of continuous hysteresis; afterwards we shall also deal with discontinuous hysteresis.

It is convenient to assume that any point of the hysteresis region $\mathcal{L}$ is accessible to the pair (u, w). In typical cases, the system is controlable, that is, by means of a suitable choice of the input function u the system can be driven from any initial point of $\mathcal{L}$ to any final point of $\mathcal{L}$. In any case we assume that the evolution of w is uniquely determined by that of u; this will be made precise by formulating the concept of hysteresis operator.

In this simplified setting we are assuming that at any instant t the state of the system is completely characterized by the pair $(u(t), w(t))$. This is a severe restriction, and actually fails in several examples of major physical

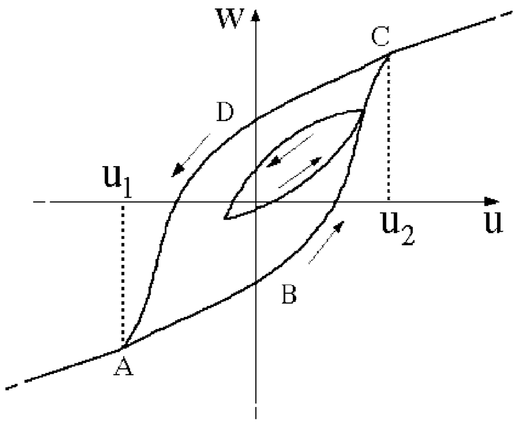


FIGURE 1.1 Continuous hysteresis loop.

interest, as we shall see. Later on we shall encounter models that also account for the evolution of internal variables.

Memory

At any instant t, $w(t)$ depends on the previous evolution of u and on the initial state of the system; we can express this as follows:

$$w(t) = \big[\mathcal{F}(u, w^0)\big](t) \quad \forall t \in [0, T]. \tag{1.1}$$

We assume that

$$(u(0), w^0) \in \mathcal{L}, \quad \big[\mathcal{F}(u, w^0)\big](0) = w^0.$$

Here $\mathcal{F}(\cdot, w^0)$ represents an operator that acts among suitable spaces of time-dependent functions, for any fixed w^0. We also assume that $\mathcal{F}(\cdot, w^0)$ is causal: for any $t \in [0, T]$, the output $w(t)$ is independent of $u_{|[t,T]}$, i.e.,

$$u_1\big|_{[0,t]} = u_2\big|_{[0,t]} \;\Rightarrow\; \big[\mathcal{F}(u_1, w^0)\big](t) = \big[\mathcal{F}(u_2, w^0)\big](t). \tag{1.2}$$

1.1.2 RATE-INDEPENDENCE

We require the path of the pair $(u(t), w(t))$ to be invariant with respect to any increasing diffeomorphism $\varphi : [0, T] \to [0, T]$, i.e.,

$$\mathcal{F}(u \circ \varphi, w^0) = \mathcal{F}(u, w^0) \circ \varphi \quad \text{in } [0, T]; \tag{1.3}$$

in other terms, for any fixed w^0,

$$\mathcal{F}(\cdot, w^0) : u \mapsto w \quad \Rightarrow \quad \mathcal{F}(\cdot, w^0) : u \circ \varphi \mapsto w \circ \varphi.$$

This means that at any instant t, $w(t)$ only depends on $u([0, t])$ and on the order in which values have been attained before t. We name this property *rate-independence*, and regard it as the main characteristic of hysteresis. It is essential for representing hysteresis graphically in the (u, w)-plane, like in Fig. 1.1, without the need of relating the different branches to any specific time-law of the input $u(t)$. In particular, this entails frequency-independence: if the input function u is periodic, the w versus u relation does not depend on the frequency. (However, notice that here a frequency analysis via the classic integral transformations does not seem natural, since hysteresis is a nonlinear phenomenon.)

The above definition of hysteresis includes cases in which the output anticipates the input, instead of lagging behind it; for instance, this occurs in univariate elasto-plasticity, if strain and stress are respectively represented on the abscissa and ordinate axes. However, the orientation of hysteresis loops is not intrinsic, for it depends on the choice of which variable is regarded as input and which one as output, and in many cases either choices are admissible.

Although most typical examples of hysteresis phenomena exhibit hysteresis loops, the occurrence of loops should not be regarded as an essential feature of hysteresis. Indeed, one can easily conceive rate-independent models in which no loop occurs (cf. (1.8) below). In some cases the hysteresis region is unbounded, as we shall see for some relevant examples in Section 1.3. Rate-dependent loops may also occur; for instance, electric losses due to eddy currents cannot be ascribed to hysteresis.

For the hysteresis operator $\mathcal{F}$ it is also natural to assume the following *semigroup property*:

$$\begin{cases} \forall (u, w^0) \in \mathrm{Dom}(\mathcal{F}), \forall [t_1, t_2] \subset]0, T], \\ \text{setting } w(t_1) := \big[\mathcal{F}(u, w^0)\big](t_1), \\ \big[\mathcal{F}(u, w^0)\big](t_2) = \big[\mathcal{F}(u(t_1 + \cdot), w(t_1))\big](t_2 - t_1). \end{cases} \tag{1.4}$$

Nondegenerate hysteresis relations are irreversible, that is, they are not invariant for time-reversal; in fact the pair (u, w) moves along branching and merging paths. Hysteresis is typically associated with dissipation; for instance in ferromagnetism and in elasto-plasticity, in periodic processes, at any cycle an amount of energy proportional to the area of the region encircled by the hysteresis loop is dissipated.

1.1.3 Memory Sequences

For any piecewise monotone input function u, on account of rate-independence, at any instant $\tilde{t} \in]0, T]$ the output of a hysteresis operator is determined by the relative maxima and minima of u in the time interval $[0, \tilde{t}]$, namely, by the (finite) sequence of values attained by u at the instants $\{t_i\}$ ($t_i < \tilde{t}$) at which u had inverted its monotonicity. $\{u(t_i)\}$ is accordingly named a *memory sequence*; it contains all the information that is needed to determine the output of any hysteresis operator, on account of the rate-independence of the latter. Anyway, it is known that (either finite or infinite) sequences of local maxima and minima like that may fail to exist even for continuous functions.

One then defines *reduced memory sequences* as follows. For any $u \in C^0([0, T])$ and any $\tilde{t} \in]0, T]$, let t_1 equal the last instant at which $|u|$ attains its maximum in $[0, \tilde{t}]$, and let u_1 equal the corresponding value of u. For definiteness, let us assume that $u(t_1) > 0$. Let then t_2 equal the last instant at which $|u|$ attains its minimum in $[t_1, \tilde{t}]$, and u_2 equal that minimum. Next let t_3 equal the last instant at which $|u|$ attains its maximum in $[t_2, \tilde{t}]$, and let u_3 equal that maximum; and go on alternating local maxima and local minima of u. In this way two (either finite or infinite) sequences $\{t_j\}$ and $\{u(t_j)\}$ are constructed:

$$
\begin{aligned}
&0 \leqslant t_1 < t_2 < \cdots < t_j < \cdots < \tilde{t} \\
&u(t_2) < \cdots < u(t_{2j}) < \cdots < u(\tilde{t}) \\
&u(\tilde{t}) < \cdots < u(t_{2j+1}) < \cdots < u(t_1),
\end{aligned}
\tag{1.5}
$$

and, if the number of steps is infinite,

$$
t_j \nearrow \tilde{t}, \qquad u(t_{2j}) \nearrow u(\tilde{t}), \qquad u(t_{2j+1}) \searrow u(\tilde{t}) \qquad \text{as } j \to \infty. \tag{1.6}
$$

For some hysteresis operators (but not for all), at any instant the output is determined by the reduced memory sequence of the input.

1.1.4 Hysteresis And Monotonicity

The monotonicity of the output versus input relation plays an important role in the analysis of many nonlinear operators. There are several forms of monotonicity, and for memory operators the range of possibilities is even larger.

The standard L^2-monotonicity property

$$
\int_0^T [\mathcal{F}(u_1) - \mathcal{F}(u_2)](u_1 - u_2)\,\mathrm{d}t \geqslant 0 \quad \forall u_1, u_2 \in \mathrm{Dom}(\mathcal{F}) \tag{1.7}
$$

is too strong a requirement for hysteresis operators. The following simple counterexample should convince the reader that a rate-independent operator is monotone with respect to the usual scalar product of $L^2(0,T)$ only if it is reduced to a superposition operator, namely, only if it is the form $\mathcal{F}(u) = \varphi \circ u$ for some function $\varphi : \mathbf{R} \to \mathbf{R}$. In other terms, no genuine hysteresis operator is L^2-monotone. (An example of the devastating power of rate-independence...)

Let

$$\mathcal{F} : W^{1,1}(0,T) \times \mathbf{R} \to W^{1,1}(0,T) : \big(u, w^0\big) \mapsto w$$

be defined by means of the following Cauchy problem

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^+ \quad \text{in }]0,T[, \qquad w(0) = w^0 \tag{1.8}$$

(here $x^+ := (x + |x|)/2$ for any $x \in \mathbf{R}$). Causality and rate-independence are straightforward, thus $\mathcal{F}$ is a hysteresis operator; actually, this is an especially simple example of the Duhem operator, cf. Section 1.2. Let us fix any $T > 3\pi/2$ and set

$$u_1(t) := \sin t \quad \forall t \in [0, 3\pi/2], \qquad u_1(t) := -1 \quad \forall t \in [3\pi/2, T],$$
$$u_2(t) := 0 \qquad \forall t \in [0,T];$$

cf. Fig. 1.2. Setting $w_i := \mathcal{F}(u_i, 0)$ for $i = 1, 2$, we have

$$[w_1(t) - w_2(t)][u_1(t) - u_2(t)] = -1 \qquad \forall t \geqslant 3\pi/2.$$

For T large enough the inequality (1.1.7) then fails.

This construction can easily be extended to virtually any nondegenerate hysteresis operator. It suffices to select

(i) two points (u', w'), (u'', w'') such that $(w'' - w')(u'' - u') < 0$, and
(ii) an input function u_1 and a $\tilde{t} > 0$ such that

$$u_1(0) = u', \quad u_1(t) = u'' \quad \forall t \geqslant \tilde{t}, \quad \big[\mathcal{F}(u_1, w')\big](\tilde{t}) = w''.$$

(That is, the input u_1 takes the system from (u', w') at $t = 0$ to (u'', w'') at $t = \tilde{t}$.)

Finally, let us set $u_2(t) \equiv u'$ in $\mathbf{R}^+$, so that $\mathcal{F}(u_2, w') \equiv w'$. This yields

$$\int_{\tilde{t}}^{T} \big(\big[\mathcal{F}(u_1, w')\big] - \big[\mathcal{F}(u_2, w')\big]\big)(u_1 - u_2)\,\mathrm{d}t$$
$$= (T - \tilde{t})(w'' - w')(u'' - u') < 0 \qquad \forall T > \tilde{t};$$

if T is large enough, we then get

$$\int_0^T \big(\big[\mathcal{F}(u_1, w')\big] - \big[\mathcal{F}(u_2, w')\big]\big)(u_1 - u_2)\,\mathrm{d}t < 0.$$

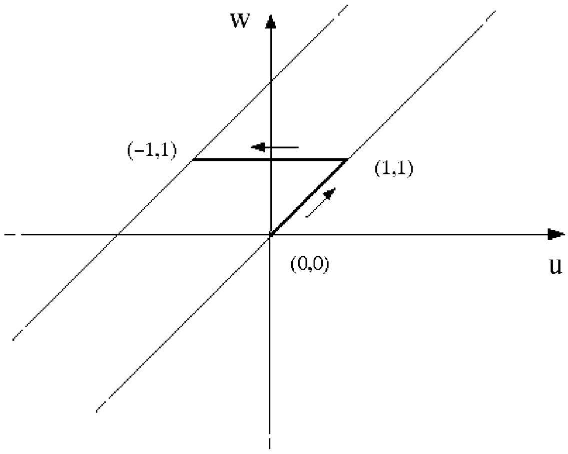


FIGURE 1.2 Counterexample to L^2-monotonicity: the pair (u, w) moves from $(0, 0)$ to $(-1, 1)$ via $(1, 1)$ in the time interval $[0, 3\pi/2]$, and stays there for $t > 3\pi/2$. If T is large enough, then $\int_0^T (w_1 - w_2)(u_1 - u_2)\,\mathrm{d}t < 0$.

1.1.5 Other Monotonicity Properties

Several hysteresis operators are *order-preserving*:

$$\begin{aligned}&\forall (u_1, w_1^0), (u_2, w_2^0) \in \mathrm{Dom}(\mathcal{F}), \forall t \in]0, T],\\ &u_1 \leqslant u_2 \text{ in } [0, t], w_1^0 \leqslant w_2^0 \quad \Rightarrow \quad \big[\mathcal{F}(u_1, w_1^0)\big](t) \leqslant \big[\mathcal{F}(u_2, w_2^0)\big](t).\end{aligned} \tag{1.9}$$

The following property of *piecewise monotonicity* preservation (more briefly, piecewise monotonicity) seems especially appropriate for hysteresis operators:

$$\begin{aligned}&\forall (u, w^0) \in \mathrm{Dom}(\mathcal{F}), \forall [t_1, t_2] \subset [0, T],\\ &\text{if } u \text{ is either nondecreasing or nonincreasing in } [t_1, t_2],\\ &\text{then the same holds for } \mathcal{F}(u, w^0);\end{aligned} \tag{1.10}$$

i.e.,

$$u, w := \mathcal{F}(u, w^0) \in W^{1,1}(0, T) \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}w}{\mathrm{d}t} \geqslant 0 \quad \text{a.e. in }]0, T[. \tag{1.11}$$

This simply means that hysteresis branches are nondecreasing.

Let us say that a hysteresis operator $\mathcal{F}$ is piecewise antimonotone whenever $-\mathcal{F}$ is piecewise monotone. Piecewise monotonicity (or antimonotonicity) is a necessary but nonsufficient condition for invertibility of a hysteresis operator; that is, any hysteresis operator is invertible only if it is either piecewise monotone or piecewise antimonotone, but neither property suffices to guarantee the invertibility, even if monotonicity (or antimonotonicity) is strict.

It is easy to see that, for any hysteresis operator, order preservation entails piecewise monotonicity. The converse fails: Duhem operators (cf. Section 1.2) exhibit counterexamples. One might also wonder whether piecewise monotonicity entails rate-independence. The following counterexample, due to P. Krejčí, answers this question in the negative:

$$[\mathcal{F}(u)](t) = tu(t) - \int_0^t u(\tau)\mathrm{d}\tau \quad \forall u \in C^0([0,T]). \tag{1.12}$$

For any $u \in W^{1,1}(0,T)$,

$$w := \mathcal{F}(u) \in W^{1,1}(0,T), \quad w(t) = \int_0^t \tau \frac{\mathrm{d}u}{\mathrm{d}\tau}(\tau)\,\mathrm{d}\tau \quad \text{in } [0,T];$$

thus (1.1.11) holds. However $\mathcal{F}$ is rate-dependent (thus it is not a hysteresis operator).

1.1.6 Rate-dependent Corrections

The definition of rate-independence excludes any viscous-type memory, such as that represented by time-convolution. However, even in typical hysteresis phenomena, like ferromagnetism, ferroelectricity, and plasticity, memory effects are not purely rate-independent, for hysteresis is coupled with (what we might loosely define as) viscosity-type effects. One might tentatively represent the output as the sum of hysteretic and viscous components:

$$w = \mathcal{F}_{\mathrm{tot}}(u) := \mathcal{F}_{\mathrm{hys}}(u) + \mathcal{F}_{\mathrm{vis}}(u). \tag{1.13}$$

This is a rather crude extension. Other expressions might be considered, too: for instance, one might consider a relaxation dynamics of the form

$$w + \varepsilon_1 \frac{\mathrm{d}w}{\mathrm{d}t} = \mathcal{F}(u) + \varepsilon_2 \frac{\mathrm{d}u}{\mathrm{d}t}\,, \tag{1.14}$$

$\varepsilon_1, \varepsilon_2$ being relaxation constants. In Section 1.11 we shall discuss examples of this rate-dependent correction in connection with hysteresis in porous media filtration.

Usually rate-dependent effects get larger as the rate increases, and vanish as the rate vanishes. Therefore hysteresis is more evident at slow regimes, whereas it may be dominated by viscosity at fast regimes. In applications hysteresis appears only if the time-scale is sufficiently slow, and we are induced to regard as hysteretic those phenomena for which the rate-independent component prevails at typical regimes.

Several continuous hysteresis operators are constructed as follows: first the operator is defined for piecewise monotone (equivalently, piecewise linear) input functions. A uniform continuity property is then established; this allows one to extend the operator by continuity to a complete function space, usually either $C^0([0,T])$ or $W^{1,1}(0,T)$. This extension preserves several properties: causality, rate-independence, piecewise monotonicity, and so on.

1.1.7 Hysteresis Relations

So far we have assumed that the evolution of the variable w is determined by that of u, and accordingly regarded u as input and w as output. However this is not always the case, and we outline a more general point of view, in which the functions u and w are set on equal ground.

We deal with functions u and w that are elements of some Banach space, B, of functions of time (e.g., $B = C^0([0,T])$ or $B = W^{1,1}(0,T)$). We assume that we are given a functional $\Phi : B^2 \times [0,T] \to \mathbf{R}$, and constrain a pair $(u,w) \in B^2$ by setting

$$\Phi(u,w,t) = 0 \quad \forall t \in [0,T]. \tag{1.15}$$

In general $\Phi(u,w,t)$ depends not only on $u(t)$ and $w(t)$ at the instant t, but also on the evolution of both variables; namely, $\Phi(u,w,\cdot)$ is a memory functional. We say that (1.15) is a hysteresis relation if Φ is causal and rate-independent. Here by causality we mean that

$$\begin{aligned}&\forall t \in [0,T],\ u_1\big|_{[0,t]} = u_2\big|_{[0,t]}, w_1\big|_{]t,T]} = w_2\big|_{]t,T]}\\ &\quad\Rightarrow\quad \Phi(u_1,w_1,t) = \Phi(u_2,w_2,t);\end{aligned} \tag{1.16}$$

thus $\Phi(u,w,t)$ does not depend on $u\big|_{]t,T]}$ and $w\big|_{]t,T]}$. By rate-independence we mean that, for any increasing smooth diffeomorphism $\varphi : [0,T] \to [0,T]$,

$$\Phi(u \circ \varphi, w \circ \varphi, t) = \Phi(u,w,\varphi(t)) \quad \forall t \in [0,T]; \tag{1.17}$$

this generalizes (1.3). Any hysteresis operator $\mathcal{F} : B \times \mathbf{R} \to B$ trivially determines the hysteresis relation

$$\Phi(u,w,t) := [\mathcal{F}(u,w(0))](t) - w(t) \quad \forall t \in [0,T], \forall (u,w) \in B^2. \tag{1.18}$$

If we allow the operator Φ to be multivalued, then we get a wide class of hysteresis models.

Hysteresis can also be represented without using the notion of hysteresis operator (or relation). In Sections 1.4 and 1.5, dealing with discontinuous hysteresis and with micromagnetism, we shall see two alternative formulations.

1.1.8 HISTORICAL NOTE

Rate-dependent memory has been known to mathematicians for a long time: Volterra's pioneering studies date back to the beginning of the last century. On the other hand the history of hysteresis (i.e., rate-independent memory) is quite short: mathematical developments have been lagging behind those of physicists and engineers. Of course, mathematics has been used in works of applied scientists on hysteresis, but there it occurred more as calculus than in the form of functional analysis.

Apparently, it was only in 1966 that hysteresis was first given a functional approach. This was due to an engineering student: in [11,12] R. Bouc modeled several hysteresis phenomena, regarding hysteresis as a map between function spaces. The Bouc hysteresis operator is briefly reviewed in Section 1.2.

In 1970 M.A. Krasnosel'skiĭ and co-workers [13] first formulated (what now is called) the Prandtl–Ishlinskiĭ model in terms of hysteresis operators. Then Krasnosel'skiĭ, Pokrovskiĭ and others conducted a systematic analysis of the mathematical properties of these operators. In the period 1970–80, they published a number of papers, several of which also appeared in English translation. This formed the basis for their 1983 monograph [1], which was translated into English in 1989. In the 1980s some western applied mathematicians and mathematical physicists also began to study hysteresis models issued from applications, especially PDEs with hysteresis. In recent years, research on models of hysteresis phenomena has been progressing, see e.g. the monographs [1–9].

The notion of hysteresis operator (or relation) raises several questions: the adequacy of these operators to represent specific applicative phenomena, the analysis of their properties (continuity in various functional spaces, construction of the closure of discontinuous hysteresis operators, and so on), their characterization, the study of their operator structure, the identification of parameters, and so on. Special attention needs to be paid to both ordinary equations and PDEs that include these operators. Something has been done, but much more is left to do.

1.2 THE DUHEM MODEL

In this section we outline a hysteresis model, which seems to be due to the nineteenth-century physicist P. Duhem [15]; this model has an especially simple analytic representation, but also exhibits some drawbacks which reduce its applicability. We also mention a related model of hysteresis due to R. Bouc [11].

For any differentiable input function $u(t)$ and any initial value w^0 of w, we define the output function $w(t)$ to be the solution of the following initial value problem:

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} = g_1(u,w)\Big(\dfrac{\mathrm{d}u}{\mathrm{d}t}\Big)^+ - g_2(u,w)\Big(\dfrac{\mathrm{d}u}{\mathrm{d}t}\Big)^- \quad \text{in }]0,T[\\ w(0) = w^0, \end{cases} \tag{1.19}$$

where we set $x^+ := (|x|+x)/2$ and $x^- := (|x|-x)/2$ for any $x \in \mathbf{R}$. Here g_1 and g_2 are prescribed (nonnegative) functions, and are assumed to be so regular that the Cauchy problem (1.19) has one and only one solution; cf. Theorem 1.2.1 below.

As we saw in Section 1.1, irreversibility is one of the main features of hysteresis; we thus assume that (1.19) only holds for increasing time, and impose the constraint $\mathrm{d}t > 0$ throughout this section. Multiplying both members of $(1.19)_1$ by $\mathrm{d}t$, the positive and negative parts are then preserved, and we formally get the equivalent equation

$$\frac{\mathrm{d}w}{\mathrm{d}u} = \begin{cases} g_1(u,w) & \text{if } \mathrm{d}u > 0 \\ g_2(u,w) & \text{if } \mathrm{d}u < 0 \end{cases} \quad \text{in }]0,T[. \tag{1.20}$$

In this way we have eliminated time, and rate-dependence has become evident. Under appropriate regularity conditions, two systems of curves are obtained in the (u,w)-plane by integrating the fields g_1 and g_2; these curves respectively represent the paths of evolution of the pair (u,w) for increasing and decreasing u. These curves may span either the whole plane $\mathbf{R}^2$ or just a subset.

By setting

$$\begin{cases} \operatorname{sign}_0(x) := -1 & \text{if } x < 0, \\ \operatorname{sign}_0(0) := 0, & \\ \operatorname{sign}_0(x) := 1 & \text{if } x > 0, \end{cases} .$$

$$\begin{cases} g(u,w,-1) := g_2(u,w), & \\ g(u,w,0) := 0, & \forall (u,w) \in \mathbf{R}^2, \\ g(u,w,1) := g_1(u,w), & \end{cases}$$

we can also write (1.20) in the form

$$\frac{\mathrm{d}w}{\mathrm{d}u} = g\Big(u, w, \mathrm{sign}_0\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)\Big) \qquad \text{in }]0, T[\ \ (\mathrm{d}t > 0), \tag{1.21}$$

or equivalently,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = h\Big(u, w, \Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)\Big) \qquad \text{in }]0, T[\ \ (\mathrm{d}t > 0); \tag{1.22}$$

here the function h is positively homogeneous of degree one with respect to its third argument. These equivalent representations of the equation (1.19) illustrate a connection between the sign function, positive homogeneity of degree one, and rate-independence: these are basic ingredients of the analytic representation of hysteresis, and in the sequel we shall encounter other examples of this interplay.

In this model at any instant t the state is characterized by the pair $(u(t), w(t))$ without any internal variable. This is not always satisfactory for applications, as is shown by the following example. Let us move from the origin of the (u, w)-plane and, by a suitable choice of the input function u, come back to the origin along a closed curve. According to (1.19), the system should *forget* this process; the same should occur for any other closed loop in the (u, w)-plane. But, for several ferromagnetic materials experimental tests are at variance with this prediction. This suggests that for those materials the state is not completely characterized by the pair (u, w), and that the evolution of one or more interior parameters should also be described by the model. In the next sections we shall see some examples of hysteresis models that include internal variables. Nevertheless, some physicists and engineers use relations of the form (1.19) to represent ferromagnetic hysteresis, maybe because of its simplicity; one can then expect that the drawback we just pointed out might be quantitatively negligible for some materials and for certain processes.

If $g_1, g_2 \geqslant 0$ then the Duhem operator $\mathcal{D}$ is piecewise monotone, cf. (1.10); but simple examples show that it needs not be order-preserving, that is,

$$u_1 \leqslant u_2 \quad \text{in } [0, T] \quad \not\Rightarrow \quad \mathcal{D}(u_1, w^0) \leqslant \mathcal{D}(u_2, w^0) \quad \text{in } [0, T]. \tag{1.23}$$

Moreover, in general the output of the Duhem operator is not determined by the reduced memory sequence of the input, defined in Section 1.1. (We leave the search for simple counterexamples to the reader.)

1.2.1 CONFINEMENT

Dealing with (1.19) we allowed the pair (u, w) to vary in the whole $\mathbf{R}^2$. A more interesting model is obtained if (u, w) is confined to a region $\mathcal{L} \subset \mathbf{R}^2$

that is comprised between the graphs of two monotone continuous functions. This can be represented as follows. Let us assume that we are given two functions

$$\gamma_\ell, \gamma_r : \mathbf{R} \to [-\infty, +\infty] \text{ continuous and nondecreasing}, \quad \gamma_r \leqslant \gamma_\ell, \tag{1.24}$$

and set

$$J(u) := [\gamma_r(u), \gamma_\ell(u)], \qquad \varphi(u,w) := I_{J(u)}(w) \qquad \forall (u,w) \in \mathbf{R}^2$$

(see the Appendix for the definition of the indicator function $I_{J(u)}$). We then couple the rate-independent differential inclusion

$$\frac{\mathrm{d}w}{\mathrm{d}t} \in \varphi(u,w) + g_1(u,w)\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^+ - g_2(u,w)\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^- \qquad \text{in }]0,T[\ \ (\mathrm{d}t > 0) \tag{1.25}$$

with the initial condition $(1.19)_2$. By Proposition A.9, the inclusion (1.25) is equivalent to the following variational inequality (still with $\mathrm{d}t > 0$):

$$\begin{cases} w \in J(u) \ \text{ in }]0,T[; \quad \forall v \in J(u) \ \text{ in }]0,T[, \\ \Big[\dfrac{\mathrm{d}w}{\mathrm{d}t} - g_1(u,w)\Big(\dfrac{\mathrm{d}u}{\mathrm{d}t}\Big)^+ + g_2(u,w)\Big(\dfrac{\mathrm{d}u}{\mathrm{d}t}\Big)^-\Big](w-v) \leqslant 0 \qquad \text{in }]0,T[. \end{cases} \tag{1.26}$$

The differential equation (1.22) can easily be extended to vectors. For any $N > 1$, it suffices to replace the scalar function $h : \mathbf{R}^3 \to \mathbf{R}$ by a vector function $\vec{h} : (\mathbf{R}^N)^3 \to \mathbf{R}^N$ that is positively homogeneous of degree one with respect to its third argument.

1.2.2 Some Properties of the Duhem Operator

Here we just state a result, and refer to [8; Chapter V] for details.

THEOREM 1.2.1. *Assume that g_1, g_2 are continuous, and that*

$$|g_i(u,w_1) - g_i(u,w_2)| \leqslant L(u)|w_1 - w_2| \qquad \forall u, w_1, w_2 \in \mathbf{R} \ (i = 1,2), \tag{1.27}$$

with $L : \mathbf{R} \to \mathbf{R}^+$ continuous. Then, for any $u \in W^{1,1}(0,T)$ and any $w^0 \in \mathbf{R}$, there exists a unique $w := \mathcal{D}(u, w^0) \in W^{1,1}(0,T)$ that fulfils (1.19).

The Duhem operator $\mathcal{D}(\cdot, w^0)$ is strongly continuous in $W^{1,p}(0,T)$ for any $p \in [1, +\infty]$, and is strongly and weakly continuous in $C^1([0,T])$.

Even under smoothness assumptions on the prescribed functions g_1 and g_2, in general the Duhem operator is not continuous with respect to the weak topology of $W^{1,p}(0,T)$ for any $p \geqslant 1$. Actually, even small-amplitude oscillations of the input field u may trigger large drifts of the output w. For instance, for any $T > 0$, let

$$g_1 \equiv 1 \quad \text{and} \quad g_2 \equiv 0 \quad \text{in } \mathbf{R}^2, \qquad u_n(t) = \frac{\sin(nt)}{n} \quad \forall t \in [0,T];$$

then

$$u_n \to 0 \qquad \text{weakly star in } W^{1,\infty}(0,T) \text{ and uniformly in } [0,T].$$

For any $w^0 \in \mathbf{R}$, the limit input $u(t) \equiv 0$ is transformed by $\mathcal{D}$ into the constant output $w(t) \equiv w^0$, but, setting $w_n = \mathcal{D}(u_n, w^0)$,

$$w_n(t) = w^0 + \int_0^t (\cos n\tau)^+ \mathrm{d}\tau \not\rightharpoonup w^0 \qquad \text{weakly star in } W^{1,\infty}(0,T).$$

This simple example also shows that in general the Duhem operator $\mathcal{D}(\cdot, w^0)$ cannot be extended by continuity to $C^0([0,T])$. For this operator, counterexamples of this sort are not exceptional; actually they can easily be constructed whenever in the interior of the hysteresis region (here the whole $\mathbf{R}^2$) g_1 and g_2 are two nonidentical continuous functions.

It may even occur that, for a smooth input u with unbounded variation, the output w diverges in zero time. For instance, if

$$g_1 \equiv 1 \quad \text{and} \quad g_2 \equiv 0 \quad \text{in } \mathbf{R}^2, \qquad u(t) = t\sin(1/t) \quad \forall t \in [0,T],$$

then

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^+ = \left[\sin\left(\frac{1}{t}\right) - \frac{1}{t}\cos\left(\frac{1}{t}\right)\right]^+ \qquad \forall t > 0 \;\; (\mathrm{d}t > 0),$$

which is not integrable in any right neighborhood of $t = 0$.

In general the Duhem operator has no continuous extension either to $C^0([0,T])$ or to $BV(0,T)$. However, it can be extended to a unique continuous operator acting in $C^0([0,T]) \cap BV(0,T)$ equipped with the metric

$$\mathrm{d}(u,v) := \max_{[0,T]} |u - v| + \left| \int_{[0,T]} |\mathrm{d}u| - \int_{[0,T]} |\mathrm{d}v| \right| \qquad \forall u, v \in C^0([0,T]) \cap BV(0,T).$$

We refer to [8; Chapter V] for the latter statements, too.

In conclusion, despite of its simplicity and of the appeal of its analytic representation, the functional properties of the Duhem model are not very satisfactory; in particular, it is rather disturbing that in general the associated hysteresis operator cannot be extended by continuity to $C^0([0,T])$. This restricts its application, and excludes several models governed by differential equations. Nevertheless the Duhem model has some elements of interest; for instance, it is one of the few models of hysteresis of which a closed analytic representation is known.

1.2.3 THE BOUC MODEL

R. Bouc was one of the very first pioneers of mathematical research in hysteresis. Already in the 1960s, as a PhD student at the Ecole Polytechnique of Paris, he proposed and studied the following operator:

$$[\mathcal{B}(u)](t) := au(t) + \int_0^t f\Big(\int_s^t |u'(\tau)|\,\mathrm{d}\tau\Big)\varphi(u(s))u'(s)\,\mathrm{d}s \qquad \forall t \in [0,T]. \tag{1.28}$$

Bouc applied this model to several problems of engineering interest, and studied its analytic properties also in connection with ordinary differential equations; cf. e.g. [11,12]. Here u is an absolutely continuous input, a is a positive constant, f and φ are prescribed nonnegative continuous functions, with f nonincreasing. It is easy to see that $\mathcal{B}$ is causal and rate-independent, namely, a hysteresis operator. $\mathcal{B}$ is also piecewise monotone in the sense of the definition (1.10), but it is not order-preserving in general, cf. (1.9). If $f \in C^1([0,+\infty[)$, it is not difficult to see that $\mathcal{B}$ is continuous in $W^{1,1}(0,T)$.

In the particular case of

$$f(v) := A\mathrm{e}^{-\alpha v} \qquad \forall v \geqslant 0 \ (A, \alpha: \text{ constants} > 0), \tag{1.29}$$

direct computation shows that $\mathcal{B}$ is equivalent to a Duhem operator [11]. Other choices of the function f can represent effects of accommodation, namely, changes of the hysteresis loop by repeated cycling and convergence to a limit loop.

This model has not (yet) been studied by mathematicians, despite of its potentialities.

1.3 RHEOLOGICAL MODELS OF ELASTO-PLASTICITY

In this section we introduce plastic hysteresis. This is one of the most typical examples of hysteresis phenomena, was the first to be studied, and suggests the most simple examples of hysteresis operators; moreover the underlying convexity entails a number of interesting properties. This then looks as a good starting point to illustrate the interplay between physical and mathematical aspects of hysteresis.

1.3.1 Constitutive Relations

Several phenomena of engineering interest can be modeled via the following procedure.

(i) The state of the system is first characterized by means of a (usually finite) set of *state variables*. For instance, strain and stress in continuum mechanics, electric tension and electric current in electricity, electric field and electric displacement in electrostatics, magnetic field and magnetic induction in magnetostatics, the four latter fields in electromagnetism, and so on.
(ii) Some fundamental laws, the validity of which is not restricted to specific materials, are then formulated. Examples include the momentum balance and the laws of mass and energy conservation in continuum mechanics, the Kirschhoff laws of electricity, the laws of electrostatics, the laws of magnetostatics, the system of Maxwell equations in electromagnetism, the energy and entropy balances in thermodynamics, and so on.
(iii) The specific behavior of different materials is represented via *constitutive relations*, which may also exhibit memory effects, and in some cases hysteresis.

Here we concentrate our attention upon the third issue. Constitutive laws are sometimes formulated via the construction of *ideal bodies*, known as rheological models in continuum mechanics, as electric circuital models in electricity, as magnetic circuital models in magnetism, and so on. These are:

(i) phenomenologic models, and are not intended to represent the fine-scale structure of materials;
(ii) *lumped-parameter* models, that is, they do not account for space dependence; nevertheless, they can also be used to study space-distributed systems;
(iii) scalar models, but can be extrapolated to tensors, as we shall see.

The construction proceeds as follows. One first introduces a small class of *elementary models* that are meant to represent basic properties: e.g., elasticity, viscosity, plasticity in continuum mechanics; resistance, inductance, capacitance in electricity, and so on. *Composite ideal bodies* are then built by assembling these elements according to a restricted set of rules, which can be represented via series (or cascade) and parallel arrangements. For each material, one then tries to select an ideal body capable of representing its constitutive behavior, and to identify the parameters occurring in the corresponding constitutive equation.

Each elementary model is described by an equation, which contains one or more parameters. The constitutive behavior of the composed model is determined by that of its elements, and the corresponding equation is derived from those of its constituents by means of simple composition rules that now we outline.

Composition Rules of Rheological Models

Here we deal with continuum mechanical models, regard the strain and stress tensors as state variables, and denote them by ε and σ, respectively. Rheological models typically relate an applied force with the corresponding deformation and/or with the time-derivative of the latter. They are representative of strain and stress on the basis of the following identifications:

$$\text{force} \leftrightarrow \text{stress } (= \sigma); \qquad \text{deformation} \leftrightarrow \text{strain } (= \varepsilon).$$

Here we deal with univariate models; however the relations we derive will be extrapolated to tensors. Serial and parallel combinations have an intuitive meaning, and are also prone to graphic representation. Let us consider a composite rheological model that consists of two or more (either elementary or composite) submodels, which are arranged either in series or in parallel, cf. Fig. 1.3. Let us denote the stress (strain, respectively) of its components by σ_i (ε_i, respectively), for $i = 1, 2, \ldots$, and the stress (strain, respectively) of the composite system by $\tilde{\sigma}$ ($\tilde{\varepsilon}$, respectively). Let us see the main properties of these operations.

(i) *Combinations in Series.* The constitutive elements are subjected to the same force, and this also coincides with the force acting on the composite model. The deformation of the latter is instead equal to the sum of the deformations of its elements, cf. Fig. 1.3(a):

$$\tilde{\sigma} = \sigma_1 = \sigma_2 = \cdots, \qquad \tilde{\varepsilon} = \varepsilon_1 + \varepsilon_2 + \cdots. \tag{1.30}$$

(ii) *Combinations in Parallel.* The properties of force and deformation are here exchanged, cf. Fig. 1.3(b):

$$\tilde{\sigma} = \sigma_1 + \sigma_2 + \cdots, \qquad \tilde{\varepsilon} = \varepsilon_1 = \varepsilon_2 = \cdots. \tag{1.31}$$

The two latter characterizations will be used to extend the operations of combination in series and in parallel to tensors.

