Mathematical Music Theory—Status Quo 2000

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Abstract

We give an overview of mathematical music theory as it has been developed in the past twenty years. The present theory includes a formal language for musical and musicological objects and relations. This language is built upon topos theory and its logic. Various models of musical phenomena have been developed. They include harmony (function theory, cadences, and modulations), classical counterpoint (Fux rules), rhythm, motif theory, and the theory of musical performance. Most of these models have also been implemented and evaluated in computer applications. Some models have been tested empirically in neuro-musicology and the cognitive science of music. The mathematical nature of this modeling process canonically embedds the given historical music theories in a variety of fictitious theories and thereby enables a qualification of historical reality against potential variants. As a result, the historical realizations often turn out to be some kind of "best possible world" and thus reveals a type of "anthropic principle" in music.

These models use different types of mathematical approaches, such as—for instance—enumeration combinatorics, group and module theory, algebraic geometry and topology, vector fields and numerical solutions of differential equations, Grothendieck topologies, topos theory, and statistics. The results lead to good simulations of classical results of music and performance theory. There is a number of classifiaction theorems of determined categories of musical structures.

The overview concludes by a discussion of mathematical and musicological challenges which issue from the investigation of music by mathematics, including the project of "Grand Unification" of harmony and counterpoint and the classification of musical performance.

Introduction

This is the second status quo report on mathematical music theory. The first was written exactly ten years ago for the Deutsche Mathematiker-Vereinigung [37], and ten years after my first steps into mathematical music theory [31]. The former report essentially paralleled with the book *Geometrie der Töne* [34], whose title reflects the theoretical approach of that time: The central concern was not logic but geometry, i.e., the investigation of categories of local and global compositions which formalize the relevant objects and relations for harmony (cadence and modulation), counterpoint (Fux rules), melody, rhythm, large musical forms (in particular the classical sonata theory), including their classification, and the paradigmatic semiotics of musical structures as described by Ruwet [57] and Nattiez [49].

This approach included satisfactory theorems which model modulation, counterpoint, and string quartet theory in coincidence with the classical knowledge and tradition, which yield classification for some interesting global structures [34, 37], and which have been operationalized in music composition software [35] and corresponding CDs [5, 6, 7, 36]. It was however incomplete and too narrow in its concept framework for many musical problems. Here are some critical points:

- The Yoneda point of view was not properly developed. This defect became virulent after Noll's reconstruction of Riemann harmony [50].
- The development of the music platform RUBATO® for analysis and performance [39, 42] enforced a critical review of music data models for universal purposes from score representation to performance [38] and the definition of an extended concept framework whose elements were described in [38, 62] and implemented in RUBATO®'s PrediBase DBMS.
- The complexity of musical performance asked for concepts and methods from differential geometry, such as vector fields and their integration (ordinary differential equations and associated numerical Runge-Kutta-Fehlberg methods), Lie derivatives, and characteristics methods in partial differential equations.
- The differenciation between mathematical fiction and musical facticity had to be explicated and led to the concept of textual and paratextual predicates [43, 44]. At this point, logical and geometric perspectives were forced to unite. This approach is centered around topos-theoretic

construction of musical predicates by means of logical and geometric operations and also targets at the design of universal composition tools. Presently, several research groups (e.g. TU Berlin, IRCAM Paris, U Osnabrück, UNAM Mexico, ETH Zürich, U Zürich) are collaborating in the theoretical and software design of these extensions.

So this second status report will center around the most important improvements and extensions of the theory since the early nineties. Again, this report is paralleled by an upcomping book *The Topos of Music* [45] with extensive discussions of the old and new topics. A history of mathematical music theory has however not been written, and this report is just a flash on the ongoing process.

The report first deals with mathematical models in music, discussing the methodological background, then illustrating it by three classical models: modulation, counterpoint, and performance. The second section introduces the concept framework of forms and denotators, including their operationalization on the RUBATO® workstation and the Galois theory of concepts. Thirdly, we discuss the central category of local and global compositions with general Yoneda 'addresses', i.e., domains of presheaves. This leads to Grothendieck topologies and sheaves of affine functions which are essential for classification purposes. This latter subject is dealt with in the fourth section. We discuss enumeration theory of musical objects and algebraic schemes whose points parametrize isomorphism classes of global compositions. Based on a substantial isomorphism between harmonic and contrapuntal structures, we give a preview in the fifth section of what future research in mathematical music theory could (and should) envisage.

The fact that this report has been realized under the excellent organization of the Universidad Nacional Autónoma de México and Emilio Lluis-Puebla, president of the Mathematical Society of México, is also a sign that mathematical music theory has transcended its original Swiss roots and has attended international acceptance. At this point, I would like to acknowledge all my collaborators and colleagues for their continuous support and encouragement.

1 Models

1.1 What Are Models?

Basically, mathematical models of musical phenomena and their musicological reflexions are similar to corresponding models of physical phenomena.

The difference is that music and musicology are not phenomena of exterior nature, but of interior, human nature. To begin with, there is a status of music structures and correponding conceptual fields, together with compositions in that area, and the modeler first has to rebuild this data in a precise concept framework of mathematical quality. Next, the historical material selection in music and musicology (scales, interval qualities, for example) has to be paralleled in the mathematical concept framework by a selection of instances. Here, the historical genesis is contrasted by the systematic definition and selection of a priori arbitrari instances. After this positioning act, the musical and/or musicological process type (such as a modulation or cadence or contrapuntal movement) has to be rephrased in terms of the mathematical concept framework. With this in mind, the historically grown construction and analysis rules of that determined process have to be modeled on the level of mathematics. This means that the formal process restatement must be completed by structure theorems (including the proofs, a strong change of paradigm!), and then, by use of such theorems, the grown rules must be deduced in the mathematical concept framework.

The typical property of mathematical models in music is this: To enable a quasi-automatic generalization to situations where the classical music theory for which the model was constructed has no answer. In the case of modulation which originally was modeled for major scales, the generalization extends to arbitrary 7-tone scales. This is due to the a priori systematic concept framework of mathematics. Once a bunch of concepts and structures has been set up, there is no reason whatsoever to stick to the historical material selection, the genericity of precise concepts and theorems enables a broader perspective which pure historicity cannot offer.

The property of extensibility of a mathematical model relocates the existing music theory (which it models) in a field of potential, fictitious theories. This puts the historically grown facticity into a relation with the potential 'worlds of music'. The purely historic justification of existing modulation rules, for example, does not give us reasons for this choice, and this makes the purely historical approach a poor knowledge basis: We know that something is the case, but not why, and why other possibilities are not. In contrast, the mathematical approach gives us a field of potential theories wherein the actual one can be asked for its possible special properties with respect to non-existing variants. This differentia specifica is a remarkable advantage of mathematical methodology against the historical approach of musicology which cannot embed the facts in a viariety of fictions and thereby understand the selection of what is against what is not.

This evokes Leibniz idea that the existing world is the best of all possible worlds: Is the existing music theory the best possible choice? Or is it at least a distinguished one? In cosmology, this idea has been restated under the title of the "anthropic principle" [3]. It says that the physical laws are the best possible for the existence of humans, more precisely (and less radically), it is the theorem stating that a slight variation of the fundamental constants, such as the gravitational constant, or the electric charge of electrons and protons, would make any higher molecular complexity as it is necessary for the carbon-based biochemistry impossible.

1.2 Modulation

The historically first model in mathematical music theory dealt with tonal modulation, more precisely: with Arnold Schönberg's model of a tripartite modulation process from tonality X to tonality Y, as it is described in the classical treatise on harmony [58]. The process parts are the following, exemplified for $X = C$-major, $Y = F$-major:

A. Neutralization of the old tonality X, neutral degrees of X are presented, for example I_C, VI_C.

B. The pivotal root progression degrees (German: "Fundamentschritte der Modulation") are played to enforce the turning movement towards the new tonality, for example degrees II_F, IV_F, VII_F.

C. The new tonality (F-major in our example) is evidenced by a set of cadence degrees, for example II_F, V_F.

In [58], such transition processes are described for a set of tonality couples, but not for all possible couples: These omitted couples are dealt with by a chain of at least two successive modulations through intermediate tonalities. Also is the construction of the core steps, i.e., the pivotal degrees, not independent of the specific constellation, it is rather an ad hoc argumentation. Moreover, the concepts are quite fuzzy, as usual in musicology. Finally, one cannot infer, how such an argumentation should deal with non-European tonalities. So there is the mathematical modeling enterprise as described above, on the level of musicological theory. Besides that, the model must also be tested on the corpora of compositions where there is a certain chance to recognize such modulation processes. But let us get off on the theoretical level first and comment on the experimental work later.