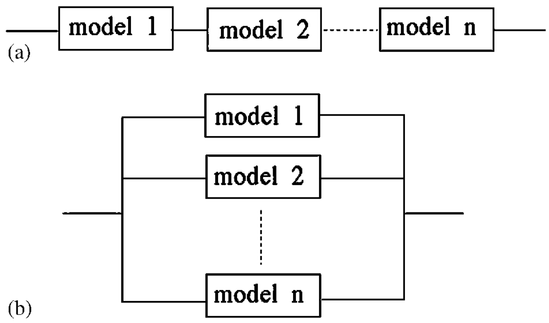


FIGURE 1.3 Arrangements in series (**a**) and in parallel (**b**). They correspond to (1.30) and (1.31), respectively.

The dual character of these arrangements is evident. These rules are extended in a natural way to combinations in parallel and in series of infinitely many elements; in that case sums are replaced either by series, or more generally by integrals with respect to a prescribed density measure.

There exists a third basic composition rule: the feedback arrangement. This is especially interesting, for combined with nonlinearity it can generate hysteresis [8; Sections I.4, II.4]. However, feedback is rather different from series and parallel arrangements, and does not enter the present discussion; we then postpone it to Section 1.7.

As we have mentioned above, these models can be applied to several phenomena. In electricity, for instance, series and parallel arrangements correspond to the rules (1.30) and (1.31), with ε replaced by the tension, V, and σ by the current, J. In electrostatics, ε is replaced by the electric field, E, and σ by the electric displacement, D. In magnetostatics, ε is replaced by the magnetic field, H, and σ by the magnetic induction, B. And so on.

Elementary Rheological Behaviors

First we introduce some elementary rheological laws for multivariate systems. We shall then introduce corresponding univariate rheological models, and use them to build more general ones. Finally we shall extrapolate the latter to multivariate systems. Schematically we shall proceed as follows:

introduction of (tensor) laws $\rightarrow$
definition of (scalar) rheological elements $\rightarrow$
series and parallel combinations of rheological elements $\rightarrow$
extension to tensors.

We introduce the assumption of small (infinitesimal) deformations, so that we can identify Euler and Lagrange coordinates, Piola and Cauchy stress tensors, σ, and use the linearized strain tensor, ε. In the formulation of the basic constitutive laws of continuum mechanics, we distinguish between the spheric components of the strain and stress tensors,

$$\varepsilon_{(s)} := \frac{1}{3}\sum_{i=1}^{3} \varepsilon_{ii} I, \qquad \sigma_{(s)} := \frac{1}{3}\sum_{i=1}^{3} \sigma_{ii} I$$

(here by I we denote the identity 3×3-tensor), and their deviatoric components,

$$\varepsilon_{(d)} := \varepsilon - \varepsilon_{(s)}, \qquad \sigma_{(d)} := \sigma - \sigma_{(s)}. \tag{1.32}$$

The spaces of spheric and deviatoric tensors are orthogonal. The tensor $\varepsilon_{(d)}$ is symmetric, since this holds for ε (by definition) as well as for $\varepsilon_{(s)}$. It is a well-known consequence of the principle of conservation of angular momentum that the tensor $\sigma_{(d)}$ is also symmetric. Thus $\varepsilon_{(d)}$ and $\sigma_{(d)}$ are both elements of the linear space of symmetric 3×3 deviatoric tensors, that we denote by D_s.

A linear elastic relation is classically assumed between the spheric components of σ and ε:

$$\sigma_{(s)} = a\varepsilon_{(s)} \quad (a: \text{ constant } > 0). \tag{1.33}$$

On the other hand, several models have been proposed to relate the deviatoric components; the main ones involve $\sigma_{(d)}$, $\varepsilon_{(d)}$ and $\mathrm{d}\varepsilon_{(d)}/\mathrm{d}t$. Here we confine ourselves to the following two classes, that are rate-independent; the same then holds for their series and parallel combinations.

(i) *Linear Elasticity.* This is represented by the following equality, which also accounts for linearity of the spheric components,

$$\sigma = A\varepsilon \qquad \text{i.e.} \qquad \sigma_{ij} = \sum_{\ell,m=1}^{3} A_{ij\ell m}\varepsilon_{\ell m} \quad \text{for } i,j = 1,2,3. \tag{1.34}$$

Here A is a positive-definite matrix which fulfils the symmetry conditions

$$A_{ij\ell m} = A_{ijm\ell} = A_{\ell m ij} \qquad \text{for } i,j,\ell,m = 1,2,3.$$

For isotropic materials, denoting the Kronecker tensor by δ_{ij}, one can show that there exist two positive constants, λ and μ, named *Lamé moduli*, such that

$$A_{ij\ell m} = \lambda\delta_{\ell m}\delta_{ij} + 2\mu\delta_{i\ell}\delta_{jm} \qquad \text{for } i,j,\ell,m = 1,2,3.$$

In this case (1.34) reads

$$\sigma_{ij} = \lambda \sum_{\ell=1}^{3} \varepsilon_{\ell\ell}\delta_{ij} + 2\mu\varepsilon_{ij} \qquad \text{for } i, j = 1, 2, 3. \tag{1.35}$$

(ii) *Rigid Perfect Plasticity.* This can be represented by the following inclusion:

$$\sigma_{(d)} \in \left(\partial I_K\right)^{-1} \left(\frac{\mathrm{d}\varepsilon_{(d)}}{\mathrm{d}t}\right), \tag{1.36}$$

(see the Appendix for this notation) or equivalently,

$$\frac{\mathrm{d}\varepsilon_{(d)}}{\mathrm{d}t} \in \partial I_K(\sigma_{(d)}); \tag{1.37}$$

here K is a closed convex subset of D_s and $K \ni 0$. This set is known as the *yield criterion*, and its selection has been at the focus of studies on plasticity since the pioneering works of Tresca, Saint Vénant, Lévy, von Mises.

The relation (1.36) may be regarded as a limit case of nonlinear viscosity, that is characterized by an inclusion of the form

$$\sigma_{(d)} \in g\left(\frac{\mathrm{d}\varepsilon_{(d)}}{\mathrm{d}t}\right), \tag{1.38}$$

for a maximal monotone function $g : D_s \to 2^{D_s}$ (the set of the parts of D_s). Some of the following developments can be extended to this more general setting; however there is a fundamental difference between viscosity and plasticity: the latter is rate-independent, at variance with the former. Accordingly viscosity is not a hysteresis phenomenon.

According to (1.37) the shear deformation, $\varepsilon_{(d)}$, is unchanged as long as the shear stress, $\sigma_{(d)}$, lies in the interior of K. As $\sigma_{(d)}$ attains the boundary of K, the material can flow; more precisely, $\varepsilon_{(d)}$ can increase along any oriented direction of the normal cone to K (cf. the Appendix). This flow is plastic, since removing the shear stress the deformation is not removed. More precisely, the flow is perfectly plastic, for it occurs under constant shear stress, without strain-hardening (see below). It is easy to see that the deviatoric relation (1.36) is consistent with the spheric relation (1.33).

Denoting by I_K^* the convex conjugate of the function I_K, by (A.15) we have $\partial I_K^* = \left(\partial I_K\right)^{-1}$. By Proposition A.9, the inclusions (1.36) and (1.37) are respectively equivalent to the following variational inequalities

$$\sum_{i,j=1}^{3} \sigma_{(d)ij}\Big(\frac{\mathrm{d}\varepsilon_{(d)ij}}{\mathrm{d}t} - v_{ij}\Big) \geqslant I_K^*\Big(\frac{\mathrm{d}\varepsilon_{(d)ij}}{\mathrm{d}t}\Big) - I_K^*(v) \qquad \forall v \in D_s, \tag{1.39}$$

$$\sigma_{(d)} \in K, \qquad \sum_{i,j=1}^{3} \frac{\mathrm{d}\varepsilon_{(d)ij}}{\mathrm{d}t}(\sigma_{(d)ij} - v_{ij}) \geqslant 0 \qquad \forall v \in K. \tag{1.40}$$

1.3.2 ELASTIC AND PLASTIC UNIVARIATE ELEMENTS

Consistently with (1.34), we define a (linear) *elastic elementary model*, that we denote by E, which is characterized by the univariate rheological law

$$\sigma = a\varepsilon \qquad (a: \text{ constant } > 0). \tag{1.41}$$

This rheological model is often graphically represented by a spring, cf. Fig. 1.4(a).

We define the multivalued sign function as follows:

$$\begin{cases} \text{sign}\,(x) := \{-1\} & \text{if } x < 0, \\ \text{sign}\,(0) := [0, 1], & \\ \text{sign}\,(x) := \{1\} & \text{if } x > 0, \end{cases} \tag{1.42}$$

and introduce another elementary model, which is characterized by the univariate rheological law

$$\sigma \in b\,\text{sign}\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right) \qquad (b: \text{ constant } > 0). \tag{1.43}$$

We regard this as a scalar model of rigid perfect plasticity, denote it by P, and refer to it as a *plastic element*. Indeed this inclusion is of the form (1.36), since sign $= \big(\partial I_{]-1,1[}\big)^{-1}$.

The inclusion (1.43) may represent the resistance which opposes the motion of a heavy body sliding along a horizontal surface, according to Coulomb's model of dry friction. This is also consistent with the usual interpretation of plasticity as an effect of internal friction. Accordingly, this rheological model is often graphically represented by a sliding block, cf. Fig. 1.4(b).

Stop

Now we introduce two (mutually dual) elementary rheological models, named stop and play, that can be used to construct more sophisticated

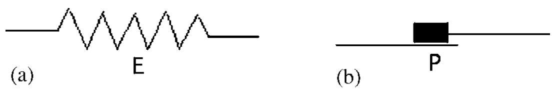


FIGURE 1.4 Graphic representation of the elastic (a) and plastic (b) elementary models.

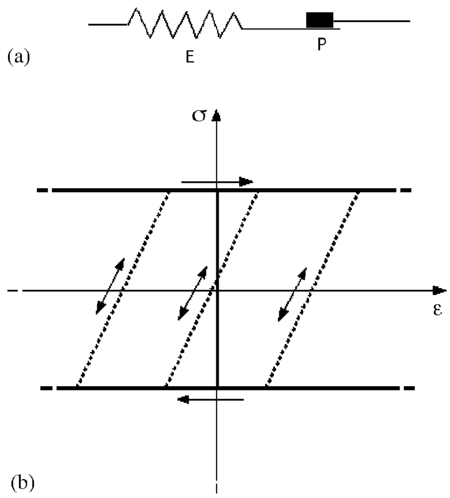


FIGURE 1.5 Prandtl's model of elastoplasticity (or stop), $E-P$, in (a); corresponding σ versus ε relation in (b).

models of elasto-plasticity. We define them in the univariate setting; the formalism of convex calculus makes their tensor extension straightforward.

The classic Prandtl model of elasto-plasticity, also named stop, can be represented by coupling a linear elastic element, E, in series with a rigid perfectly-plastic element, P. Denoting series and parallel arrangements respectively by the symbols '$-$' and '$|$', this model can be represented by the rheological formula $E - P$; cf. Fig. 1.5.

By (1.41) and (1.43), denoting by ε the total displacement and by σ the (signed) force intensity, one easily derives the following variational inequality

$$|\sigma|\leqslant b, \qquad \left(a\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} - \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)(\sigma - v)\geqslant 0 \qquad \forall v, |v|\leqslant b. \tag{1.44}$$

PROPOSITION 1.3.1 (*[8; Chapter II]*). *For any $\varepsilon \in W^{1,1}(0,T)$ and $\sigma^0 \in [-b,b]$, there exists one and only one $\sigma := \mathcal{G}(\varepsilon,\sigma^0) \in W^{1,1}(0,T)$ that fulfils (1.44) and such that $\sigma(0) = \sigma^0$.*

The operator $\mathcal{G}(\cdot,\sigma^0)$ is continuous in $W^{1,1}(0,T)$, and can be extended to a Lipschitz-continuous operator in $C^0([0,T])$.

Alternatively, one may first define $\mathcal{G}$ for piecewise linear inputs, check that this operator is Lipschitz-continuous with respect to the uniform metric, and then extend it to $C^0([0,T])$ [1,3,5,8,16].

Defining the sign graph as in (1.42), (1.44) can also be written as a differential inclusion:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} + \operatorname{sign}^{-1}\left(\frac{\sigma}{b}\right) \ni a\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}. \tag{1.45}$$

The transformation $\varepsilon \mapsto \sigma$ is rate-independent. In order to check this, first notice that in (1.45) it is implicitly assumed that $\mathrm{d}t > 0$, since hysteresis is irreversible; moreover

$$\lambda \operatorname{sign}^{-1}(\sigma) = \operatorname{sign}^{-1}(\sigma) \qquad \forall \lambda > 0.$$

Multiplying both members of (1.45) by $\mathrm{d}t$, formally we then get the equivalent condition

$$\mathrm{d}\sigma + \operatorname{sign}^{-1}\left(\frac{\sigma}{b}\right) \ni a\mathrm{d}\varepsilon,$$

which is obviously rate-independent.

This model can account for elasto-plasticity without strain-hardening. Indeed it represents a material that, starting from $\sigma = \varepsilon = 0$, behaves elastically as long as $|\sigma| < b$. As σ attains the threshold b, it flows plastically under constant stress. This is interpreted by stating that the deformation consists of an elastic and a plastic part, and that for $\sigma = b$ the plastic part may increase, whereas the elastic part remains unchanged. This flow is plastic, for if the stress is decreased, the plastic part of the deformation is not removed, until σ attains the opposite threshold, $-b$.

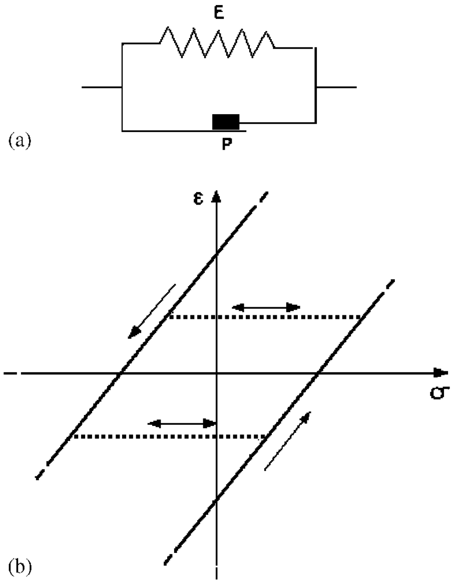
Play

This model is the dual of the stop, and can be represented by $E|P$, namely, a linear elastic element E coupled in parallel with a rigid perfectly-plastic element P, cf. Fig. 1.6(a):

By (1.41) and (1.43), denoting by σ the total applied force and by ε the displacement, σ and ε are then related by the following variational inequality:

$$|\sigma - a\varepsilon| \leqslant b, \qquad \frac{\mathrm{d}\varepsilon}{\mathrm{d}t}(\sigma - a\varepsilon - v) \geqslant 0 \qquad \forall v,\ |v| \leqslant b, \tag{1.46}$$

(a)



(b)

FIGURE 1.6 Prager's model of perfect plasticity with strain hardening (or play), $E|P$, in (a); corresponding ε versus σ relation in (b).

which is equivalently to the differential inclusion

$$b \text{ sign}\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right) + a\varepsilon \ni \sigma. \tag{1.47}$$

PROPOSITION 1.3.2 (*[8; Chapter II]*). *For any $\sigma \in W^{1,1}(0,T)$ and $\varepsilon^0 \in \mathbf{R}$ such that $|\varepsilon^0 - a\sigma(0)| \leqslant b$, there exists one and only $\varepsilon := \mathcal{E}(\sigma, \varepsilon^0) \in W^{1,1}(0,T)$ which fulfils (1.46) and such that $\varepsilon(0) = \varepsilon^0$.*

The operator $\mathcal{E}(\cdot, \varepsilon^0)$ is continuous in $W^{1,1}(0,T)$, and can be extended to a Lipschitz-continuous operator in $C^0([0,T])$.

By the homogeneity property of the sign graph, formally the inclusion (1.47) can also be written as

$$b\,\mathrm{sign}(\mathrm{d}\varepsilon) + a\varepsilon \ni \sigma,$$

by which rate-independence is evident.

This model can account for rigid plasticity with strain-hardening. Indeed, starting from $\sigma = \varepsilon = 0$, the material behaves rigidly as long as $|\sigma| < b$. Whenever σ increases beyond the threshold b it may flow, the deformation being proportional to $\sigma - b$; thus the larger is the deformation, the larger is the required stress. This flow is plastic, since if the stress is decreased, the deformation is not removed until $\sigma - a\varepsilon$ attains the opposite threshold $-b$.

At variance with the stop, the play operator has a regularizing effect, that we describe by the next proposition.

PROPOSITION 1.3.3. *Let us denote by $\mathcal{E}$ the play operator characterized by (1.46). For any $\sigma \in C^0([0,T])$ and any $\varepsilon^0 \in \mathbf{R}$ such that $|\varepsilon^0 - a\sigma(0)| \leqslant b$,*

$$\sigma \in C^0([0,T]) \quad \Rightarrow \quad \mathcal{E}(\sigma, \varepsilon^0) \in C^0([0,T]) \cap BV(0,T). \tag{1.48}$$

PROOF. Let us set $\varepsilon := \mathcal{E}(\sigma, \varepsilon^0)$. A simple inspection of Fig. 1.6(b), shows that ε can increase (decrease, respectively), only as the pair (σ, ε) lies on the right (left, respectively) border of the hysteresis strip. Any uniformly continuous function $t \mapsto \sigma(t)$ can have an infinite number of oscillations, but just a finite number of them may have amplitude $\geqslant 2b$. The pair (σ, ε) can then commute just a finite number of times, if any, from the ε-ascending line to the ε-descending straight line or conversely. Therefore $t \mapsto \varepsilon(t)$ has just a finite number of either nondecreasing or nonincreasing branches, i.e., it is piecewise monotone. The time variation of $t \mapsto \varepsilon(t)$ is then finite. □

On account of this result, the occurrence of a play operator in a differential equation may increase the regularity of the solution. This is an example of the smoothing character of hysteresis.

Plays and stops are closely related each other. Let us denote by $\mathcal{G}$ and $\mathcal{E}$ the stop and play operators, characterized by (1.45) and (1.47), respectively; for the sake of simplicity, let us assume that $a = b = 1$. Denoting the identity by $\mathcal{I}$, and omitting initial values, it is not difficult to check that

$$\mathcal{E} + \mathcal{G} = \mathcal{I}, \qquad 2(\mathcal{I} + \mathcal{E})^{-1} = (\mathcal{I} + \mathcal{G}). \tag{1.49}$$

Generalized Play

Let us set $\tilde{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$ and assume that

$$\begin{cases} \gamma_\ell, \gamma_r \text{ are maximal monotone (possibly multivalued) functions} \\ \mathbf{R} \to 2^{\tilde{\mathbf{R}}}, \text{ such that } \inf \gamma_r(u) \leqslant \sup \gamma_\ell(u) \quad \forall u \in \mathbf{R} \end{cases} \tag{1.50}$$

(by $2^{\tilde{\mathbf{R}}}$ we denote the set of parts of $\tilde{\mathbf{R}}$). The (possibly discontinuous and multivalued) generalized play operator, that we outline in Fig. 1.7(a),

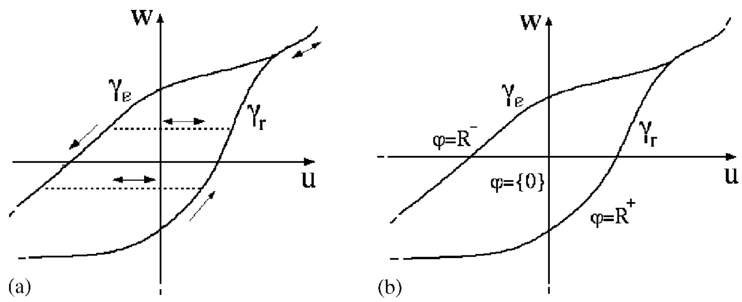


FIGURE 1.7 Discontinuous generalized play in (a) and associated auxiliary function φ in (**b**).

corresponds to the following rate-independent differential inclusion:

$$\frac{\mathrm{d}w}{\mathrm{d}t} \in \varphi(u, w) := \begin{cases} \tilde{\mathbf{R}}^- & \text{if } w \in \gamma_\ell(u) \setminus \gamma_r(u) \\ \{0\} & \text{if } \sup \gamma_r(u) < w < \inf \gamma_\ell(u) \\ \tilde{\mathbf{R}}^+ & \text{if } w \in \gamma_r(u) \setminus \gamma_\ell(u) \\ \tilde{\mathbf{R}} & \text{if } w \in \gamma_r(u) \cap \gamma_\ell(u) \end{cases} \tag{1.51}$$

a.e. in $]0, T[$, cf. Fig. 1.7(b). Here we set $\tilde{\mathbf{R}}^+ := \mathbf{R}^+ \cup \{+\infty\}$ and $\tilde{\mathbf{R}}^- := \mathbf{R}^- \cup \{-\infty\}$.

More precisely, if γ_r and γ_ℓ are Lipschitz-continuous, then the generalized play operator transforms any pair $(u, \xi) \in W^{1,1}(0, T) \times \mathbf{R}$ into the unique function $w \in W^{1,1}(0, T)$ such that $w(0)$ equals the projection of ξ onto $[\gamma_r(u(0)), \gamma_\ell(u(0))]$ and (1.51) is satisfied.

This operator can be extended to $C^0([0, T]) \times \mathbf{R}$ by continuity, and is equivalent to a variational inequality. For any $u \in \mathbf{R}$ let us set $K(u) := [\gamma_r(u), \gamma_\ell(u)]$ and denote by $I_{K(u)}$ the corresponding indicator function (cf. the Appendix). Then $\varphi(u, w) = -\partial I_{K(u)}(w)$, and (1.51) is equivalent to

$$w \in K(u), \qquad \frac{\mathrm{d}w}{\mathrm{d}t}(w - v) \leqslant 0 \qquad \forall v \in K(u), \text{ a.e. in }]0, T[. \tag{1.52}$$

The following property is especially useful in the analysis of differential equations that include a generalized play, see Section 1.9. Here we use the Heaviside function:

$$\hat{H}(\xi) := 0 \quad \text{if } \xi \leqslant 0, \qquad \hat{H}(\xi) := 1 \quad \text{if } \xi > 0,$$

and the sign function $\mathrm{sign}(\xi) := \hat{H}(\xi) - \hat{H}(-\xi)$, for any $\xi \in \mathbf{R}$.

THEOREM 1.3.4 (*Hilpert's Inequality [17]*). *Let* $(u_i, w_i^0) \in W^{1,1}(0,T) \times \mathbf{R}$ $(i = 1, 2)$, *h be any measurable function* $[0,T] \to \mathbf{R}$ *such that* $h \in \hat{H}(u_1 - u_2)$ *a.e. in* $]0,T[$, *and denote by* $\mathcal{E}$ *a generalized play operator. Setting* $w_i := \mathcal{E}(u_i, w_i^0)$ *and* $\bar{w} := w_1 - w_2$, *then*

$$\frac{\mathrm{d}\bar{w}}{\mathrm{d}t} h \geqslant \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{w}^+\right) \qquad \textit{a.e. in }]0,T[. \tag{1.53}$$

Hence, if $s \in \operatorname{sign}(u_1 - u_2)$ a.e. in $]0,T[$,

$$\frac{\mathrm{d}\bar{w}}{\mathrm{d}t} s \geqslant \frac{\mathrm{d}}{\mathrm{d}t}|\bar{w}| \qquad \text{a.e. in }]0,T[. \tag{1.54}$$

PROOF. For any measurable function $k :]0,T[\to \mathbf{R}$,

$$k \in \hat{H}(\bar{w}) \quad \text{a.e. in }]0,T[\quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{w}^+\right) = \frac{\mathrm{d}\bar{w}}{\mathrm{d}t} k \quad \text{a.e. in }]0,T[.$$

To prove (1.53) it then suffices to show that k can be chosen such that

$$\frac{\mathrm{d}\bar{w}}{\mathrm{d}t}(h - k) \geqslant 0 \qquad \text{a.e. in }]0,T[. \tag{1.55}$$

This can be checked by distinguishing the different cases that may occur at any instant t:

if $\bar{w} > 0$ and $\bar{u} := u_1 - u_2 > 0$, then $h = k$ and (1.55) is fulfilled;

if $\bar{w} > 0$ and $\bar{u} \leqslant 0$, then by examining Fig. 1.7 one can see that $\mathrm{d}\bar{w}/\mathrm{d}t \leqslant 0$, hence (1.55) is fulfilled;

if $\bar{w} = 0$, then we can take $k = h$ so that (1.55) is fulfilled;

if $\bar{w} < 0$ and $\bar{u} > 0$, then again Fig. 1.7 shows that $\mathrm{d}\bar{w}/\mathrm{d}t \geqslant 0$, and (1.55) is fulfilled;

if $\bar{w} < 0$ and $\bar{u} \leqslant 0$, then $h = k$ and (1.55) is fulfilled. □

Tensor Extension

Let us denote the space of symmetric 3×3 tensors by $\mathbf{R}_s^9$. Although so far we have assumed ε and σ to be scalars, the previous developments can easily be extended to tensors, or rather to the deviatoric components of the strain and stress tensors (for we still assume a linear relation between the spheric components $\varepsilon_{(s)}$ and $\sigma_{(s)}$). The graph sign is then replaced by $(\partial I_K)^{-1}$, K being a closed, convex subset of D_s such that $0 \in K$. After Proposition A.9, $(\partial I_K)^{-1}$ coincides with the subdifferential of the support function of K; that is, $(\partial I_K)^{-1} = \partial \sigma_K$ $(= \partial I_K^*)$. The function σ_K is positively homogeneous of degree one, cf. Proposition A.12; this property is at the basis of rate-independence. In the multivariate setting the operations of series and parallel arrangement have no meaning in themselves; however, one may associate them to the rules (1.30) and (1.31), and extend them to tensors.

1.3.3 PRANDTL–ISHLINSKIĬ MODELS

Above we introduced a model of elasto-plasticity without strain-hardening, and one of perfect-plasticity with strain-hardening; it is then natural to search for a model of elasto-plasticity with strain-hardening. Moreover, the state of any of the above models is characterized by the pair (ε, σ), whereas a number of tests on elasto-plastic materials indicates the occurrence of internal variables.

An answer to these exhigences is given by a (scalar) model Prandtl [18] proposed in 1928, and Ishlinskiĭ [19] rediscovered in 1944, that is obtained by coupling a family of stops in parallel. The stop–play duality then suggests construction of a dual model by coupling a family of plays in series. These two classes will be respectively named *Prandtl–Ishlinskiĭ models of stop- and play-type*. The corresponding operators can be represented by systems of (possibly infinite) variational inequalities analogous to (1.44) and (1.46); the extension to tensors will be straightforward.

Here we illustrate a generalization of those models, in which possibly infinite families of elements are considered, along the lines of [8; Chapter III]. In order to construct a (scalar) Prandtl–Ishlinskiĭ operator of stop-type, first we need a density over the set of all stops. As any stop is characterized by two positive parameters, a, b, this density can be represented by a positive and finite measure, μ, over the set

$$\mathcal{P} := \{\rho := (a, b) \in \mathbf{R}^2 : a, b > 0\}. \tag{1.56}$$

Let us denote by $\tilde{\varepsilon}$ and $\tilde{\sigma}$ the strain and stress of the composite model. For any $\tilde{\varepsilon} \in W^{1,1}(0, T)$ and any family $\{\sigma_\rho^0\}_{\rho \in \mathcal{P}}$ of admissible initial stresses, by (1.45) that model corresponds to an infinite system of variational inequalities, or equivalently of inclusions:

$$\begin{cases} \dfrac{d\sigma_\rho}{dt} + \operatorname{sign}^{-1}\left(\dfrac{\sigma}{b}\right) \ni a\dfrac{d\tilde{\varepsilon}}{dt} & \mu\text{-a.e. in } \mathcal{P} \\ \sigma_\rho(0) = \sigma_\rho^0 & \mu\text{-a.e. in } \mathcal{P} \\ \tilde{\sigma} = \displaystyle\int_{\mathcal{P}} \sigma_\rho d\mu(\rho). & \end{cases} \tag{1.57}$$

By Proposition 1.3.1 this is equivalent to

$$\tilde{\sigma} = \int_{\mathcal{P}} \mathcal{G}_\rho\left(\tilde{\varepsilon}, \sigma_\rho^0\right) d\mu(\rho) =: \tilde{\mathcal{G}}_\mu\left(\tilde{\varepsilon}, \{\sigma_\rho^0\}_{\rho \in \mathcal{P}}\right). \tag{1.58}$$

$\tilde{\mathcal{G}}_\mu$ is a hysteresis operator, that we name *Prandtl–Ishlinskiĭ operator of stop-type*. If μ is nonnegative this operator is piecewise monotone, but not order preserving, in the sense of (1.9), (1.10).

At any instant t the state of the system is uniquely determined by $\{\tilde{\varepsilon}(t)\} \cup \{\sigma_\rho(t)\}_{\rho\in\mathcal{P}}$. The σ_ρ's play the role of internal variables; in particular they must be specified to define the initial state of the system. Let us denote by $\mathcal{M}(\mathcal{P})$ the set of μ-measurable functions $\mathcal{P} \to \mathbf{R}$. In the univariate setting, under regularity hypotheses on the data, $\tilde{\mathcal{G}}_\mu$ can be extended to a continuous operator $C^0([0,T]) \times \mathcal{M}(\mathcal{P}) \to C^0([0,T])$.

We now consider the dual procedure, and arrange a family of plays in series. Let μ be a prescribed positive finite measure over $\mathcal{P}$, and still denote by $\tilde{\varepsilon}$ and $\tilde{\sigma}$ the stress and the strain of the composite model. For any $\tilde{\sigma} \in W^{1,1}(0,T)$ and any family $\{\varepsilon^0_\rho\}_{\rho\in\mathcal{P}}$ of admissible initial strains, by (1.47) this model also corresponds to an infinite system of variational inequalities, or equivalently of inclusions:

$$\begin{cases} b\,\mathrm{sign}\left(\dfrac{\mathrm{d}\varepsilon_\rho}{\mathrm{d}t}\right) + a\varepsilon_\rho \ni \tilde{\sigma} & \mu\text{-a.e. in } \mathcal{P} \\ \varepsilon_\rho(0) = \varepsilon^0_\rho & \mu\text{-a.e. in } \mathcal{P} \\ \tilde{\varepsilon} = \displaystyle\int_{\mathcal{P}} \varepsilon_\rho \mathrm{d}\mu(\rho), & \end{cases} \tag{1.59}$$

and by Proposition 1.3.2 this is equivalent to

$$\tilde{\varepsilon} = \int_{\mathcal{P}} \mathcal{E}_\rho(\tilde{\sigma}, \varepsilon^0_\rho)\,\mathrm{d}\mu(\rho) =: \tilde{\mathcal{E}}_\mu\left(\tilde{\sigma}, \{\varepsilon^0_\rho\}_{\rho\in\mathcal{P}}\right). \tag{1.60}$$

$\tilde{\mathcal{E}}_\mu$ is a hysteresis operator, that we name *Prandtl–Ishlinskiĭ model of play-type*. At any instant t, the state of the system is determined by $\{\tilde{\sigma}(t)\} \cup \{\varepsilon_\rho(t)\}_{\rho\in\mathcal{P}}$; here the ε_ρ's play the role of internal variables.

Both models can be extended to tensors; to this aim it suffices to replace the multivalued function 'sign' by ∂I^*_K, namely, the subdifferential of the conjugate of the indicator function of a closed convex set $K \subset D_s$.

We saw that stops and plays are related by (1.49). The next statement establishes an analogous property for scalar Prandtl–Ishlinskiĭ models of stop- and play-type.

PROPOSITION 1.3.5 (*[20], cf. also [8; Section III.6]*). *For any $\alpha > 0$ and any nonnegative finite Borel measure μ over $\mathcal{P}$, there exist $\beta > 0$ and a nonnegative finite Borel measure ν over $\mathcal{P}$ such that (omitting initial values)*

$$(\alpha I + \tilde{\mathcal{E}}_\mu)^{-1} = \beta I + \tilde{\mathcal{G}}_\nu. \tag{1.61}$$

1.3.4 Elasto-plasticity with Strain-Hardening without Internal Variables

Here we briefly illustrate two equivalent and mutually dual rheological models that account for elasto-plasticity with strain-hardening, contain no internal variable (at variance with the Prandtl–Ishlinskiĭ models), and generalize both the stop and the play. As it is typical of the approach based on rheological models, first we deal with the univariate setting and then we extend our results to tensors.

Let us fix three positive constants k, α, β $(\alpha < \beta)$, set $K := [-k, k]$, and consider the differential inclusion

$$\partial I_K(\sigma - \alpha\varepsilon) \ni \beta\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} - \frac{\mathrm{d}\sigma}{\mathrm{d}t} \qquad \text{in }]0, T[. \tag{1.62}$$

As $\lambda\partial I_K = \partial I_K$ for any $\lambda \geqslant 0$, the relation between σ and ε defined by (1.62) is causal and invariant with respect to any increasing C^∞-diffeomorphism $\varphi : [0, T] \to [0, T]$, thus it is a hysteresis relation. This inclusion is equivalent to the following variational inequality:

$$\begin{cases} \sigma - \alpha\varepsilon \in K \qquad \text{in }]0, T[\\ \Big(\beta\dfrac{\mathrm{d}\varepsilon}{\mathrm{d}t} - \dfrac{\mathrm{d}\sigma}{\mathrm{d}t}\Big)(\sigma - \alpha\varepsilon - v) \geqslant 0 \qquad \forall v \in K, \text{ in }]0, T[, \end{cases} \tag{1.63}$$

which generalizes both (1.44) and (1.46).

Proposition 1.3.6 ([21]). *Let $0 < \alpha < \beta$, $K := [-k, k]$ $(k > 0)$, and two elastic elements E_1 and E_2 and a rigid perfectly-plastic element P be respectively characterized by the rheological equations*

$$\sigma = \alpha\varepsilon, \qquad \sigma = (\beta - \alpha)\varepsilon, \qquad \sigma \in (\partial I_K)^{-1}\Big(\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\Big). \tag{1.64}$$

Then (1.62) is the rheological equation of the model $E_1|(E_2 - P)$, and represents elasto-plasticity with (linear) kinematic strain-hardening, cf. Fig. 1.8.

The simple argument is based on repeated application of the rules (1.30), (1.31) of series and parallel arrangements.

No internal variable occurs in this model. As $\alpha \to 0$ $(\beta \to +\infty$, respectively) in (1.62), the stop (play, respectively) model is retrieved. As $k \to 0$ $(k \to +\infty$, respectively), (1.62) is reduced to $\sigma = \alpha\varepsilon$ $(\sigma = \beta\varepsilon$, respectively), which represents the elastic model E_1 $(E_1|E_2$, respectively).

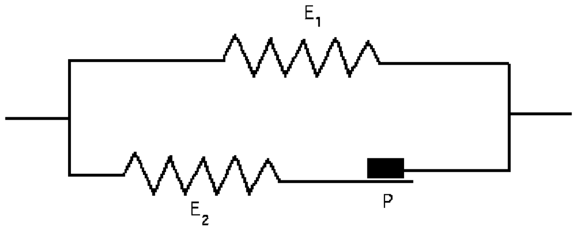


FIGURE 1.8 First model of elastoplasticity with (linear) kinematic strain hardening: $E_1|(E_2 - P)$.

PROPOSITION 1.3.7 (*[21], see also [5; Section I.1]*). *Let α, β, K be as above, and $\sigma^0 \in K$, ε^0 be such that $\sigma^0 - \alpha\varepsilon^0 \in K$. Then:*

(i) For any $\varepsilon \in W^{1,1}(0,T)$ such that $\varepsilon(0) = \varepsilon^0$, there exists one and only one $\sigma \in W^{1,1}(0,T)$ that fulfils (1.62) and such that $\sigma(0) = \sigma^0$. This defines a Lipschitz-continuous hysteresis operator

$$\tilde{\mathcal{G}} : \mathrm{Dom}(\tilde{\mathcal{G}}) \subset W^{1,1}(0,T) \times \mathbf{R} \to W^{1,1}(0,T) : (\varepsilon, \sigma^0) \mapsto \sigma. \tag{1.65}$$

(ii) Dually (but not symmetrically), for any $\sigma \in W^{1,1}(0,T)$ such that $\sigma(0) = \sigma^0$, there exists one and only one $\varepsilon \in W^{1,1}(0,T)$ that fulfils (1.62) and such that $\varepsilon(0) = \varepsilon^0$. This defines a Lipschitz-continuous hysteresis operator

$$\tilde{\mathcal{E}} : \mathrm{Dom}(\tilde{\mathcal{E}}) \subset W^{1,1}(0,T) \times \mathbf{R} \to W^{1,1}(0,T) : (\sigma, \varepsilon^0) \mapsto \varepsilon. \tag{1.66}$$

(iii) The operators $\tilde{\mathcal{G}}$, $\tilde{\mathcal{E}}$ can both be extended by continuity to $C^0([0,T])$; these extensions are Lipschitz-continuous with respect to the uniform norm, and piecewise monotone (cf. (1.10)).

The Dual Model

Setting $a := \alpha^{-1}$, $b := \beta^{-1}$ and $\tilde{K} := aK$, (1.62) also reads

$$\partial I_{\tilde{K}}(a\sigma - \varepsilon) \ni \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} - b\frac{\mathrm{d}\sigma}{\mathrm{d}t} \qquad \text{in }]0,T[. \tag{1.67}$$

PROPOSITION 1.3.8 (*[21]*). *Let a, b, h be three positive constants, $b < a$, and set $H := [-h,h]$, $\tilde{K} := (a-b)H$. Let two elastic elements $\tilde{E}_1$ and $\tilde{E}_2$ and a rigid perfectly-plastic element $\tilde{P}$ be respectively characterized by the rheological*

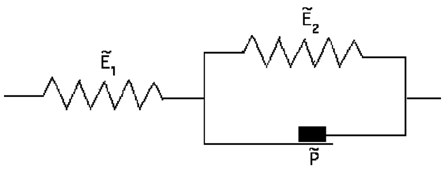


FIGURE 1.9 Second model of elastoplasticity with (linear) kinematic strain hardening: $\tilde{E}_1 - (\tilde{E}_2|\tilde{P})$.

equations

$$\varepsilon = b\sigma, \qquad \varepsilon := (a-b)\sigma, \qquad \sigma = (\partial I_H)^{-1}\Big(\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\Big). \tag{1.68}$$

Then (1.67) is the rheological equation of the model $\tilde{E}_1 - (\tilde{E}_2|\tilde{P})$, and also represents elasto-plasticity with (linear) kinematic strain-hardening, cf. Fig. 1.9.

The latter is a classic model of kinematic strain-hardening, and is due to Prager [22]. Other conclusions analogous to those of the previous model can be drawn.

Prandtl–Ishlinskiĭ-Type Models

Either of above two models can be used as a building block for a more general model, along the lines of the Prandtl–Ishlinskiĭ constructions. For instance one can consider a family $\{(\alpha_\rho, \beta_\rho, K_\rho) : \rho \in \mathcal{P}\}$ as in Proposition 1.3.6; each of these triplets determines a model of the form $E_{1\rho}|(E_{2\rho} - P_\rho)$, and by Proposition 1.3.7 this determines a play-type and a stop-type operator. These models can then be arranged either in series or in parallel, and this corresponds to Prandtl–Ishlinskiĭ models of play-type and stop-type, respectively. An analogous procedure can be applied to the dual model. As (1.62) and (1.67) are equivalent, it is clear that equivalences also occur among the latter operators; however we shall not enter that issue here.

Tensor Extension

The two latter models can be extended to the multivariate setting, by setting (1.62) ((1.67), respectively) for K ($\tilde{K}$, respectively) equal to closed convex subsets of D_s (the linear space of symmetric 3×3 deviators). By the orthogonality of the spaces of deviatoric and spheric tensors, it is not difficult to see that this entails a relation of the same form for $\sigma_{(d)}$ and $\varepsilon_{(d)}$, as well as a linear elastic relation between $\sigma_{(s)}$ and $\varepsilon_{(s)}$. Propositions 1.3.6, 1.3.7 and

1.3.8 hold also in this more general setting, with the exception of part (iii) of Proposition 1.3.7.

Coupling with PDEs

The models of this section are prone to be coupled with the dynamic equation

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} - \nabla\cdot\sigma = \vec{f} \qquad \text{in } \Omega_T \ \big(\text{here } \ (\nabla\cdot\sigma)_i := \textstyle\sum_j \partial\sigma_{ij}/\partial x_j \ \ \forall i\big) \tag{1.69}$$

for space-distributed systems. Corresponding initial- and boundary-value problems for elastoplasticity have been studied via variational inequalities in a large number of works; here we just refer to the monographs [23,24]. That approach has been applied to Prandtl–Ishlinskiĭ models e.g. in [8; Chapter VIII]; Prandtl–Ishlinskiĭ operators have also been used to study wave propagation by Krejčí in [5,25–29]; see also [3].

Stop, plays and Prandtl–Ishlinskiĭ models have been at the basis of early mathematical investigations on hysteresis, e.g. [1], and are still among the most studied models of scalar hysteresis. As we have seen in this section, many properties can be derived for the corresponding hysteresis operators by formulating them in terms of variational inequalities, and by applying techniques of convex analysis, that we briefly review in the Appendix.

A large literature deals with elasto-plasticity; see e.g. the monographs of Besseling and Van der Giessen [30], Duvaut and Lions [23], Halphen and Salencon [31], Han and Reddy [32], Haupt [33], Hill [34], Lemaitre and Chaboche [35], Lubliner [36], Nečas and Hlaváček [24], Prager [22], Prager and Hodge [37], Simo and Hughes [38], Washizu [39].

1.4 DISCONTINUOUS HYSTERESIS

The so-called (*delayed*) *relay* is the most simple model of discontinuous hysteresis. In this section we review its definition, specify the functional framework, and extend it to vectors. We also introduce the Preisach model and its vector extension, and refer to the discussion of Section 1.6 on population-type models of hysteresis. A more detailed presentation may be found in [8; Chapter VI] and in [40].

1.4.1 Scalar Relay

Let us fix any pair $\rho := (\rho_1, \rho_2) \in \mathbf{R}^2$, $\rho_1 < \rho_2$. For any continuous function $u : [0, T] \to \mathbf{R}$ and any $\xi \in \{-1, 1\}$, let us set $X_u(t) := \{\tau \in]0, t] : u(\tau) = \rho_1$

or $\rho_2\}$ and

$$w(0) := \begin{cases} -1 & \text{if } u(0) \leqslant \rho_1, \\ \xi & \text{if } \rho_1 < u(0) < \rho_2, \\ 1 & \text{if } u(0) \geqslant \rho_2, \end{cases} \tag{1.70}$$

$$w(t) := \begin{cases} w(0) & \text{if } X_u(t) = \emptyset, \\ -1 & \text{if } X_u(t) \neq \emptyset \text{ and } u(\max X_u(t)) = \rho_1, \\ 1 & \text{if } X_u(t) \neq \emptyset \text{ and } u(\max X_u(t)) = \rho_2, \end{cases} \quad \forall t \in]0, T], \tag{1.71}$$

cf. Fig. 1.10. Thus $w(t) = \pm 1$ for a.a. t, and this function is measurable. Notice that any continuous function $u : [0, T] \to \mathbf{R}$ is uniformly continuous, hence it can oscillate no more than a finite number of times between the two thresholds ρ_1, ρ_2; therefore $w \in BV(0, T)$. By setting $h_\rho(u, \xi) := w$, an operator

$$h_\rho =: C^0([0, T]) \times \{-1, 1\} \to BV(0, T)$$

is thus defined. This operator is obviously causal and rate-independent, namely, it is a hysteresis operator; it is also piecewise monotone and order preserving.

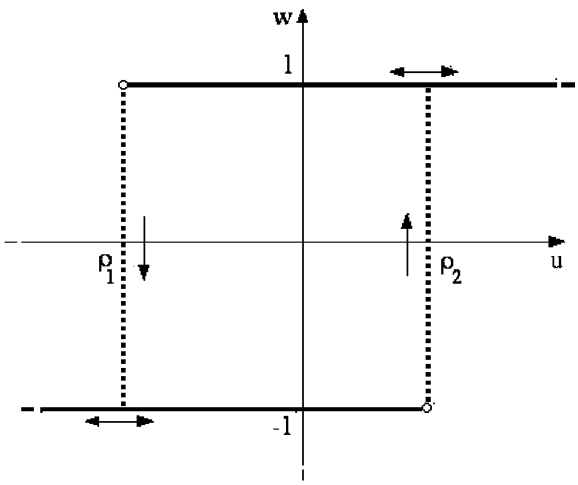


FIGURE 1.10 Relay operator.

Closure

It is evident that the relay operator is not closed as an operator $h_\rho : C^0([0,T]) \to L^1(0,T)$. We then introduce the (multivalued) *completed relay operator* k_ρ. For any $u \in C^0([0,T])$ and any $\xi \in [-1,1]$, we set $w \in k_\rho(u,\xi)$ if and only if w is measurable in $]0,T[$,

$$w(0) := \begin{cases} -1 & \text{if } u(0) < \rho_1, \\ \xi & \text{if } \rho_1 \leqslant u(0) \leqslant \rho_2, \\ 1 & \text{if } u(0) > \rho_2, \end{cases} \tag{1.72}$$

and, for any $t \in]0,T]$,

$$w(t) \in \begin{cases} \{-1\} & \text{if } u(t) < \rho_1, \\ [-1,1] & \text{if } \rho_1 \leqslant u(t) \leqslant \rho_2, \\ \{1\} & \text{if } u(t) > \rho_2, \end{cases} \tag{1.73}$$

$$\begin{cases} \text{if } u(t) \neq \rho_1, \rho_2, & \text{then } w \text{ is constant in a neighborhood of } t \\ \text{if } u(t) = \rho_1, & \text{then } w \text{ is nonincreasing in a neighborhood of } t \\ \text{if } u(t) = \rho_2, & \text{then } w \text{ is nondecreasing in a neighborhood of } t. \end{cases} \tag{1.74}$$

Thus $-1 \leqslant w(t) \leqslant 1$ for a.a. t. Notice that $w \in BV(0,T)$ for any $u \in C^0([0,T])$, because of the argument we saw for h_ρ. Thus

$$k_\rho =: C^0([0,T]) \times [-1,1] \to 2^{BV(0,T)}.$$

The graph of k_ρ in the (u,w)-plane invades the whole rectangle $[\rho_1, \rho_2] \times [-1,1]$, cf. Fig. 1.11. One can show that this operator is the closure of h_ρ with respect to the strong topology of $C^0([0,T])$ and the sequential weak star topology of $BV(0,T)$ [8; Section VI.1]. The use of k_ρ is then especially convenient in problems for PDEs.

Reformulation of the Scalar Relay

In view of the coupling with PDEs, cf. Section 1.10, we reformulate the completed relay operator, k_ρ. The conditions (1.73) and (1.74) are respectively equivalent to

$$\begin{cases} |w| \leqslant 1 \\ (w-1)(u-\rho_2) \geqslant 0 \\ (w+1)(u-\rho_1) \geqslant 0 \end{cases} \qquad \text{a.e. in }]0,T[, \tag{1.75}$$

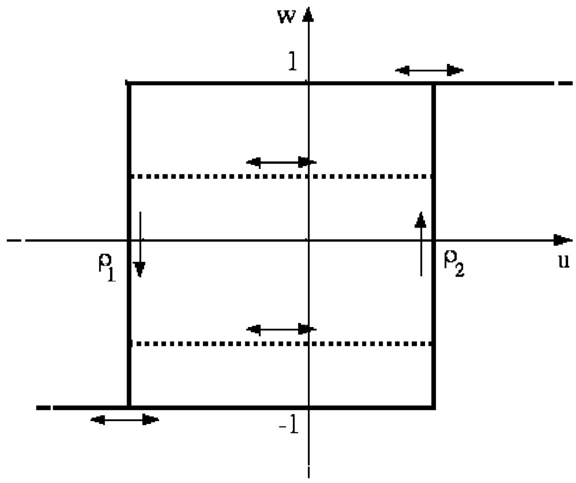


FIGURE 1.11 Completed relay operator. Here the pair (u, w) can attain any value of the rectangle $[\rho_1, \rho_2]\times[-1, 1]$. w is locally nonincreasing (nondecreasing, respectively) if $u(t) = \rho_1$ ($u(t) = \rho_2$, respectively); w is locally constant if $\rho_1 < u(t) < \rho_2$.

$$\int_0^t u\,\mathrm{d}w = \int_0^t \rho_2\,\mathrm{d}w^+ - \int_0^t \rho_1\,\mathrm{d}w^- =: \Psi_\rho(w, [0, t]) \qquad \forall t \in]0, T] \quad (1.76)$$

(these are Stieltjes integrals) [41,42]. The condition (1.73) entails that $u\,\mathrm{d}w \leqslant \rho_2 \mathrm{d}w^+ - \rho_1 \mathrm{d}w^-$, whence $\int_0^t u\,\mathrm{d}w \leqslant \Psi_\rho(w, [0, t])$, independently from the dynamics of the pair (u, w) through the rectangle $[\rho_1, \rho_2]\times[-1, 1]$; the opposite inequality is then equivalent to (1.76).

In conclusion, the system (1.73), (1.74) is equivalent to (1.75) coupled with the inequality

$$\int_0^t u\,\mathrm{d}w \geqslant \Psi_\rho(w, [0, t]) \qquad \forall t \in]0, T]. \quad (1.77)$$

1.4.2 VECTOR RELAY

Let us denote by $\theta := (\theta_1, \theta_2)$ the angular coordinates (longitude and co-latitude, say) on the unit sphere, S^2; by an obvious identification, we can then assume that θ ranges over S^2. For any $\rho := (\rho_1, \rho_2) \in \mathbf{R}^2$ with $\rho_1 < \rho_2$ and any $\theta \in S^2$, we introduce the vector-relay operator:

$$\vec{h}_{(\rho,\theta)} : C^0([0, T])^3 \times \{\pm 1\} \to BV(0, T)^3 : (\vec{u}, \xi) \mapsto h_\rho(\vec{u}\cdot\vec{\theta}, \xi)\vec{\theta}. \quad (1.78)$$

Its closure in natural function spaces, $\vec{k}_{(\rho,\theta)}$, is simply obtained by replacing the scalar relay h_ρ with its completion k_ρ in (1.78), and is multivalued.

Reformulation of the Vector Relay

The characterization (1.72), (1.75), (1.77) of scalar hysteresis operators can easily be extended to vectors. For any $(\vec{u}, \xi) \in C^0([0, T])^3 \times [-1, 1]$ and any (ρ, θ) as above, by (1.72), (1.75) and (1.77) we have $\vec{w} \in \vec{k}_{(\rho,\theta)}(\vec{u}, \xi)$ if and only if $\vec{w}(t) = w(t)\vec{\theta}$ for any t, where the scalar function w is such that

$$w(0) = \begin{cases} -1 & \text{if } \vec{u}(0)\cdot\vec{\theta} < \rho_1 \\ \xi_{(\rho,\theta)} & \text{if } \rho_1 \leqslant \vec{u}(0)\cdot\vec{\theta} \leqslant \rho_2 \\ 1 & \text{if } \vec{u}(0)\cdot\vec{\theta} > \rho_2, \end{cases} \tag{1.79}$$

$$\begin{cases} (w(t) - 1)\big(\vec{u}(t)\cdot\vec{\theta} - \rho_2\big) \geqslant 0 \\ (w(t) + 1)\big(\vec{u}(t)\cdot\vec{\theta} - \rho_1\big) \geqslant 0 \qquad \forall t \in [0, T], \\ |w(t)| \leqslant 1 \end{cases} \tag{1.80}$$

$$\int_0^t \vec{u}\cdot\vec{\theta}\,\mathrm{d}w \geqslant \Psi_\rho(w, [0, t]) \qquad \forall t \in]0, T]. \tag{1.81}$$

This formulation of the vector-relay operator can be extended to space-distributed systems, just by assuming that it is fulfilled pointwise in space. Let us set $\Omega_t := \Omega \times]0, t[$ for any $t > 0$, and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $C^0(\overline{\Omega_t})$ and $C^0(\overline{\Omega_t})'$. As $\mathrm{d}w = \mathrm{d}w^+ - \mathrm{d}w^-$ and $|\mathrm{d}w| = \mathrm{d}w^+ + \mathrm{d}w^-$, (1.81) can then be extended as follows:

$$\begin{aligned} &\left\langle \vec{u}\cdot\vec{\theta}, \frac{\partial w(\rho,\theta)}{\partial \tau} \right\rangle \geqslant \frac{\rho_2 + \rho_1}{2} \int_\Omega \big[w(\rho, \theta, x, t) - w^0(\rho, \theta, x)\big]\,\mathrm{d}x \\ &+ \frac{\rho_2 - \rho_1}{2} \left\| \frac{\partial w(\rho,\theta)}{\partial \tau} \right\|_{C^0(\overline{\Omega_t})'} \left(=: \int_{\bar{\Omega}} \Psi_\rho(w(\rho,\theta), [0, t]) \right) \qquad \forall t \in]0, T]. \end{aligned} \tag{1.82}$$

1.4.3 THE PREISACH MODEL

Maybe this is the most powerful scalar model of hysteresis among those that are known so far (which are not many; more would be wellcome...) This model was proposed by the physicist F. Preisach in 1935 [43] to represent scalar ferromagnetism; it was then also applied to model hysteresis in porous-media filtration as well as other hysteresis phenomena. Here we introduce the corresponding hysteresis operator, and review its main properties.

First, let us define the so-called Preisach (half-)plane to be the set of thresholds of all relay operators

$$\mathcal{P} := \left\{ \rho = (\rho_1, \rho_2) \in \mathbf{R}^2 : \rho_1 < \rho_2 \right\}. \tag{1.83}$$

Let us denote by $\mathcal{R}$ the family of Borel measurable functions $\mathcal{P} \to \{-1, 1\}$, and by $\{\xi_\rho\}$, or just ξ, a generic element of $\mathcal{R}$. For any finite (signed) Borel measure μ over $\mathcal{P}$, let us then define the Preisach operator

$$\begin{aligned} &\mathcal{H}_\mu : C^0([0,T]) \times \mathcal{R} \to L^\infty(0,T), \\ &\left[\mathcal{H}_\mu(u,\xi)\right](t) := \int_{\mathcal{P}} \left[h_\rho(u,\xi_\rho)\right](t)\, \mathrm{d}\mu(\rho) \qquad \forall t \in [0,T]. \end{aligned} \tag{1.84}$$

This construction may be regarded as a sort of *spectral resolution* of a class of hysteresis operators, that we shall characterize afterwards. The operator $\mathcal{H}_\mu$ is causal and rate-independent, namely, it is a hysteresis operator; if $\mu \geqslant 0$, $\mathcal{H}_\mu$ is also piecewise monotone and order preserving. These properties are clearly inherited from the relay operator.

Memory Maps

Let us fix any $u \in C^0([0,T])$, any $\xi \in \mathcal{R}$, and let $w_\rho \in k_\rho(u, \xi_\rho)$ μ-a.e. in $\mathcal{P}$. On the basis of the definition of the relays h_ρ, it is easy to see that for any $t \in]0,T]$,

$$\begin{cases} \text{if } \rho_1 > u(t) & \text{then } w_\rho(t) = -1 \\ \text{if } \rho_2 < u(t) & \text{then } w_\rho(t) = 1 \\ \text{if } \rho_1 < u(t) < \rho_2 & \text{then } w_\rho(t) \text{ depends on } u|_{[0,t]} \text{ and on } \xi_\rho. \end{cases} \tag{1.85}$$

Let us set

$$\begin{aligned} &A_w^-(t) := \{\rho \in \mathcal{P} : w_\rho(t) = -1\}, \qquad A_w^+(t) := \{\rho \in \mathcal{P} : w_\rho(t) = 1\}, \\ &B_w(t) := \partial A_w^-(t) \cap \partial A_w^+(t). \end{aligned} \tag{1.86}$$

At any t, the curve $B_w(t)$ intersects the line $\rho_1 = \rho_2$ at the point $(u(t), u(t))$. Because of (1.85), as u increases (decreases, respectively) in time, B_w moves up (to the left, respectively), cf. Fig. 1.12. It is then easy to check that for any $t \in [0,T]$, $B_w(t)$ is a maximal antimonotone graph in $\mathcal{P}$, provided that this holds for $t = 0$. This graph will be called a *memory map*; it contains the complete information on the state at the instant t of all relays that do not sit on the the map itself. At any instant the output is retrieved by integrating the values of all relays with respect to the prescribed *Preisach measure* μ. This integral is determined by $B_w(t)$ provided that $\mu(B_w(t)) = 0$; obviously

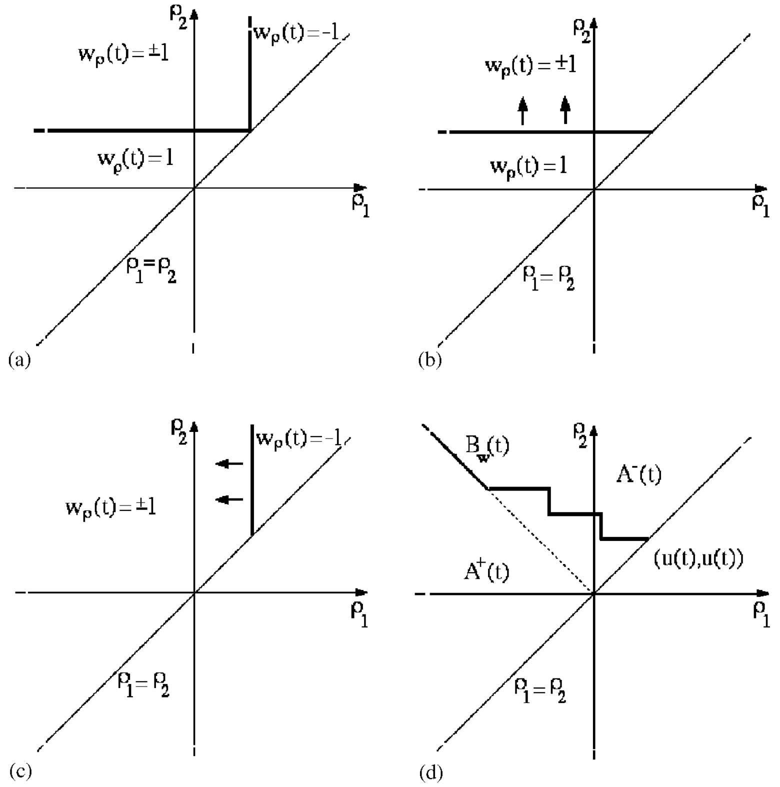


FIGURE 1.12 In (a) the possible states of relays $w_\rho := h_\rho(u, \xi_\rho)$ are shown. As u increases (decreases, respectively) certain relays are switched to the state 1 (-1, respectively), cf. (b) and (c) respectively. The regions characterized by the states -1 and 1 are then separated by a maximal antimonotone graph, $B_w(t)$, provided that the same holds at $t = 0$; cf. (d).

this condition is fulfilled whenever μ is not singular with respect to the ordinary bidimensional Lebesgue measure.

Let us assume that $B_w(0)$ coincides with the bisectrix of the second quadrant:

$$B_w(0) = B^v := \{\rho \in \mathcal{P} : \rho_1 + \rho_2 = 0\}.$$

This relay configuration represents the state of a virginal ferromagnetic material, namely, a system which has never experienced any hysteresis process.

Let us apply any input $u \in C^0([0, T])$, and set $m(t) = \max\{|u(\tau)| : \tau \in [0, t]\}$ for any $t \in [0, t]$. One can show that at any instant t the memory map

$B_w(t)$ is then the union of the half-line $\{\rho \in \mathcal{P} : -\rho_1 = \rho_2 > m(t)\}$ with an either finite or countable family of segments; each of these segments is parallel to one of the axes ρ_1 and ρ_2, and they can only accumulate in a neighborhood of the line $\{\rho \in \mathcal{P} : \rho_1 = \rho_2\}$.

Conversely, starting from the virginal state, any graph of the form, that we just outlined can be attained by applying a suitable input $u \in C^0([0, T])$. It should be noticed that not any memory map B may be attained starting from the virginal state B^v, since parts of B^v lying on the line $\rho_1+\rho_2 = 0$ once deleted cannot be restored. However, if a maximal antimonotone graph B in $\mathcal{P}$ contains a half-line included in the bisectrix $\{\rho \in \mathcal{P} : -\rho_1 = \rho_2\}$, then it can be approximated in the Hausdorff distance by a sequence of graphs having a staircase shape in correspondence to diagonal parts of B. Each of these approximating graphs is obtained by applying a continuous input function having oscillations of decreasing amplitude.

There is a strict analogy between the above approximation procedure and the *alternate demagnetization* process, which is used to demagnetize magnetic heads for instance. The latter consists of applying an alternating current of decreasing amplitude, which generates a magnetic field also having oscillations of decreasing amplitude.

By the above construction, it is easy to see that for any input function $u \in C^0([0, T])$ and at any instant, defining the memory sequence $\{u(t_j)\}$ as in (1.5),

the sequence $\{u(t_j)\}$ determines the memory map $B_w(t)$, and conversely;

the input u determines the memory map $B_w(t)$, and $B_w(t)$
determines the output $[\mathcal{H}_\mu(u, \xi_v)](t)$ via integration with respect to μ.

The Preisach model allows for a unified treatment of several (scalar) hysteresis models. This is conveniently illustrated by means of the Preisach plane. The (ordinary) play operator is equivalent to a Preisach operator characterized by a measure that is uniformly distributed on a straight line parallel to the bisectrix of the first and third quadrants (as this measure is infinite, some modifications are needed in the above formulation, in particular integration over $\mathcal{P}$ should be replaced by the Cauchy principal value). More generally, a generalized play is obtained if the measure is (possibly nonuniformly) distributed along the graph of a strictly increasing single-valued function $\rho_1 \mapsto \rho_2$.

Thus ordinary plays and generalized plays are Preisach operators. By (1.49) one can then represent stop operators in terms of Preisach operators, too. The same also applies to Prandtl–Ishlinskiĭ operators of both types, for they are averages of plays and stops.

Continuity Properties of the Preisach Operator

We summarize several continuity results in the next statement. Here by $\mathcal{B}$ we denote the set of maximal antimonotone graphs in $\mathcal{P}$.

THEOREM 1.4.2 (*[44], [8; Section IV.3]*). *Let μ be a finite Borel measure over $\mathcal{P}$ and $\xi \in \mathcal{R}$. Then:*

(i)
$$\mathcal{H}_\mu(u, \xi) \in C^0([0, T]) \qquad \forall u \in C^0([0, T]) \tag{1.87}$$
if and only if (setting $\mu := 0$ outside $\mathcal{P}$)
$$|\mu|\,(\mathbf{R} \times \{r\}) = |\mu|\,(\{r\} \times \mathbf{R}) = 0 \qquad \forall r \in \mathbf{R}. \tag{1.88}$$

(ii) If $\mathcal{H}_\mu(\cdot, \xi)$ operates in $C^0([0, T])$, then it is also continuous in that space.

(iii) $\mathcal{H}_\mu(\cdot, \xi)$ is uniformly continuous in $C^0([0, T])$ if and only if $|\mu(B)| = 0$ for any $B \in \mathcal{B}$.

(iv) $\mathcal{H}_\mu(\cdot, \xi)$ is Lipschitz-continuous in $C^0([0, T])$ with Lipschitz constant L if and only if

$$|\mu\{(\rho_1+\alpha_1, \rho_2+\alpha_2) : (\rho_1, \rho_2) \in B, \sqrt{\alpha_1^2 + \alpha_2^2} < \varepsilon\}| \leqslant L\varepsilon \qquad \forall B \in \mathcal{B}, \forall \varepsilon > 0. \tag{1.89}$$

Under appropriate conditions on the Preisach measure μ, $\mathcal{H}_\mu(\cdot, \xi)$ operates in the Sobolev spaces $W^{1,p}(0, T)$ $(1 \leqslant p \leqslant +\infty)$, or in the Hölder spaces $C^{0,\nu}([0, T])$ $(0 < \nu \leqslant 1)$, or in $C^0([0, T]) \cap BV(0, T)$ [44]. Other conditions on μ guarantee the existence of the inverse operator $\mathcal{H}_\mu^{-1}$, and yield its continuity in the spaces above.

Vector Preisach Model

The extension of the Preisach model to vectors is of special interest for applications. This can be constructed by integrating a family of vector relays with respect to a suitable measure ν, that is defined over the four parameters that determine vector relays: $\rho_1, \rho_2, \theta_1, \theta_2$. At the expense of some generality, one can also use the following procedure [6,7,45–47]:

(i) project the input onto a generic direction, $\vec{\theta}$;
(ii) apply to it a scalar Preisach operator, which may depend on $\vec{\theta}$;
(iii) average these outputs with respect to a prescribed weight function, $\gamma(\vec{\theta})$.

Characterization of Preisach Operators

For any measure μ, the Preisach operator $\mathcal{H}_\mu$ fulfils the *congruency* and *wiping-out* properties, that we now outline. Let $a, b \in \mathbf{R}$ $(b \neq 0)$, $T \gg 1 + \pi$,

$\xi_i \in \mathcal{R}$, and

$$u_i \in C^0([0,T]), \qquad u_i(t) = a + b\cos(t-1) \quad \forall t \in [1,T] \quad (i = 1,2). \quad (1.90)$$

Thus u_i is periodic of period 2π in $[1+\pi, T]$. By means of the definition of the relay operator, one can check that the same holds for $w_i := \mathcal{H}_\mu(u_i, \xi_i)$. The pair (u_i, w_i) then moves along (possibly degenerate) so-called minor hysteresis loops, at the interior to the major loop. The following holds:

(i) *Congruency Property.* The function $w_1 - w_2$ is constant in $[1+\pi, T]$; that is, minor hysteresis loops are congruent via vertical translations in the (u, w)-plane.

(ii) *Wiping-out Property* (also named *return-point* property). Let u_1 and u_2 be as in (1.90), and set

$$A_i := \{t \in [0,1] : u_i(t) \notin [a-b, a+b]\} \qquad (i = 1,2).$$

It is easy to see that, if $A_1 = A_2$ and $u_1 = u_2$ in A_1, then $w_1 = w_2$ in $[1+\pi, T]$. (Notice that u_1 and u_2 need not coincide in $[0,1] \setminus A_1$.) This can easily be checked on the basis of the properties of the memory map, i.e. of relays, and means that any past oscillation of u at the interior of a larger one has no influence on the memory: small cycles in the (u, w)-plane are wiped out by larger ones.

The characterization theorem states that, under minor restrictions that we omit,

a hysteresis operator fulfils the congruency and wiping-out properties if and only if it is a Preisach operator. (1.91)

This result was first established by Mayergoyz [48,49]; a rigorous mathematical argument was then provided by Brokate [50].

Identification of the Preisach Measure

As the measure μ characterizes the Preisach operator, the identification of μ is a key step for the effective use of this model in applicative problems. As can easily be checked via the memory map, by applying a suitable input and measuring the corresponding output, it is easy to evaluate the measure of an arbitrary square of the Preisach plane $\mathcal{P}$ having sides parallel to the axes. By this procedure one can effectively approximate the Preisach measure μ. An algorithm based on an appropriate selection of the input functions allows one to reduce the number of measurements [6–8,51].

The Preisach Model and Ferromagnetism

The Preisach model is often applied to represent the magnetization versus magnetic field relation in a ferromagnetic body formed by an aggregate of single-domain particles. Although the properties that can be derived from the Preisach model are in good qualitative agreement with the physical evidence, for several ferromagnetic materials there are quantitative discrepancies. This also applies to the vector Preisach model. Physicists and engineers then proposed several variants of the originary Preisach model, in order to provide a more adequate model of ferromagnetic hysteresis. For instance, Della Torre [52] replaced the relation $M = \mathcal{H}_\mu(H)$ by

$$M = \mathcal{H}_\mu(H + \alpha M) \qquad (\alpha \text{ being a constant} > 0); \tag{1.92}$$

this also defines a hysteresis operator, $\mathcal{H}_{\mu,\alpha} : H \mapsto M$; this is known as the *moving Preisach model*. Other models are reviewed e.g. in [4,6,7].

In the next section we illustrate a classic model of vector ferromagnetic hysteresis, that deals with a finer length-scale than that the Preisach model is supposed to represent.

1.5 MICROMAGNETISM

In this section we outline the theory of *micromagnetism*, and introduce the *Landau–Lifshitz equation*. In order to account for dissipation due to magnetic inclusions, we then amend this equation, and derive a purely rate-independent mesoscopic model of ferromagnetic hysteresis.

Spontaneous Magnetization

Even a small magnetic field is capable of inducing a large magnetization in a ferromagnetic body below a temperature (named Curie's temperature) that is characteristic of the material. In 1907 Weiss [53] explained this phenomenon by assuming that any ferromagnetic material exhibits a *spontaneous magnetization*, $\vec{M} = \vec{M}(x)$, even if no magnetic field is applied. Since magnets can look macroscopically demagnetized, Weiss assumed that the body breaks up into small uniformly magnetized regions (named *magnetic domains*), which may be magnetized along different directions. Under these conditions, by applying a small magnetic field one may produce a large macroscopic magnetization either by modifying the domain configuration, or by rotating the domain magnetization.

The origin of spontaneous magnetization remained unclear, until in 1928 Heisenberg explained it on the basis of quantum mechanics. In 1932 Bloch assumed that magnetic domains are separated by thin transition layers (*Bloch walls*), where the magnetization rotates smoothly. Afterwards several kinds of walls and more exotic configurations were conjectured and experimentally observed; see e.g. [54] for a detailed account. In 1935

Landau and Lifshitz [55] proposed a mesoscopic quantitative model, and on that basis Brown [56–58] introduced a theory that is now known as *micromagnetism* (or micromagnetics),* that we synthetically review below.

In their work of 1935 Landau and Lifshitz also proposed representing the evolution of magnetization via a simple equation, that accounts for the domain-wall dynamic and relaxation towards (metastable) equilibrium. That equation is rate-dependent, and thus cannot represent hysteresis, as we defined it in Section 1.1. Moreover, it does not account for rate-independent dissipation due to magnetic inclusions. We shall propose a modified equation, to eliminate these drawbacks.

Maxwell's Equations

We deal with processes in a ferromagnetic material that occupies a bounded domain Ω of $\mathbf{R}^3$ in a time interval $]0,T[$, and set $\Omega_T := \Omega\times]0,T[$, $\mathbf{R}^3_T := \mathbf{R}^3\times]0,T[$. We denote the magnetic field by $\vec{H}$, the magnetization by $\vec{M}$, and the magnetic induction by $\vec{B}$; in Gaussian units, these fields are related by the condition $\vec{B} = \vec{H} + 4\pi\vec{M}$. We also denote the electric field by $\vec{E}$, the electric displacement by $\vec{D}$, the electric current density by $\vec{J}$, the electric charge density by $\hat{\rho}$, and the speed of light in vacuum by c. The Maxwell laws read

$$c\nabla\times\vec{H} = 4\pi\vec{J} + \frac{\partial\vec{D}}{\partial t}, \qquad c\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \text{in } \mathbf{R}^3_T, \tag{1.93}$$

$$\nabla\cdot\vec{B} = 0, \qquad \nabla\cdot\vec{D} = 4\pi\hat{\rho} \qquad \text{in } \mathbf{R}^3_T. \tag{1.94}$$

These equations must be coupled with appropriate constitutive relations. For instance, we may assume that the material is homogeneous, isotropic, and is surrounded by vacuum; the electric conductivity σ and the dielectric permeability ε are then constant in Ω, whereas $\sigma = 0$ and $\varepsilon = 1$ outside Ω. Ohm's law then reads

$$\vec{J} = \sigma\big(\vec{E} + \vec{g}\big) \qquad \text{in } \mathbf{R}^3_T, \tag{1.95}$$

where $\vec{g}$ represents a prescribed applied electromotive force. Moreover,

$$\vec{D} = \varepsilon\vec{E} \qquad \text{in } \mathbf{R}^3_T. \tag{1.96}$$

These equations must be coupled with initial conditions for $\vec{D}$ and $\vec{B}$, and with suitable decay conditions at infinity. Even if we were just interested

*This theory deals with phenomena that occur on the length-scale of the μm, which nowadays is labeled as *mesoscopic*.

into the evolution of the ferromagnet, the Maxwell equations should not be confined to the region Ω, due to difficulties in prescribing boundary conditions for the electromagnetic fields. This is strictly related to the action at distance which characterizes electromagnetic phenomena.

Magnetostatic Equations

In case of slowly varying fields, the system of Maxwell equations (1.93), (1.94) can be replaced by the magnetostatic equations

$$c\nabla\times\vec{H} = 4\pi\vec{J}, \qquad \nabla\cdot\vec{B} = 0 \qquad \text{in } \mathbf{R}^3. \tag{1.97}$$

We assume that the current density field $\vec{J}$ is a divergence-free datum; $\vec{J}$ may include not only conduction currents but also Ampère currents, which account for the presence of magnets outside Ω. The magnetic field $\vec{H}$ can be decomposed into the sum of an applied field $\vec{H}_{\text{app}}$ induced by $\vec{J}$, plus a demagnetizing field $\vec{H}_{\text{dem}}$ determined by $\vec{M}$. The system (1.97) is equivalent to the following equations:

$$\nabla\cdot\vec{H}_{\text{app}} = 0, \qquad c\nabla\times\vec{H}_{\text{app}} = 4\pi\vec{J} \qquad \text{in } \mathbf{R}^3, \tag{1.98}$$

$$\nabla\cdot\left(\vec{H}_{\text{dem}} + 4\pi\vec{M}\right) = 0, \qquad \nabla\times\vec{H}_{\text{dem}} = \vec{0} \qquad \text{in } \mathbf{R}^3. \tag{1.99}$$

Henceforth it will be assumed that any field defined just in Ω is extended with vanishing value outside Ω.