In the first steps, one makes the concepts of "tonality", "degree", "cadence" precise. Then, one should model the modulation mechanism, and last, one has to prove theorems which yield the pivotal degrees in process part B. Since this model has been described on several occasions [31, 32, 34, 48], we shall be very sketchy and only mark the cornerstones of the modeling operation[1]. For the tonalities, one takes a seven-element scale $S \subset \mathbb{Z}_{12}$ of pitch classes and covers S by seven triadic degrees $I_S, II_S, \ldots VII_S$ which are three-element subsets with each an intermediate pitch class between the first and second, and between the second and third degree pitch. For the C-major scale $S = C$, this gives us the classical triadic degrees. By definition, a tonality $S^{(3)}$ is a scale S, together with its covering (3) by triadic degrees. For the given modulation problem, we consider the translation orbit $Dia^{(3)}$ of the C-major tonality $C^{(3)}$. For a given couple $S^{(3)}, T^{(3)}$, the modulation mechanism is the datum of a symmetry $S^{(3)} \to T^{(3)}$, i.e., a translation or an inversion on the ambient space $\mathbb{Z}_{12}$ which carries the first tonality onto the second. The cadence concept is grasped by minimal subsets of triadic coverings such that only the respective scales contain these degrees as their degree subsets. In $Dia^{(3)}$, there are five such *minimal cadential sets*, i.e., $\{II_S, III_S\}, \{III_S, IV_S\}, \{IV_S, V_S\}, \{II_S, V_S\}, \{VII_S\}$. So finally, a modulation from $S^{(3)}$ to $T^{(3)}$ in $Dia^{(3)}$ is a quatruple $(S^{(3)}, T^{(3)}, g, c)$ where $g : S^{(3)} \to T^{(3)}$ is a modulation symmetry, and c is one of the five minimal cadential sets for the target tonality.

The last point of this model is the calculation of the pivotal degrees. This is achieved by what we call a "modulation quantum". This is a subset $M \subset \mathbb{Z}_{12}$ such that

1. g is an inner symmetry of the quantum;
2. the quantum contains all degrees of the cadence c;
3. $M \cap T$ is rigid, i.e., has no translation or inversion symmetry as inner symmetry and is covered by degrees of $T^{(3)}$;
4. M is minimal with properties 1. and 2.

So a modulation quantum 'materializes' the modulation symmetry (much like quanta in physics materialize forces), contains enough elements to express a cadence for the target tonality, has its trace $M \cap T$ covered by target tonality degrees and determines uniquely its associated symmetry (this follows from rigidity) and is a minimal such candidate (economical condition). If such a quantum exists, we shall (by definition!) recover the pivotal degrees from the triadic covering $(M \cap T)^{(3)}$ of the trace $M \cap T$ by degrees of $T^{(3)}$.

[1]A detailed and mathematically generalized discussion is also contained in [45].

A modulation which has a quantum is called *quantized*. The main theorem now has to guarantee the existence of quantized modulations. This is the alias of the historically grown rule canon in the mathematical model. This theorem in fact guarantees quantized modulations for all couples in $Dia^{(3)}$, and the pivotal degrees coincide with the pivotal degrees in Schönberg's treatise wherever he considers direct modulations (see [34, section 5.5.2]).

The present mathematical model has the advantage that it can also be performed on any seven-element scale, and any translation class of that scale as a modulation domain. So the modulation model immerges the classical case $Dia^{(3)}$ in a variety of modulation scenarios which have never been dealt with in historical contexts. In [48], this extension has been calculated by computer programs (including explicit lists of modulation quanta and pivotal degrees) and commented. That extension exhibits a very special position of the common scales in European harmony which we summarize as follows (see [48] for complete results):

- Among the modulation domains of rigid triadic tonalities, the maximum of 226 quantized modulations occurs for the harmonic minor scale.
- Among modulation domains of non-rigid tonalities, the maximum of 114 quantized modulations occurs for the melodic minor scale. Among those scales with quantized modulations for all couples of their modulation domains, the minimum of 26 quantized modulations occurs for the diatonic major scale.

Besides this "anthropic principle" for modulation, the model and its extension also apply to just tuning pitch spaces, and there, where the mathematics is quite different since one works in $\mathbb{Z}^n$, one also has good results, see [53, 45]. But the model and its extension also apply to compositions of tonal character. Of course, the historical context seems to be a critical point here since not every composer would compose in the framework of Schönberg's harmony. However, the mathematical model is not a poietic model, i.e., it does not claim that the composer has used its approach to set his/her modulations. The mathematical model is more like a model in physics: The phenomena are there (in our case: the compositions), and we have to describe their structure as well as possible, ignoring whether the creator of the universe has ever used our mathematics, our logic or our conceptual model of physical processes. In this spirit a number of successful interpretations of modulatory processes, among them the hitherto poorely understood modulation architecture of Beethoven's op.106 ("Hammerklavier"), have been realized, see [34]. A reconstruction of the first movement of Beethoven's op.106 in terms of analogous structures, replacing the minor seventh chord and its satellite structures in op.106 by the augmented triad and its corresponding satellite structures, has been realized in [32].

1.3 Counterpoint

The mathematical model of counterpoint [34] was first used in the context of neurophysiological investigations via Depth-EEG [40], where we tested the perception of consonances and dissonances in limbic and auditory structures of the human brain. In that research project, classical European theories—following Johann Joseph Fux [23] as a typical reference—were our objectives. However, the model later, with the thesis of Jens Hichert [27], turned out to have a similar extension to other interval dichotomies, and again, it turned out that the European choice was an exemplification of a "anthropic principle".

We shall only sketch the core structures here to illustrate the modeling methodology. Some more technical details are given in section 5 below. This counterpoint model starts from a specific 6-by-6-element dichotomy K/D of the twelve interval quantities modulo octave which are modeled as elements of $\mathbb{Z}_{12}$, i.e., prime = 0, minor second = 1, etc., major seventh = 11. So the classical contrapuntal dichotomy is $D = \{0, 3, 4, 7, 8, 9\}/K = \{1, 2, 5, 6, 10, 11\}$. This dichotomy has a unique autocomplementarity symmetry $AC(x) = 5x + 2$, i.e., $AC(K) = D$. In this theory, such dichotomies are called *strong dichotomies*. There are six types (i.e., affine orbits) of strong dichotomies. If we draw the dichotomies as partitions of the discrete torus $\mathbb{Z}_3 \times \mathbb{Z}_4 \xrightarrow{\sim} \mathbb{Z}_{12}$ given by the Sylow decomposition of $\mathbb{Z}_{12}$ (in fact the torus of minor and major thirds!), then it turns out that the classical dichotomy K/D has a maximal separation of its parts on the torus among the six strong dichotomy types. It has a remarkable antipode dichotomy which has its parts mixed up more than any other strong type, this is the *major* dichotomy $I/J = \{2, 4, 5, 7, 9, 11\}/\{0, 1, 3, 6, 8, 10\}$ whose first part are exactly the proper intervals of the major scale when measured from the tonic!

For each strong dichotomy, the results of Hichert enable a new and historically fictitious counterpoint rule set. These six 'worlds of counterpoint' are quite fascinating for several reasons, one of which we shall now make more explicit. It deals with the seven-element scale in which the counterpoint rules are realized[2]. If one looks for the diatonic scales (those having

[2]Moreover, but this is not our main concern here, the rule of forbidden paralles of fifth

only semi-tone and whole-tone intervals for successive notes) where the freedom of choice of a successor interval to a given interval is maximal in Fux counterpoint (dichotomy K/D), then the major scale is best, and it has no cul-de-sac, i.e., it is always possible to proceed from one consonant interval to another such interval under the given rules. The latter result is, by the way, a fact which has never been demonstrated in a logically consistent way in musicology... And the major scale has cul-de-sacs only for the major dichotomy I/J. Among the scales with seven tones without cul-de-sac for the major dichotomy, no European scales appear! However, there is a scale $K^* = \{0, 3, 4, 7, 8, 9, 11\}$ without cul-de-saces for I/J. It is nearly a "mela" (No. 15 = {0,1,3,4,7,8,9}), i.e., a basic scale for Indian ragas. And it is very similar to the consonant half K of the Fux dichotomy.