1.5.1 MAGNETIC ENERGY MINIMIZATION

At the length-scale of about 10^{-5} cm, the ferromagnetic behaviour can be described by expressing the magnetic energy as a functional of the mesoscopic magnetization field M, and then minimizing that functional.

Let us represent the ferromagnet as an array of magnetic moments of prescribed modulus, $\mathcal{M}$, and variable orientation. According to the above mentioned Heisenberg theory, the ferromagnetic behavior is due to the occurrence of a strong coupling among electronic spins, which induces neighboring moments to be (almost) parallel. By averaging over the mesoscopic length-scale, the constraint on the magnitude is then preserved:

$$|\vec{M}| = \mathcal{M} \qquad \text{in } \Omega. \tag{1.100}$$

The magnetic coupling and the nonconvexity of this constraint are at the basis of the ferromagnetic behavior, in particular of hysteresis.

Magnetic Energy

For the sake of simplicity, henceforth we neglect *magnetostriction*, namely, mechanical effects related to magnetization. We also assume that the material is a (single grain) perfect crystal; however, this restriction will be removed later. The magnetic (free) energy can be represented as a functional of the magnetization, $\vec{M}$, and consists of the following contributions.

(i) *Exchange Energy*. The force which locally tends to align electronic spins can be represented by a term which penalizes the space derivatives of $\vec{M}$:

$$\mathcal{E}_{\mathrm{ex}}(\vec{M}) := \frac{1}{2}\sum_{i,j=1}^{3} a_{ij}\int_{\Omega}\frac{\partial\vec{M}}{\partial x_i}\cdot\frac{\partial\vec{M}}{\partial x_j}\,\mathrm{d}x; \tag{1.101}$$

here $\{a_{ij}\}$ is a positive definite, symmetric 3×3-tensor.

(ii) *Anisotropy Energy*. This has the form

$$\mathcal{E}_{\mathrm{an}}(\vec{M}) := \int_{\Omega}\varphi(\vec{M})\,\mathrm{d}x. \tag{1.102}$$

The minimization of this term accounts for the tendency of $\vec{M}$ to point in one or more directions of easy magnetization. The function $\varphi : \mathbf{R}^3 \to \mathbf{R}$ depends on the crystal structure of the ferromagnet. We assume that φ is convex and smooth; this convexity hypothesis is not very restrictive, as any function of $|\vec{M}|$ may be added to $\varphi(\vec{M})$, because of the constraint on the modulus. Near the Curie temperature, as a first approximation we can assume that φ is quadratic, e.g., $\varphi(\vec{M}) = c(M_x^2 + M_y^2)$, c being a positive constant.

(iii) *Magnetic Field Energy*. We saw that the field $\vec{H}$ can be represented as the sum of the applied field $\vec{H}_{\mathrm{app}} = \vec{H}_{\mathrm{app}}(\vec{J})$ and the demagnetizing field $\vec{H}_{\mathrm{dem}} = \vec{H}_{\mathrm{dem}}(\vec{M})$, cf. (1.98), (1.99). The energy that is stored in the field $\vec{H}$ is then the sum of two terms:

$$\mathcal{E}_{\mathrm{field}}(\vec{M}) := \mathcal{E}_{\mathrm{app}}(\vec{M}) + \mathcal{E}_{\mathrm{dem}}(\vec{M}) = -\int_{\Omega}\vec{H}_{\mathrm{app}}\cdot\vec{M}\,\mathrm{d}x + \frac{1}{8\pi}\int_{\mathbf{R}^3}|\vec{H}_{\mathrm{dem}}(\vec{M})|^2\,\mathrm{d}x. \tag{1.103}$$

Magnetic Energy Functional

By assembling the above contributions, we get the magnetic energy functional:

$$
\begin{aligned}
\mathcal{E}_{\text{mag}}(\vec{M}) &:= \mathcal{E}_{\text{ex}}(\vec{M}) + \mathcal{E}_{\text{an}}(\vec{M}) + \mathcal{E}_{\text{field}}(\vec{M}) \\
&= \int_\Omega \Big(\frac{1}{2} \sum_{i,j=1}^{3} a_{ij} \frac{\partial \vec{M}}{\partial x_i} \cdot \frac{\partial \vec{M}}{\partial x_j} + \varphi(\vec{M}) - \vec{H}_{\text{app}} \cdot \vec{M} \Big) \mathrm{d}x + \frac{1}{8\pi} \int_{\mathbf{R}^3} \big| \vec{H}_{\text{dem}}(\vec{M}) \big|^2 \, \mathrm{d}x.
\end{aligned}
\tag{1.104}
$$

Henceforth we shall assume that the applied field $\vec{H}_{\text{app}}$ is prescribed. Any thermodynamically stable magnetic configuration fulfils the following variational principle:

$$
\mathcal{E}_{\text{mag}}(\vec{M}) = \inf \big\{ \mathcal{E}_{\text{mag}}(\vec{v}) : \vec{v} \in H^1(\Omega)^3, |\vec{v}| = \mathcal{M} \text{ a.e. in } \Omega \big\}.
$$

By the direct method of the calculus of variations, one can see that this problem has a solution. The H^1-coerciveness entails that the limit of any minimizing sequence is compact in $L^2(\Omega)^3$, and then fulfils the nonconvex constraint on the modulus (1.100). In general uniqueness fails, because of nonconvexity. For instance, several stable stationary configurations may occur if $\vec{H}_{\text{app}} = \vec{0}$ identically in Ω, provided that the specimen is not so small to be reduced to a single domain. Relative minimizers (i.e., states that minimize the functional $\mathcal{E}_{\text{mag}}$ just with respect to variations of $\vec{M}$ that are small in the norm of $H^1(\Omega)^3$) represent metastable states, and are physically acceptable. Their stability, that is, the time-scale by which they will decay to a lower minimum of $\mathcal{E}_{\text{mag}}$, depends on the temperature and on the deepness of the potential well they sit in. The (either absolute or relative) minimization of the functional $\mathcal{E}_{\text{mag}}$ accounts for the onset of the domain structure, as it is briefly illustrated in the caption of Fig. 1.13.

Domain Walls

Magnetic domains are separated by transition zones (walls) across which magnetization rotates smoothly, cf. Fig. 1.14. The layer thickness is a compromise between the tendency of the exchange energy to enlarge the layers, and that of the anisotropy energy $\mathcal{E}_{\text{an}}(\vec{M})$ to reduce their thickness, in order to align the field to a preferred direction, z. If either anisotropy is too small or the specimen is not large enough, a single domain appears. In the bulk of the ferromagnet the field is essentially parallel to z, whereas near the boundary it tends to stay tangential to the boundary itself, in order to reduce the field energy, $\mathcal{E}_{\text{field}}(\vec{M})$, cf. (1.103). Thus in thin films the domain magnetization is parallel to the film itself. Note that the demagnetizing

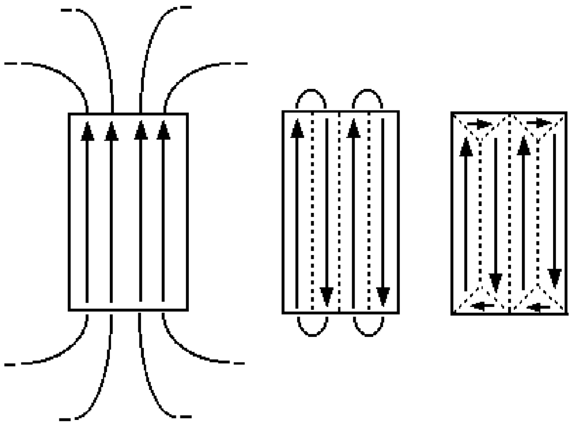


FIGURE 1.13 Along the lines of [59], here it is illustrated how progressive modifications of the magnetization of a ferromagnet may reduce the functional $\mathcal{E}_{\mathrm{mag}}$. The first configuration generates a large exterior magnetic field. The second one reduces that field at the expense of the formation of domains, which are separated by 180°-walls. The third one introduces so-called closure domains by forming 90°-walls; this modification eliminates the exterior field almost completely. The broken lines represent domain walls, whereas the solid lines are force lines of the field $\vec{M}$ inside the ferromagnet, of the field $\vec{H}$ outside.

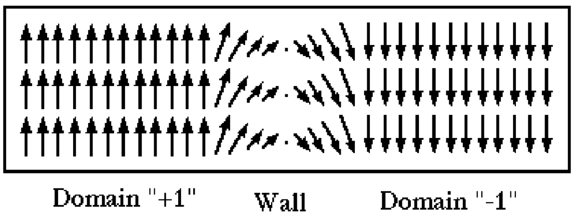


FIGURE 1.14 180° domain wall. More precisely, this a *hybrid Bloch–Néel wall*: the field $\vec{M}$ rotates in the plane of the page, and also in an orthogonal plane (it then appears shorter, because of perspective).

field $\vec{H}_{\mathrm{dem}}$ determines a nonlocal feedback: $\vec{M}$ depends on $\vec{H}_{\mathrm{dem}}$ via the above minimization principle, and in turn the latter depends on the former via the magnetostatic equations.

This theory accounts for the occurrence of different types of walls. For instance, let us consider a wall separating two domains in which the magnetization attains two opposite values parallel to the wall itself (a so-called 180°-wall). $\vec{M}$ then rotates through the wall matching smoothly the two opposite values. This rotation can occur in two fundamental modes: either parallel to the plane of the wall (*Bloch wall*), or orthogonal to that plane (*Néel wall*). The first mode is energetically convenient in the bulk, whereas the second mode is preferred near the boundary, for it allows the magnetization to remain parallel to the boundary itself. Actual walls have an intermediate structure, and more exotic structures may also occur, cf. Fig. 1.14. In thin films Néel walls may only occur.

In general the coefficient of the exchange energy is fairly small, and for certain ferromagnetic materials anisotropy is large. In that case the coefficients of the exchange and anisotropy energies can be represented as proportional to ε and ε^{-1}, respectively. As $\varepsilon \to 0$ a sharp interface is then obtained; this can be justified analytically via a Γ-limit operation [60].

1.5.2 THE LANDAU–LIFSHITZ EQUATION

In their seminal work [55] of 1935, Landau and Lifshitz proposed to represent micromagnetic evolution via the following equation:

$$\frac{\partial \vec{M}}{\partial t} = \lambda_1 \vec{M} \times \vec{H}^e - \lambda_2 \vec{M} \times (\vec{M} \times \vec{H}^e) \qquad \text{in } \Omega_T, \tag{1.105}$$

where the *effective magnetic field* $\vec{H}^e$ is defined as

$$\vec{H}^e := \vec{H} - \varphi'(\vec{M}) + \sum_{i,j=1}^{3} a_{ij} \frac{\partial^2 \vec{M}}{\partial x_i \partial x_j} \Big(= -\frac{\partial}{\partial \vec{M}} \mathcal{E}_{\text{mag}}(\vec{M}) \Big) \qquad \text{in } \Omega_T \tag{1.106}$$

(by $\frac{\partial}{\partial \vec{M}}$ we denote the Fréchet, or variational, derivative). λ_1 and λ_2 are positive phenomenological coefficients; typically $\lambda_1 > \lambda_2$, in some cases $\lambda_1 \gg \lambda_2$. We shall refer to the system (1.105), (1.106) as the *Landau–Lifshitz equation*.

This dynamic is equivalently expressed by the *Gilbert equation*

$$\frac{\partial \vec{M}}{\partial t} = \mu_1 \vec{M} \times \Big(\vec{H}^e - \frac{\mu_2}{\mu_1} \frac{\partial \vec{M}}{\partial t} \Big) \qquad \text{in } \Omega_T. \tag{1.107}$$

A simple calculation shows that the two pairs of constants (λ_1, λ_2) and (μ_1, μ_2) are related by the following transformation formulae:

$$\begin{cases} \lambda_1 = \dfrac{\mu_1}{1 + \mu_2^2 \mathcal{M}^2} \\ \lambda_2 = \dfrac{\mu_1 \mu_2}{1 + \mu_2^2 \mathcal{M}^2} \end{cases} \qquad \text{or equivalently} \qquad \begin{cases} \mu_1 = \dfrac{\lambda_1^2 + \lambda_2^2 \mathcal{M}^2}{\lambda_1} \\ \mu_2 = \dfrac{\lambda_2}{\lambda_1} \, . \end{cases}$$

(1.105) is the most simple relaxation dynamic for a magnet that is free to rotate and is subjected to a magnetic field $\vec{H}^e$. In fact, multiplying (1.105) scalarly by $\vec{M}$, one can see that $|\vec{M}|$ is constant in time. For a magnetic material surrounded by a diamagnetic medium, the following boundary condition holds:

$$\vec{M}\times\sum_{i,j=1}^{3} a_{ij}\frac{\partial\vec{M}}{\partial x_i}\nu_j = \vec{0} \qquad \text{on } \partial\Omega\times]0,T[. \tag{1.108}$$

Energy Balance

The vector

$$-\frac{1}{\mathcal{M}^2}\vec{M}\times\big(\vec{M}\times\vec{H}^e\big) = \vec{H}^e - \frac{1}{\mathcal{M}^2}\big(\vec{M}\cdot\vec{H}^e\big)\vec{M} \tag{1.109}$$

is the projection of the driving force $\vec{H}^e$ onto the tangent plane at the point $\vec{M}$ to the sphere with center $\vec{0}$ and radius $\mathcal{M}$. By (1.105), this term drives $\vec{M}$ to move towards $\vec{H}^e$ and is dissipative. The vector $\vec{M}\times\vec{H}^e$ lies in the same tangent plane and is orthogonal to $\vec{M}\times(\vec{M}\times\vec{H}^e)$; it induces $\vec{M}$ to rotate around $\vec{H}^e$ by forming a constant angle (precession) and with angular velocity proportional to $|\vec{H}^e|$; this contribution is not dissipative. As a result of the composition of these two forces, under a constant $\vec{H}^e$, $\vec{M}$ asymptotically converges to $\vec{H}^e$ along a nonplanar spiral on the spheric surface of radius $\mathcal{M}$. The relaxation time is proportional to $1/\lambda_2$. By (1.105) and (1.106),

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathrm{mag}}\big(\vec{M}\big) = \int_\Omega \frac{\partial\vec{M}}{\partial t}\cdot\vec{H}^e\,\mathrm{d}x = \lambda_2\int_\Omega \big|\vec{M}\times\vec{H}^e\big|^2\,\mathrm{d}x, \tag{1.110}$$

which yields the energy balance equation

$$\mathcal{E}_{\mathrm{mag}}\big(\vec{M}(t)\big) + \lambda_2\int_0^t \mathrm{d}\tau\int_\Omega \big|\vec{M}\times\vec{H}^e\big|^2\,\mathrm{d}x = \mathcal{E}_{\mathrm{mag}}\big(\vec{M}(0)\big). \tag{1.111}$$

Quasi-stationary Landau–Lifshitz equation

The larger is λ_2, the faster is the relaxation; in the limit as $\lambda_2 \to +\infty$, (1.105) yields

$$\vec{M}\times\vec{H}^e = \vec{0} \qquad \text{in } \Omega_T. \tag{1.112}$$

This means that the vectors $\vec{M}$ and $\vec{H}^e$ are pointwise parallel, and only entails that $\vec{M}$ is a stationary point of the magnetic energy functional $\mathcal{E}_{\mathrm{mag}}$

under the constraint $|\vec{M}| = \mathcal{M}$. The dissipation condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathrm{mag}}(\vec{M})\leqslant 0 \tag{1.113}$$

must then be required explicitly.

1.5.3 Modified Landau–Lifshitz Equation

The Landau–Lifshitz equation is rate-dependent: if $\big(\vec{H}(x,t), \vec{M}(x,t)\big)$ is a solution of (1.105) and (1.106), and $s : \mathbf{R}^+ \to \mathbf{R}^+$ is a diffeomorphism, then in general $\big(\vec{H}(x,s(t)), \vec{M}(x,s(t))\big)$ does not solve the same system. On the other hand, ferromagnetic hysteresis cycles are rate-independent for a fairly wide range of frequencies. This suggests that relaxation towards (metastable) equilibrium occurs on a shorter time-scale. Analytically, this can be represented by rewriting (1.105) of the form

$$\gamma\frac{\partial\vec{M}}{\partial t} = \tilde{\lambda}_1\vec{M}\times\vec{H}^e - \tilde{\lambda}_2\vec{M}\times\big(\vec{M}\times\vec{H}^e\big) \qquad \text{in } \Omega_T, \tag{1.114}$$

and then passing to the limit as the relaxation time γ vanishes. However, due to the nonlinearity of this equation, difficulties arise in performing the limit procedure; indeed in this way the time regularity of $\vec{M}$ is lost, hence the field $\vec{M}$ might exhibit uncontrolled oscillations in time.

Let us then revisit the physics of the problem. So far we have not accounted for the occurrence of *magnetic inclusions*, like impurities, dislocations, and so on. These inclusions induce a sort of internal friction which opposes the motion of domain walls, and thus contributes to dissipation. In fact domain walls are pinned by defects; unpinning requires a sufficiently strong effective magnetic field, and occurs by a small dissipative jump. This also accounts for the well-known *Barkhausen noise*. This process can be regarded as rate-independent, and is responsible for most of hysteresis dissipation in ferromagnetic materials that exhibit a domain structure. For instance, steel may be regarded as iron with uniformly distributed inclusions of carbon and other substances on a microscopic length-scale; it turns out that hysteresis is much more pronounced in this material than in pure iron.

Here we describe the effect of inclusions on magnetic dynamics in a way that is reminiscent of Coulomb's dry friction. Along the lines of [61] we couple the equation

$$\frac{\partial\vec{M}}{\partial t} = \lambda_1\vec{M}\times\vec{H}^E - \lambda_2\vec{M}\times\big(\vec{M}\times\vec{H}^E\big) \qquad \text{in } \Omega_T, \tag{1.115}$$

with the *modified effective magnetic field*

$$\vec{H}^E := \vec{H} - \varphi'(\vec{M}) + \sum_{i,j=1}^{3} a_{ij} \frac{\partial^2 \vec{M}}{\partial x_i \partial x_j} - \vec{Z} \ \left(= \vec{H}^e - \vec{Z}\right), \\ \text{where } \vec{Z} \in \partial\psi\left(\frac{\partial \vec{M}}{\partial t}\right). \tag{1.116}$$

Here $\partial\psi$ is the subdifferential of a prescribed convex function $\mathbf{R}^3 \to \mathbf{R}^+$, that is positively homogeneous of degree one: $\psi(a\vec{v}) = a\psi(\vec{v})$ for any $a > 0$ and any $\vec{v} \in \mathbf{R}^3$. By Proposition A.12, ψ is then a support function, i.e., it is of the form

$$\psi = I_K^* \qquad \text{where } K \subset \mathbf{R}^3 \text{ is closed and convex, and } 0 \in K;$$

(1.116) is thus equivalent to a variational inequality. The most simple choice of ψ is to take it proportional to the modulus function: $\psi(\vec{v}) = \eta|\vec{v}|$, η being a positive constant; in this case K is a ball with center at the origin. A more general ψ may account for anisotropy. Nonuniformity of the mesoscopic distribution of inclusions might also be represented by allowing ψ to depend explicitly upon x. As it occurs for similar equations, the fact that the function $\partial\psi$ is multivalued does not prevent the process from selecting a unique $\vec{Z} \in \partial\psi(\partial\vec{M}/\partial t)$.

We shall refer to the system (1.115), (1.116) as the *modified Landau–Lifshitz Equation*. (1.115) might also be written in the following equivalent Gilbert-like form, cf. (1.107):

$$\frac{\partial \vec{M}}{\partial t} = \mu_1 \vec{M} \times \left(\vec{H}^E - \frac{\mu_2}{\mu_1} \frac{\partial \vec{M}}{\partial t}\right) \qquad \text{in } \Omega_T. \tag{1.117}$$

Energetic Interpretation

As

$$\int_\Omega \frac{\partial \vec{M}}{\partial t} \cdot \vec{H}^E \, \mathrm{d}x = \int_\Omega \frac{\partial \vec{M}}{\partial t} \cdot \left(\vec{H}^e - \vec{Z}\right) \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{\mathrm{mag}}(\vec{M}) - \int_\Omega \psi\left(\frac{\partial \vec{M}}{\partial t}\right) \mathrm{d}x, \tag{1.118}$$

multiplying (1.115) by $\partial\vec{M}/\partial t$ we get the energy balance equation

$$\mathcal{E}_{\mathrm{mag}}\left(\vec{M}(t)\right) + \int_0^t \mathrm{d}\tau \int_\Omega \left[\lambda_2 \left|\vec{M} \times \vec{H}^E\right|^2 + \psi\left(\frac{\partial \vec{M}}{\partial t}\right)\right] \mathrm{d}x = \mathcal{E}_{\mathrm{mag}}\left(\vec{M}(0)\right). \tag{1.119}$$

Similar to (1.105), equation (1.115) represents a relaxation dynamic of $\vec{M}$ under the action of the field $\vec{H}^E$. Because of the $\partial\psi$-term, $\vec{M}$ can change only if $\vec{H}^E$ lies on the border of K. Sometimes this setting is ascribed to occurrence of an *activation energy*.

The system (1.115), (1.116) accounts for

(i) precession because of the λ_1-term;
(ii) relaxation (i.e., rate-dependent dissipation) because of the λ_2-term;
(iii) a component of hysteresis (i.e., rate-independent dissipation) related to the nonconvexity of the potential. This is due to the combined effect of the nonconvex constraint (1.100), and of anisotropy;
(iv) a friction-type component of hysteresis, due to magnetic inclusions. This is related to the lack of Gâteaux differentiability of the functional ψ, cf. the Appendix.

The larger the projection of the input field $\vec{H}$ onto the tangent plane at $\vec{M}$ to the sphere of radius $\mathcal{M}$, the larger is the effective field $\vec{H}^e$, the farther the system is from equilibrium, and the larger is the relaxation rate $\partial\vec{M}/\partial t$. On the other hand $\vec{Z}$ (cf. (1.116)) is confined to K. Friction effects are more evident at low rates, whereas at high rates they are dominated by relaxation. The presence of the $\partial\psi$-term increases the stability of the system, and enlarges the class of metastable states.

Quasi-stationary Modified Landau–Lifshitz Equation

Let us rewrite the modified Landau–Lifshitz equation in the form (1.114), with $\vec{H}^E$ in place of $\vec{H}^e$. By passing to the limit as the relaxation time γ vanishes, we get

$$\tilde{\lambda}_1 \vec{M} \times \vec{H}^E - \tilde{\lambda}_2 \vec{M} \times \big(\vec{M} \times \vec{H}^E\big) = \vec{0} \qquad \text{in } \Omega_T.$$

If $(\tilde{\lambda}_1, \tilde{\lambda}_2) \neq (0,0)$, this is equivalent to the quasi-stationary modified Landau–Lifshitz equation

$$\vec{M} \times \vec{H}^E = \vec{0} \qquad \text{in } \Omega_T. \tag{1.120}$$

This equation means that there exists $\vec{H}^E$ as in (1.116), and that this field is either parallel or antiparallel to $\vec{M}$ pointwise in Ω_T. This condition is rate-independent but not stationary, for $\vec{H}^E$ contains the term $\vec{Z}$, which depends on $\partial\vec{M}/\partial t$. (1.120) does not entail dissipation; we then require it explicitly, in analogy with (1.113):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathrm{mag}}(\vec{M}) + \int_\Omega \psi\Big(\frac{\partial \vec{M}}{\partial t}\Big)\,\mathrm{d}x \leqslant 0. \tag{1.121}$$

In conclusion, the system (1.120), (1.121) represents a mesoscopic model of ferromagnetic hysteresis, which can be coupled either with the Maxwell system or with the magnetostatic equations.

A number of monographs deal with ferromagnetism, e.g. Aharoni [62], Bertotti [2], Brown [56–58], Chikazumi and Charap [63], Cullity [64], Della Torre [4], Doering [65], Herpin [66], Hubert and Schäfer [54], Jiles [67], Kneller [68], Kronmüller [69], Maugin [70], Mayergoyz [6,7], Morrish [71]. The Landau–Lifshitz equation has been studied either independently or coupled with the Maxwell laws, see e.g. [61,72–89].

1.6 MODELS OF HYSTERESIS WITH INTERNAL VARIABLES

Population Models

The Prandtl–Ishlinskiĭ and Preisach models of hysteresis, which are respectively used to represent elasto-plastic and ferromagnetic hysteresis, account for the occurrence of internal variables, and are both formulated in terms of populations. In this section we outline and discuss a larger class of population models, whose applicability is not restricted to any specific physical phenomenon, and which can also be applied to nonhysteretic phenomena. In the case of ferromagentism, we shall see that it is not clear how a natural vector extension of the Preisach model might be consistent with the Maxwell system. We then propose an alternative approach, that is issued from the theory of *two-scale homogenization*, which was introduced by Allaire [90] and Nguetseng [91].

The Prandtl–Ishlinskiĭ and Preisach models are constructed by assembling a population of elementary models, either plays or stops or relays, which have common characteristics and only differ for the value of their characteristic parameters. Each of these constitutive elements is assumed to be subjected to the same input, which also coincides with the input of the composite model; we shall see that this assumption is crucial. These elements are also assumed to be reciprocally independent: each one then yields an output which does not depend either on the state or on the output of the others. These elementary outputs are then summed to produce the output of the global model.

1.6.1 Scalar Setting

First Model: the Particle-Density Approach

We deal with a material which occupies a Euclidean domain Ω, and assume that a population of (not better specified) *particles* sits at each coarse-scale point $x \in \Omega$. These particles may be characterized by their either elasto-plastic or magnetic behavior: here this choice is immaterial. We first assume that all fields have a fixed direction, and replace vectors by components in that direction, so that we can deal with a scalar setting. For the time being, we also neglect dependence on the space variable x. We assume that each of these particles is characterized by a function f_ρ, with ρ ranging through some set $\mathcal{P}$ of parameters. For a large part of this discussion we can also neglect memory effects; we thus assume that the f_ρ's are reduced to real functions. Actually, most of these developments also apply to phenomena without memory.

As different particles may correspond to a same ρ, we represent the space density of these different ρ's by means of a (positive) measure μ on $\mathcal{P}$. One may think that each ρ characterizes a family of undistinguishable particles, which coexist at any coarse-scale point x, and whose behavior is characterized by a function, f_ρ, which transforms an input variable, u, into an output variable, w. For instance, in elasto-plasticity u and w are the strain and stress, or conversely; in ferromagnetism as u and w we can take the magnetic field and the magnetization, respectively. Our basic constitutive relation is then

$$w(\rho) = f_\rho(u(\rho)) \qquad \forall \rho \in \mathcal{P}. \tag{1.122}$$

We represent by $\mu(\rho)$ the space density of the family of particles characterized by the parameter ρ, and introduce two key hypotheses: we assume that, at each $x \in \Omega$,

(i) all the particles are subjected to the same input, u; (1.122) then reads $w(\rho) = f_\rho(u) \;\; \forall \rho \in \mathcal{P}$; (1.123)

(ii) the total output equals the sum of the outputs of the different particles: $w = \int_{\mathcal{P}} w(\rho)\, \mathrm{d}\mu(\rho)$. (1.124)

These hypotheses obviously entail the following relation:

$$w = \int_{\mathcal{P}} f_\rho(u)\, \mathrm{d}\mu(\rho) =: \mathcal{F}_\mu(u) \qquad \text{at any } x \in \Omega. \tag{1.125}$$

Notice that this condition is just expressed in terms of the fields u, w, and keeps no trace of the underlying distributions $\rho \mapsto (u(\rho), w(\rho))$. This model and the corresponding operator, $\mathcal{F}_\mu$, are indeed characterized by the family $\{f_\rho : \rho \in \mathcal{P}\}$ and by the measure μ.

Second Model: Two-scale Approach

The occurrence of a population distributed on a fine scale suggests the possibility of applying a homogenization approach to the previous model.

To each coarse-scale point $x \in \Omega$ we associate a reference set, $\mathcal{Y} := [0, 1[$ say, that we equip with the Lebesgue measure. We assume that a particle sits at each $y \in \mathcal{Y}$, and transforms an input field $\tilde{u}(y)$ into an output $\tilde{w}(y)$; $y \in \mathcal{Y}$ thus represents a fine-scale variable. As the parameter ρ depends on the specific particle, there exists a function $\hat{\rho} : \mathcal{Y} \to \mathcal{P}$ such that $\rho = \hat{\rho}(y)$. Thus $\tilde{u}(y) = u(\hat{\rho}(y))$ and $\tilde{w}(y) = w(\hat{\rho}(y))$; henceforth we shall omit the tildes. The constitutive law (1.122) is here replaced by

$$w(y) = f_{\hat{\rho}(y)}(u(y)) \qquad \forall y \in \mathcal{Y}. \tag{1.126}$$

The assumptions (1.123) and (1.124) then read

$$w(y) = f_{\hat{\rho}(y)}(u) \qquad \forall y \in \mathcal{Y}, \qquad w = \int_{\mathcal{Y}} w(y)\, \mathrm{d}y, \tag{1.127}$$

whence

$$w = \int_{\mathcal{Y}} f_{\hat{\rho}(y)}(u)\, \mathrm{d}y =: \mathcal{G}_{\hat{\rho}}(u) \qquad \forall x \in \Omega. \tag{1.128}$$

As in (1.125), here the variables x and t have been omitted, and in (1.128) there is no trace of the underlying distributions $y \mapsto (u(y), w(y))$. This model and the corresponding function $\mathcal{G}_{\hat{\rho}}$ are characterized by the family $\{f_\rho : \rho \in \mathcal{P}\}$ and by the function $\hat{\rho} : \mathcal{Y} \to \mathcal{P}$.

Comparison

The two approaches above are equivalent, and $\mathcal{F}_\mu = \mathcal{G}_{\hat{\rho}}$, if and only if μ is the image of the ordinary three-dimensional Lebesgue measure, λ, by the mapping $\hat{\rho} : \mathcal{Y} \to \mathcal{P}$, that is,

$$\mathcal{F}_\mu = \mathcal{G}_{\hat{\rho}} \quad \Leftrightarrow \quad \mu = \hat{\rho}(\lambda). \tag{1.129}$$

Setting $|A| := \lambda(A)$ $(= \int_A \mathrm{d}x)$ for any measurable set $A \subset \mathbf{R}^3$, this condition on the measure μ means that $\mu(A) = |\hat{\rho}^{-1}(A)|$ for any A, or equivalently (setting $\mathrm{d}y = \mathrm{d}\lambda(y)$ in the integral, as usual)

$$\int_{\mathcal{P}} \varphi(\rho)\, \mathrm{d}\mu(\rho) = \int_{\mathcal{Y}} \varphi(\hat{\rho}(y))\, \mathrm{d}y \qquad \forall \varphi \in L^1(\mathcal{P}; \mu). \tag{1.130}$$

The two-scale approach may be regarded as underlying the particle-density model, and thus looks more fundamental. It provides a detailed account of the fine-scale structure: for any coarse-scale point x and any input u, it yields the fine-space field $w = w(y)$. The particle-density model, on the other hand, requires less information, for it only involves the density measure of the field $w = w(\rho)$, parameterized by the variable $\rho \in \mathcal{P}$.

Nevertheless, under condition (1.130), the two models are equivalent, for the final outcome of both is a relation between the coarse-scale fields u and w, with no trace of the underlying fine-scale field. We shall see that this equivalence is strictly related to the hypotheses (1.123).

There is an obvious analogy between the particle density and *Young's parameterized measures* [92–94], which indeed are also image measures. It may be noticed that the two-scale representation is more precise; on the other hand parameterized measures seem to be a more ductile tool, with a wider spectrum of applications.

Coupling with PDEs (Scalar Case)

Here we assume that Ω is univariate, replace it by an interval $]a, b[$, and set $Q :=]a, b[\times]0, T[$. We couple the w versus u constitutive relation with a PDE,

$$\frac{\partial}{\partial t}(u + w) - \frac{\partial^2 u}{\partial x^2} = g \qquad \text{in } Q, \tag{1.131}$$

$g = g(x, t)$ being a prescribed source field. This equation may represent the Maxwell–Ohm equations without displacement current for a univariate conducting material (with normalized coefficients); in this case u and w respectively denote the magnetic field, H, and the magnetization, M.

Let us couple the coarse-scale equation (1.131) with a coarse-scale constitutive relation between the fields u and w: let us prescribe either $w = \mathcal{F}_\mu(u)$ or $w = \mathcal{G}_{\hat\rho}(u)$ pointwise in $]a, b[$, cf. (1.125), (1.128). This problem can be treated also if $\mathcal{F}_\mu$ and $\mathcal{G}_{\hat\rho}$ are replaced by hysteresis operators. Under natural assumptions on the data, it is not difficult to prove the existence of a solution such that

$$u \in H^1\big(0, T; L^2(a, b)\big) \cap L^\infty\big(0, T; H^1(a, b)\big) =: X, \qquad w \in L^\infty(Q). \tag{1.132}$$

cf. e.g. [8; Chapter IX] and Section 1.9 below. The argument is based on approximation by time discretization. An energy-type balance can be derived by multiplying the approximate equation by the approximate solution, u_τ, τ being the time-step. This yields a uniform estimate for u_τ in X. Therefore there exists $u \in X$ such that, possibly extracting a subsequence, $u_\tau \to u$ weakly star in X. Because of the compactness of the injection $\tilde{X} \to L^2\big(a, b; C^0([0, T])^3\big)$, possibly extracting a further subsequence,

$$u_\tau \to u \qquad \text{in } C^0([0, T]), \text{ a.e. in }]a, b[. \tag{1.133}$$

As we saw, several hysteresis operators are either continuous on $C^0([0, T])$ or discontinuous but closed; either property suffices to pass to the limit in the hysteresis relation, and thus to conclude the proof of the existence of a solution for our problem.

We conclude that in the scalar setting a coarse-scale formulation of the Maxwell–Ohm equations coupled with the population model can be justified analytically.

1.6.2 VECTOR SETTING

Now we extend the two models above to vectors. At any coarse-scale point x of the domain Ω of $\mathbf{R}^3$, we still represent the system as an aggregate of particles. We still denote the parameter set by $\mathcal{P}$, although in general it will be different from that of the univariate setting (as it is the case for the vector Preisach model). We represent the input–output relation in the form

$$\vec{w} = \vec{f}_\rho(\vec{u}) \qquad \forall \rho \in \mathcal{P}, \tag{1.134}$$

and still assume that $\mathcal{P}$ is equipped with a density measure μ. Under the assumptions (1.123) and (1.124), the input $\vec{u}$ and the output $\vec{w}$ are related as follows

$$\vec{w} = \int_{\mathcal{P}} \vec{f}_\rho(\vec{u})\, \mathrm{d}\mu(\rho) =: \vec{\mathcal{F}}_\mu(\vec{u}) \qquad \text{at any } x \in \Omega, \tag{1.135}$$

similarly to what we saw in the scalar setting, cf. (1.125). The corresponding two-scale model is also analogous to the scalar one: at any $x \in \Omega$, we associate a reference set, $\mathcal{Y} := [0, 1[^3$, that we equip with the Lebesgue measure. We then prescribe a mapping $\hat{\rho} : \mathcal{Y} \to \mathcal{P}$. As above, for any $y \in \mathcal{Y}$, $\hat{\rho}(y)$ determines the mapping $f_{\hat{\rho}(y)}$, which characterizes the behavior of the particle sitting at the point y. The hypotheses (1.123) and (1.124) then entail that

$$\vec{w}(x) = \int_{\mathcal{Y}} \vec{f}_{\hat{\rho}(y)}(\vec{u}(x))\, \mathrm{d}y =: \vec{\mathcal{G}}_{\hat{\rho}}(\vec{u}(x)) \qquad \text{for a.a. } x \in \Omega. \tag{1.136}$$

It is clear that these two models are equivalent, and $\vec{\mathcal{F}}_\mu = \vec{\mathcal{G}}_{\hat{\rho}}$, if and only if μ is the image of the Lebesgue measure by the mappings $\hat{\rho}$, cf. (1.129), (1.130).

Coupling with PDEs (Vector Case)

We shall see that in the vector case the analysis of the Maxwell–Ohm equations is more problematic than in the scalar setting.

Let us consider electromagnetic processes of a three-dimensional ferromagnetic metal, that we represent by a Euclidean domain Ω. Let us denote the magnetic field by $\vec{u}$ and the magnetization by $\vec{w}$. The system of the

Maxwell–Ohm equations without displacement current (with normalized coefficients) yields

$$\frac{\partial}{\partial t}(\vec{u}+\vec{w}) + \nabla\times\nabla\times\vec{u} = \vec{g} \qquad \text{in } \Omega_T := \Omega\times]0,T[\ \ (\nabla\times := \text{curl}), \quad (1.137)$$

$\vec{g} = \vec{g}(x,t)$ being a given function. As an alternative one might deal with the corresponding system of first-order equations, namely the Ampère and Faraday laws; this would have no effect on the conclusions we draw below.

Let us couple the coarse-scale equation (1.137) with a coarse-scale constitutive relation between $\vec{u}$ and $\vec{w}$; let us then prescribe either $\vec{w} = \vec{\mathcal{F}}_\mu(\vec{u})$ or $\vec{w} = \vec{\mathcal{G}}_{\hat{\rho}}(\vec{u})$ pointwise in Ω_T. Natural regularity conditions for the solution of (1.137) reads as follows:

$$\vec{u} \in H^1\big(0,T;L^2(\Omega)^3\big)\cap L^\infty\big(0,T;L^2_{\text{rot}}(\Omega)^3\big) =: \tilde{X}, \qquad \vec{w} \in L^\infty(\Omega_T)^3. \quad (1.138)$$

In analogy to what we saw for the scalar setting, a suitable time-discretized problem has a solution, and a uniform-in-τ estimate in $\tilde{X}$ can be derived for the approximate field $\vec{u}_\tau$. Therefore there exists $\vec{u} \in \tilde{X}$ such that, possibly extracting a subsequence,

$$\vec{u}_\tau \to \vec{u} \qquad \text{weakly star in } \tilde{X}, \text{ as } \tau \to 0. \quad (1.139)$$

However, space oscillations of increasing frequency are not excluded in this limit, since the injection $\tilde{X} \to L^2_{\text{loc}}(\Omega; C^0([0,T])^3)$ is not compact. It is then not clear how one might pass to the limit in the hysteresis relation, and thus prove the existence of a solution to our problem.

We conclude that in the vector setting we are not able to justify analytically a coarse-scale formulation of the Maxwell–Ohm equations coupled with the population model. These difficulties are strictly related to the occurrence of the double-curl operator in the equation (1.137).

Homogenization Approach

The above discussion suggests that the field $\vec{u}(x,t)$ might have a nontrivial fine-scale space structure, which should be accounted for in the formulation of the problem. This induces us to amend the hypothesis (1.123), and to apply the viewpoint of homogenization theory. Let us denote the mesoscopic length-scale by ε. Dealing with the two-scale-type vector model, we already defined the mapping $\hat{\rho} : \mathcal{Y} \to \mathcal{P}$. Now we extend that mapping to $\mathbf{R}^3$ by $\mathcal{Y}$-periodicity, and consider the following $\vec{w}$ versus $\vec{u}$ relation:

$$\vec{w}(x) = \vec{f}_{\hat{\rho}(x/\varepsilon)}(\vec{u}(x)) \qquad \text{for a.a. } x \in \Omega. \quad (1.140)$$

This represents the constitutive behavior of a periodic ferromagnetic material, which in each period $\varepsilon\mathcal{Y}$ reproduces the arrangement of magnetic particles that we represented by the function $\hat{\rho}(y)$. At this point one should couple this constitutive relation with the Maxwell system, and study the limit as ε vanishes. (In these notes we do not enter that issue.)

Comparison between the Prandtl–Ishlinskiĭ and Preisach Models

As we anticipated, the above discussion is not restricted to ferromagnetism, and might also be applied to a number of phenomena. For instance, in elasto-plasticity one can assume the (linearized) strain and the stress as state variables. In Section 1.3 we formulated two basic models: the play, and the stop; the corresponding constitutive relations can be expressed in terms of variational inequalities. By means of parallel and series arrangements of these elements, we then constructed the Prandtl–Ishlinskiĭ models, which can be represented by systems of variational inequalities.

Along the lines of the discussion of this section, we distinguish a fine-scale description (two-scale approach) from that based on the image measure (particle-density model). The latter is at the basis of the Prandtl–Ishlinskiĭ model, and provides a less detailed description than the former. There is an obvious analogy between relays and the Preisach model on one hand, and plays or stops and the Prandtl–Ishlinskiĭ models on the other.

However, these two settings exhibit a relevant difference. For ferromagnetism we have just seen the difficulties that arise in coupling the Maxwell–Ohm equations with a population model expressed in terms of the image measure (thus using a coarse-scale constitutive law). On the other hand, in the case of elasto-plasticity well-posedness has been proved for the dynamic equation coupled with the Prandtl–Ishlinskiĭ models (that is a coarse-scale constitutive law, too) [88; Chapter VII]. This can be ascribed to the intrinsic convex structure plays and stops have, at variance with relays; indeed plays and stops can be represented via variational inequalities.

1.7 GENESIS OF HYSTERESIS AND COUPLING WITH PDEs

1.7.1 Hysteresis and Nonmonotonicity

Discontinuous hysteresis relations can be approximated by introducing a time-relaxation term into equations that contain a nonmonotone function. In this section we couple equations of this type with some especially simple

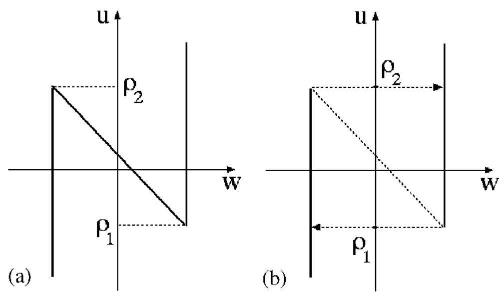


FIGURE 1.15 Nonmonotone u versus w relation (1.142) in (a). Associated relaxation dynamics approximating a hysteresis behavior in (b).

ODEs and PDEs, and deal with the limit behavior as the relaxation constant vanishes.

By inverting a nonmonotone relation, a multivalued correspondence is obtained; this represents a *feedback* mechanism. A hysteresis relation is then derived by assuming a suitable dynamic. For instance, let us fix any $\rho_1, \rho_2 \in \mathbf{R}$ with $\rho_1 < \rho_2$, and consider the nonmonotone relation of Fig. 1.15(a):

$$u \in \alpha(w) := \begin{cases}]-\infty, \rho_2] & \text{if } w = -1, \\ \frac{1}{2}[\rho_1 + \rho_2 - (\rho_2 - \rho_1)w] & \text{if } -1 < w < 1, \\ [\rho_1, +\infty[& \text{if } w = 1. \end{cases} \tag{1.141}$$

Let us fix a constant $\varepsilon > 0$, approximate (1.141) by means of the relaxation dynamics

$$\varepsilon \frac{\mathrm{d}w_\varepsilon}{\mathrm{d}t} + \alpha(w_\varepsilon) \ni u \qquad \text{for } t > 0, \tag{1.142}$$

and set

$$\lambda(w) := \frac{1}{2}[(\rho_1 - \rho_2)w + \rho_1 + \rho_2] \qquad \forall w \in \mathbf{R}.$$

Let us also denote by $\partial I_{[-1,1]}$ the subdifferential of the indicator function of the interval $[-1, 1]$ (cf. the Appendix), so that $\alpha = \lambda + \partial I_{[-1,1]}$. The inclusion (1.142) is then equivalent to the following variational inequality

$$\begin{cases} -1 \leqslant w_\varepsilon \leqslant 1 \\ \left(\varepsilon \dfrac{\mathrm{d}w_\varepsilon}{\mathrm{d}t} + \lambda(w_\varepsilon) - u\right)(w_\varepsilon - v) \leqslant 0 \quad \forall v \in [-1, 1] \end{cases} \qquad \text{for } t > 0. \tag{1.143}$$

The two vertical branches of the graph of α are attractors for this dynamic, whereas the oblique segment is a repulsor. Let us assume that the evolution of u is prescribed, and make some heuristic remarks. If at the initial instant the pair (u, w) does not lie on the oblique branch, then after a transient $w \simeq \pm 1$ and $u \simeq z$ for some $z \in \alpha(w)$.

If $w(t) = -1$, as u increases beyond the upper threshold ρ_2, by (1.142) w increases until $w \simeq 1$. Similarly, if $w(t) = 1$, as u decreases below the lower threshold ρ_1, then w decreases to $w \simeq -1$. The smaller is ε, the faster is this dynamic; see Fig. 1.15(b). Therefore, as ε vanishes, (1.142) tends to a discontinuous hysteresis relation between u and w; more precisely, in the limit we get a relation of the form

$$w \in \bar{h}_\rho(u, w^0) \qquad \text{for } t > 0. \tag{1.144}$$

The multivalued operator $\bar{h}_\rho$ acts from $C^0([0, T])$ to $2^{BV(0,T)}$ for any $T > 0$; it essentially differs from h_ρ (we defined in Section 1.4) in that, if $u(t) = \rho_1$ ($u(t) = \rho_2$, respectively) at some instant t, then $[\bar{h}_\rho(u, w^0)](t)$ can attain either -1 or 1 [88; Section XI.1].

This analysis rests on the assumption that the evolution of u is prescribed; if this restriction is removed, a more complex behavior may occur. We now couple the relaxation dynamic (1.142) with a differential equation that involves both u and w, and see whether the feedback that w exerts on u via this equation is able to modify the above dynamic.

Coupling with ODEs

(i) Let $f \in L^1_{\text{loc}}(\mathbf{R}^+)$ be a given function, and consider the system

$$\begin{cases} \dfrac{\mathrm{d}u_\varepsilon}{\mathrm{d}t} + w_\varepsilon = f \\ \varepsilon \dfrac{\mathrm{d}w_\varepsilon}{\mathrm{d}t} + \alpha(w_\varepsilon) \ni u_\varepsilon \end{cases} \qquad \text{for } t > 0 \ (\varepsilon\text{: constant } > 0). \tag{1.145}$$

The function $u = \lim_{\varepsilon \to 0} u_\varepsilon$ is necessarily continuous. We claim that then the feedback exerted on u by w via the differential equation $(1.145)_1$ does not modify the above picture; as $\varepsilon \to 0$ we thus get

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + w = f \\ w \in \bar{h}_\rho(u, w^0) \end{cases} \qquad \text{for } t > 0. \tag{1.146}$$

For instance, let us start from $u_\varepsilon(0) < \rho_2$, $w_\varepsilon(0) = -1$, and let $f = f_0$ (constant) > -1. Then u_ε increases, until it overtakes ρ_2. At that point,

w_ε quickly moves up, and attains a value $\simeq 1$ in a time of order of $1/\varepsilon$. It might be objected that, as w_ε increases, $\mathrm{d}u_\varepsilon/\mathrm{d}t = f - w_\varepsilon$ may change sign, so that u_ε may invert its direction and thus decrease below the threshold ρ_2. But, if ε is small enough, this cannot stop the increase of w_ε. Indeed, as w_ε increases, the threshold $\frac{1}{2}[\rho_1 + \rho_2 - (\rho_2 - \rho_1)w_\varepsilon]$ at which $\mathrm{d}w_\varepsilon/\mathrm{d}t$ changes sign moves to values $< \rho_2$; and the smaller is ε, the faster is this motion.

The latter argument depends on the shape of the graph of α in a neighborhood of the turning points $(-1, \rho_2)$ and $(1, \rho_1)$; if α were modified so to have a horizontal tangent at those points instead of a corner, then the discussion would be more delicate. (We refrain from entering these details here.)

Thus $f - w_\varepsilon \simeq f_0 - 1$ after a transient; if $f_0 < 1$, then u_ε inverts its direction, and decreases until $u_\varepsilon \leqslant \rho_1$. Therefore if $-1 < f_0 < 1$ then the pair $(u_\varepsilon, w_\varepsilon)$ keeps on cycling: u oscillates between ρ_1 and ρ_2, and w jumps back and forth between -1 and 1. This construction is used to model several oscillatory behaviors in physics, biology, engineering, and so on.

(ii) Let us now consider a system of the form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(u_\varepsilon + w_\varepsilon) + w_\varepsilon = f \\ \varepsilon\dfrac{\mathrm{d}w_\varepsilon}{\mathrm{d}t} + \alpha(w_\varepsilon) \ni u_\varepsilon \end{cases} \qquad \text{for } t > 0. \tag{1.147}$$

If f is locally bounded, the function $u + w = \lim_{\varepsilon\to 0}(u_\varepsilon + w_\varepsilon)$ is continuous, whereas u needs not be so. Let us then set $\tilde{\alpha}(v) := \alpha(v) + v$ for any $v \in \mathbf{R}$, and rewrite $(1.147)_2$ in the equivalent form

$$\varepsilon\frac{\mathrm{d}w_\varepsilon}{\mathrm{d}t} + \tilde{\alpha}(w_\varepsilon) \ni u_\varepsilon + w_\varepsilon \qquad \text{for } t > 0 \ (\varepsilon\text{: constant } > 0), \tag{1.148}$$

so that the function $u_\varepsilon + w_\varepsilon$ here plays the role that before was of u_ε. If the function $\tilde{\alpha}$ is nondecreasing, i.e. $\rho_2 - \rho_1 \leqslant 2$, then no hysteresis occurs as $\varepsilon \to 0$. If instead $\tilde{\alpha}$ is decreasing in $]-1, 1[$, i.e. $\rho_2 - \rho_1 > 2$, then hysteresis appears; in the limit we then get

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(u + w) + w = f \\ w \in \mathcal{F}(u + w, w^0) \end{cases} \qquad \text{for } t > 0. \tag{1.149}$$

Here $\mathcal{F}$ is a multivalued hysteresis operator: $\mathcal{F}$ is analogous to k_ρ of (1.72)–(1.74), but its thresholds are different.

Coupling with PDEs

(iii) Let Ω be a bounded domain of $\mathbf{R}^N$ $(N \geqslant 1)$, f be a prescribed function $\Omega \times \mathbf{R}^+ \to \mathbf{R}$. Let us set $\Delta := \sum_{i=1}^N \partial^2/\partial x_i^2$, consider a system of the form

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon + w_\varepsilon = f \\ \varepsilon \dfrac{\partial w_\varepsilon}{\partial t} + \alpha(w_\varepsilon) \ni u_\varepsilon \end{cases} \quad \text{in } \Omega, \text{ for } t > 0, \tag{1.150}$$

and couple it with initial and boundary conditions for u_ε. If f is regular enough, by analogy with the ODE system (1.145), as $\varepsilon \to 0$ we may expect to get

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + w = f \\ w \in k_\rho(u, w^0) \end{cases} \quad \text{in } \Omega, \text{ for } t > 0. \tag{1.151}$$

(iv) Let us now consider an equation of the form

$$\frac{\partial}{\partial t}(u_\varepsilon + w_\varepsilon) - \Delta u_\varepsilon = f \qquad \text{in } \Omega, \text{ for } t > 0. \tag{1.152}$$

If this is coupled with a maximal monotone relation $w_\varepsilon = a(u_\varepsilon)$, the corresponding initial- and boundary-value problem is well posed under mild assumptions [95,96]. The same holds also if (1.152) is coupled with the relaxation dynamics (1.142), for any $\varepsilon > 0$; but in this case, as ε vanishes, the analysis of the asymptotic behavior of the pair $(u_\varepsilon, w_\varepsilon)$ is nontrivial. Actually, here one cannot expect that the evolution of u_ε uncouples from that of w_ε, since the feedback exerted by w_ε on u_ε might modify the above picture.† Therefore *a priori* it is not obvious that by passing to the limit in the system (1.142), (1.152) one would get

$$\begin{cases} \dfrac{\partial}{\partial t}(u + w) - \Delta u = f \\ w \in k_\rho(u, w^0) \end{cases} \quad \text{in } \Omega, \text{ for } t > 0. \tag{1.153}$$

An Alternative Approach

Now we provide an alternative interpretation of the relaxation dynamics (1.142), and introduce a different approach, along the lines of [97]. We still assume that the function α is as in (1.141), although our discussion might easily be extended to other nonmonotone functions.

† This would be the case even if (1.142) were coupled with a simple equation like $u + bw = g$, where b is a constant $> (\rho_2 - \rho_1)/2$ and g is a prescribed function of (x, t).

Let us consider a space-distributed system the state of which is characterized by two scalar variables, u and w. By u we shall denote a continuous quantity, such as temperature, or the component of the magnetic field along a fixed direction, and so on. On the other hand, $w = \pm 1$ will represent two admissible phases; e.g., solid and liquid, or up and down orientation of magnetization in a uniaxial ferromagnet. The variables u and w will be related by a constitutive law like (1.141), that we couple with the equation

$$\frac{\partial}{\partial t}(u + w) - \Delta u = f \qquad \text{in } \Omega, \text{ for } t > 0; \tag{1.154}$$

this may represent either the energy balance for solid–liquid systems, or (for a univariate system) the coupling of the Ohm and Maxwell laws without displacement current. However, here we shall not refer to any specific applicative setting.

Now we assume that at the small length-scale the two phases are well separated, but that at the coarse length-scale they may look to be finely mixed. On the latter scale we then represent the phase by a new variable $\tilde{w}$, which we derive from w via a local average, or by convolution with a compactly supported *mollifier*. Accordingly $\tilde{w}$ will range in the set $[-1, 1]$; values of $\tilde{w} \in]-1, 1[$ may be interpreted as representing mixtures, with a fraction $\frac{1}{2}(1 + \tilde{w})$ ($\frac{1}{2}(1 - \tilde{w})$, respectively) of particles in the state $w = 1$ ($w = -1$, respectively).(Here we use the term *particle* rather freely, referring to a small aggregate of elements.) Thus w and $\tilde{w}$ can respectively be regarded as fine-scale (or mesoscopic) and coarse-scale (or macroscopic) phase variables.

In the framework of the two-scale model that we are outlining, as long as $\rho_1 < u < \rho_2$ the particles that are in either state $w = \pm 1$ remain unchanged. The same then occurs for the average value $\tilde{w}$; as long as $\rho_1 < u < \rho_2$, the coarse scale variable $\tilde{w}$ then constantly attains some value of $[-1, 1]$. This may represent the behavior of $\tilde{w}$ in the critical cases in which either

(i) u drops below ρ_2 while $\tilde{w}$ is rapidly moving towards 1, or symmetrically

(ii) u increases beyond ρ_1 while $\tilde{w}$ is rapidly moving towards -1.

On the basis of the above discussion, we set

$$\varphi(u, \tilde{w}) := \begin{cases} 1 - \tilde{w} & \text{if } u > \rho_2, \\ 0 & \text{if } \rho_1 \leqslant u \leqslant \rho_2, \\ \tilde{w} - 1 & \text{if } u < \rho_1, \end{cases} \qquad \forall (u, \tilde{w}) \in \mathbf{R}^2; \tag{1.155}$$

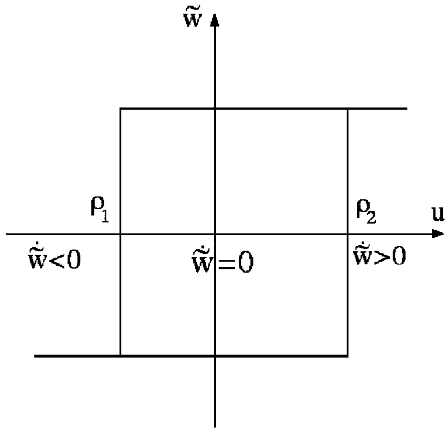


FIGURE 1.16 Outline of the coarse-scale relaxation dynamics $\varepsilon \mathrm{d}\tilde{w}/\mathrm{d}t = \varphi(u, \tilde{w})$ corresponding to φ as in (1.155).

we then replace the fine-scale relaxation equation (1.142) by the (nonequivalent) coarse-scale dynamic (cf. Fig. 1.16)

$$-1 \leqslant \tilde{w} \leqslant 1, \qquad \varepsilon \frac{\partial \tilde{w}}{\partial t} = \varphi(u, \tilde{w}) \qquad \text{in } Q. \tag{1.156}$$

A More General Setting

Let us assume that

$$\begin{cases} \gamma_\ell, \gamma_r \text{ are maximal monotone (possibly multivalued) functions} \\ \mathbf{R} \to 2^{\mathbf{R}} \setminus \{\emptyset\} \text{ such that } \inf \gamma_r(u) \leqslant \sup \gamma_\ell(u) \ \ \forall u \in \mathbf{R}, \text{ cf. Fig. 1.17(a),} \end{cases} \tag{1.157}$$

and set

$$\varphi(u, \tilde{w}) := [\tilde{w} - \inf \gamma_r(u)]^- - [\tilde{w} - \sup \gamma_\ell(u)]^+ \qquad \forall (u, \tilde{w}) \in \mathbf{R}^2; \tag{1.158}$$

thus

$$\varphi(u, \tilde{w}) = 0 \quad \Leftrightarrow \quad \inf \gamma_r(u) \leqslant \tilde{w} \leqslant \sup \gamma_\ell(u) \qquad \forall (u, \tilde{w}) \in \mathbf{R}^2.$$

As ε vanishes in the corresponding relaxation equation (1.156), we get the dynamic of Fig. 1.17(b).

The equation (1.156), with φ as in (1.155) or (1.158), can be coupled with the equation

$$\frac{\partial}{\partial t}(u + \tilde{w}) - \Delta u = f \qquad \text{in } \Omega, \text{ for } t > 0. \tag{1.159}$$

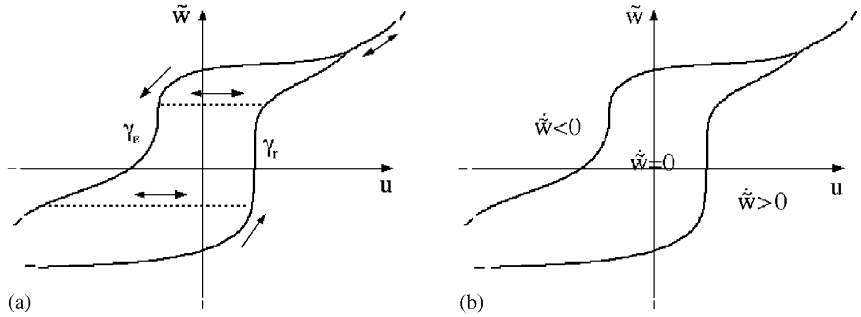


FIGURE 1.17 Function φ of (1.158) in (a); corresponding coarse-scale relaxation dynamics $\varepsilon \mathrm{d}\tilde{w}/\mathrm{d}t = \varphi(u, \tilde{w})$ in (b).

In [97] corresponding initial- and boundary-value problems were formulated, and the behavior of the solution $(u, \tilde{w})$ as ε vanishes was studied.

1.8 A TRANSPORT EQUATION WITH HYSTERESIS

In this section we introduce PDEs with hysteresis operators. We shall use typical notation of the analysis of P.D.E.s in Sobolev spaces, see e.g. [3,5,8] and references therein. We then deal with a simple example: we prove well-posedness and asymptotic stability of an initial- and boundary-value problem for a semilinear transport equation in a single dimension of space.

1.8.1 PDEs with Hysteresis

First we extend the definition of hysteresis operator to space-distributed systems. Let Ω be a domain of $\mathbf{R}^N$ ($N \geqslant 1$) and T be a positive constant. We assume that

$$\mathcal{F} : C^0([0, T]) \times \mathbf{R} \to C^0([0, T]) \text{ is a continuous hysteresis operator,} \tag{1.160}$$

cf. (1.2) and (1.3), and define the corresponding space-distributed operator $\tilde{\mathcal{F}}$ as follows:

$$\begin{aligned} &\forall u \in L^1_{\mathrm{loc}}\big(\Omega; C^0([0, T])\big), \forall w^0 \in L^1_{\mathrm{loc}}(\Omega), \text{ for a.a. } x \in \Omega, \\ &\forall t \in [0, T], \ \big[\tilde{\mathcal{F}}\big(u, w^0\big)\big](x, t) := \big[\mathcal{F}\big(u(x, \cdot), w^0(x)\big)\big](t). \end{aligned} \tag{1.161}$$

We thus regard the space variable x as a parameter, and neglect any space interaction between neighboring points, for no space derivative occurs in (1.161). This can also be extended to multivalued hysteresis operator. Henceforth we identify $\tilde{\mathcal{F}}$ with $\mathcal{F}$, and omit the tilde.

Let A be an elliptic operator. Here are some basic examples of PDEs with hysteresis, in $\Omega \times]0, T[$:

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] + Au = f, \tag{1.162}$$

$$\frac{\partial u}{\partial t} + Au + \mathcal{F}(u) = f, \tag{1.163}$$

$$\frac{\partial^2}{\partial t^2}[u + \mathcal{F}(u)] + Au = f, \tag{1.164}$$

$$\frac{\partial^2 u}{\partial t^2} + Au + \frac{\partial}{\partial t}\mathcal{F}(u) = f, \tag{1.165}$$

$$\frac{\partial^2 u}{\partial t^2} + Au + \mathcal{F}(u) = f, \tag{1.166}$$

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] + \frac{\partial u}{\partial x} = f \qquad (\text{for } N = 1), \tag{1.167}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \mathcal{F}(u) = f \qquad (\text{for } N = 1). \tag{1.168}$$

Classification of PDEs with Hysteresis

The standard classification of nonlinear PDEs can be extended to equations that contain hysteresis operators.

In Section 1.1 we saw that any scalar hysteresis operator, $\mathcal{F}$, is reduced to a (possibly multivalued) superposition operator on any time interval in which the input function is monotone (either nondecreasing or nonincreasing). Let us denote by $S_{\mathcal{F}}$ this class of superposition operators; in several examples of applicative interest, these operators are associated with nondecreasing functions (we named this property piecewise monotonicity of $\mathcal{F}$). We say that a scalar equation that contains $\mathcal{F}$ is either parabolic or hyperbolic whenever it would be so if the operator $\mathcal{F}$ were replaced by any element of $S_{\mathcal{F}}$. These equations are nonlinear, and by the same criterion we also extend the usual definition of semilinearity, quasilinearity and full nonlinearity.

These definitions can be extended to vector hysteresis operators, provided that whenever a vector input function evolves monotonically along any fixed (possibly x-dependent) direction, the vector output is reduced to a (possibly multivalued) superposition operator. This property is fulfilled, for instance, by the vector-relay and the vector Preisach operator. The same applies to tensor hysteresis operators, like those of Prandtl–Ishlinskiĭ.

For instance, the equations (1.162), (1.163) are parabolic, and (1.164)–(1.168) are hyperbolic. On the other hand (1.162), (1.164), (1.167) are quasilinear, and (1.163), (1.165), (1.166), (1.168) are semilinear.

1.8.2 Transport Equation with Hysteresis in Classical Spaces

In this and the next sections we shall deal with some of the above equations. In the remainder of this section we study the semilinear first-order equation (1.168), along the lines of [8; Section X.4]; this example is rather simple, and gives us the opportunity to introduce some of the techniques that are currently used for the analysis of PDEs with hysteresis.

First we fix any $T > 0$, set $Q_t := \mathbf{R}^+ \times]0, t[$ for any $t > 0$ and $Q := Q_T$, denote by $C^0_b(\mathbf{R}^+)$ the Banach space of bounded continuous functions $\mathbf{R}^+ \to \mathbf{R}$ equipped with the uniform norm, and define $C^0_b(\bar{Q})$ similarly. We assume that (1.160) holds and that

$$f \in C^0_b(\bar{Q}), \quad u^0, w^0 \in C^0_b(\mathbf{R}^+), \quad v^0 \in C^0([0, T]), \quad u^0(0) = v^0(0); \tag{1.169}$$

by integrating the equation (1.168) along the characteristic lines $x - t =$ constant, we now formulate an initial- and boundary-value problem in integral form in the space of continuous functions.

Problem 1.8.1. *Find $u \in C^0_b(\bar{Q})$ such that, setting*

$$w(x, t) := \big[\mathcal{F}\big(u(x, \cdot), w^0(x)\big)\big](t) \qquad \forall (x, t) \in \bar{Q}, \tag{1.170}$$

one has

$$u(x, t) = \int_0^t [f - w](x - t + \tau, \tau)\, \mathrm{d}\tau + \begin{cases} u^0(x - t) & \text{if } 0 < t \leqslant x \\ v^0(t - x) & \text{if } 0 < x < t \leqslant T \end{cases} \tag{1.171}$$

for any $(x, t) \in \bar{Q}$.

Theorem 1.8.1 (*Well-posedness*). *Let (1.160), (1.169) hold, and assume that*

$$\begin{aligned} &\exists L, \hat{L} > 0 : \forall v_1, v_2 \in C^0([0, T]), \forall \xi_1, \xi_2 \in \mathbf{R}, \forall t \in]0, T], \\ &\max_{[0,t]} |\mathcal{F}(v_1, \xi_1) - \mathcal{F}(v_2, \xi_2)| \leqslant L \max_{[0,t]} |v_1 - v_2| + \hat{L}|\xi_1 - \xi_2|, \end{aligned} \tag{1.172}$$

$$\forall x > 0, \forall v \in C^0([0, T]), \text{ if } v(0) = u^0(x) \text{ then } \big[\mathcal{F}\big(v, w^0(x)\big)\big](0) = w^0(x). \tag{1.173}$$

Then Problem 1.8.1 is well posed. More precisely, for $i = 1, 2$, *let* u_i^0, w_i^0, v_i^0, f_i *be as in (1.169), and denote by* u_i *the corresponding solution of Problem 1.8.1. Set* $\tilde{u} := u_1 - u_2$, *define* $\tilde{u}^0, \tilde{w}^0, \tilde{v}^0, \tilde{f}$ *similarly, and set*

$$M(\tilde{u}^0, \tilde{v}^0, t) := \max\left\{\|\tilde{u}^0\|_{C_b^0(\mathbf{R}^+)}, \|\tilde{v}^0\|_{C^0([0,t])}\right\};$$

then

$$\|\tilde{u}\|_{C_b^0(\bar{Q}_t)} \leqslant e^{Lt}\Big(\int_0^t \|\tilde{f}\|_{C_b^0(\bar{Q}_\tau)}\,\mathrm{d}\tau + \hat{L}t\|\tilde{w}^0\|_{C_b^0(\mathbf{R}^+)} + M(\tilde{u}^0, \tilde{v}^0, t)\Big) \qquad \forall t \in]0, T]. \tag{1.174}$$

Proof. For $i = 1, 2$, let us take any $z_i \in C_b^0(\bar{Q})$ and set

$$w_i(x, t) := \big[\mathcal{F}\big(z_i(x, \cdot), w_i^0(x)\big)\big](t) \qquad \forall (x, t) \in \bar{Q}, \tag{1.175}$$

$$u_i(x, t) = \int_0^t [f_i - w_i](x - t + \tau, \tau)\,\mathrm{d}\tau + \begin{cases} u_i^0(x - t) & \text{if } 0 < t \leqslant x \\ v_i^0(t - x) & \text{if } 0 < x < t \leqslant T. \end{cases} \tag{1.176}$$

Setting $\tilde{w} := w_1 - w_2$ and $\tilde{z} := z_1 - z_2$, by (1.172) we have

$$\|\tilde{w}\|_{C_b^0(\bar{Q}_t)} \leqslant L\|\tilde{z}\|_{C_b^0(\bar{Q}_t)} + \hat{L}\|\tilde{w}^0\|_{C_b^0(\mathbf{R}^+)} \qquad \forall t \in]0, T];$$

(1.176) then yields

$$\begin{aligned} \|\tilde{u}\|_{C_b^0(\bar{Q}_t)} &\leqslant \int_0^t \big(\|\tilde{f}\|_{C_b^0(\bar{Q}_\tau)} + \|\tilde{w}\|_{C_b^0(\bar{Q}_\tau)}\big)\,\mathrm{d}\tau + M(\tilde{u}^0, \tilde{v}^0, t) \\ &\leqslant \int_0^t \big(\|\tilde{f}\|_{C_b^0(\bar{Q}_\tau)} + L\|\tilde{z}\|_{C_b^0(\bar{Q}_\tau)}\big)\,\mathrm{d}\tau + \hat{L}t\|\tilde{w}^0\|_{C_b^0(\mathbf{R}^+)} + M(\tilde{u}^0, \tilde{v}^0, t) \\ &\qquad \forall t \in]0, T]. \end{aligned} \tag{1.177}$$

In particular for any u^0, w^0, v^0, f as in (1.169) we have

$$\|\tilde{u}\|_{C_b^0(\bar{Q}_t)} \leqslant L\int_0^t \|\tilde{z}\|_{C_b^0(\bar{Q}_\tau)}\,\mathrm{d}\tau \qquad \forall t \in]0, T].$$

If $t < 1/L$ then the mapping $z \mapsto u$ is a strict contraction in $C_b^0(\bar{Q}_t)$, hence it has a fixed point. Reiterating the argument step by step in time, we conclude that Problem 1.8.1 has a solution. (1.177) then holds with $\tilde{u}$ in place of $\tilde{z}$, and (1.174) follows by the Gronwall lemma. □

1.8.3 Formulation in Sobolev Spaces

We assume that (1.160) is fulfilled, that

$$u^0, w^0 \in L^1(\mathbf{R}^+), \quad v^0 \in L^1(0,T), \quad f \in L^1(Q), \tag{1.178}$$

and reformulate the above problem in differential form. This will allow us to improve the regularity of the solution, by using some of the most common techniques for the analysis of PDEs with hysteresis operators.

Problem 1.8.2. *Find $u \in W^{1,1}_{\text{loc}}(Q)$ such that, setting*

$$w(x,t) := \big[\mathcal{F}\big(u(x,\cdot), w^0(x)\big)\big](t) \qquad \forall t \in [0,T], \textit{for a.a. } x > 0, \tag{1.179}$$

one has

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + w = f \qquad \textit{a.e. in } Q, \tag{1.180}$$

$$u(x,0) = u^0(x) \qquad \textit{for a.a. } x > 0, \tag{1.181}$$

$$u(0,t) = v^0(t) \qquad \textit{for a.a. } t \in]0,T[. \tag{1.182}$$

As $W^{1,1}(Q) \subset L^1\big(\mathbf{R}^+; C^0([0,T])\big)$, (1.179) is meaningful, and $w(x,\cdot) \in C^0([0,T])$ for a.a. $x > 0$. The functions u and w will be regarded as elements of spaces of functions of t parameterized by the variable x, and also as elements of spaces of functions of x parameterized by the variable t. Occasionally this will be made clearer by displaying indices x and t in the expression of function spaces.

Theorem 1.8.2 (*Existence, Uniqueness and Regularity*). *Let $1 < p < +\infty$, assume that (1.160) holds, and*

$$\begin{aligned}&\forall (v,\xi) \in \mathrm{Dom}(\mathcal{F}), \forall [t_1,t_2] \subset [0,T],\\ &\textit{if } v \textit{ is either nondecreasing or nonincreasing in } [t_1,t_2],\\ &\textit{then the same holds for } \mathcal{F}(v,\xi),\end{aligned} \tag{1.183}$$

$$\begin{aligned}&\exists M, N > 0 : \forall (v,\xi) \in C^0([0,T]) \times \mathbf{R}, \forall t \in [0,T],\\ &\big|\big[\mathcal{F}(v,\xi)\big](t)\big| \leqslant |\xi| + M|v(t)| + N.\end{aligned} \tag{1.184}$$

Then Problem 1.8.2 has one and only one solution; moreover

$$u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x} \in L^\infty_t\big(0,T; L^p_x(\mathbf{R}^+)\big) \cap L^\infty_x\big(\mathbf{R}^+; L^p_t(0,T)\big), \tag{1.185}$$

$$
\begin{aligned}
u \in\, & W_t^{1,\infty}\big(0,T;L_x^p(\mathbf{R}^+)\big)\cap L_x^\infty\big(\mathbf{R}^+;W_t^{1,p}(0,T)\big)\\
&\cap L_t^\infty\big(0,T;W_x^{1,p}(\mathbf{R}^+)\big)\cap W_x^{1,\infty}\big(\mathbf{R}^+;L_t^p(0,T)\big),
\end{aligned}
\tag{1.186}
$$

$$
w \in L_x^\infty\big(\mathbf{R}^+;W_t^{1,p}(0,T)\big). \tag{1.187}
$$

If (1.172) is fulfilled, then the solution is also unique.