So the counterpoint model not only exhibits a variety of fictitious counterpoint theories which could very well yield new, interesting counterpoint compositions. It also relates the existent counterpoint theory of the Fux dichotomy K/D to its antipode, the major dichotomy I/J, through the scales where the counterpoint has to be inserted, and thereby to a far-out music structure such as the melas from Indian raga tradition. It is not clear whether these intercultural relations can be made more realistic or whether they remain fictitious. Here, more research must be done. But it becomes evident that the extension of mathematical models could open not only new perspectives of historical developments, it could also unfold new perspectives of cultural specializations.

1.4 Performance

The author's first steps in performance modeling were made 1989-1994 while programming the commercial musical composition software *presto*® for Atari computers [35]. In *presto*®'s "AgoLogic" subroutine, a hierarchy of polygonal tempo curves can be defined and edited. The program uses the definition of musical tempo as a piecewise continuous map $T : \mathbb{R} \to \mathbb{R}_+$ on the positive reals of symbolic time E, measured in quarters Q and with values in the positive reals, measuring the tempo $T(E)$ at symbolic (score) time E in units of quarters per minute, Q/Min, say. Mathematically, the tempo is the inverse derivative of the physical time e as a function of symbolic time E, as a function of symbolic time, i.e., $T(E) = (de/dE)^{-1}(E)$. The program uses the calculation of physical time via the evident integration of $1/T(E)$. The hierarchical tempo structure implements the fact that musical tempo is

is valid, and the coincidence with the Fux rules is extremely high, statistically speaking, the difference is less than 10^{-8}, see [50, II.4.3] for a precise argumentation.

not the same for all notes at a given score time. Rather is the tempo layered in a tree of successive refinements of local tempi. Typically, this looks like this: We are given a 'mother tempo' curve T_{mother}, defined on the closed symbolic time interval $[E_0, E_1]$. In a homophonic piano piece, this could be the global tempo which is played by the left hand. If the right hand should play a Chopin rubato during a subinterval $[E_{00}, E_{01}]$ of $[E_0, E_1]$, then the tempo of the right hand will deviate from the mother tempo in this interval. However, at the start and end times E_{00}, E_{01}, we ask the hands to coincide. So the daughter tempo $T_{daughter}$ of the right hand should have the same integral as the left hand with its mother tempo, i.e.,

$$\int_{E_{00}}^{E_{01}} 1/T_{daughter} = \int_{E_{00}}^{E_{01}} 1/T_{mother}.$$

By use of adaptation algorithms, the tempo hierarchy subroutine in the *presto*® software enables the graphically-interactive construction of such daughter curves, including an arbitrary number of sisters and of genealogical depth for daughters, granddaughters, great-granddaughters etc. This means that interpretative time is encoded in a ramified tree of genealogical refinement of local tempi.

This first approach was successful on the time level. Therefore, the SNSF grant (1992-196) for the RUBATO® project [38, 42] was designed to extend this approach to other parameters, such as pitch, duration, loudness, glissandi, and crescendi. But the *presto*® approach also had no rationale for shaping the tempo hierarchy, except intuitive graphical interaction. So the RUBATO® project had to deal with the question of constructing operators for shaping performance from a more analytical point of view.

The basic extension of tempo curves to higher parameter spaces is this: The performance is described by a performance mapping $\wp$ from the n-dimensional real space $\mathbb{R}^{EHLD\dots}$ of n symbolic parameters, onset E, pitch H, loudness L, duration D, etc. to the n-dimensional real space $\mathbb{R}^{ehld\dots}$ of n physical parameters, onset e, pitch h, loudness l, duration d, etc. Locally on the score, we suppose that $\wp$ is a diffeomorphism on an n-dimensional cube C, applied to a finite number of score events which are contained in this cube. So for a symbolic event X (uppercase), $x = \wp(X)$ (lowercase) denotes the associated physical performance event.

The extension of the tempo concept is given by the inverse vector field $Z_\wp$ of the constant diagonal field $\Delta(x) = \Delta = (1, \dots 1)$ on the physical space, i.e., $Z_\wp(X) = (J\wp(X))^{-1}(\Delta)$ with the Jacobian $J\wp(X)$. This defines a *performance field* associated with the performance map $\wp$. The value $x = \wp(X)$ can be calculated as follows (still generalizing the situation for tempo): suppose that the performance is known for a 'initial set' $I \subset C$ of symbolic events. Suppose also that the integral curve $\int_X Z_\wp$ of $Z_\wp$ through X hits I at the initial point X_0, and for the curve parameter time t. Then we have

$$x = \wp(X) = \wp(X_0) - t\Delta. \quad (1)$$

So the performance map can actually be *defined* from a performance field Z on the cube domain C, together with an initial performance map $\wp_I : I \to \mathbb{R}^{ehld\ldots}$. On the tempo level, the initial performance is the moment where the conductor lowers the baton to initialize the performance, and on the pitch level, the initial performance encodes the concert pitch!

This generalization is only possible by use of the generic mathematical concept framework of differential geometry. And it has the great advantage that it includes a very fine shaping tool for musical performance: performance vector fields! This meets the philosophy of performance as an effort of "infinite subtlety", as it was established by Theodor W. Adorno and Walter Bejamin [2]. Moreover, the shaping operators of performance can now be defined as operators which act on given performance fields and depend upon parameters which are typically available from data of harmonic, rhythmical, or motivic analysis of the underlying score. This would also meet Adorno's principle of a performance which is based on understanding the score's logical structure.

At present, there is no general system of performance operators. Several operators have been implemented on the RUBATO® platform, and they have been tested for classical scores, such as Bach's *Kunst der Fuge* (see [59] for a very interesting performance of *Contrapunctus III*). The most general type of operators are linear operators in the analytical parameters as well as in the given performance field. See [38, 41, 11] for this subject. The formal setup of this operator type follows these lines: We are given the analytical information in form of a "weight", i.e., a function $\Lambda : I \to \mathbb{R}$. This is what the analytical moduli of RUBATO® in fact do calculate. Then, we are given an affine endomorphism Dir of the symbolic parameter space $\mathbb{R}^{EHLD\ldots}$. Given the mother performance field Z, we have a new field

$$Z_{\Lambda,Dir} = Z - L_Z\Lambda.Dir, \quad (2)$$

where L is the Lie derivative. By the method of characteristics in partial differential equations it can be shown [38, Vol.I,p.214] that this type of operator englobes all known shaping operators in the implementations on the RUBATO® workstation.

Summarizing, the mathematical model of performance is a canonical generalization of the very special, and musically too narrow, situation known for tempo. And even in that special case has musicology never achieved a valid definition of tempo which exceeds the medieval level of a locally constant velocity (!) [45].

2 Concepts

The conceptual extension enforced by research in mathematical music theory is a dramatic process which led to new problems in musicology, knowledge representation theory, and mathematics.

2.1 Generalization of Common Structures

Ten years ago, the geometric approach to music theory was nothing more than a common mathematization of music(ologic)al objects in the sense that one dealt with categories of local and global compositions. A local composition is a pair (K, M), where K is a (usually finite) subset of a module M over a commutative ring A, whereas a morphism $f : (K, M) \to (L, N)$ between two local compositions is a set map $f : K \to L$ which extends to an affine homomorphism $F : M \to N$, i.e., $F(m) = n + F_0(m)$, a translation by n in the codomain plus a A-linear homomorphism $F_0 : M \to N$. A global composition is defined via a finite covering of a set K by charts K_i which are in bijection with supports of local compositions (L_i, M_i), including transition isomorphisms of local compositions induced by the pairwise chart intersections. Morphisms are the evident maps which are locally chart morphisms [34].

But this setup was too special for two main reasons. Firstly, the development of data base management systems for music research software had to cover more general musical objects, not just local or global compositions. For instance, the objects had to carry names, had to be defined in a recursive way in order to enable hierarchical concepts, and had to admit completely heterogeneous types, such as products, coproducts, lists, etc. In this environment, local and global compositions turned out to be too tightly related to naive mathematical objects. Secondly, new constructions of musicological objects required more general points than just elements of modules: For instance, new developments in harmony [50] require local compositions K where the elements of K are affine morphisms $k : B \to M$ on a domain module B instead of the classical case $B = 0$ which evidently covers the elements of the codomain module M. Thirdly, the recursive constructions turned out to include circular constructions, a completely new situation which also mathematically has some serious implications: this is the subject of the conceptual Galois theory we want to sketch below. Therefore, the following general framework was created, a framework which is rooted in the modern topos theory rather than in classical algebra and geometry.

2.2 Forms

Forms are the structure type which mimic a generic space concept[3] . They are based on the category **Mod** of (left) modules over associative, rings[4] with identity. The morphisms of this category are the diaffine morphisms. This means that if M, N are modules over rings R, S, respectively, a diaffine morphism $f : M \to N$ is the composition $f = e^n \circ f_0$ of a dilinear morphism f_0 with respect to a ring homomorphism $r : R \to S$ and a translation e^n on the codomain N. The morphism set from M to N is denoted by $M@N$. The category of presheaves over **Mod** is denoted by $\mathbf{Mod}^@$; in particular, the representable presheaf of a module M is denoted by $@M$. More generally, for any presheaf F in $\mathbf{Mod}^@$, its value at module M will be denoted by $M@F$. In the context of $\mathbf{Mod}^@$, we shall call a module an *address*, a terminology which stresses the Yoneda philosophy, stating that the isomorphism class of a module is determined by the system $@M$ of all the 'perspectives' it takes when 'observed' from all possible addresses. Recall [29] that $\mathbf{Mod}^@$ is a topos whose subobject classifier Ω evaluates to $M@\Omega = \{S|\ S = \text{sieve in } M\}$. Its exponential Ω^F for a presheaf F evaluates to $M@\Omega^F = \{S|\ S = \text{subfunctor of } @M \times F\}$, and for a representable $F = @N$, we have $M@\Omega^{@N} \xrightarrow{\sim} (M \times N)@\Omega$, the set of sieves in $M \times N$. For a subfunctor $S \subset @M \times F$, an address B, and a morphism $f : B \to M$, we write $f@S = \{(f, s)|\ (f, s) \in B@S\}$, i.e., $B@S = \coprod_{f \in B@M} f@S$.

To construct the formal setup of forms, we consider the set $Mono\mathbf{Mod}^@$ of monomorphisms in $\mathbf{Mod}^@$. We further consider the set

$$Types = \{\mathbf{Simple}, \mathbf{Syn}, \mathbf{Limit}, \mathbf{Colimit}, \mathbf{Power}\}$$

of form types. We then need the free monoid $Names = \langle UNICODE \rangle$ over the $UNICODE$ alpabet [5]. We next need the set $Dia(Names)$ of all diagram schemes with vertices in $Names$. More precisely, a diagram scheme over $Names$ is a finite directed multigraph whose vertices are elements of

[3] Fro further motivations, see [44].

[4] The empty module (!) is included in this category to guarantee universal constructions.

[5] This is the current extension of the $ASCII$ alphabet code to non-European letters

$Names$, and whose arrows $i : A \to B$ are triples (i, A, B), with $i = 1, \ldots$ natural numbers to identify arrows for given vertices.

Next, consider the set $Dia(Names/\mathbf{Mod}^@)$ of diagrams on $Dia(Names)$ with values in $\mathbf{Mod}^@$. Such a diagram is a map

$$dia : D \to \mathbf{Mod}^@$$

which with every vertex of D associates a functor and with every arrow associates a natural transformation between corresponding vertex functors. So $i : A \to B$ is mapped to the natural transformation $dia(i) : dia(A) \to dia(B)$.

With these notations, we can define a *semiotic of forms* as follows:

Definition 1 *A* semiotic of forms *is a set map*

$$sem : FORMS \to Types \times Mono\mathbf{Mod}^@ \times Dia(Names/\mathbf{Mod}^@)$$

defined on a subset $FORMS \subset Names$ with the following properties (i) *to* (iv)*. To ease language, we use the following notations and terminology:*

- *An element $F \in FORMS$ is called a* form name*, and the pair (F, sem) a* form *(if sem is clear, the form is identified with its name)*
- $pr_1 \cdot sem(F) = t(F)$ *(=type of F)*
- $pr_2 \cdot sem(F) = id(F)$ *(= identifier of F)*
- $domain(id(F)) = fun(F)$ *(= functor or "space" of F)*
- $codomain(id(F)) = frame(F)$ *(= frame or "frame space" of F)*
- $pr_3 \cdot sem(F) = coord(F)$ *(= coordinator of F)*

Then these properties are required:

(i) *The empty word $\emptyset$ is not a member of $FORMS$*

(ii) *Within the coordinator of F, if $t(F) \neq$* **Simple***, the vertices of the diagram are form names, i.e. elements of $FORMS$*

(iii) *For any vertex X of the coordinator diagram $coord(F)$, we have*

$$coord(F)(X) = fun(X)$$

(iv) *If the type $t(F)$ is given, we have the following for the corresponding frames:*

- *For* **Syn** *and* **Power***, the coordinator has exactly one vertex G and no arrows, i.e. $coord(F): G \to fun(G)$, what means that in theses cases, the coordinator is determined by a form name G.*

  Further, for **Syn***, we have $frame(F) = fun(G)$, and for* **Power***, we have $frame(F) = \Omega^{fun(G)}$.*
- *For* **Limit** *and* **Colimit***, the coordinator is any diagram $coord(F)$. For* **Limit***, we have the frame $frame(F) = lim(coord(F))$, and for* **Colimit***, we have the frame $frame(F) = colim(coord(F))$.*
- *For type* **Simple***, the coordinator has the unique vertex $\emptyset$, and a value $coord(F): \emptyset \to @M$ for a module M, or, in a more sloppy notation: $coord(F) = M$.*

Given a form semiology, we shall denote a form by the symbol

$$F \underset{id(F):fun(F)\to frame(F)}{\longrightarrow} t(F)(coord(F))$$

and omit the identifier if it is the identity functor. We also write

$$F \underset{id(F)}{\longrightarrow} \mathbf{Limit}(F_0, F_1, \ldots F_n) \text{ and } F \underset{id(F)}{\longrightarrow} \mathbf{Colimit}(F_0, F_1, \ldots F_n)$$

if the diagram reduces to the discrete set of forms $F_0, F_1, \ldots F_n$.

Given two forms F, G in a semiotic of forms *sem*, a morphism $f: F \to G$ is just a natural transformation $f: fun(F) \to fun(G)$. Hence every semiotic of forms defines its category $Forms_{sem}$ of forms.

2.3 Conceptual Galois Theory

The general problem of existence and size of form semiotics, i.e., the extent of the $FORMS$ set, maximal candidates of such sets, gluing such sets together along compatible intersections, etc., is far from being settled. We shall not pursue this interesting and logically essential branch for reasons of space. The least one should say is that *regular* forms, i.e., those forms which are built from simple forms by transfinite recursion, may be supposed to be included in a form semiotics without further danger concerning logical consistency.

Example 1 For non-negative integers m, n, consider the forms

$$OnMod_m \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}_m)$$
$$PiMod_n \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}_n)$$
$$OnPiMod_{m,n} \underset{Id}{\longrightarrow} \mathbf{Limit}(OnMod_m, PiMod_n)$$
$$IntMod_{m,n} \underset{Id}{\longrightarrow} \mathbf{Limit}(\mathbb{D})$$

with $\mathbb{Z}$-modules $\mathbb{Z}_m$ and $\mathbb{Z}_n$ as coordinators, and with the diagram $\mathbb{D} =$

$$OnMod_m \xleftarrow{pr_1} OnPiMod_{m,n} \xrightarrow{pr_2} OnMod_n$$

associated with the canonical projections onto the forms $OnMod_m, OnMod_n$. The name "OnMod" symbolizes "onset modulo..." whereas "PiMod" symbolizes "pitch modulo...", i.e., ordinary pitch classes. We see that the last form's diagram is just the condition that we should take the fiber product over onset, i.e., the simultaneity of two events in pitch and onset; this is a way to encode an interval of simultaneous note events.