Proof. *(i) Approximation.* Let us fix any $m \in \mathbf{N}$, set $h := T/m$ and

$$
u_m^0 := u^0, \quad w_m^0 := w^0, \quad f_m^n := f(\cdot, nh) \qquad \text{for } n = 1, \ldots, m.
$$

We approximate our problem via implicit time-discretization. For any family $\{v_m^n\}_{n=1,\ldots,m}$ of functions $\mathbf{R} \to \mathbf{R}$, let us set:

$$
\begin{aligned}
&v_m := \text{ time-interpolate of } v_m^0, \ldots, v_m^m, \text{ a.e. in } \Omega,\\
&\bar{v}_m(\cdot, t) := v_m^n \quad \text{a.e. in } \Omega, \forall t \in](n-1)h, nh], \text{ for } n = 1, \ldots, m.
\end{aligned}
\tag{1.188}
$$

Problem 1.8.2$_m$. *Find $u_m^n \in W^{1,p}(\mathbf{R}^+)$ such that, for $n = 1, \ldots, m$, setting*

$$
w_m^n(x) := \big[\mathcal{F}\big(u_m(x,\cdot), w^0(x)\big)\big](nh) \quad \textit{for a.a. } x > 0, \tag{1.189}
$$

one has

$$
\frac{u_m^n - u_m^{n-1}}{h} + \frac{\mathrm{d}u_m^n}{\mathrm{d}x} + w_m^n = f_m^n \qquad \textit{a.e. in } \mathbf{R}^+, \tag{1.190}
$$

$$
u_m^n(0) = v^0(nh). \tag{1.191}
$$

Existence of an approximate solution can be proved step by step. For any n, at the nth step $u_m^0,\ldots, u_m^{n-1}$ are known. By the causality of $\mathcal{F}$, $w_m^n(x)$ then only depends on $u_m^n(x)$; that is, there exists a Caratheodory function $g_m^n : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$ such that (1.189) reads

$$
w_m^n(x) = g_m^n(u_m^n(x), x) \qquad \text{for a.a. } x > 0. \tag{1.192}
$$

By (1.183) and (1.184), $g_m^n(\cdot, x)$ is nondecreasing and affinely bounded. The Cauchy problem (1.190)–(1.192) then has one and only one solution in $\mathbf{R}^+$.

(ii) A Priori *Estimates.* Let us set

$$
z_m^0 = f(\cdot, 0) - \frac{\mathrm{d}u^0}{\mathrm{d}x} - w^0, \qquad z_m^n := \frac{u_m^n - u_m^{n-1}}{h} \qquad \text{for } n = 1, \ldots, m, \tag{1.193}
$$

$$
a_{m,p-1}(v) := \min\{|v|^{p-2}, m\}v, \quad A_{m,p}(v) := \int_0^t a_{m,p-1}(\xi)\,\mathrm{d}\xi \quad \forall v \in \mathbf{R}, \forall m \in \mathbf{N}. \tag{1.194}
$$

By taking the incremental ratio in time in (1.190), we have

$$\frac{z_m^n - z_m^{n-1}}{h} + \frac{dz_m^n}{\mathrm{d}x} + \frac{w_m^n - w_m^{n-1}}{h} = \frac{f_m^n - f_m^{n-1}}{h} \qquad \text{a.e. in } \mathbf{R}^+, \text{ for } n = 1, ..., m. \tag{1.195}$$

Multiplying this equation by $a_{m,p-1}(z_m^n)$, and integrating in $]0, x[$ we have

$$\int_0^x \big(A_{m,p}(z_m^n(x)) - A_{m,p}(z_m^{n-1}(x))\big)\,\mathrm{d}x + \big(A_{m,p}(z_m^n(x)) - A_{m,p}(z_m^n(0))\big) + \int_0^x \frac{w_m^n - w_m^{n-1}}{h} a_{m,p-1}(z_m^n)\,\mathrm{d}x = \int_0^x \frac{f_m^n - f_m^{n-1}}{h} a_{m,p-1}(z_m^n)\,\mathrm{d}x \tag{1.196}$$

for a.a. $x > 0$, for any n; by (1.183) the third addendum is nonnegative. By summing for $n = 1, ..., \ell$ for any $\ell \in \{1, ..., m\}$, we then get

$$\int_0^x \big(A_{m,p}(z_m^\ell(x)) - A_{m,p}(z^0(x))\big)\,\mathrm{d}x + \sum_{n=1}^{\ell} \big(A_{m,p}(z_m^\ell(x)) - A_{m,p}(z_m^\ell(0))\big) \leqslant \sum_{n=1}^{\ell} \Big\| \frac{f_m^n - f_m^{n-1}}{h} \Big\|_{L^p(0,x)} \|z_m^n\|_{L^p(0,x)}^{p-1} \qquad \text{for a.a. } x > 0, \forall \ell.$$

Notice that $z_m^\ell(0) := [v^0(\ell h) - v^0(\ell h - h)]/h$ for any ℓ. A simple calculation then yields

$$\Big\| \frac{\partial u_m}{\partial t} \Big\|_{L_t^\infty(0,T;L_x^p(\mathbf{R}^+))}, \Big\| \frac{\partial u_m}{\partial x} \Big\|_{L_x^\infty(\mathbf{R}^+;L_t^p(0,T))} \leqslant C_1 \tag{1.197}$$

(by $C_1, C_2, ...$ we denote suitable constants independent of m); hence, by (1.184),

$$\|w_m\|_{L^\infty(Q)} \leqslant C_2. \tag{1.198}$$

(iii) Limit Procedure. The equation (1.190) also reads

$$\frac{\partial u_m}{\partial t} + \frac{\partial \bar{u}_m}{\partial x} + \bar{w}_m = \bar{f}_m \qquad \text{a.e. in } \mathbf{R}^+ \times]0, T[. \tag{1.199}$$

By (1.197) and by comparing the terms of this equation, we get

$$\Big\| \frac{\partial u_m}{\partial x} \Big\|_{L_t^\infty(0,T;L_x^p(\mathbf{R}^+))}, \Big\| \frac{\partial u_m}{\partial t} \Big\|_{L_x^\infty(\mathbf{R}^+;L_t^p(0,T))} \leqslant C_3. \tag{1.200}$$

By (1.197), (1.198) and (1.200) there exist u, w such that, possibly taking $m \to \infty$ along a subsequence,

$$u_m \to u, \ \frac{\partial u_m}{\partial t} \to \frac{\partial u}{\partial t}, \ \frac{\partial u_m}{\partial x} \to \frac{\partial u}{\partial x} \qquad \text{weakly star in } L_t^\infty\big(0, T; L_x^p(\mathbf{R}^+)\big) \cap L_x^\infty\big(\mathbf{R}^+; L_t^p(0, T)\big), \tag{1.201}$$

$$w_m \to w \qquad \text{weakly star in } L^\infty_x\big(\mathbf{R}^+; W^{1,p}_t(0,T)\big). \tag{1.202}$$

By passing to the limit in (1.199) we get (1.180). By (1.201), $u_m \to u$ uniformly in $[0, T]$ for a.a. $x > 0$; by (1.160) this entails

$$\mathcal{F}\big(u_m, w^0\big) \to \mathcal{F}\big(u, w^0\big) \qquad \text{uniformly in } [0,T], \text{ a.e. in } \mathbf{R}^+.$$

As w_m is the linear interpolate of $w_m(\cdot, nh) := \big[\mathcal{F}\big(u_m, w^0\big)\big](\cdot, nh)$ $(n = 1, ..., m)$, then

$$w_m \to \mathcal{F}\big(u, w^0\big) \qquad \text{uniformly in } [0,T], \text{ a.e. in } \mathbf{R}^+.$$

By (1.202) we conclude that $w = \mathcal{F}(u)$ a.e. in Q. Thus u solves Problem 1.8.1.

(iv) Uniqueness. By (1.186), $u \in C^0_b(\bar{Q})$ and u solves Problem 1.8.1; by Theorem 1.8.1 the solution is then unique. (This might also easily be proved directly on Problem 1.8.2.) □

Remarks

(i) Existence of a solution of Problem 1.8.2 can easily be established also if the piecewise monotonicity property (1.183) is replaced by that of Lipschitz-continuity (1.172). In this case one also gets $w \in L^\infty_x\big(\mathbf{R}^+; W^{1,p}_t(0,T)\big)$.

(ii) If the drift term $\partial u/\partial x$ is replaced by $v(x)\partial u/\partial x$ for a prescribed nonincreasing positive function $v \in W^{1,\infty}(\mathbf{R}^+)$, no difficulty arises in extending Theorem 1.8.2. Other extensions are also possible, e.g. $\mathbf{R}^+$ can be replaced either by $\mathbf{R}$ or by a bounded interval.

(iii) Let $\mathbf{R}^+$ be replaced by a domain $\Omega \subset \mathbf{R}^N$, $\vec{v} : \Omega \to \mathbf{R}^N$ be a given function, and consider the equation

$$\frac{\partial u}{\partial t} + \vec{v}(x)\cdot\nabla u + \mathcal{F}(u) = f \qquad \text{in } \Omega\times]0,T[. \tag{1.203}$$

If the field $\vec{v}$ is integrable, then one can integrate this equation along each integral line, retrieve a family of equations in a single space variable of the form (1.168), and thus derive well-posedness results.

1.8.4 LARGE-TIME BEHAVIOR

Problems 1.8.1 and 1.8.2 can be extended to t ranging in the whole $\mathbf{R}^+$, just by assuming that u solves the corresponding problem in any finite interval $[0, T]$; in that case we shall say that u solves the problem in $\mathbf{R}^+$.

Because of rate-independence, for any hysteresis operator $\mathcal{F}$ as in (1.160) it is easy to see that

$$\mathcal{F} : C_b^0(\mathbf{R}^+) \times \mathbf{R} \to C_b^0(\mathbf{R}^+) \text{ and is continuous,} \tag{1.204}$$

$$\begin{aligned}&\forall(v, \xi) \in C_b^0(\mathbf{R}^+) \times \mathbf{R},\\ &\exists \lim_{t\to+\infty} v(t) \in \mathbf{R} \quad \Rightarrow \quad \exists \lim_{t\to+\infty} \big[\mathcal{F}(v, \xi)\big](t) \in \mathbf{R}.\end{aligned} \tag{1.205}$$

Let us set $Q_\infty := (\mathbf{R}^+)^2$.

Proposition 1.8.3. *Assume that (1.160), (1.172), (1.173) hold, and*

$$u^0, w^0, v^0 \in C^0(\mathbf{R}^+), \quad f : \mathbf{R}^+ \to C_b^0(\mathbf{R}^+) \textit{ is strongly continuous.} \tag{1.206}$$

Then Problem 1.8.1 has one and only one solution in $\mathbf{R}^+$.

This is a straightforward consequence of Theorem 1.8.1. Let us now come to the large-time behavior of the solution of Problem 1.8.2.

Proposition 1.8.4. *Let* $1 < p < +\infty$, *(1.160), (1.172), (1.173) hold, and*

$$u^0, w^0, v^0 \in W^{1,p}(\mathbf{R}^+), \qquad f \in W_t^{1,p}\big(\mathbf{R}^+; L_x^p(\mathbf{R}^+)\big), \tag{1.207}$$

$$\exists b > 0 : \forall u \in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^+), \ \frac{\partial \mathcal{F}(u, w^0)}{\partial t}\Big|\frac{\partial u}{\partial t}\Big|^{p-2}\frac{\partial u}{\partial t} \geqslant b\Big|\frac{\partial u}{\partial t}\Big|^p \quad \textit{a.e. in } \mathbf{R}^+. \tag{1.208}$$

Then Problem 1.8.2 has one and only one solution in $\mathbf{R}^+$*; moreover*

$$\frac{\partial u}{\partial t} \in L_t^\infty\big(\mathbf{R}^+; L_x^p(\mathbf{R}^+)\big) \cap L_x^\infty\big(\mathbf{R}^+; L_t^p(\mathbf{R}^+)\big), \tag{1.209}$$

$$\frac{\partial u}{\partial t} \to 0 \qquad \textit{strongly in } L^p(\mathbf{R}^+), \textit{ as } t \to +\infty. \tag{1.210}$$

Proof. Existence and uniqueness of the solution in $\mathbf{R}^+$ follow from Theorem 1.8.2.

By passing to the limit as $h \to 0$ in (1.196), by (1.208) we have

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^x \Big|\frac{\partial u}{\partial t}(\xi, t)\Big|^p \mathrm{d}\xi + \frac{1}{p}\Big(\Big|\frac{\partial u}{\partial t}(x, t)\Big|^p - \Big|\frac{\mathrm{d}v^0}{\mathrm{d}t}(t)\Big|^p\Big) + b\int_0^x \Big|\frac{\partial u}{\partial \tau}\Big|^p \mathrm{d}\xi\\ &\leqslant \frac{2}{bp}\int_0^x \Big|\frac{\partial f}{\partial \tau}\Big|^p \mathrm{d}\xi + \frac{b(p-1)}{2p}\int_0^x \Big|\frac{\partial u}{\partial \tau}\Big|^p \mathrm{d}\xi \qquad \text{for a.a. } x > 0, \forall t > 0,\end{aligned}$$

whence, multiplying by p,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^x \left|\frac{\partial u}{\partial t}(\xi,t)\right|^p \mathrm{d}\xi + \left|\frac{\partial u}{\partial t}(x,t)\right|^p + \frac{b(p+1)}{2}\int_0^x \left|\frac{\partial u}{\partial \tau}\right|^p \mathrm{d}\xi \\
&\leqslant \frac{2}{b}\int_0^x \left|\frac{\partial f}{\partial \tau}\right|^p \mathrm{d}\xi + \left|\frac{\mathrm{d}v^0}{\mathrm{d}t}(t)\right|^p \qquad \text{for a.a. } x>0 \text{ and a.a. } t>0.
\end{aligned}
\tag{1.211}
$$

As the right side is integrable with respect to time in $\mathbf{R}^+$, Lemma 1.8.5 below yields (1.209) and (1.210). □

LEMMA 1.8.5. *Let $f \in L^1(\mathbf{R}^+)$, $f \geqslant 0$, and a, C be positive constants. Let Y be the family of nonnegative functions $y \in W^{1,1}(\mathbf{R}^+)$ such that*

$$
y(0)\leqslant C, \qquad y'(t) + ay(t) \leqslant f(t) \qquad \text{for a.a. } t>0. \tag{1.212}
$$

Then

$$
\begin{aligned}
&y(t) \leqslant C + \int_0^{+\infty} f(\tau)\,\mathrm{d}\tau \qquad \forall t>0,\\
&y(t) \to 0 \qquad \text{as } t \to +\infty, \text{ uniformly in } Y.
\end{aligned}
\tag{1.213}
$$

PROOF. By the formula of variation of constants we have

$$
y(t) \leqslant e^{-at}y(0) + \int_0^t e^{-a(t-\tau)} f(\tau)\,\mathrm{d}\tau \leqslant C + \int_0^{+\infty} f(\tau)\,\mathrm{d}\tau =: M \qquad \forall t>0,
$$

$$
\begin{aligned}
y(t) &\leqslant e^{-a(t-T)}y(T) + \int_T^t e^{-a(t-\tau)} f(\tau)\,\mathrm{d}\tau \\
&\leqslant e^{-a(t-T)}M + \int_T^t f(\tau)\,\mathrm{d}\tau \qquad \forall T, t>0, T<t.
\end{aligned}
$$

This yields (1.213). □

1.9 A QUASILINEAR PARABOLIC PDE WITH HYSTERESIS

In this section we deal with a quasilinear parabolic equation that contains a continuous hysteresis operator. We prove well-posedness, study the large-time behavior of the solution, deal with the corresponding periodic problem, and finally prove the existence of a solution for a parabolic equation with hysteresis in the coefficient of the elliptic term.

1.9.1 FORMULATION AND EXISTENCE RESULT

Let Ω be a bounded Lipschitz domain of $\mathbf{R}^N$ ($N \geqslant 1$), T be a positive constant, and set $\Omega_T := \Omega \times]0, T[$. We assume that

$$\mathcal{F} : C^0([0,T]) \times \mathbf{R} \to C^0([0,T]) \quad \text{is a continuous hysteresis operator,} \tag{1.214}$$

and identify it with $\tilde{\mathcal{F}}$, cf. (1.161). We fix a function $f : \Omega_T \to \mathbf{R}$, and couple the equation

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] - \Delta u = f \qquad \text{in } \Omega_T \tag{1.215}$$

with suitable initial and boundary conditions. For the sake of simplicity, here we assume the homogeneous Dirichlet condition, and set $V := H_0^1(\Omega)$. We identify the (real) Hilbert space $H := L^2(\Omega)$ with its topological dual H'; in turn H can then be identified with a subspace of V', as V is a dense subspace of H with continuous injection. We thus get the Hilbert triplet

$$V \subset H = H' \subset V' \quad \text{with continuous and dense injections.}$$

We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V, and define the linear and continuous operator $A : V \to V'$ by

$$\langle Au, v \rangle := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \qquad \forall u, v \in V.$$

Finally, we assume that

$$u^0, w^0 \in H, \qquad f \in L^2(0, T; V'). \tag{1.216}$$

PROBLEM 1.9.1. *Find $u : \Omega \to C^0([0,T])$ measurable such that $u \in L^2(0,T;V)$ and, setting*

$$w(x,t) := \big[\mathcal{F}\big(u(x,\cdot), w^0(x)\big)\big](t) \qquad \forall t \in [0,T], \textit{for a.a. } x \in \Omega, \tag{1.217}$$

$w \in L^2(0,T;H)$, $u + w \in H^1(0,T;V')$, and

$$\frac{\partial}{\partial t}(u + w) + Au = f \qquad \textit{in } V', \ \textit{a.e. in }]0,T[, \tag{1.218}$$

$$(u + w)|_{t=0} = u^0 + w^0 \qquad \textit{in } V'. \tag{1.219}$$

For $N = 1$ this problem is a simplified model of scalar ferromagnetism, with normalized coefficients. (1.218) can be derived from the Maxwell equations, neglecting the displacement current term and assuming a linear relation between the electric field, E, and the electric current density, J. The operator $\mathcal{F}$ then represents the (scalar) hysteretic relation between the magnetic field, H, and the magnetization, M.

Theorem 1.9.1 (*Existence*). *Assume that (1.214) is fulfilled, and*

$$\begin{aligned}&\forall(v,\xi)\in \mathrm{Dom}(\mathcal{F}),\ \forall[t_1,t_2]\subset[0,T],\\ &\textit{if } v \textit{ is either nondecreasing or nonincreasing in } [t_1,t_2],\\ &\textit{then the same holds for } \mathcal{F}(v,\xi),\end{aligned} \tag{1.220}$$

$$\begin{aligned}&\exists M, N\in \mathbf{R}^+ : \forall(v,\xi)\in C^0([0,T])\times\mathbf{R},\ \forall t\in[0,T],\\ &|[\mathcal{F}(v,\xi)](t)|\leqslant|\xi|+M|v(t)|+N,\end{aligned} \tag{1.221}$$

$$\begin{aligned}&\textit{for a.a. } x\in\Omega,\ \forall v\in C^0([0,T]),\\ &\textit{if } v(0)=u^0(x) \textit{ then } \big[\mathcal{F}\big(v,w^0(x)\big)\big](0)=w^0(x),\end{aligned} \tag{1.222}$$

$$\begin{aligned}&u^0\in V,\quad w^0\in H,\\ &f=f_1+f_2,\quad f_1\in L^2(0,T;H),\quad f_2\in W^{1,1}(0,T;V').\end{aligned} \tag{1.223}$$

Then Problem 1.9.1 has a solution such that

$$u\in H^1(0,T;H)\cap L^\infty(0,T;V),\qquad w\in L^2\big(\Omega;C^0([0,T])\big). \tag{1.224}$$

Moreover the norm of (u, w) *in the latter spaces is bounded by a constant that only depends on the data* u^0, w^0, f *via the respective norms in* V, H, $L^2(0,T;H)+W^{1,1}(0,T;V')$.

Proof. We use the argument of Theorem 1.1 of [8; Chapter IX], which is based on approximation by implicit time-discretization, derivation of *a priori* estimates, and passage to the limit by compactness.

(i) Approximation. We fix any $m\in\mathbf{N}$, set $h := T/m$ and

$$\begin{cases} f_{1m}^n(x) := \dfrac{1}{h}\displaystyle\int_{(n-1)h}^{nh} f_1(x,t)\,\mathrm{d}t \quad \text{for a.a. } x\in\Omega,\quad f_{2m}^n := f_2(nh),\\ f_m^n := f_{1m}^n + f_{2m}^n,\quad u_m^0 := u^0,\quad w_m^0 := w^0,\qquad \text{for } n=1,\dots,m.\end{cases} \tag{1.225}$$

Defining time-interpolate functions as in (1.188), we now approximate our problem via implicit time-discretization.

PROBLEM 1.9.1$_m$. *For $n = 1, ..., m$, find $u_m^n \in V$ such that, setting*

$$w_m^n(x) := \big[\mathcal{F}\big(u_m(x,\cdot), w^0(x)\big)\big](nh) \qquad \text{for } n = 1, ..., m, \text{ for a.a. } x \in \Omega, \tag{1.226}$$

one has

$$\frac{u_m^n - u_m^{n-1}}{h} + \frac{w_m^n - w_m^{n-1}}{h} + Au_m^n = f_m^n \qquad \text{in } V', \text{ for } n = 1, ..., m. \tag{1.227}$$

Existence of an approximate solution can be proved step by step. For any n, at the nth step $u_m^0, ..., u_m^{n-1}$ are known a.e. in Ω. By the causality of $\mathcal{F}$, w_m^n then only depends on u_m^n, i.e.,

$$w_m^n(x) = g_m^n(u_m^n(x), x) \qquad \text{for a.a. } x \in \Omega,$$

for a suitable Caratheodory function $g_m^n : \mathbf{R} \times \Omega \to \mathbf{R}$. By (1.220), (1.221), for a.a. $x \in \Omega$ the function $g_m^n(\cdot, x)$ is nondecreasing and affinely bounded. Existence of a solution of the elliptic equation (1.227) then follows.

(ii) A Priori *Estimates.* Let us multiply equation (1.227) by $u_m^n - u_m^{n-1}$, and sum for $n = 1, ..., \ell$, for any $\ell \in \{1, ..., m\}$. By (1.220),

$$\left(w_m^n - w_m^{n-1}\right)\left(u_m^n - u_m^{n-1}\right) \geqslant 0 \qquad \text{a.e. in } \Omega, \text{ for } n = 1, ..., m;$$

hence

$$\begin{aligned} &h\sum_{n=1}^{\ell}\int_\Omega \left|\frac{u_m^n - u_m^{n-1}}{h}\right|^2 + \frac{1}{2}\int_\Omega \left(|\nabla u_m^\ell|^2 - |\nabla u^0|^2\right) \mathrm{d}x \\ &\leqslant \sum_{n=1}^{\ell} {}_{V'}\langle f_m^n, u_m^n - u_m^{n-1}\rangle_V \qquad \forall \ell. \end{aligned} \tag{1.228}$$

As

$$\begin{aligned} \sum_{n=1}^{\ell} {}_{V'}\langle f_m^n, u_m^n - u_m^{n-1}\rangle_V \leqslant \quad & \tfrac{1}{2}\|f_1\|^2_{L^2(0,T;H)} + \tfrac{h}{2}\sum_{n=1}^{\ell}\int_\Omega \left(\frac{u_m^n - u_m^{n-1}}{h}\right)^2 \mathrm{d}x \\ & + C_1\|f_2\|^2_{W^{1,1}(0,T;V')} + \tfrac{1}{2}\max\nolimits_{n=0,...,\ell}\|u_m^n\|_V^2 \quad \forall \ell \end{aligned} \tag{1.229}$$

(by $C_1, C_2, ...$ we denote suitable constants independent of m), by a standard calculation we get

$$\|u_m\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leqslant C_2. \tag{1.230}$$

Hence u_m is uniformly bounded in $L^2\big(\Omega; C^0([0,T])\big)$, and by (1.221) we have

$$\|w_m\|_{L^2(\Omega;C^0([0,T]))} \leqslant C_3. \tag{1.231}$$

Notice that (1.227) also reads

$$\frac{\partial}{\partial t}(u_m + w_m) + A\bar{u}_m = \bar{f}_m \qquad \text{in } V', \ \text{a.e. in }]0,T[. \tag{1.232}$$

(iii) Passage to the Limit. By the estimates (1.230) and (1.231) there exist u and w such that, possibly taking $m \to \infty$ along a subsequence,

$$\begin{aligned} u_m \to u \quad & \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V), \\ w_m \to w \quad & \text{weakly star in } L^2\big(\Omega; L^\infty(0,T)\big). \end{aligned} \tag{1.233}$$

By taking $m \to \infty$ in (1.227) we then get (1.218); (1.219) is also easily derived. As

$$H^1(0,T;H) \cap L^\infty(0,T;V) \subset L^2\big(\Omega; C^0([0,T])\big) \text{ with compact injection,}$$

by $(1.233)_1$ possibly extracting a further subsequence

$$u_m(x,\cdot) \to u(x,\cdot) \qquad \text{uniformly in } [0,T], \text{ for a.a. } x \in \Omega. \tag{1.234}$$

By the continuity of $\mathcal{F}$, cf. (1.214), this entails that $\mathcal{F}\big(u_m, w^0\big) \to \mathcal{F}\big(u, w^0\big)$ uniformly in $[0,T]$ a.e. in Ω. As $w_m(x,\cdot)$ is the linear interpolate of $w_m(x,nh) := \big[\mathcal{F}\big(u_m, w^0\big)\big](x,nh)$ $(n = 0,\dots,m)$ for a.a. x, $w_m \to \mathcal{F}\big(u, w^0\big)$ uniformly in $[0,T]$ a.e. in Ω. By $(1.233)_2$ we then conclude that $w = \mathcal{F}(u)$ a.e. in Ω_T. The final statement is a straightforward consequence of the above estimates. □

1.9.2 UNIQUENESS

We assume that

$$\begin{aligned} &\gamma_\ell, \gamma_r : \mathbf{R} \to \mathbf{R} \ \text{ are two Lipschitz-continuous} \\ &\text{and affinely bounded curves, } \ \gamma_r \leqslant \gamma_\ell \text{ pointwise,} \end{aligned} \tag{1.235}$$

and denote by $\mathcal{F}$ the corresponding generalized play operator, cf. (1.51). The next result carries over to a larger class of hysteresis operators, which also includes several (either continuous or discontinuous) Preisach operators, cf. Section 1.4.

THEOREM 1.9.2 (*Dependence on the Data and Uniqueness [17]*). *Assume that (1.235) is fulfilled, let*

$$u_i^0 \in V, \quad w_i^0 \in H \quad (i = 1,2), \quad f_1 - f_2 \in L^1(\Omega_T), \tag{1.236}$$

and set

$$w_i^0 := \min\big\{\max\{w_i^0, \gamma_r(u_i^0)\}, \gamma_\ell(u_i^0)\big\} \qquad \textit{a.e. in } \Omega \, (i = 1,2). \tag{1.237}$$

Let $u_1, u_2 \in H^1(0,T;H) \cap L^\infty(0,T;V)$ be corresponding solutions of Problem 1.9.1 (which exist after Theorem 1.9.1), and define w_1, w_2 as in (1.217). Then

$$\begin{aligned}&\int_\Omega \big[(u_1-u_2)^+(x,t) + (w_1-w_2)^+(x,t)\big]\mathrm{d}x \\ &\leqslant \int_\Omega \big[(u_1^0-u_2^0)^+ + (w_1^0-w_2^0)^+\big]\mathrm{d}x + \int_0^t \mathrm{d}\tau \int_\Omega (f_1-f_2)^+(x,\tau)\mathrm{d}x,\end{aligned} \tag{1.238}$$

for any $t \in [0,T]$; an analogous inequality holds if the positive part is replaced either by the negative part or by the absolute value. Therefore the solution of Problem 1.9.1 is unique, and depends monotonically and Lipschitz-continuously on the data u^0, w^0, f.

Proof. We follow Hilpert's argument of [17]. First let us approximate the Heaviside function,

$$\hat{H}(\xi) := 0 \quad \text{if } \xi \leqslant 0, \qquad \hat{H}(\xi) := 1 \quad \text{if } \xi > 0,$$

by setting $\hat{H}_m(\xi) := \max\{\min\{m\xi, 1\}, 0\}$ for any $\xi \in \mathbf{R}$ and any $m \in \mathbf{N}$.

Note that $w_1, w_2 \in H^1(0,T;H)$, as γ_ℓ and γ_r are Lipschitz-continuous. Let us write (1.218) for $i = 1, 2$, take the difference of these equations, multiply it by $\hat{H}_m(u_1-u_2)$, and integrate it in $\Omega_t := \Omega \times]0,t[$ for a.a. $t \in]0,T[$. By the monotonicity of $\hat{H}_m$, we get

$$\begin{aligned}&\iint_{\Omega_t} \Big(\frac{\partial}{\partial\tau}(u_1-u_2) + \frac{\partial}{\partial\tau}(w_1-w_2)\Big)\hat{H}_m(u_1-u_2)\,\mathrm{d}x\mathrm{d}\tau \\ &\leqslant \iint_{\Omega_t} (f_1-f_2)\,\hat{H}_m(u_1-u_2)\,\mathrm{d}x\mathrm{d}\tau \qquad \text{for a.a. } t \in]0,T[.\end{aligned}$$

Notice that $\hat{H}_m(u_1-u_2) \to \psi = \hat{H}(u_1-u_2)$ a.e. in Ω_T as $m \to \infty$. By passing to the limit in the latter inequality we then infer that

$$\iint_{\Omega_t} \Big(\frac{\partial}{\partial\tau}(u_1-u_2) + \frac{\partial}{\partial\tau}(w_1-w_2)\Big)\psi\,\mathrm{d}x\mathrm{d}\tau \leqslant \iint_{\Omega_t} (f_1-f_2)^+\,\mathrm{d}x\mathrm{d}\tau \quad \text{for a.a. } t \in]0,T[. \tag{1.239}$$

Moreover

$$\Big(\frac{\partial}{\partial\tau}(u_1-u_2)\Big)\psi = \frac{\partial}{\partial\tau}\big[(u_1-u_2)^+\big] \qquad \text{a.e. in } \Omega_T,$$

and the Hilpert inequality (1.53) yields

$$\Big(\frac{\partial}{\partial\tau}(w_1-w_2)\Big)\psi \geqslant \frac{\partial}{\partial\tau}\big[(w_1-w_2)^+\big] \qquad \text{a.e. in } \Omega_T.$$

The inequality (1.239) then yields

$$\iint_{\Omega_t} \frac{\partial}{\partial\tau}\big((u_1-u_2)^+ + (w_1-w_2|)^+\big)\,\mathrm{d}x\mathrm{d}\tau \leqslant \iint_{\Omega_t} (f_1-f_2)^+\,\mathrm{d}x\mathrm{d}\tau \quad \text{for a.a. } t\in]0,T[,$$

and integrating in time we get (1.234). □

1.9.3 Large Time Behaviour

THEOREM 1.9.3 (*Uniform Asymptotic Stability in V*). *Assume that* (1.214)*,* (1.220) *and* (1.221) *hold, that*

$$\begin{cases} \exists L>0: \forall v\in C^0([0,+\infty[), \forall\xi\in\mathbf{R}, \forall[t_1,t_2]\subset[0,+\infty[, \\ v \textit{ affine in } [t_1,t_2] \;\Rightarrow\; |\big[\mathcal{F}(v,\xi)\big](t_2)-\big[\mathcal{F}(v,\xi)\big](t_1)|\leqslant L|v(t_2)-v(t_1)|, \end{cases} \tag{1.240}$$

$$\begin{aligned} &u^0\in V, \quad w^0\in H,\\ &f=f_1+f_\infty, \quad f_1\in L^2(0,+\infty;H), \quad f_\infty\in V', \end{aligned} \tag{1.241}$$

and set $u_\infty := A^{-1}f_\infty$ $(\in V)$*. Then there exists one and only one*

$$u\in\big(H^1(0,+\infty;H)\cap L^\infty(0,+\infty;V)\big)+u_\infty \tag{1.242}$$

that solves Problem 1.9.1 for any $T>0$*. Moreover*

$$u(\cdot,t)\to u_\infty := A^{-1}f_\infty \qquad \textit{strongly in } V, \textit{ as } t\to+\infty, \tag{1.243}$$

uniformly as u^0 *(*f_1*, respectively) ranges in any bounded subset of* V *(*$L^2(0,+\infty;H)$*, respectively).*

Finally,

$$w\in L^2_{\mathrm{loc}}(0,+\infty;H), \qquad \frac{\partial w}{\partial t}\in L^2(0,+\infty;H), \tag{1.244}$$

and there exists $w_\infty\in H$ *such that*

$$w(\cdot,t)\to w_\infty \qquad \textit{in measure in } \Omega, \textit{ as } t\to+\infty. \tag{1.245}$$

PROOF. First notice that by rate-independence (1.220) and (1.221) hold for any $T>0$.

Setting $\tilde u := u-u_\infty$ (1.218) also reads

$$\frac{\partial}{\partial t}(\tilde u + w) + A\tilde u = f_1 \qquad \text{in } V', \text{ a.e. in }]0,T[, \tag{1.246}$$

Let us multiply this equation by $\partial\tilde{u}/\partial t$, and integrate over Ω. By (1.220)

$$\frac{\partial w}{\partial t}\frac{\partial \tilde{u}}{\partial t} = \frac{\partial w}{\partial t}\frac{\partial u}{\partial t} \geqslant 0 \qquad \text{a.e. in } \Omega_T,$$

and we get

$$\int_\Omega \left|\frac{\partial \tilde{u}}{\partial t}\right|^2 \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla \tilde{u}|^2 \mathrm{d}x = \int_\Omega f_1 \frac{\partial \tilde{u}}{\partial t}\mathrm{d}x \leqslant \frac{1}{2}\int_\Omega \left(|f_1|^2 + \left|\frac{\partial \tilde{u}}{\partial t}\right|^2\right)\mathrm{d}x \quad \text{a.e. in }]0,T[. \tag{1.247}$$

By (1.240), $|\partial w/\partial t| \leqslant L|\partial \tilde{u}/\partial t|$. Multiplying (1.246) by $\tilde{u}$ we then have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\tilde{u}|^2 \mathrm{d}x + \int_\Omega |\nabla \tilde{u}|^2 \mathrm{d}x &= \int_\Omega \left(f_1 - \frac{\partial w}{\partial t}\right)\tilde{u}\,\mathrm{d}x \leqslant \int_\Omega \left(|f_1| + L\left|\frac{\partial \tilde{u}}{\partial t}\right|\right)|\tilde{u}|\,\mathrm{d}x \\ &\leqslant \int_\Omega \left(\frac{1}{2a}|f_1|^2 \mathrm{d}x + \frac{L^2}{2a}\left|\frac{\partial \tilde{u}}{\partial t}\right|^2 + a|\tilde{u}|^2\right)\mathrm{d}x \qquad \text{a.e. in }]0,T[,\ \forall a > 0.\end{aligned} \tag{1.248}$$

By the Poincaré inequality there exists a constant $a > 0$ such that

$$a\int_\Omega |\tilde{u}|^2 \mathrm{d}x \leqslant \frac{1}{2}\int_\Omega |\nabla \tilde{u}|^2\,\mathrm{d}x; \tag{1.249}$$

multiplying (1.248) by $a/(2L^2)$ and adding it to (1.247), we then get

$$\begin{aligned}&\frac{1}{4}\int_\Omega \left|\frac{\partial \tilde{u}}{\partial t}\right|^2 \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{a}{2L^2}|\tilde{u}|^2 + |\nabla \tilde{u}|^2\right)\mathrm{d}x + \frac{a}{4L^2}\int_\Omega |\nabla \tilde{u}|^2 \mathrm{d}x \\ &\leqslant \frac{2L^2+1}{4L^2}\|f_1\|_H^2 \qquad \text{a.e. in }]0,T[.\end{aligned}$$

This yields (1.242). By setting

$$y(t) := \int_\Omega \left(\frac{a}{2L^2}|\tilde{u}(x,t)|^2 + |\nabla \tilde{u}(x,t)|^2\right)\mathrm{d}x \qquad \text{for a.a. } t \in]0,T[,$$

for a suitable constant $\alpha > 0$ we also get

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) + \alpha y(t) \leqslant \frac{2L^2+1}{2L^2}\|f_1(\cdot,t)\|_H^2 \qquad \text{for a.a. } t \in]0,T[.$$

As the right side is integrable in $]0.+\infty[$, by Lemma 1.8.5 $y(t) \to 0$ as $t \to +\infty$; (1.243) thus holds.