But *circular*, i.e., non-regular forms do not exist automatically, nor are they uniquely defined. For example, defining a form

$$F \underset{I}{\longrightarrow} \mathbf{Power}(F)$$

is equivalent to selecting any monomorphism $I : G \rightarrowtail \Omega^G$, and setting $fun(F) = G$. To elaborate canonical monomorpisms, consider a set $S \subset A@G$ for a presheaf G. This defines a subfunctor $S^@ \subset @A \times F$ which in the morphism $f : B \to A$ takes the value $f@S^@ = \{f\} \times S.f$. Since we have $Id_M@S^@ = \{Id_M\} \times S$, S is recovered by $S^@$. This defines a presheaf monomorphism

$$?^@ : 2^G \rightarrowtail \Omega^G$$

on the presheaf 2^G of all subsets $2^{A@G}$ at address A. When combined with the singleton monomorphism $sing : G \rightarrowtail Fin(G) : x \mapsto \{x\}$ with the codomain presheaf $Fin(G) \subset 2^G$ of all finite subsets (per address), we have this chain

$$G \rightarrowtail Fin(G) \rightarrowtail 2^G \rightarrowtail \Omega^G$$

of monomorphisms. A number of common circular forms can be constructed by use of the following proposition ([47]):

Proposition 1 *Let H be a presheaf in $\mathbf{Mod}^@$. Then there are presheaves X and Y in $\mathbf{Mod}^@$such that*

$$X \xrightarrow{\sim} Fin(H \times X) \text{ and}$$
$$Y \xrightarrow{\sim} H \times Fin(Y).$$

Example 2 It is common to consider sound events which share a specific grouping behavior, for example when dealing with arpeggios, trills or larger groupings such as they are considered in Schenker or in Jackendoff-Lerdahl theory [28]. We want to deal with this phenomenon in defining *MakroEvent* forms. Put generically, let *Basic* be a form which describes a sound event type, for example the above event type $Basic = OnPiMod_{m,n}$. We then set

$$Makro_{Basic} \underset{f:F\xrightarrow{\sim}Fin(FK)\rightarrowtail\Omega^{FK}}{\longrightarrow} \mathbf{Power}(Knot_{Basic})$$

with $F = fun(Makro_{Basic}), FK = fun(Knot_{Basic})$
and the limit form

$$Knot_{Basic} \underset{Id}{\longrightarrow} \mathbf{Limit}(Basic, Makro_{Basic}),$$

a form definition which by the above proposition yields existing forms.

The typical situation here is an existing form semiotic *sem* and a bunch of ‘equations’ $E_{F_1,F_2,\ldots F_n}(F)$ which contain the form names $F_1, F_2, \ldots F_n$ already covered by *sem*, and the new form name F. The equations are just form definitions, using different types and other ingredients which specify forms. The existence of an extended semiotics sem' which fits with these equations is a kind of algebraic field extension which solves the equations E. This type of conceptual Galois theory should answer the question about all possible solutions and their symmetry group, i.e., the automorphisms of sem' over *sem*. No systematic account of these problems has been given to the date, but in view of the central role of circular forms in any field of non-trivial knowledge bases [4], the topic asks for serious research.

2.4 Denotators

The level of forms is still not the substance we are looking for. The substance is what is called a denotator. More precisely, given an address A and a form F, a *denotator* is a quatruple $Name : A \rightsquigarrow F(c)$, consisting of a string D (in $UNICODE$), its name, its address A, its form F, and its coordinates $c \in A@fun(F)$. So a denotator is a kind of substance point, sitting in its form-space, and fixed on a determined address. This approach is really a restatement of Aristotelian principles according to which the real thing is a substance plus its "instanciation" in a determined form space. Restating the above coordinates as a morphism $c : @A \to fun(F)$ on the representable contravariant functor $@A$ of address A by the Yoneda lemma, the "pure substance" concept crystallizes on the representable functor $@A$, the "pure form" on the functor $fun(F)$, and the "real thing" on the morphism between pure substance and pure form.

In classical mathematical music theory [37], denotators were always special zero-addressed objects in the following sense: If M is a non-empty R-module, and if $0 = 0_{\mathbb{Z}}$ is the zero module over the integers, we have the well-known bijection $0@M \xrightarrow{\sim} M$, and the elements of M may be identified with zero-addressed points of M. Therefore, a *local composition* from classical mathematical music theory, i.e., a finite set $K \subset M$, is identified with a denotator $K^* : 0 \rightsquigarrow Loc(M)(K)$, with form

$$Loc(M) \underset{Fin([M]) \rightarrowtail \Omega^{[M]}}{\longrightarrow} \mathbf{Power}([M])$$

and $[M] \longrightarrow \mathbf{Simple}(M)$.

Evidently, this approach relates to approaches to set theory, such as Aczel's hyperset theory [1] which reconsiders the set theory as developed and published by Finsler[6] in the early twenties of the last century [12, 13]. The present setup is a generalization on two levels (besides the functorial setup): It includes circularity on the level of forms and circularity on the level of denotators. For instance, the above circular form named $Makro_{Basic}$ enables denotators which have infinite descent in their knot sets. Similar constructs intervene for frequency modulation denotators, see [45, 43].

The denotator approach evidently fails to cover more connotative strata of the complex musical sign system. But it is shown in music semiotics [43, section 1.2.2] that the highly connotative Hjelmslev stratification of music can be construed by successive connotational enrichment around the core system of denotators. This is the reason why the naming "denotator" was chosen: Denotators are the denotative kernel objects.

2.5 The RUBATO Enterprise

So far, the language of forms and denotators seems to live exclusively in the esotheric universe of mathematics. Fortunately, this is not true: On the

[6] It is not clear whether Aczel is aware of this pioneer who is more known for his works in differential geometry ("Finsler spaces").

contrary, this language was developed under constraints of object-oriented programming in a research grant (1992-1996) of the Swiss national Science Foundation, directed by the author, and targetted at the construction of a software platform for analysis and performance of musical scores. After the programming work, executed by the author's assistant Oliver Zahorka and the author in C and Objective C language, the platform was named RUBATO®; several research reports [38], papers [39], and free software units [42] are available in this context. The Swiss RUBATO® project was restartet in 1998 in a German research grant of the Vokswagen Foundation by a research group under direction of Thomas Noll [52] and is being ported to the Mac OS X platform on the basis of the Objective C language. Presently, the research group of the author at the Computer Science Department of the University of Zurich is also developing new software units of the RUBATO® platform, based on Java/Java3D language. The latter implements the full functorial version of forms and denotators.

Independently of its different realizations on current software environments, the idea of such a platform is this: To implement a database management system (called *PrediBase* in the RUBATO® terminology) which is based on the form and denotator data model, to implement a frame software (RUBATO®) which incorporates PrediBase, and to extend the frame software by dynamically loadable program units (such a unit is called RUBETTE® in the RUBATO® terminology) which implement different tasks for musical analysis, performance, composition, logical constructions, navigation, etc. So the entire platform is a universal, and indefinitely extendible tool for musicology in its valid form of an exact, operationalized science, where the experimental paradigm can be (and has been [59, 14]) performed on an objective, distributed level of international collaboration. For details on this concept, see [44].

3 Local and Global Compositions

Local and global compositions are a ind of musical variety and describe the core theory of musical objects. Their cateogries share important properties which are also basic to the topos-theoretic evolution of the entire theory.

3.1 Categories of Local Compositions

Although the category of all denotators is defined [45], we shall focus on the classically prominent subcategory of local compositions. These are the denotators $D : A \rightsquigarrow F(x)$ whose form F is of power type. More precisely, we shall consider A-addressed denotators with coordinates $x \subset @A \times fun(S)$, where form S is called the *ambient space*[7] of D. If there is a set $X \subset A@S$ such that $x = X^{@}$, the local composition is said to be *objective*, otherwise, we call it *functorial*.

Given two local compositions $D : A \rightsquigarrow F(x), E : B \rightsquigarrow G(y)$, a morphism $f/\alpha : D \to E$ is a couple $(f : x \to y, \alpha \in A@B)$, consisting of a morphism of presheaves f and an address change α such that there is a form morphism $h : S \to T$ which makes the diagram of presheaves

$$\begin{array}{ccc} x & \longrightarrow & @A \times S \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle @\alpha \times h} \\ y & \longrightarrow & @B \times T \end{array}$$

commute. This defines the *category Loc of local compositions*. If both, D, E are objective with $x = X^{@}, y = Y^{@}$, one may also define morphisms on the sets X, Y by the expressions $f/\alpha : X \to Y$ (forgetting about the names) which means that $f : X \to Y$ is a set map such that there is a form morphism $h : S \to T$ which makes the diagram

$$\begin{array}{ccc} X & \longrightarrow & A@S \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle A@h} \\ Y.\alpha & \longrightarrow & A@T \end{array}$$

of sets commute. This defines the *category ObLoc of objective local compositions*. Every objective morphism $f/\alpha : X \to Y$ induces a functorial morphism $f^{@}/\alpha : x \to y$ in an evident way. This defines a functor

$$?^{@} : ObLoc \to Loc$$

This functor is fully faithful. Moreover, each functorial local composition x (again forgetting about names) gives rise to its *objective trace* $X = x_{@}$ where $\{Id_A\} \times X = Id_A@x$. If we fix the address A and restrict to the identity $\alpha = Id_A$ as address change, we obtain subcategories $ObLoc_A, Loc_A$ and a corresonding fully faithful embedding $?^{@}_{A} : ObLoc_A \to Loc_A$. In this context, the objective trace canonically extends to a left inverse functor $?_{@A}$ of $?^{@}_{A}$. Moreover

Proposition 2 *The morphisms* $?_{@A}$ *and* $?^{@}_{A}$ *build an adjoint pair* $?^{@}_{A} \dashv ?_{@A}$.

[7] If no confusion is likely, we identify S with $fun(S)$.

For more algebraic calculations, such as Grothendieck topologies and Cech cohomology, one has to restrict to special subcategories. We shall therefore also look at the "address" category ${}^R\mathbf{Mod}$ of left R-modules with R-affine morphisms for a given commutative ring R. In this category, the set of morphisms from module M to module N is denoted by $M@_R N$. We denote the corresponding category of (objective) local compositions by ${}^R Loc$ (${}^R ObLoc$) Proposition 2 is also valid mutatis mutandis for ${}^R Loc_A$.

3.2 Finite Completeness

So, on a fixed address, objective and associated functorial local compositions are quite the same. But there is a characteristic difference when allowing address change. This relates to universal constructions:

Theorem 1 [45] *The categories Loc and ${}^R Loc$ are finitely complete.*

If we admit general address changes, the subcategory of objective local compositions is not finitely complete, there are examples [45] of musically meaningful diagrams $E \to D \leftarrow G$ of objective local compositions whose fiber product $E \times_D G$ is not objective[8]. Therefore address change—which is the portal to the full Yoneda point of view—enforces functorial local compositions if one insists on finite completeness. This latter requirement is however crucial if, for example, Grothendieck topologies must be defined (see below).

The dual situation is less simple: There are no general colimits in *Loc*. This is the reason why global compositions, i.e., 'manifolds' defined by (finite) atlases whose charts are local compositions, have been introduced to mathematical music theory [31, 37].

3.3 Categories of Global Compositions

More precisely, given an address A, an *objective global composition* G^I is a set G which is covered by a finite atlas I of subsets G_i which are in bijection to A-addressed objective local compositions $H_i \subset A@F_i$ with transition isomorphisms $f_{i,j}/Id_A$ on the inverse images of the intersections $G_i \cap G_j$. A *functorial global composition* at this address is a presheaf in $\mathbf{Mod}^@$, together with a finite covering by subsheaves G_i which are isomorphic to functorial local compositions $H_i \subset @A \times F_i$ with transition isomorphisms $f_{i,j}/Id_A$ on the inverse images of the intersections $G_i \cap G_j$. Suppose we are given two global objective (functorial) compositions G^I at address A, with atlas $(G_i)_I$,

[8] The right adjointness of the objective trace functor for fixed addresses only guarantees preservation of limits for fixed addresses.

and U^J at address B, with atlas $(U_j)_J$. A morphism $(f^\iota/\alpha) : G^I \to U^J$ is is a morphism f of the underlying sets (presheaves), together with an address-change $\alpha : A \to B$, and a map $\iota : I \to J$ such that

- $f(G_i) \subset U_{\iota(i)}$, all $i \in I$,
- the induced morphisms on the charts are morphisms of objective (functorial) local compositions unter the address-change α.

This defines the category *ObGlob* (*Glob*) of objective (functorial) global compositions. The functorialization process described for local compositions works also globally to yield an injection

$$?^@ : ObGlob \rightarrowtail Glob$$

The significant difference of this concept from mathematical manifolds is that the covering $(G_i)_I$ is part of the global composition, i.e., no passage to the limit of atlas refinements is admitted. For music this is a semiotically important information since the covering of a musical composition is a significant part of its understanding [34]. In fact, a typical construction of global compositions starts with a local composition and then covers its functor by a familiy of subfunctors, together with the induced atlas of the canonical restrictions, the result is called an *interpretation*. The absence of colimits in *Loc* can be restated in terms that there are global compositions which are not isomorphic to interpretations, see [37, 45] for criteria of interpretability in terms of flasque sheaves of affine functions.

From this general definition, several specialization are derived for more specific usage. First, algebraic applications are more feasible in the smaller categories ^{R}*ObLoc* and ^{R}*Glob* of objective and functorial global compositions defined on the category $\mathbf{Mod}_R$ of R-modules and R-affine morphisms over a commutative ring R instead of **Mod**. Again, we have a completeness theorem:

Theorem 2 [45] *The categories Glob and* ^{R}*Glob are finitely complete.*

For Grothendieck topologies, one better works with a more mathematical manifold concept of global compositions. This regards uniquely the morphism concept. Two morphisms $f^\iota/\alpha, g^\kappa/\beta : G^I \to H^J$ are *mathematically equivalent* iff $f = g$, so just consider set maps (natural transformations for functorial global compositions) f such that there is an address change and a covering map which extends f to a morphism in *ObGlob* (*Glob*) (or corresponding categories ^{R}*ObGlob*, ^{R}*Glob*). This equivalence defines coarser categories which we index by "μ" for "mathematical":

$${}_{\mu}Glob,\ {}_{\mu}ObGlob,\ {}^{R}_{\mu}Glob,\ {}^{R}_{\mu}ObGlob.$$

Theorem 3 *The categories ${}_{\mu}Glob$, ${}^{R}_{\mu}Glob$ are finitely complete.*

Observe that the "mathematical" categories have the same objects as the original ones, only the morphisms are 'blurred'. So the mathematical categories are half way between the original musical manifold (morphism) concept and the purely mathematical manifold (morphism) concept.

3.4 Grothendieck Topology

Because of theorem 3, we may define a Grothendieck (pre)topology, the *finite cover* topology, on ${}_{\mu}Glob$ and on ${}^{R}_{\mu}Glob$ via covering families. Its covering families for a global composition G^I are finite collections of morphisms $(H_k^{J_k} \to G^I)_k$ which generate the functor of G^I.

Proposition 3 [45] *The finite cover topology on ${}_{\mu}Glob$ and ${}^{R}_{\mu}Glob$ is subcanonical. For a fixed address A, the finite cover topology on ${}_{\mu}Glob_A$ and ${}^{R}_{\mu}Glob_A$ is subcanonical.*

Various Čech cohomology groups (in the sense of Verdier [25, exposé V]) can be associated to covering families of the finite cover Grothendieck topology [45, chapter 19].

3.5 Sheaves of Functions

We now want to look at affine functions on functorial global compositions. We fix an address module A over the commutative ring R and work in the category ${}^{R}_{\mu}Glob_A$ which is finitely complete, much like ${}^{R}_{\mu}Glob$.

To define affine functions on a global composition G^I in ${}^{R}_{\mu}Glob_A$, consider the objective composition $\hat{A} = \widehat{A@_R R}$. We claim that the set $Hom(G^I, \hat{A})$ is canonically provided with a R-module structure. For any R-module M, we in fact have the sum morphism

$$@+ : @_R M \times @_R M \to @_R M$$

which is induced by the sum homomorphism in M. We also have the scalar dilatation morphism

$$@\lambda : @_R M \to @_R M$$

for $\lambda \in R$, induced by the scalar multiplication on M. It is easily seen that these two natural transformations for $M = R$ induce morphisms

$$\hat{+} : \hat{A} \times \hat{A} \to \hat{A}, \hat{\lambda} : \hat{A} \to \hat{A} \tag{3}$$

which turn the functor $\hat{A}$ into a module-valued functor, and therefore, the morphism set $Hom(G^I, \hat{A})$ into a R-module. Call this module $Hom(G^I, \hat{A}) = \Gamma(G^I)$ the *module of affine functions on* G^I. We therefore have a representable contravariant functor

$$\Gamma : \ {}^R_\mu Glob_A \to \ {}_R\mathbf{Mod} \tag{4}$$

which associates the R-module $\Gamma(G^I)$ of global affine functions with a given global composition G^I. By proposition 3, we have

Theorem 4 *The presheaf of affine functions* (4) *is a sheaf in the finite cover topology.*

The sheaf of affine functions will be used in the classification theory to construct global compositions from free global compositions.