By (1.240) and (1.242) w has the regularity (1.244). Finally, by (1.243), (1.244) and by the continuity of $\mathcal{F}$, we infer that there exists $w_\infty \in H$ such that (1.245) holds. □

1.9.4 Periodic Problem

We introduce a periodic problem associated with Problem 1.9.1, with a nonhomogeneous Dirichlet condition. Let $\mathcal{F}$ fulfil (1.214) and

$$f \in L^2(0,T;V'). \tag{1.250}$$

PROBLEM 1.9.2. *Find*

$$\begin{aligned} &u \in L^2\big(\Omega; C^0([0,T])\big) \cap L^2(0,T;V),\\ &w \in L^2\big(\Omega; C^0([0,T])\big) \cap L^2(0,T;H), \end{aligned} \tag{1.251}$$

such that $u + w \in H^1(0,T;V')$ *and*

$$w(x,t) = \big[\mathcal{F}\big(u(x,\cdot), w(x,0)\big)\big](t) \qquad \forall t \in [0,T],\ \textit{for a.a. } x \in \Omega, \tag{1.252}$$

$$\frac{\partial}{\partial t}(u+w) + Au = f \qquad \textit{in } V',\ \textit{a.e. in }]0,T[, \tag{1.253}$$

$$u(\cdot,0) = u(\cdot,T), \quad w(\cdot,0) = w(\cdot,T) \qquad \textit{a.e. in } \Omega. \tag{1.254}$$

THEOREM 1.9.4 (*[98]*). *Let $\mathcal{F}$ be the generalized play operator associated with two curves γ_ℓ, γ_r such that*

$$\begin{aligned} &\gamma_\ell, \gamma_r : \mathbf{R} \to \mathbf{R} \textit{ are Lipschitz-continuous and nondecreasing};\\ &\gamma_r \leqslant \gamma_\ell \textit{ in } \mathbf{R};\ \gamma_\ell(\xi) > 0\ \forall \xi \geqslant 0,\ \gamma_r(\xi) < 0\ \forall \xi \leqslant 0, \end{aligned} \tag{1.255}$$

and assume that

$$f = f_1 + f_2, \quad f_1 \in L^2(0,T;H), \quad f_2 \in W^{1,1}(0,T;V'), \quad f_2(0) = f_2(T), \tag{1.256}$$

$$\exists f_*, f^* \in V' : \qquad f_* \leqslant f_2 \leqslant f^* \qquad \textit{in } \mathcal{D}'(\Omega), \forall t \in [0,T]. \tag{1.257}$$

Then Problem 1.9.2 has minimal and maximal solutions with respect to the pointwise ordering. That is, denoting by $\mathcal{S}$ the set of all solutions of this problem,

$$\begin{aligned} &\exists (u_*, w_*), (u^*, w^*) \in \mathcal{S} : \forall (u,w) \in \mathcal{S},\\ &u_* \leqslant u \leqslant u^*, \quad w_* \leqslant w \leqslant w^* \quad \textit{a.e. in } \Omega_T. \end{aligned} \tag{1.258}$$

Moreover, these minimal solutions and a maximal solution have the regularity

$$u \in H^1(0,T;H) \cap L^\infty(0,T;V), \qquad w \in H^1(0,T;H). \tag{1.259}$$

PROOF. We follow the argument of [98], and split it into several steps.

(i) Let us first set $X := \big\{(u,w) \in V \times H : \gamma_r(u) \leqslant w \leqslant \gamma_\ell(u) \text{ a.e. in } \Omega\big\}$; this set is partially ordered with respect to the natural ordering

$$(u_1, w_1) \preceq (u_2, w_2) \quad \Leftrightarrow \quad u_1 \leqslant u_2, \quad w_1 \leqslant w_2 \quad \text{a.e. in } \Omega.$$

For any $(u^0, w^0) \in X$ let us denote by $\mathcal{S}$ the solution operator, that maps (u^0, w^0) to the solution

$$u \in H^1(0,T;H) \cap L^\infty(0,T;V), \quad w \in L^2\big(\Omega; C^0([0,T])\big)$$

of Problem 1.9.1 corresponding to the initial values (u^0, w^0) and to the source term f, cf. (1.224). Let us then define the Poincaré mapping $\mathcal{P} : X \to X : (u^0, w^0) \mapsto (u(\cdot, T), w(\cdot, T))$.

(ii) We claim that for any sequence $\{(u_n^0, w_n^0)\}$ in X and any $(u^0, w^0) \in X$, if

$$u_n^0 \to u^0 \quad \text{weakly in } V, \qquad w_n^0 \to w^0 \quad \text{weakly in } H,$$

then, setting $(u_n(\cdot, T), w_n(\cdot, T)) := \mathcal{P}(u_n^0, w_n^0)$ and $(u(\cdot, T), w(\cdot, T)) := \mathcal{P}(u^0, w^0)$,

$$u_n(\cdot, T) \to u(\cdot, T) \quad \text{weakly in } V, \qquad w_n(\cdot, T) \to w(\cdot, T) \quad \text{weakly in } H.$$

After the final statement of Theorem 1.9.1, the uniform estimates that we derived in the proof of that theorem apply to the sequences $\{u_n\}$ and $\{w_n\}$. Moreover, as (1.255) entails the Lipschitz-continuity of $\mathcal{F}$, the uniform boundedness of $\{u_n\}$ in $H^1(0,T;H)$ yields the same property for $\{w_n\}$. The stated convergences then follow.

(iii) Let $z_*, z^* \in V$ be such that

$$Az_* = f_*, \qquad Az^* = f^* \qquad \text{in } V'.$$

Then $(z_*, \gamma_r(z_*))$ ($(z^*, \gamma_\ell(z^*))$, respectively) is a (constant in time) solution of Problem 1.9.1 associated with the source term f_* (f^*, respectively). By the monotonicity of A^{-1}, (1.257) entails that

$$z_* \leqslant z^* \qquad \text{a.e. in } \Omega. \tag{1.260}$$

(iv) We claim that the sequences $\{\mathcal{P}^n(z_*, \gamma_r(z_*))\}$ and $\{\mathcal{P}^n(z^*, \gamma_\ell(z^*))\}$ are respectively nondecreasing and nonincreasing in $V \times H$, and are bounded in that space. Moreover,

$$(z_*, \gamma_r(z_*)) \preceq \mathcal{P}^n(z_*, \gamma_r(z_*)) \preceq \mathcal{P}^n(z^*, \gamma_\ell(z^*)) \preceq (z^*, \gamma_\ell(z^*)) \quad \forall n. \tag{1.261}$$

In order to prove this statement, let us set $(u_*, w_*) := \mathcal{S}(z_*, \gamma_r(z_*))$ and $(u^*, w^*) := \mathcal{S}(z^*, \gamma_\ell(z^*))$. By (1.257), (1.260) and Theorem 1.9.2,

$$\begin{aligned}&(z_*, \gamma_r(z_*)) \preceq (u_*(\cdot, T), w_*(\cdot, T)) =: \mathcal{P}(z_*, \gamma_r(z_*))\\ &\preceq (u^*(\cdot, T), w^*(\cdot, T)) =: \mathcal{P}(z^*, \gamma_\ell(z^*)) \preceq (z^*, \gamma_\ell(z^*)).\end{aligned}$$

By applying the operator $\mathcal{S}$ to all these terms, by Theorem 1.9.2 we get

$$\begin{aligned}&\mathcal{P}(z_*, \gamma_r(z_*)) \preceq \mathcal{P}^2(z_*, \gamma_r(z_*)) \preceq \cdots \preceq \mathcal{P}^n(z_*, \gamma_r(z_*)) \preceq \cdots\\ &\preceq \mathcal{P}^n(z^*, \gamma_\ell(z^*)) \preceq \cdots \preceq \mathcal{P}^2(z^*, \gamma_\ell(z^*)) \preceq \mathcal{P}(z^*, \gamma_\ell(z^*)),\end{aligned}$$

i.e., (1.261).

(v) By the final part of Theorem 1.9.1 the sequences $\{\mathcal{P}^n(z_*, \gamma_r(z_*))\}$ and $\{\mathcal{P}^n(z^*, \gamma_\ell(z^*))\}$ are bounded in $V \times H$. Therefore there exist (u^0_*, w^0_*), $(u^{0*}, w^{0*}) \in X$ such that

$$\mathcal{P}^n(z_*, \gamma_r(z_*)) \to (u^0_*, w^0_*), \quad \mathcal{P}^n(z^*, \gamma_\ell(z^*)) \to (u^{0*}, w^{0*}) \qquad \text{weakly in } V \times H.$$

By (1.261) then

$$\mathcal{P}(u^0_*, w^0_*) = (u^0_*, w^0_*) \preceq \mathcal{P}(u^{0*}, w^{0*}) = (u^{0*}, w^{0*}).$$

The solutions (u_*, w_*) and (u^*, w^*) of Problem 1.9.1, that respectively correspond to the initial data (u^0_*, w^0_*) and (u^{0*}, w^{0*}), are then respectively minimal and maximal solutions of Problem 1.9.2 in the sense of (1.258). The boundedness of the initial data finally entails the regularity (1.259). □

1.9.5 An Equation with Hysteresis in a Coefficient

Let now a function k be given such that

$$k \in C^0(\mathbf{R}), \quad \exists k_{(1)}, k_{(2)} \in \mathbf{R} : \forall \xi \in \mathbf{R},\ 0 < k_{(1)} \leqslant k(\xi) \leqslant k_{(2)}. \tag{1.262}$$

We are interested into the following equation with hysteresis in a coefficient:

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] - \nabla \cdot \big[k(\mathcal{F}(u))\, \nabla u\big] = f \qquad \text{in } \Omega_T. \tag{1.263}$$

This is a simplified version of an equation we shall encounter in Secction 1.11, dealing with hysteresis in fluid flow through porous media. The analysis of (1.263) does not seem easy, and even the apparently simpler equation

$$\frac{\partial u}{\partial t} - \nabla \cdot \big[k(\mathcal{F}(u))\, \nabla u\big] = f \qquad \text{in } \Omega_T \tag{1.264}$$

looks rather challenging. If the coefficient were of the form $k(u)$, one would apply the classic Kirchhoff transformation:

$$K : u \mapsto U := \int_0^u k(\xi)\, \mathrm{d}\xi,$$

so that, $\nabla\cdot\big[k(u)\,\nabla u\big] = \Delta U$. As K is invertible, one would then replace u by U in the PDE. However, it is not clear how a Kirchhoff-type transformation might be applied whenever hysteresis occurs in the k versus u constitutive relation. We then suggest simplifying the problem as follows, by inserting a regularizing rate-dependent memory effect. We fix a kernel φ such that

$$\varphi \in C^1_c(\mathbf{R}^N), \quad \varphi \geqslant 0, \quad \int_{\mathbf{R}^N} \varphi(\xi)\,\mathrm{d}\xi = 1, \tag{1.265}$$

denote the space convolution by

$$(f * \varphi)(x) := \int_{\mathbf{R}^N} f(x-y)\varphi(y)\,\mathrm{d}y \qquad \forall x \in \mathbf{R}^N, \forall f \in L^1_{\mathrm{loc}}(\mathbf{R}^N),$$

and deal with the regularized equation

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] - \nabla\cdot\big[k\big(\mathcal{F}(u)*\varphi\big)\,\nabla u\big] = f \qquad \text{in } \Omega_T. \tag{1.266}$$

(In the convolution, functions defined in Ω_T will automatically be extended with vanishing value outside Ω_T.) For any $g \in L^\infty(\Omega)$, let us define the linear and continuous operator $A_g : V \to V'$ by

$$\langle A_g u, v\rangle := \int_\Omega g \nabla u\cdot\nabla v\,\mathrm{d}x \qquad \forall u, v \in V.$$

Problem 1.9.3. *Find $u : \Omega \to C^0([0,T])$ measurable such that $u \in L^2(0,T;V)$ and, setting*

$$w(x,t) := \big[\mathcal{F}\big(u(x,\cdot), w^0(x)\big)\big](t) \qquad \forall t \in [0,T], \textit{for a.a. } x \in \Omega, \tag{1.267}$$

$$g := k\big(\mathcal{F}(u)*\varphi\big) \qquad \textit{a.e. in } \Omega_T, \tag{1.268}$$

$w \in L^2(0,T;H)$, $u + w \in H^1(0,T;V')$, and

$$\frac{\partial}{\partial t}(u+w) + A_g u = f \qquad \textit{in } V', \ \textit{a.e. in }]0,T[, \tag{1.269}$$

$$(u+w)|_{t=0} = u^0 + w^0 \qquad \textit{in } V'. \tag{1.270}$$

Theorem 1.9.6 (*Existence*). *Assume that the hypotheses of Theorem 1.9.1 are fulfilled, as well as (1.262) and (1.265). Then Problem 1.9.3 has a solution such that (1.224) holds.*

Outline of the Proof. This argument is partially analogous to that of Theorem 1.9.1; here we just illustrate the main differences. In the approximate problem here we set

$$g_m^n := k(w_m^n * \varphi) \qquad \text{a.e. in } \Omega, \text{ for } n = 1, ..., m, \tag{1.271}$$

and replace (1.227) by

$$\frac{u_m^n - u_m^{n-1}}{h} + \frac{w_m^n - w_m^{n-1}}{h} + A_{(g_m^{n-1})}\, u_m^n = f_m^n \qquad \text{in } V', \text{ for } n = 1, ..., m. \tag{1.272}$$

Existence of a solution $\{u_m^n\}_{n=1,...,m}$ can then be proved step by step via the argument that we used for Problem 1.9.1$_m$. In order to derive *a priori* estimates, let us multiply equation (1.272) by $\left(u_m^n - u_m^{n-1}\right)/g_m^{n-1}$ $(\in V)$, and sum for $n = 1, ..., \ell$, for any $\ell \in \{1, ..., m\}$. As

$$\begin{aligned}\|\nabla g_m^{n-1}\|_{L^\infty(\Omega)^N} &= \Big\| \int_{\mathbf{R}^N} k(w_m^{n-1}(x-y))\; \nabla\varphi(y)\,\mathrm{d}y \Big\|_{L^\infty(\Omega)^N} \\ &\leqslant k_{(2)} \|\nabla\varphi\|_{C^1(\mathbf{R}^N)} \leqslant C_4 \qquad \forall n,\end{aligned}$$

we have

$$\begin{aligned}\langle A_{(g_m^{n-1})} u_m^n, \frac{u_m^n - u_m^{n-1}}{g_m^{n-1}} \rangle &= \int_\Omega g_m^{n-1} \nabla u_m^n \cdot \nabla \frac{u_m^n - u_m^{n-1}}{g_m^{n-1}}\,\mathrm{d}x \\ &= \int_\Omega \nabla u_m^n \cdot \nabla\left(u_m^n - u_m^{n-1}\right)\mathrm{d}x - \int_\Omega \nabla u_m^n \cdot \frac{\nabla g_m^{n-1}}{g_m^{n-1}} \left(u_m^n - u_m^{n-1}\right)\mathrm{d}x \\ &\geqslant \frac{1}{2}\int_\Omega \left(|\nabla u_m^n|^2 - |\nabla u_m^{n-1}|^2\right)\mathrm{d}x - \frac{C_4}{k_{(1)}} \int_\Omega |\nabla u_m^n|\; |u_m^n - u_m^{n-1}|\,\mathrm{d}x;\end{aligned}$$

moreover, as $ab \leqslant ca^2 + (4c)^{-1}b^2$ for any $a, b, c \in \mathbf{R}$ $(c \neq 0)$,

$$\frac{C_4}{k_{(1)}} \int_\Omega |\nabla u_m^n|\; |u_m^n - u_m^{n-1}|\,\mathrm{d}x \leqslant \frac{C_4^2 h k_{(2)}}{k_{(1)}^2} \int_\Omega |\nabla u_m^n|^2\,\mathrm{d}x + \frac{h}{4k_{(2)}} \int_\Omega \Big|\frac{u_m^n - u_m^{n-1}}{h}\Big|^2 \mathrm{d}x.$$

By the equation (1.272) we then get

$$\begin{aligned}&\frac{h}{k_{(2)}} \sum_{n=1}^{\ell} \int_\Omega \Big|\frac{u_m^n - u_m^{n-1}}{h}\Big|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega \left(|\nabla u_m^\ell|^2 - |u^0|^2\right)\mathrm{d}x \\ &\quad \leqslant \frac{C_4^2 h k_{(2)}}{(k_{(1)})^2} \sum_{n=1}^{\ell} \int_\Omega |\nabla u_m^n|^2\,\mathrm{d}x + \frac{h}{4k_{(2)}} \sum_{n=1}^{\ell} \int_\Omega \Big|\frac{u_m^n - u_m^{n-1}}{h}\Big|^2 \mathrm{d}x \\ &\quad\quad + h \sum_{n=1}^{\ell} {}_{V'}\langle f_m^n, u_m^n - u_m^{n-1}\rangle_V\,.\end{aligned} \tag{1.273}$$

The latter term can be estimated similarly to (1.229); a standard calculation based on the Gronwall lemma then yields an *a priori* estimate like (1.230). The remainder follows as in the proof of Theorem 1.9.1. □

A similar result holds if in (1.268) the convolution in space is replaced by a convolution in time, cf. Section 1.11.

A Semilinear Parabolic Equation with Hysteresis

Results are also known for initial- and boundary-value problems governed by semilinear equations like

$$\frac{\partial u}{\partial t} - \Delta u + \mathcal{F}(u) = f \quad \text{in } \Omega_T. \tag{1.274}$$

If the hysteresis operator $\mathcal{F}$ is Lipschitz-continuous in $C^0([0, T])$, existence and uniqueness can be proved via the contraction mapping principle, without assuming any monotonicity property just as it is natural for semilinear equations [8; Chapter X]. The asymptotic behavior as $t \to +\infty$ can also be studied by means of classic techniques.

Parabolic Equations with Discontinuous Hysteresis

Several results of this section can be extended to quasilinear and semilinear parabolic equations like (1.215) and (1.274) with a discontinuous hysteresis operator $\mathcal{F}$. In this case one of the main difficulties stays in the formulation of the operator $\mathcal{F}$ itself. For the relay and Preisach operators, this can be achieved via the approach of Section 1.4 (cf. the weak formulation (1.75)–(1.77)); in Section 1.10 we shall apply this approach to a quasilinear hyperbolic equation of the second order. For continuous Preisach operators one can then use either the formulation in terms of hysteresis operators, or the latter one.

1.10 A QUASILINEAR HYPERBOLIC PDE WITH HYSTERESIS

In this section we deal with an initial- and boundary-value problem for a quasilinear hyperbolic equation of second order that contains a hysteresis operator, $\mathcal{F}$:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}[u + \mathcal{F}(u)] - \Delta u = f & \text{in } \Omega_T, \\ [u + \mathcal{F}(u)]_{t=0} = u^0 + w^0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial t}[u + \mathcal{F}(u)]\Big|_{t=0} = u^1 + w^1 & \text{in } \Omega. \end{cases} \tag{1.275}$$

If $\mathcal{F}$ is replaced by a (nonlinear) superposition operator without memory, existence of a solution is only known under severe restrictions on the nonlinearity. On the other hand if $\mathcal{F}$ is a Prandtl–Ishlinskiĭ hysteresis operator of play-type, cf. Section 1.3, the problem can be reduced to a system of variational inequalities, and well-posedness can be proved without much effort [8; Chapter VII].

Along the lines of [41], here we assume that $\mathcal{F}$ is a relay operator, cf. Section 1.4, and prove existence of a weak solution, by means of a technique that can easily be extended to (either continuous or discontinuous) Preisach operators and to more general equations as well. In this case, because of the discontinuity of the hysteresis relation, the equation $(1.275)_1$ is the weak formulation of a *free boundary problem*. In fact, if $w = \pm 1$ a.e. in Ω_T, under regularity conditions the space-sets respectively characterized by $w = 1$ and $w = -1$ are separated by an unknown moving front (or *free boundary*).

We still denote by Ω a bounded Lipschitz domain of $\mathbf{R}^N$, fix any $T > 0$, and set $\Omega_T := \Omega \times]0, T[$. We also set

$$F := \int_0^t f(\cdot, \tau)\, \mathrm{d}\tau + u^1 + w^1 \qquad \text{in } \Omega_T, \tag{1.276}$$

so that, integrating $(1.276)_1$ in time, we get

$$\frac{\partial}{\partial t}(u + w) - \Delta \int_0^t u(\cdot, \tau)\, \mathrm{d}\tau = F \qquad \text{in } \Omega_T. \tag{1.277}$$

We define the Hilbert spaces H, V, and the operator $A : V \to V'$ as in Section 1.9, assume that

$$u^0, w^0 \in H, \qquad F \in L^2(0, T; V'), \tag{1.278}$$

and formulate an initial- and boundary-value problem for Eqn. $(1.275)_1$.

Problem 1.10.1. *Find* $U \in L^2(0, T; V) \cap H^1(0, T; H)$ *and* $w \in L^\infty(\Omega_T)$ *such that, setting* $u := \partial U/\partial t$,

$$U = 0 \quad \textit{a.e. in } (\Omega \times \{0\}) \cup (\partial\Omega \times]0, T[), \tag{1.279}$$

$$|w| \leqslant 1 \quad \textit{a.e. in } \Omega_T, \qquad \frac{\partial w}{\partial t} \in C^0(\overline{\Omega_T})', \tag{1.280}$$

$$\iint_{\Omega_T} \Big((u^0 - u + w^0 - w) \frac{\partial v}{\partial t} + \nabla U \cdot \nabla v \Big) \mathrm{d}x \mathrm{d}\tau = \int_0^T \langle F, v \rangle\, \mathrm{d}\tau \tag{1.281}$$
$$\forall v \in H^1(0, T; H) \cap L^2(0, T; V), \ v(\cdot, T) = 0,$$

$$\begin{cases} (w-1)(u-\rho_2) \geqslant 0 \\ (w+1)(u-\rho_1) \geqslant 0 \end{cases} \qquad \textit{a.e. in } \Omega_T, \tag{1.282}$$

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left[u(x,t)^2 - u^0(x)^2 + |\nabla U(x,t)|^2\right] \mathrm{d}x + \int_\Omega \Psi_\rho(w(x,\cdot),[0,t]) \\ &\leqslant \int_0^t \langle F, u\rangle \,\mathrm{d}\tau \qquad \textit{for a.a. } t \in]0,T[. \end{aligned} \tag{1.283}$$

Interpretation

Eqn. (1.281) entails

$$\frac{\partial}{\partial t}(u+w) + AU = F \qquad \text{in } V', \text{ a.e. in }]0,T[. \tag{1.284}$$

By differentiating in time and setting $f := \partial F/\partial t$ we get $(1.275)_1$; by integrating in time the latter and comparing with (1.281), the initial condition $(1.275)_2$ is also derived in the sense of traces of $H^1(0,T;V')$.

Whenever $u \in L^2(0,T;V)$, we can multiply (1.284) by $u = \partial U/\partial t$, and integrate in space and time; it is easy to see that (1.283) then reads

$$\int_0^t \langle \frac{\partial}{\partial \tau}(u+w), u\rangle \,\mathrm{d}\tau \geqslant \frac{1}{2}\int_\Omega \left[u(x,t)^2 - u^0(x)^2\right]\mathrm{d}x + \int_{\bar\Omega} \Psi_\rho(w,[0,t]) \quad \text{for a.a. } t \in]0,T[. \tag{1.285}$$

This inequality is meaningful, but this derivation is rigorous only if $u \in L^2(0,T;V)$, and this regularity property is far from being obvious for Problem 1.10.1. (1.283) may then be regarded as a weak formulation of the dissipation condition (1.76) (i.e. (1.74)) a.e. in Ω. On the other hand, the system (1.282) is equivalent to the confinement condition (1.75) (i.e. (1.73)) a.e. in Ω_T. In Section 1.4 we saw that (1.75), (1.76) and the initial condition (1.72) (here rewritten with w^0 in place of ξ) are equivalent to the hysteresis relation

$$w \in k_\rho(u, w^0) \qquad \text{a.e. in } \Omega. \tag{1.286}$$

We conclude that Problem 1.10.1 is the weak formulation of an initial- and boundary-value problem associated to the system (1.284), (1.286).

Notice that the second order equation $(1.275)_1$ may equivalently be replaced by a system of two first-order equations, with unchanged results.

Applications

For $N = 1$, Problem 1.10.1 can represent processes in a univariate insulating ferrimagnetic material; these materials are actually characterized by a rectangular hysteresis loop. The equation $(1.275)_1$ can be derived from the Maxwell equations, assuming that the displacement field D is proportional to the electric field E, that there is no electric current, i.e. $J \equiv 0$. The magnetic field H and the magnetization field M are here denoted by u and w, respectively.

This model can also be extended to univariate conducting ferromagnetic materials; in this case the Ohm law yields $J = \sigma E$, and a term proportional to $\partial(u+w)/\partial t$ must be inserted on the left side of $(1.275)_1$; this corresponds to occurrence of a term proportional to $u + w$ in the time-integrated equation (1.284). Displaying coefficients the equation reads

$$\varepsilon \frac{\partial^2}{\partial t^2}(u+w) + 4\pi\sigma \frac{\partial}{\partial t}(u+w) - c^2 \frac{\partial^2 u}{\partial x^2} = \text{ prescribed field} \qquad \text{in } \Omega_T. \quad (1.287)$$

The electric conductivity σ is so large that (denoting the dielectric constant by ε) the term $\varepsilon\partial^2 B/\partial t^2$ is not dominated by $\sigma\partial B/\partial t$ only for rapidly variable fields (i.e., high frequencies).

Problem 1.10.1 can also represent evolution in a univariate insulating ferroelectric material; in this case u and w represent the electric field E and the polarization field P, respectively. As we already pointed out for equation $(1.275)_1$, this does not apply to $N > 1$, for in that case u and w are vector variables, and the operator $-\Delta$ should be replaced by curl^2.

THEOREM 1.10.1 (*[41]*). *Let (1.278) be fulfilled, and*

$$F = F_1 + F_2, \quad F_1 \in L^1(0,T;H), \quad F_2 \in W^{1,1}(0,T;V'). \qquad (1.288)$$

Then Problem 1.10.1 has a solution (U, w) such that

$$U \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V). \qquad (1.289)$$

PROOF.

(i) Approximation. Let us fix any $m \in \mathbf{N}$ and set $h := T/m$,

$$\begin{cases} F_{1m}^n(x) := \dfrac{1}{h}\displaystyle\int_{(n-1)h}^{nh} F_1(x,t)\,\mathrm{d}t \quad \text{for a.a. } x \in \Omega, \quad F_{2m}^n := F_2(nh), \\ F_m^n := F_{1m}^n + F_{2m}^n, \quad u_m^0 := u^0, \quad w_m^0 := w^0, \quad \text{for } n = 1, \dots, m, \end{cases} \qquad (1.290)$$

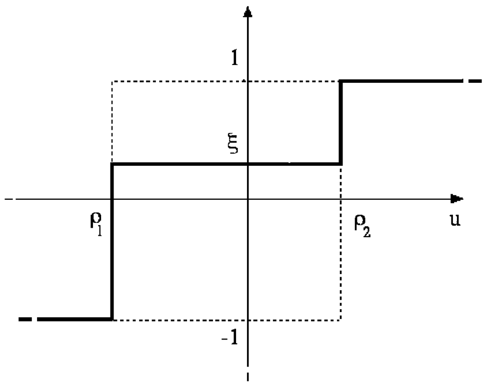


FIGURE 1.18 Graph of the multivalued function $G_\rho(\cdot, \xi)$ for a fixed $\xi \in [-1, 1]$.

$$G_\rho(v, \xi) := \begin{cases} \{-1\} & \text{if } v < \rho_1 \\ [-1, \xi] & \text{if } v = \rho_1 \\ \{\xi\} & \text{if } \rho_1 < v < \rho_2 \\ [\xi, 1] & \text{if } v = \rho_2 \\ \{1\} & \text{if } v > \rho_2 \end{cases} \qquad \forall (v, \xi) \in \mathbf{R} \times [-1, 1], \quad (1.291)$$

cf. Fig. 1.18. Let us define time-interpolate functions as in (1.188).

Problem 1.10.1$_m$. *Find $u_m^n \in V$ and $w_m^n \in H$ $(n = 1, ..., m)$ such that, for any n,*

$$w_m^n \in G_\rho(u_m^n, w_m^{n-1}) \qquad \textit{a.e. in } \Omega, \tag{1.292}$$

$$\frac{u_m^n - u_m^{n-1}}{h} + \frac{w_m^n - w_m^{n-1}}{h} + h \sum_{j=1}^{n} A u_m^j = F_m^n \quad \textit{in } V'. \tag{1.293}$$

Existence of an approximate solution can easily be proved step by step, as G_ρ is maximal monotone.
(ii) A Priori *Estimates.* Let us multiply the equation (1.293) by hu_m^n, and sum for $n = 1, ..., \ell$, for any $\ell \in \{1, ..., m\}$. Setting $U_m^n := h \sum_{j=1}^{n} u_m^j$ a.e. in Ω for

$n = 1, ..., m$, by (1.291) we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left[(u_m^\ell)^2 - (u^0)^2 + |\nabla U_m^\ell|^2\right] \mathrm{d}x + \int_\Omega \Psi_\rho(w_m, [0, \ell h])\, \mathrm{d}x \\
&\leqslant h \sum_{n=1}^{\ell} \langle F_m^n, u_m^n \rangle \leqslant \|F_1\|_{L^1(0,T;H)} \max_{n=0,\ldots,\ell} \|u_m^n\|_H \\
&+2\|F_2\|_{W^{1,1}(0,T;V')} \max_{n=0,\ldots,\ell} \|U_m^n\|_V \qquad \text{for } \ell = 1, ..., m.
\end{aligned} \tag{1.294}$$

A simple calculation then yields

$$\begin{aligned}
&\|U_m\|_{W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V)}\,, \|\Psi_\rho(w_m, [0, t])\|_{L^\infty(0,T;L^1(\Omega))} \\
&\leqslant \text{constant (independent of } m).
\end{aligned} \tag{1.295}$$

(iii) Limit Procedure. By the above estimates, there exist U, w such that, as $m \to \infty$ along a suitable sequence,

$$U_m \to U \quad \text{weakly star in } W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V), \tag{1.296}$$

$$w_m \to w \quad \text{weakly star in } L^\infty(\Omega_T), \tag{1.297}$$

$$\frac{\partial w_m}{\partial t} \to \frac{\partial w}{\partial t} \quad \text{weakly star in } C^0(\overline{\Omega_T})'. \tag{1.298}$$

Setting $U_m := \int_0^t \bar{u}_m(\cdot, \tau)\mathrm{d}\tau$ a.e. in Ω_T, (1.293) and (1.294) also read

$$\frac{\partial}{\partial t}(u_m + w_m) + AU_m = \bar{F}_m \qquad \text{in } V', \text{ a.e. in }]0, T[, \tag{1.299}$$

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left[\bar{u}_m(x, t)^2 - u^0(x)^2 + |\nabla \bar{u}_m(x, t)|^2\right]\mathrm{d}x + \int_\Omega \Psi_\rho(w_m, [0, t])\, \mathrm{d}x \\
&\leqslant \int_0^t \langle \bar{F}_m, \bar{u}_m \rangle \mathrm{d}\tau \qquad \text{for a.a. } t \in]0, T[;
\end{aligned} \tag{1.300}$$

by passing to the limit in (1.299) and to the inferior limit in (1.300), we then get (1.281) and (1.283). Finally, (1.292) entails

$$\left.\begin{aligned}
&\iint_{\Omega_T} (\bar{w}_m - 1)\, (\bar{u}_m - \rho_2)\, \varphi(x, t)\, \mathrm{d}x\mathrm{d}t \geqslant 0 \\
&\iint_{\Omega_T} (\bar{w}_m + 1)\, (\bar{u}_m - \rho_1)\, \varphi(x, t)\, \mathrm{d}x\mathrm{d}t \geqslant 0
\end{aligned}\right. \quad \forall \varphi \in \mathcal{D}(\Omega_T),\ \varphi \geqslant 0, \tag{1.301}$$

and (1.296)–(1.298) allow one to pass to the limit in these inequalities, because of the following lemma. □

LEMMA 1.10.2 (*[41]*). *If two sequences* $\{u_m\}$, $\{w_m\}$ *are such that*

$$\begin{aligned} &u_m \to u \qquad \text{weakly in } L^2(\Omega_T) \cap H^{-1}(0,T;V), \\ &w_m \to w \qquad \text{weakly star in } L^\infty(\Omega_T), \\ &\left\| \frac{\partial w_m}{\partial t} \right\|_{L^1(\Omega_T)} \leqslant \text{constant}, \end{aligned} \tag{1.302}$$

then

$$\iint_{\Omega_T} u_m w_m \,\mathrm{d}x\mathrm{d}t \to \iint_{\Omega_T} uw \,\mathrm{d}x\mathrm{d}t. \tag{1.303}$$

This statement can be proved via Banach-space interpolation [41].

The convergence (1.298) stems from regularizing properties of the relay operator we met in Section 1.4; this plays a key role in the proof of the above existence result, which indeed has no analog for quasilinear hyperbolic equations without hysteresis. Equation $(1.275)_1$ turns out to be one of the few known examples in which analysis is made easier by occurrence of hysteresis.

Uniqueness of the solution of Problem 1.10.1 is an open question.

Extensions

Problem 1.10.1 and Theorem 1.10.1 can be extended into two main directions:

(i) the relay operator can be replaced by the either continuous or discontinuous Preisach model,
(ii) one can deal with the vector setting, i.e., with u and w ranging in $\mathbf{R}^N$.

The first extension can be pursued without much effort [8]. As for the second one, we already mentioned that processes in insulating ferrimagnetic materials fit into the above picture. Assuming that the field $\vec{D}$ is proportional to $\vec{E}$, the Maxwell system yields an equation of the form

$$\frac{\partial^2}{\partial t^2}(\vec{H} + \vec{M}) + \mathrm{curl}^2 \vec{H} = \vec{f} \tag{1.304}$$

(here written with normalized coefficients). The relation between $\vec{M}$ and $\vec{H}$ can be represented by a vector extension of the relay operator, we illustrated in Section 1.4 [45,48,49]; an initial- and boundary-value problem can be formulated in the framework of Sobolev spaces, and existence of a solution can be proved [99].

A different approach to quasilinear hyperbolic equations with hysteresis in a single dimension of space was used by Krejčí in a series of papers, see [26,5; Chapters III,IV] and references therein. Assuming the convexity of hysteresis loops and exploiting the dissipativity properties of hysteresis, he derived results of existence and uniqueness of the solution, decay as $t \to +\infty$, stability and asymptotic stability, and the existence of periodic solutions. On top of all this, he also studied the Riemann problem in detail.

1.11 HYSTERESIS IN POROUS-MEDIA FILTRATION

In this section we illustrate hysteresis in unsaturated flow through porous media, and formulate a relation between saturation and pressure that accounts for rate-dependent decay towards the hysteresis relation. We then illustrate how existence of a solution can be proved for a boundary- and initial-value problem, in which that relation is coupled with the law of mass conservation and Darcy's law; we refer to [100,120] for details.

Filtration Without Hysteresis

Unsaturated fluid flow through porous media is relevant for engineering, and has been treated in a large technical literature, cf. e.g. the monographs [101–104]. If no unsaturated flow occurs, the wet and dry regions are in contact along an *a priori* unknown surface (the *phreatic surface*). This free boundary problem was studied in a number of mathematical papers, after Baiocchi formulated it as a variational inequality and proved its well-posedness in 1972; see e.g. [105–107], the monograph [108] and references therein. As an alternative, by an approach that was proposed and studied by W. Alt, one can deal with partially saturated flow [109–112].

Quantitatively relevant hysteresis effects occur in fluid flow through porous media [101–104,113]. It seems that Poulovassilis was the first one to apply the Preisach model (under the denomination of *independent domain model*) to represent hysteresis in the dependence of saturation on pressure [114–116]; Muhalem then assumed a specific form of the Preisach density [117,118]. Apparently these hysteresis effects have not (yet) received much attention by mathematicians; as far as this author knows, the functional approach based on the notion of hysteresis operator has been applied only recently and in few papers [14,100,113,119].

Let a domain Ω of $\mathbf{R}^3$ represent a region occupied by a porous medium in communication with one or more nonstationary water reservoirs, cf. Fig. 1.19; let us fix any $T > 0$, and set $\Omega_T := \Omega \times]0, T[$.

Let us denote the saturation of the medium by s, its hydraulic conductivity by k, its porosity by φ, the pressure of the fluid (water, say) by u, its flux per unit area by $\vec{q}$, its mass density by ρ, the gravity acceleration by g,

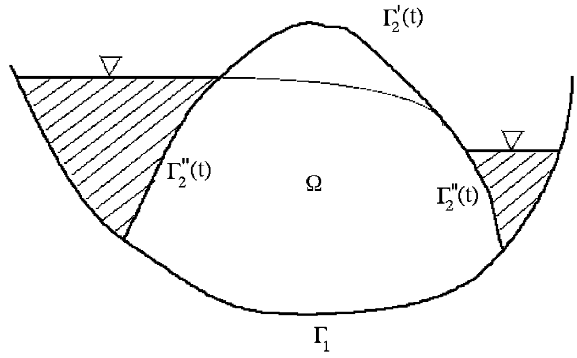


FIGURE 1.19 A porous dam, with two reservoirs and an impervious bottom.

and the vertical coordinate by z. The equation of mass conservation and the Darcy law respectively read

$$\varphi \frac{\partial s}{\partial t} + \nabla \cdot \vec{q} = 0 \qquad \text{in } \Omega_T \ (\nabla \cdot := \text{div}), \tag{1.305}$$

$$\vec{q} = -k\nabla(u + \rho g z) \qquad \text{in } \Omega_T; \tag{1.306}$$

by eliminating the field $\vec{q}$ we then get

$$\varphi \frac{\partial s}{\partial t} - \nabla \cdot [k\nabla(u + \rho g z)] = 0 \qquad \text{in } \Omega_T. \tag{1.307}$$

(For some phenomena it is possible to neglect the gravitation term; for instance this applied to two-phase flow of two immiscible fluids of comparable densities.)