4 Classification

Classification of global musical objects deals with the determination of isomorphism classes in adequate categories of global compositions, the type of objetcs which play the role of musical manifolds. It relies on two constructions: coefficient systems of affine functions and resolutions of global compositions. From the musicological point of view, this is one of the most dificult chapters of mathematical music theory since the relation between classification and musicology, in particular: esthetics, is quite implicit. Therefore classification is highly controversial in musicology. Here are three prominent reasons:

- It is commonly believed that classifying musical objects on whatever level is contrarious to the individual expressivity of compositions as they are cultivated since the Renaissance.
- Classification is misunderstood as a purely bureaucratic activity of list compilation.
- Due to a catastrophical lack of technical tools, traditional musicology has only rarely been able to control the variety of their objects.

The third point shows a disdain of detailed technical work which is psychologically comprehensible must scientifically be blamed for a major scientific retard even with respect to other humanities such as linguistics.

We have already vaporized these mystifications on several occasions, e.g. [33, 34], and we will stress one main argument again here: Classification is nothing else than the task of totally understanding an object. This is Yoneda's lemma in its full philosophical implication, in fact, the isomorphism class of an object X is equivalent to that of its contravariant Yoneda functor $Hom(-, X)$. The latter boils down to the synopsis of all perspectives $Y \to X$ (morphisms) under which X may be 'observed'. Such a result is in complete harmony with Adorno's, Valéry's, and Bätschmann's insights in the theory of arts. They state that understanding works of art means a synthesis of all their interpretative perspectives, see [34, 45] for details. Results and methods regarding classification of musical structures have already been applied to the theory of the string quartet, composition, and performance [34, 35, 36, 38].

4.1 Enumeration Theory

A first aspect of classification deals with enumeration, i.e., counting the number of isomorphism classes of determined musical structures, such as local or global composition, if this number is finite. The representative case is that of local and global objective compositions at finite addresses A and living in finite ambient modules M, such as the classical situation of $A = 0$ and $M = \mathbb{Z}_{12}$, the case considered by the American Set Theory for pitch classes modulo 12, and $A = M = \mathbb{Z}_{12}$, the case of self-addressed pitch classes investigated by Thomas Noll [50]. The most complete enumeration results have been obtained by Harald Fripertinger by use of Pólya-de-Bruijn enumeration theory [16, 17, 18, 19, 20, 21, 22].

Let us give a short overview of the main (some results being negleced, no doubt; the author apologizes for this incompleteness) historical landmarks in enumeration and listings of isomorphism classes: In 1973 Allen Forte [15] established the list of 352 orbits of chords of pitch classes under the translation group $T_{12} = e^{\mathbb{Z}_{12}}$ and the 224 orbits of chords under the group $TI_{12} = e^{\mathbb{Z}_{12}}. \pm 1$ of translations and inversions. In 1978, George Halsey and Edwin Hewitt [26] succeeded to give a recursive formula for enumeration of translation orbits of chords in finite abelian groups, and enumerating the translation orbit number for chords in cyclic groups of cardinality $n \leq 24$. In 1980, the author [31] calculated the list of 158 affine orbits of chords in $\mathbb{Z}_{12}$, the list of 26 affine orbits of three-element motives in $\mathbb{Z}_{12}^2$, and the list of 45 three-element motives in $\mathbb{Z}_5 \times \mathbb{Z}_{12}$. In 1989, Hans Straub and Egmont Köhler [60, 30] gave the list of all affine 216 four-element motive orbits in $\mathbb{Z}_{12}^2$. In his works starting from 1991 to the present, Harald Fripertinger (loc.cit.) has given enumeration formulas for chord orbit numbers in $\mathbb{Z}_n$ under T_n, TI_n, and the full affine group, also for n-phonic k-series, all-interval series, motives in $\mathbb{Z}_m \times \mathbb{Z}_n$, and Vuza canons in $\mathbb{Z}_n$. He has calculated lists of affine motive orbits in $\mathbb{Z}_{12}^2$ for motives up to cardinality 6.

The usage of classification has been annotated above, but there is one particular result which we cannot withhold from the reader: Fripertinger's formulas (in fact the cycle index polynomial) yield the impressive number of 2 230 741 522 540 743 033 415 296 821 609 381 912 affine orbits of 72-element motives in $\mathbb{Z}_{12}^2$. This is of order 10^{36} against the estimated order of 10^{11} stars in a galaxy! So the musical universe is a serious competitor against the physical universe, in its quantity as well as in its quality of a spiritual antagonist.

Let us briefly review the Pólya-de-Bruijn enumeration methods applied by Fripertinger. We typically work in the space $F = \mathbb{Z}_n$. A subset $C \subset F$ is identified with its characteristic function $\chi_C : F \to 2 = \{0, 1\}$. For a permutation g in a subgroup G of the full group $\overrightarrow{GL}(F)$ of affine automorphisms of F, we have the *cycle index* $cyc(g) = (c_1, \ldots c_f), f = card(F)$, with $c_i =$ number of cycles of cardinality i. Take the indeterminates $X_1, \ldots X_f$ and set $X^g = X_1^{c_1} \ldots X_f^{c_f}$. Then the cycle index polynomial is defined by

$$Z(G) = card(G)^{-1} \sum_G X^g.$$

Consider now "Pólya weights" $w(0), w(1) \in \mathbb{Q}[x]$ and for a characteristic function $\chi : F \to 2$, the product $\pi_w(\chi) = \prod_{t \in F} w(\chi(t))$ which is invariant under the canonical action of G on 2^F. Then, the *configuarion counting series* is defined by

$$C(G, F, w) = \sum_{2^F/G} \pi_w(\chi).$$

With these definitions, we have the following results:

- For weights $w(0) = 1, w(1) = x$, the x^k-coefficient of $C(G, F, w)$ is the number of G-orbits of k-element sets in F.

- For the constant weights $w(0) = w(1) = 1$, we have $C(G, F, w) = card(2^F/G)$.

We have the main

Theorem 5

$$C(G, F, w) = Z(G)(w(0) + w(1), w(0)^2 + w(1)^2, \ldots w(0)^f + w(1)^f)$$

and the

Corollary 1 *The cardinality of the orbit space* $2^F/G$ *is* $Z(G)(2, \ldots 2)$.

Generalizations of the main theorem by de Bruijn yield (for example) the orbit cardinalities of k-element chords in $\mathbb{Z}_{12}$ under the groups T_{12}, TI_{12}, and $\overrightarrow{GL}(\mathbb{Z}_{12})$:

$k =$	0	1	2	3	4	5	6	7	8	9	10	11	12
T_{12}	1	1	6	19	43	66	80	66	43	19	6	1	1
TI_{12}	1	1	6	12	29	38	50	38	29	12	6	1	1
$\overrightarrow{GL}(\mathbb{Z}_{12})$	1	1	5	9	21	25	34	25	21	9	5	1	1

The same result also yields formulas for orbits of (k, n)-series [19, Satz 2.2.5]. We reproduce the particularly interesting list of orbits for $n = 12$:

k	number of orbits of $(k, 12)$-series
2	6
3	30
4	275
5	2 000
6	14 060
7	83 280
8	416 880
9	1 663 680
10	4 993 440
11	9 980 160
12	(dodecaphonic series) 9 985 920

The huge number of isomorphism classes of local compositions of the most common type, and in spaces which are strong reductions of the 'real parameter spaces' modulo octave or similar periodicities, preconizes the usage of statistical methods to control the variety of cases encountered in practical analyses. Even next generation computers cannot reach the calculation power to check all possible classes. We refer to specialized papers [8, 9, 10] for this subject.

4.2 Standard Objects

General classification of local and global compositions has been attacked by the author since 1980 [31]. First descriptions of algebraic schemes whose rational points parametrize local and global compositions have been published in [32]. Those results also included recursive algorithms to calculate classes of global compositions in modules of finite length. They have been generalized to compositions of variable address [45, 46]. In the following subsections, we want to give a series of concepts and results which describe the recent classification theorems on variable addresses.