For a homogeneous medium without hysteresis, s and u are related by a constitutive law of the form

$$s(x, t) \in \beta(u(x, t)) \qquad \forall (x, t) \in \Omega_T, \tag{1.308}$$

where $\beta : \mathbf{R} \to [0, 1]$ is a (possibly multivalued) maximal monotone function, cf. Fig. 1.20. For instance, if β equals the Heaviside graph

$$H(u) := \{0\} \quad \text{if } u < 0, \quad H(0) := [0, 1], \quad H(u) := \{1\} \quad \text{if } u > 0,$$

then (1.307) and (1.308) represent unsaturated flow. In this case both (1.305) and (1.307) should be understood in the sense of distributions, and they are the weak formulation of a *free boundary problem*.

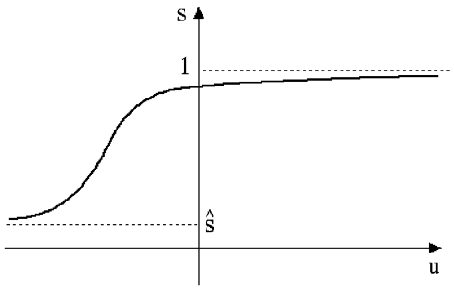


FIGURE 1.20 Saturation versus pressure constitutive relation without hysteresis.

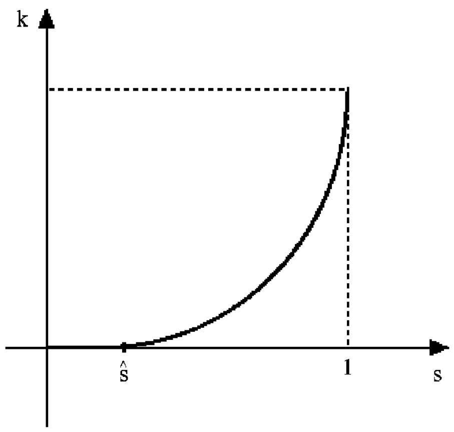


FIGURE 1.21 Hydraulic conductivity versus saturation constitutive relation.

The hydraulic conductivity can be represented as a nonnegative, nondecreasing function of saturation, cf. Fig. 1.21:

$$k(x,t) = h(s(x,t)) \qquad \forall (x,t) \in \Omega_T; \tag{1.309}$$

k is then a (possibly multivalued) function of pressure: $k \in h(\beta(u))$.

Suitable boundary conditions must be appended to equation (1.307); these include a no-flux condition on the impervious part of the boundary, and a *Signorini-type* (i.e., *obstacle-type*) condition on the part of the boundary that is in contact with the atmosphere; this also accounts for occurrence of overflow along a so-called *seepage face*. An initial condition must also be prescribed for s. A problem in the framework of Sobolev spaces can then be formulated for the system (1.307)–(1.309), and can be coupled with these initial and boundary conditions; existence of a solution was proved in [112].

Filtration With Hysteresis

The constitutive relation (1.308) is oversimplified: porous media exhibit quantitatively relevant hysteresis effects, cf. Fig. 1.22. Laboratory measurements actually indicate that at any point $x \in \Omega$ and any instant t, the saturation $s(x, t)$ depends not only on the pressure $u(x, t)$, but also on the previous evolution of u at the same point, $u(x, \cdot)$, and on the initial value $s^0(x)$; moreover this dependence is rate-independent.

Accordingly, in place of the constitutive relation (1.308) we consider a relation of the form

$$s(t) = \big[\mathcal{F}(u, s^0)\big](t) \qquad \forall t \in [0, T]; \tag{1.310}$$

here $\mathcal{F}$ is a hysteresis operator, that we assume to be continuous in $C^0([0, T]) \times \mathbf{R}$. (More generally, one might assume that $\mathcal{F}$ is the sum of a hysteresis operator and a possibly multivalued maximal monotone graph [100].) Whenever this constitutive relation is coupled with the PDE (1.307), one must account for dependence on the space variable, $x \in \Omega$; here we assume that x occurs just as a parameter:

$$s(x, t) = \big[\mathcal{F}(u(x, \cdot), s^0(x))\big](t) \qquad \forall t \in [0, T], \text{ for a.a. } x \in \Omega. \tag{1.311}$$

This excludes any space interaction in the constitutive relation.

By (1.309), hysteresis in the s versus u dependence entails occurrence of hysteresis in the k versus u relation. (1.307) can then be labeled as a PDE with hysteresis in a coefficient; the analysis of equations of this type often exhibits difficulties, cf. Section 1.9.

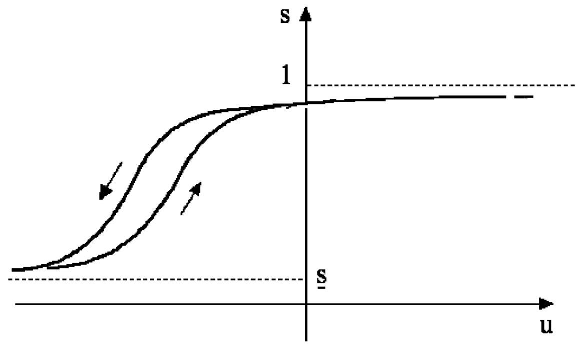


FIGURE 1.22 Saturation versus pressure constitutive relation with hysteresis.

The Issue of Existence

To prove the existence of a solution for the corresponding initial- and boundary-value problem does not seem an easy task. Following a standard procedure one might approximate this problem, derive a priori estimates, and then try to pass to the limit. Because of the occurrence of a memory operator, it seems especially convenient to use time-discretization. Let us denote the approximation parameter by $m \in \mathbf{N}$, and the approximate solution by (u_m, s_m). Uniform estimates for u_m in $L^2\big(0, T; H^1(\Omega)\big)$ can easily be derived by multiplying the approximate equation by u_m, and then integrating with respect to space and time. This yields existence of a weakly convergent subsequence, but does not grant convergence of any subsequence of u_m in $C^0([0, T])$ a.e. in Ω, and thus does not suffice to pass to the limit in the memory operator. In order to derive stronger *a priori* estimates, one might try to multiply the approximate equation by $\partial u_m/\partial t$, and again integrate with respect to space and time. But then difficulties arise in dealing with the elliptic term.

Whenever the dependence of s on u is without memory, of the form $s = \beta(u)$ with β continuous and nondecreasing, say, one can apply the classic *Kirchhoff transformation*:

$$\mathcal{K} : u \mapsto w := \int_0^u h(\beta(\xi))\, \mathrm{d}\xi, \tag{1.312}$$

so that

$$h(\beta(u))\nabla u = \nabla w \quad \text{a.e. in } \Omega_T, \qquad \nabla\cdot[h(\beta(u))\nabla u] = \Delta w \quad \text{in } \mathcal{D}'(\Omega_T).$$

As h and β are nondecreasing, $\mathcal{K}$ is a (possibly multivalued) maximal monotone operator; hence

$$s = \beta(u) \in \big(\beta \circ \mathcal{K}^{-1}\big)(w) \qquad \text{a.e. in } \Omega_T,$$

and one can couple this relation with the equation

$$\varphi \frac{\partial s}{\partial t} - \Delta w = 0 \qquad \text{in } \mathcal{D}'(\Omega_T). \tag{1.313}$$

This procedure was used in [112] to prove existence of a solution for the problem without hysteresis. However, it is not clear how the transformation (1.312) might be extended in the presence of hysteresis in the s versus u relation.

A Rate-dependent Correction

The above difficulties induce us to amend the model. First we do so by inserting a rate-dependent correction into the memory relation. Although we are not able to derive any uniform estimate on the pressure rate, we conjecture that this rate should not be too large, even on the (rather slow) time-scale of filtration phenomena. Along the lines of [100] we then propose inserting a term which penalizes high pressure rates into the s versus u constitutive relation. In this way we account for a rate-dependent component of memory, aside hysteresis.

Let us assume that the hysteresis operator $\mathcal{F}$ is piecewise monotone in the sense of (1.10); we can then invert it, and write (1.311) in the equivalent form

$$u(t) = \big[\mathcal{G}(s, u^0)\big](t) \qquad \forall t \in [0, T],$$

$\mathcal{G} := \mathcal{F}^{-1}$ being also a hysteresis operator. (Some specifications should be made as for the initial data u^0 and s^0, but here we omit these details.) We then append a time-relaxation term, and write the constitutive relation in the form

$$u(t) = \big[\mathcal{G}(s, u^0)\big](t) + \alpha\frac{\mathrm{d}s}{\mathrm{d}t}(t) =: \big[\mathcal{G}_\alpha(s, u^0)\big](t) \qquad \forall t \in [0, T], \tag{1.314}$$

where α is a *(small)* positive constant. Under natural assumptions, the rate-dependent operator $\mathcal{G}_\alpha(\cdot, u^0)$ maps $H^1(0, T)$ to $L^2(0, T)$. If $\mathcal{G}(\cdot, u^0)$ is continuous in $C^0([0, T])$, then $\mathcal{G}_\alpha$ is sequentially weakly continuous: for any sequence $\{s_n\}$ in $H^1(0, T)$,

$$\begin{aligned} &s_n \to s \quad \text{weakly in } H^1(0, T) \quad \Rightarrow \\ &\mathcal{G}_\alpha(s_n, u^0) \to \mathcal{G}_\alpha(s, u^0) \quad \text{weakly in } L^2(0, T). \end{aligned} \tag{1.315}$$

The operator $\mathcal{G}_\alpha$ can then be extended to the above space-distributed problem by inserting the dependence on the parameter x:

$$u(x, t) = \big[\mathcal{G}(s(x, \cdot), u^0(x))\big](t) + \alpha\frac{\partial s}{\partial t}(x, t) \qquad \forall t \in [0, T], \text{ for a.a. } x \in \Omega. \tag{1.316}$$

Estimation Procedure

Multiplying (1.316) by $\partial s/\partial t$ and integrating in time we have

$$\int_0^{\tilde{t}} \frac{\partial s}{\partial t} u \,\mathrm{d}t = \int_0^{\tilde{t}} \frac{\partial s}{\partial t}\big[\mathcal{G}(s(\cdot), u^0)\big](t)\,\mathrm{d}t + \alpha\int_0^{\tilde{t}} \left|\frac{\partial s}{\partial t}\right|^2 \mathrm{d}t \qquad \forall \tilde{t} \in]0, T].$$

If $\mathcal{G}$ is the inverse of a Preisach-type operator, we get an estimate of the form

$$\int_0^{\tilde{t}} \frac{\partial s}{\partial t} u \, \mathrm{d}t \geqslant \Phi(s(\tilde{t})) - \Phi(s^0) + \Psi\big(s\big|_{]0,\tilde{t}[}\big) \, \mathrm{d}t + \alpha \int_0^{\tilde{t}} \left|\frac{\partial s}{\partial t}\right|^2 \mathrm{d}t \qquad \forall \tilde{t} \in]0, T];$$

here Φ and Ψ depend on the operator $\mathcal{G} = \mathcal{F}^{-1}$; Φ is a potential function, and (loosely speaking) $\Psi\big(s\big|_{]0,\tilde{t}[}\big)$ measures the total variation of s in the interval $]0, \tilde{t}[$, cf. (1.77).

Let us now discretize the equation (1.307) in time, with time-step T/m ($m \in \mathbf{N}$), and denote the corresponding solution by (u_m, s_m). By means of the latter inequality, multiplying the discretized equation by u_m and integrating with respect to space and time, one easily derives uniform estimates of the form

$$\|u_m\|_{L^2(0,T;H^1(\Omega))}, \|s_m\|_{H^1(0,T;L^2(\Omega))} \leqslant \text{constant (independent of } m). \quad (1.317)$$

Moreover, assuming that the hysteresis operator $\mathcal{G}(\cdot, u^0)$ is Lipschitz-continuous in $C^0([0, T])$, by means of (1.316) and of the uniform estimate on u_m, one can derive a uniform estimate for s_m in $H^1(0, T; L^2(\Omega))$ [100].

By these estimates, weak convergence follows for suitable subsequences. The regularity properties of the operator $\mathcal{G}$ allow one to pass to the limit in the time-discretized version of (1.316), and thus to prove existence of a weak solution. It would then be natural to consider the behavior of the solution of our problem as the relaxation parameter α vanishes; but, as may be expected, in this limit one encounters the same difficulties that we pointed out for the purely hysteretic constitutive relation.

A Different Approach

As an alternative to the modification of the s versus u relation we just outlined, the analytic difficulties we pointed out for the system (1.307), (1.309), (1.310) can be overcome by regularizing the k versus s dependence. More specifically, let us fix a positive and decreasing function $\rho \in C^1(\mathbf{R}^+)$ such that $\int_{\mathbf{R}^+} \rho(\tau) \, \mathrm{d}\tau = 1$ (e.g., $\rho(\tau) := ae^{-a\tau}$ for any $\tau \geqslant 0$, for some $a > 0$), extend s with vanishing value for negative times, and set

$$(s*\rho)(x, t) := \int_{\mathbf{R}^+} s(x, t - \tau)\rho(\tau) \, \mathrm{d}\tau \qquad \text{for a.a. } (x, t) \in \Omega_T, \forall v \in L^1(\Omega_T); \quad (1.318)$$

we then replace (1.309) by

$$k = h(s*\rho) \qquad \text{a.e. in } \Omega_T. \quad (1.319)$$

As we remarked above, in order to pass to the limit in the hysteresis term it may be convenient to derive *a priori* estimates by multiplying the approximate equation by $\partial u_m/\partial t$ and integrating in time. By doing so one encounters the term

$$\int_0^{\tilde{t}} \mathrm{d}t \int_\Omega k_m \nabla u_m \cdot \nabla \frac{\partial u_m}{\partial t}\,\mathrm{d}x = -\frac{1}{2}\int_0^{\tilde{t}} \mathrm{d}t \int_\Omega \frac{\partial k_m}{\partial t} |\nabla u_m|^2\,\mathrm{d}x$$
$$+\frac{1}{2}\int_\Omega \big(k_m(x,\tilde{t})|\nabla u_m(x,\tilde{t})|^2 - k_m(x,0)|\nabla u^0(x)|^2\big)\,\mathrm{d}x;$$

if one can show that $s_m \geqslant$ constant $> \hat{s}$, then k_m is larger than a positive constant, cf. Fig. 1.21; moreover (this is the key point)

$$\begin{aligned} &\left\| \frac{\partial k_m}{\partial t} \right\|_{L^\infty(\Omega_T)} = \left\| \frac{\partial h(s_m * \rho)}{\partial t} \right\|_{L^\infty(\Omega_T)} \\ &\leqslant \max_{[0,1]} |h'| \sup_{\mathbf{R}^+} |\rho'| = \text{ constant (independent of } m). \end{aligned} \tag{1.320}$$

This allows one to complete the estimate procedure, and then to prove the existence of a solution for the continuous problem [120]. In this framework one can also prove that the solution is asymptotically stable; that is, if as $t \to +\infty$ the level of the reservoirs converges, then the solution (u, s) also converges to a solution of the limit stationary problem. One can also show that as the hysteresis operator degenerates into a superposition operator, the solution of the hysteresis problem tends to that of the problem without hysteresis.

1.12 CONCLUSIONS

We have dealt with mathematical models of hysteresis, and applied them to problems arising in continuum mechanics, in ferromagnetism, and in filtration through porous media.

This survey has partly reviewed the developments of [8]; this applies especially to Sections 1.1–1.3. Novelties with respect to the author's 1994 monograph include the following issues:

- (i) the concept of the hysteresis relation, as an alternative to that of the hysteresis operator (Section 1.1);
- (ii) the formulation via a single variational inequality of a model of elasto-plasticity with strain-hardening with no internal variable (final part of Section 1.3);
- (iii) the modification of the classic Landau–Lifshitz equation of micromagnetism, and the formulation of a rate-independent mesoscopic model of ferromagnetism, after [61] (Section 1.5);

(iv) the two-scale approach for the Prandtl–Ishlinskiĭ and Preisach models, based on the homogenization viewpoint (Section 1.6);
(v) an answer to the question of the origin of differential equations with hysteresis, after [97] (Section 1.7);
(vi) the classification of PDEs with hysteresis (beginning of Section 1.8);
(vii) the proof of the existence of minimal and maximal solutions of the quasilinear parabolic problem with hysteresis, after [98], and the study of an equation with hysteresis in a coefficient (Section 1.9);
(viii) the proof of the existence of a solution for a second order, quasilinear, hyperbolic equation with discontinuous hysteresis, after [41] (Section 1.10);
(ix) the study of hysteresis in porous-media filtration, via a rate-dependent correction of the hysteresis relation, after [100,120] (Section 1.11).

The mathematical analysis of hysteresis and of related PDEs offers many open questions to the researcher. One of the most evident is that so far only a few basic models of hysteresis have been discovered and studied. Confining ourselves to the scalar setting, we encounter the Duhem model and the somehow related Bouc model (the latter still demands analytic investigation); the stop, the play and the Prandtl–Ishlinskiĭ models; the relay and Preisach models. Then there are the vector extensions of these models, besides the typical vector constructions like the Landau–Lifshitz equation and its modified version we outlined in Section 1.5. Relevant issues like the identification of hysteresis operators, their classification on the basis of structural criteria, and others, are still largely open.

On the other hand, so far only few hysteresis phenomena have been studied in some detail: elasto-plasticity, ferromagnetism, undercooling and superheating, shape memory, damage and fatigue, superconductivity, and not many others. The analysis of further phenomena, or the formulation of new viewpoints for known ones, may be expected to offer a source of new models, as has already happened in the past. In any case even the analysis of the above outlined models of elastoplasticity, ferromagnetism and hysteresis in porous-medium filtration is far from complete; this especially applies to the latter. The study of these phenomena should be enriched, e.g., with the analysis of the asymptotic behavior as $t \to +\infty$, of their control, and especially with their numerical analysis.

Most of the developments we have reviewed above were based on Krasnosel'skiĭ's notion of the hysteresis operator; alternative approaches have been considered in Sections 1.4, 1.5 and 1.10, and look promising for future research.

In recent years a different approach to hysteresis has been proposed by Mielke, Theil, Levitas and other researchers, see e.g. [121–125], to model hysteresis in quasistatic evolution; that approach does not involve hysteresis operators, and is based on coupling the energy balance with a stability condition. A similar formulation has also been applied to model fracture dynamics by Francfort, Marigo, Dal Maso, Toader and others, see e.g. [126,127]. There are similarities between this model and the formulation of the relay operator of (1.79) and (1.80) above. This method looks capable of providing a rather general framework for hysteresis, and indeed applies to a large number of phenomena, although so far it has just been used to represent quasistatic evolution.

1.13 APPENDIX. ELEMENTS OF CONVEX CALCULUS

In this Appendix we review some properties of convex, lower semicontinuous functions, the Legendre–Fenchel transformation, and the notions of subdifferential, of Gâteaux differential, of support function, and of variational inequality.

Infinite dimensional spaces are a natural environment for this theory, especially in view of application to the analysis of PDEs; nevertheless here we deal with the Euclidean space $\mathbf{R}^N$, we denote by B. The finite dimensional setting indeed suffices for grasping almost all of the results of convex calculus we use in this survey; moreover it is slightly simpler, and may be closer to the interests of nonmathematical readers.

Dealing with $\mathbf{R}^N$, we might identify the dual of B (that we denote by B^*) with B; but we refrain from doing so, in order to help the reader in distinguishing the different roles these two spaces play. Anyway we shall identify B^{**}, the bidual of B, with B.

1.13.1 Convex and Lower Semicontinuous Functions

We denote the extended real line $\mathbf{R}\cup\{+\infty\}$ by $\tilde{\mathbf{R}}$. For any function $F : B \to \tilde{\mathbf{R}}$, we set

$$\mathrm{Dom}(F) := \{v \in B : F(v) < +\infty\} : \text{(effective) domain of } F, \tag{A.1}$$

$$\mathrm{epi}(F) := \{(v, a) \in B\times\mathbf{R} : F(v)\leqslant a\} : \text{epigraph of } F. \tag{A.2}$$

For any set $K \subset B$, we also define its indicator function:

$$I_K : B \to \tilde{\mathbf{R}} : v \mapsto \begin{cases} 0 & \text{if } v \in K \\ +\infty & \text{if } v \notin K. \end{cases} \tag{A.3}$$

The use of these functions is especially convenient for minimization problems, for this allows reduction of constrained problems to unconstrained ones. In fact, for any function $F : B \to \tilde{\mathbf{R}}$ and any set $K \subset B$,

$$u = \inf_K F \quad \Leftrightarrow \quad u = \inf_B \,(F + I_K). \tag{A.4}$$

A set $K \subset B$ is said to be convex if

$$\lambda v_1 + (1-\lambda)v_2 \in K \qquad \forall v_1, v_2 \in K, \forall \lambda \in \,]0,1[.$$

By convention, the empty set is also included among the convex sets. A function $F : B \to \tilde{\mathbf{R}}$ is said to be convex whenever

$$F\big(\lambda v_1 + (1-\lambda)v_2\big) \leqslant \lambda F(v_1) + (1-\lambda)F(v_2) \qquad \forall v_1, v_2 \in B, \forall \lambda \in]0,1[. \tag{A.5}$$

If the inequality (A.5) is strict for any $v_1 \neq v_2$, the function F is said to be strictly convex. A function $F : B \to \tilde{\mathbf{R}}$ is said to be lower semicontinuous if for any $a \in \mathbf{R}$ the set $\{v \in B : F(v) \leqslant a\}$ is closed. F is said to be proper if $F(B)$ is not identically equal to $+\infty$.

PROPOSITION A.1. *(i) A function $F : B \to \tilde{\mathbf{R}}$ is convex (lower semicontinuous, respectively) if and only if epi(F) is convex (closed, respectively).*
(ii) A set $K \subset B$ is convex (closed, respectively) if and only if I_K is convex (lower semicontinuous, respectively).

PROPOSITION A.2. *(i) If $\{F_i : B \to \tilde{\mathbf{R}}\}_{i \in I}$ is a family of convex (lower semicontinuous, respectively) functions, then their upper hull $F : v \mapsto \sup_{i \in I} F_i(v)$ is convex (lower semicontinuous, respectively).*
(ii) If $\{K_i\}_{i \in I}$ is a family of convex (closed, respectively) subsets of B, then their intersection $\cap_{i \in I} K_i$ is convex (closed, respectively).

Let us denote by $\Gamma(B)$ the class of functions $F : B \to \tilde{\mathbf{R}}$ that are the upper hull of a family of affine functions $B \to \mathbf{R}$.

PROPOSITION A.3. $\Gamma(B)$ *consists of the class $\Gamma_0(B)$ of convex, lower semicontinuous, proper functions, and of the function identically equal to $+\infty$.*

For any set $K \subset B$, the smallest convex and closed subset of B that contains K is convex and closed. It is named the closed convex hull of K, and is denoted by $\overline{\text{co}}(K)$.

Similarly, let us consider any function $F : B \to]-\infty, +\infty]$ which has a convex lower bound. (This assumption allows one to exclude e.g. the function $x \mapsto -\|x\|^2$.) The upper hull of all affine lower bounds of F is convex and lower semicontinuous. It is the largest lower bound of F in $\Gamma(B)$, and is named the Γ-regularized function of F. Its epigraph coincides with the closed convex hull of the epigraph of F.

1.13.2 The Legendre–Fenchel Transformation

Let $F : B \to \tilde{\mathbf{R}}$ be a proper function. The function

$$F^* : B^* \to \tilde{\mathbf{R}} : u^* \mapsto \sup_{u \in B}\{u^* \cdot u - F(u)\} \tag{A.6}$$

is named the (convex) conjugate (or polar) function of F. If F^* is proper, its conjugate function

$$F^{**} : B \to \tilde{\mathbf{R}} : u \mapsto \sup_{u^* \in B^*}\{u^* \cdot u - F^*(u^*)\} \tag{A.7}$$

is named the biconjugate (or bipolar) function of F.

Theorem A.4. *For any proper $F : B \to \tilde{\mathbf{R}}$ such that F^* is also proper,*

$$F^* \in \Gamma(B^*); \quad F^{**} \leqslant F; \quad F^{**} = F \;\Leftrightarrow\; F \in \Gamma(B); \quad (F^*)^{**} = F^*. \tag{A.8}$$

Moreover, F^{**} coincides with the Γ-regularized function of F (Fenchel–Moreau theorem).

The conjugacy transformation $F \mapsto F^*$ is a bijection between $\Gamma_0(B)$ and $\Gamma_0(B^*)$.

1.13.3 The Subdifferential

Let $F : B \to \tilde{\mathbf{R}}$ be proper. We define its subdifferential $\partial F : \mathrm{Dom}(F) \subset B \to 2^{B^*}$ (the power set) as follows:

$$\partial F(u) := \{u^* \in B^* : u^* \cdot (u - v) \geqslant F(u) - F(v), \ \forall v \in B\} \qquad \forall u \in \mathrm{Dom}(F), \tag{A.9}$$

cf. Fig. A.1. $\partial F^* : \mathrm{Dom}(F^*) \subset B^* \to 2^B$ is similarly defined:

$$\partial F^*(u^*) := \{u \in B : u \cdot (u^* - v^*) \geqslant F^*(u^*) - F^*(v^*), \ \forall v^* \in B^*\} \qquad \forall u^* \in \mathrm{Dom}(F^*). \tag{A.10}$$

$\partial F(u) = \emptyset$ is not excluded; in particular, we set $\partial F(u) := \emptyset$ for any $u \in B \setminus \mathrm{Dom}(F)$. One can then take the subdifferential even of either nonconvex or non-lower-semicontinuous functions at any point of their domain, cf. Fig. A.1.

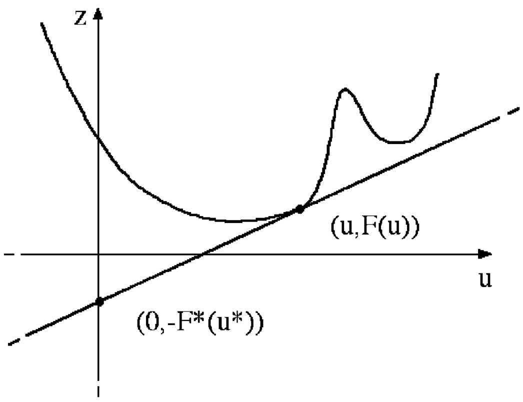


FIGURE A.1 The straight line represents a supporting hyperplane to the epigraph of F. It is characterized by the equation $z = u^* \cdot (v - u) + F(u)$, or equivalently by $z = u^* \cdot v - F^*(u^*)$, where $u^* \in \partial F(u)$.

Theorem A.5. *Let $F : B \to \tilde{\mathbf{R}}$. Then for any $u \in B$ and any $u^* \in B^*$:*

$$F(u) + F^*(u^*) \geqslant u^* \cdot u \qquad (\textit{Fenchel inequality}), \tag{A.11}$$

$$u^* \in \partial F(u) \quad \Leftrightarrow \quad F(u) + F^*(u^*) = u^* \cdot u \qquad (\textit{Fenchel equality}), \tag{A.12}$$

$$u^* \in \partial F(u) \quad \Rightarrow \quad u \in \partial F^*(u^*), \tag{A.13}$$

$$[F(u) = F^{**}(u),\ u \in \partial F^*(u^*)] \quad \Rightarrow \quad u^* \in \partial F(u), \tag{A.14}$$

$$F \in \Gamma_0(B) \quad \Rightarrow \quad \partial F^* = (\partial F)^{-1}. \tag{A.15}$$

The operator ∂F is monotone, that is,

$$(u_1^* - u_2^*) \cdot (u_1 - u_2) \geqslant 0 \qquad \forall u_i \in \mathrm{Dom}(\partial F), \forall u_i^* \in \partial F(u_i) (i = 1, 2). \tag{A.16}$$

By (A.11), the Fenchel equality (A.12) also reads as

$$u^* \in \partial F(u) \quad \Leftrightarrow \quad F(u) + F^*(u^*) \leqslant u^* \cdot u. \tag{A.17}$$

Theorem A.6 (*Rockafellar*). *Let $F_1, F_2 : B \to \tilde{\mathbf{R}}$. Then*

$$\partial F_1(u) + \partial F_2(u) \subset \partial \left(F_1 + F_2\right)(u) \qquad \forall u \in \mathrm{Dom}(F_1) \cap \mathrm{Dom}(F_2). \tag{A.18}$$

The opposite inclusion holds if F_1 and F_2 are both convex and lower semicontinuous, and either F_1 or F_2 is continuous at some point $u_0 \in \mathrm{Dom}(F_1) \cap \mathrm{Dom}(F_2)$.

The latter condition cannot be dropped. As a counter example let us take $B = \mathbf{R}$,

$$F_1(x) := +\infty \quad \forall x < 0, \quad F_1(x) := -\sqrt{x} \quad \forall x \geqslant 0, \quad F_2(x) := F_1(-x) \quad \forall x \in \mathbf{R}.$$

Then $(F_1 + F_2)(0) = 0$ and $(F_1 + F_2)(x) = +\infty$ for any $x \neq 0$; hence $\partial(F_1 + F_2)(0) = \mathbf{R}$, whereas $\partial F_1(0) + \partial F_2(0) = \emptyset + \emptyset = \emptyset$.

PROPOSITION A.7. *Let $F : B \to \tilde{\mathbf{R}}$ be convex and proper. Then F is locally Lipschitz-continuous at the interior of* Dom(F)*, and there $\partial F \neq \emptyset$.*

1.13.4 EXAMPLES

(i) The Fenchel inequality (A.11) extends the classic Young inequality: for any $p \in]1, +\infty[$,

$$\frac{1}{p}\|u\|^p + \frac{p-1}{p}\|v\|^{p/(p-1)} \geqslant u \cdot v \qquad \forall u, v \in \mathbf{R}^N; \tag{A.19}$$

the Fenchel equality (A.12) here reads

$$\frac{1}{p}\|u\|^p + \frac{p-1}{p}\|v\|^{p/(p-1)} = u \cdot v \quad \Leftrightarrow \quad v = u\|u\|^{p-2} \qquad \forall u, v \in \mathbf{R}^N. \tag{A.20}$$

(ii) Let $1 \leqslant p < +\infty$ and set $F_p(u) := \|u\|^p/p$ for any $u \in B$. If $p > 1$, then $\partial F_p(u) = \{\|u\|^{p-2}u\}$ for any $u \in B$. On the other hand, for $p = 1$

$$\partial F_1(u) = \{\|u\|^{-1}u\} \quad \forall u \in B \setminus \{0\}, \qquad \partial F_1(0) = \{v \in B^* : \|v\| \leqslant 1\}. \tag{A.21}$$

In particular, if $B := \mathbf{R}$ then $\partial F_1 = \text{sign}$, where

$$\text{sign}(x) := \{-1\} \text{ if } x < 0, \quad \text{sign}(0) := [-1, 1], \quad \text{sign}(x) := \{1\} \text{ if } x > 0. \tag{A.22}$$

PROPOSITION A.8. *For any proper, convex, lower semicontinuous function $F : B \to \tilde{\mathbf{R}}$, any $u \in B$ and any $u^* \in B^*$, the following properties are equivalent:*

$$u^* \in \partial F(u), \tag{A.23}$$

$$u \in \partial F^*(u^*), \tag{A.24}$$

$$u^* \cdot (u - v) \geqslant F(u) - F(v) \qquad \forall v \in B, \tag{A.25}$$

$$u \cdot (u^* - v^*) \geqslant F^*(u^*) - F^*(v^*) \qquad \forall v^* \in B^*, \tag{A.26}$$

$$u \cdot u^* = F(u) + F^*(u^*), \tag{A.27}$$

$$u \cdot u^* \geqslant F(u) + F^*(u^*). \tag{A.28}$$

The equivalence between (A.23) and (A.24) follows from (A.14). The inclusions (A.23) and (A.24) are equivalent to the variational inequalities (A.25) and (A.26) by definition of ∂F and ∂F^*, respectively. The equality (A.27) directly follows from the previous one, and (A.28) is equivalent to (A.27) because of (A.11).

The next statement is just a particular case of the latter one.

PROPOSITION A.9. *For any (nonempty) closed convex set $K \subset B$, any $u \in B$ and any $u^* \in B^*$, the following properties are equivalent:*

$$u^* \in \partial I_K(u), \tag{A.29}$$

$$u \in \partial I_K^*(u^*), \tag{A.30}$$

$$u \in K, \qquad u^* \cdot (u - v) \geqslant 0 \qquad \forall v \in K, \tag{A.31}$$

$$u \cdot (u^* - v^*) \geqslant I_K^*(u^*) - I_K^*(v^*) \qquad \forall v^* \in B^*, \tag{A.32}$$

$$u \in K, \qquad u \cdot u^* = I_K^*(u^*), \tag{A.33}$$

$$u \in K, \qquad u \cdot u^* \geqslant I_K^*(u^*). \tag{A.34}$$

1.13.5 THE GÂTEAUX DIFFERENTIAL

A function $F : B \to \tilde{\mathbf{R}}$ is said to be (strongly) Gâteaux differentiable at $u \in \mathrm{Dom}(F)$ if

$$\exists u^* \in B^* : \forall v \in B, \quad \frac{F(u + \lambda v) - F(u)}{\lambda} \to u^* \cdot v \quad \text{as } \lambda \to 0. \tag{A.35}$$

Such u^* is necessarily unique; it is named the (strong) Gâteaux differential of F at u, and is denoted by $F'(u)$.

PROPOSITION A.10. *Let $F : B \to \tilde{\mathbf{R}}$ be convex, and $u \in \mathrm{Dom}(F)$. If F is Gâteaux differentiable at u, then $\partial F(u) = \{F'(u)\}$. Conversely, if $\partial F(u)$ is a singleton, then it is Gâteaux differentiable at u and $\partial F(u) = \{F'(u)\}$.*

PROPOSITION A.11. *Let $F : B \to \tilde{\mathbf{R}}$ be Gâteaux differentiable at any point of* $\mathrm{Dom}(F)$*, and the set* $\mathrm{Dom}(F)$ *be convex. Then F is convex if and only if F' is monotone; that is, if and only if*

$$[F'(u_1) - F'(u_2)] \cdot (u_1 - u_2) \geqslant 0 \qquad \forall u_1, u_2 \in \mathrm{Dom}(F). \tag{A.36}$$

1.13.6 Support Functions

For any set $K \subset B$, the conjugate of the indicator function I_K,

$$I_K^* : B \to \tilde{\mathbf{R}} : u^* \mapsto \sup_{v \in K} u^* \cdot v, \tag{A.37}$$

is named the support function of the set K, and is also denoted by σ_K. For instance, $\sigma_\emptyset \equiv -\infty$ and $\sigma_B \equiv +\infty$. Assuming that all the K_i are nonempty, closed, convex subsets of B, the following formulas hold

$$\sigma_{K_1} \leqslant \sigma_{K_2} \quad \Leftrightarrow \quad K_1 \subset K_2, \tag{A.38}$$

$$\sigma_{aK_1+bK_2} = a\sigma_{K_1} + b\sigma_{K_2} \qquad \forall a, b \geqslant 0, \tag{A.39}$$

$$\sup_i \sigma_{K_i} = \sigma_{\cup_i K_i}. \tag{A.40}$$

A function $B \to]-\infty, +\infty]$ is said positively homogeneous of degree one (here written phd1) whenever $\sigma_K(\lambda u) = \lambda \sigma_K(u)$ for any $u \in B$ and any $\lambda > 0$.

Proposition A.12. *For any nonempty set $K^* \subset B^*$, the function $\sigma_{K^*} := I_{K^*}^*$ is convex and phd1. Conversely, if $\sigma : B \to]-\infty, +\infty]$ is convex and phd1, then $\sigma = \sigma_{K_\sigma^*}$, where*

$$K_\sigma^* := \{u^* \in B^* : u^* \cdot v \leqslant \sigma(v), \ \forall v \in B\} \ \ (= \mathrm{Dom}(\sigma^*)) \tag{A.41}$$

is closed, convex and nonempty.

1.13.7 Cones

A nonempty set $C \subset B$ is named a cone (with vertex at the origin) if $\lambda C = C$ for any $\lambda \geqslant 0$. For instance, any linear subspace of B is a cone. The domain of any proper phd1 function $F : B \to]-\infty, +\infty]$ is a cone in B; the graph and the epigraph of F are cones in $B \times \mathbf{R}$. For any $K \subset B$ and any $u \in \mathrm{Dom}(\partial I_K)$, $\partial I_K(u)$ is named the normal cone to K. For any nonempty set $K \subset B$, $\bigcup_{\lambda \geqslant 0} \lambda K$ is the smallest cone which contains K (i.e., the cone generated by K).

Convex analysis has been the object of a large literature. We just quote some monographs: Aubin [128], Barbu and Precupanu [129], Borwein and Lewis [130], Castaing and Valadier [131], Ekeland and Temam [132], Hiriart-Urruty and Lemarechal [133,134], Hörmander [135], Ioffe and Tihomirov [136], Kusraev and Kutateladze [137], Moreau [138], Rockafellar [139,140], Rockafellar and Wets [141], Willem [142].

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In recent years several monographs have been devoted to hysteresis; see [1,3,5,8] and [4,6,7,9] for mathematically- and physically-oriented approaches, respectively. We also mention the collective volumes [143–147], and the surveys [16,40,148–153].

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