Assumption 1 *In the following discussion of classification theory, we shall always assume that the global compositions are A-addressed for a fixed address A, objective, and that we work over the category $\mathbf{Mod}_R$ of modules over a commutative ring R. We also assume that the supports G of our global compositions G^I are* finite *sets, in other words, we are situated in the category ${}^R ObGlob_A^{finite}$.*

Let us start with the standard compositions of the theory. They represent compositions with "notes in general position", i.e. their configuration is as "free" as possible from 'occasional' coincidences. In fact, the standard composition is a geometric realization deduced from the nerve $n(G^I)$ of the composition, i.e., of its covering I, and thus depends only on combinatorial information. There is a natural projection from the standard object onto the generating composition. Here is the formal construction:

Given a module A in ${}_R\mathbf{Mod}$ and a natural number $0 \leq n$, we denote $A^{\sqcup n}$ the affine $n+1$-fold coproduct $\coprod_{n+1} A$ of A. By construction, there is an isomorphism $A^{\sqcup n} \xrightarrow{\sim} R^n \oplus A^{n+1}$. We denote the canonical basis of R^n by $(e_1, \ldots e_n)$, and for any element $a \in A$ and $0 \leq i \leq n$, we set $a_i = (0, \ldots, a, \ldots 0)$ for the $n+1$-tuple in A^{n+1} having a at its $i+1$-th position and zero else; the zero element is denoted by e_0. We have the inclusion morphisms

$$\sigma_i : A \to A^{\sqcup n} \tag{5}$$

for $0 \leq i \leq n$, with

$$\sigma_i(a) = \begin{cases} (0, a_0) & \text{(linear) if } i = 0, \\ (e_i, a_i) & \text{(affine) if } i > 0. \end{cases} \tag{6}$$

This defines a local, A-addressed composition ${}_A\Delta_n \subset A@_R A^{\sqcup n}$ which is called the *A-addressed local standard composition of dimension n*. By construction, it has the following property: If M is any R-module, and if

$s. = (s_0, \dots s_n)$ is any sequence of A-addressed points[9] in M, with associated local composition $S = \{s_0, \dots s_n\} \subset A@_R M$, then there is exactly one morphism of local compositions

$$(s.) : \ {}_A\Delta_n \to S : \sigma_i \mapsto s_i \text{ for } i = 0, \dots n. \tag{7}$$

This morphism is in fact defined by the universal property of the coproduct and is mediated by the following affine function $f : A^{\sqcup n} \to M$: Write $s_i = e^{t_i} \cdot s_{i,0}$. Then we have

$$\begin{aligned} f(e_0) &= t_0, \\ f(e_i) &= t_i - t_0 \text{ (linear) for } i > 0, \\ f(a_i) &= s_{i,0}(a) \text{ (linear) for } i \geq 0, \end{aligned}$$

and the formula $s_i = f \cdot \sigma_i$ is immediate.

To define global “free” objects among the A-addressed objective compositions with finite charts, we consider the natural weight function $\nu : n(G^I) \to \mathbb{N}$ with $\nu(\Sigma) = card(\cap\Sigma) - 1$, where we set $\cap\Sigma = \cap_{s \in \Sigma} s$. The pair $n^\star(G^I) = (n(G^I), \nu)$ is an object in the category of naturally weighted simplicial complexes whose morphims are the simplicial maps which commute with the weight functions. We shall represent the naturally weighted nerve $n^\star(G^I)$ by an isomorphic *standard representative* $n^\star$ induced by a covering of the natural interval $[0, m] = \{0, 1, 2, 3, \dots m = card(G) - 1\}$ of natural numbers.

For $n^\star$, we define the *global standard composition* ${}_A\Delta_{n^\star}$ at address A by the interpretation of the local standard composition ${}_A\Delta_m$ which is given by the present covering of $[0, m]$. We are also given a *standard atlas* of ${}_A\Delta_{n^\star}$. In fact, for any subset $q = \{t_0, \dots t_c\} \subset [0, m]$ of $c + 1$ elements, we have the canonical injection $i_q : \ {}_A\Delta_c \to \ {}_A\Delta_m$ via $\sigma_j \mapsto \sigma_{t_j}$. This defines the standard atlas.

The universal property of this global standard composition reads as follows. Take the category *Covens* of coverings of sets[10], and consider the covariant functor

$${}_ACov_{n^\star} : ObGlob_A^{finite} \to \mathbf{Sets} : G^I \mapsto Hom_{Covens}(n^\star, (G, n_0(G^I))) \tag{8}$$

where the covering $n^\star$ denotes the naturally weighted simplicial complex after forgetting about its weight. Then we have this straightforward result:

[9]In the Yoneda language, these are the morphisms in $A@_R M$.

[10]Its objects are coverings of sets X^I, its morphisms $X^I \to Y^J$ are compatible pairs of set maps $f : X \to Y, \phi : I \to J$, i.e., $f(x) \subset \phi(x)$.

Proposition 4 *The functor ${}_ACov_{n^\star}$ is representable by the standard global composition ${}_A\Delta_{n^\star}$, i.e., we have a bijection*

$$Hom_{Covens}(n^\star, (G, n_0(G^I))) \xrightarrow{\sim} Hom_{ObGlob_A^{finite}}({}_A\Delta_{n^\star}, G^I)$$

which is functorial in the A-addressed composition G^I.

In particular, if we take the standard covering $n^\star = n^\star(G^I)$ of the nerve of G^I and then the corresponding 'identity' morphism $Id : n^\star \xrightarrow{\sim} (G, n_o(G^I))$, we obtain a corresponding bijective morphism

$$res_{G^I} : \Delta_{G^I} \to G^I \tag{9}$$

with the notation $\Delta_{G^I} = {}_A\Delta_{n^\star(G^I)}$, this object and the morphism res_{G^I} being called *the resolution of* G^I. Clearly, the associated simplicial morphism $n(res_{G^I}) : n^\star(\Delta_{G^I}) \to n^\star(G^I)$ is an isomorphism, but res_{G^I} is not, in general, an isomorphism!

In particular, due to the universal propertiy of the global standard compositions, every morphism $f^\iota : G^I \to H^J$ can uniquely be lifted to a corresponding morphism res_{f^ι} of resolutions to make the diagram

$$\begin{array}{ccc} \Delta_{G^I} & \xrightarrow{res_{f^\iota}} & \Delta_{H^J} \\ {\scriptstyle res_{G^I}}\downarrow & & \downarrow{\scriptstyle res_{H^J}} \\ G^I & \xrightarrow{f^\iota} & H^J \end{array}$$

commute. We therefore have a resolution functor

$$res_A : ObGlob_A^{finite} \to ObGlob_A^{finite} \tag{10}$$

and a natural transformation

$$\delta_A : res_A \to Id_{ObGlob_A^{finite}} \tag{11}$$

The resolution of a global composition is a representation of its weighted nerve and thereby includes invariant data of the composition. But more is needed to yield a full set of invariants. The next step deals with this completion. It is related to functions on global compositions.

4.3 Global Compositions from Coefficient Systems

Recall that a global affine function on a global composition G^I is a morphism $f : G^I \to A@_R R$. We know that the set of global affine functions builds a R-module $\Gamma(G^I)$ under pointwise addition and scalar multiplication. Moreover, the retraction $f \cdot g^\iota$ of an affine function $f : G^I \to A@_R R$ via a morphism $g^\iota : H^J \to G^I$ is an affine function, and the retracted function module is a submodule of $\Gamma(H^J)$.

To control such submodules, we give an alternative description of such function modules in terms of coefficient systems from sheaf theory [24]. Given a global composition G^I, reconsider the category $n(G^I)$ of its abstract nerve.

Definition 2 *A (R-)module complex over G^I is a covariant functor (a coefficient system in Godement's terminology)*

$$M : n(G^I) \to \ {}_R\mathbf{Mod} \tag{12}$$

into the category ${}_R\mathbf{Mod}$ of R-modules and R-affine morphisms, with transition morphisms

$$M_{\sigma,\tau} : M(\sigma) \to M(\tau)$$

of modules for the simplex inclusions (morphisms) $\sigma \subset \tau$.

As usual in sheaf theory, we put $\Gamma M = \lim_{n(G^I)} M(\sigma)$ and call this the set of global sections.

Example 3 Since any morphism of global compositions $\overset{\iota}{f}/\alpha : G^I \to H^J$ yields a natural transformation $n(\overset{\iota}{f}/\alpha) : n(G^I) \to n(H^J)$, every module complex M over H^J induces a module complex on $\overset{\iota}{f}/\alpha \star M$ over G^I, with

$$\overset{\iota}{f}/\alpha \star M(\sigma) = M(n(\overset{\iota}{f}/\alpha)(\sigma)).$$

Example 4 If M is any R-module, the *constant module complex* of M is the complex with $M(\sigma) = M$ for all simplexes and identity transition. Observe that its global sections are in bijection with the set M^c, if $N(G^I)$ has c connected components.

We now review global affine functions as patchworks of affine functions on charts of atlases. Suppose that we are given a global composition G^I. For a simplex σ of $n(G^I)$, we have the canonical local composition $\cap\sigma$, and a morphism $f : \cap\sigma \to A@_R R$ is an affine function on $\cap\sigma$. The set $n\Gamma(\sigma)$ of these functions is provided with the structure of a R-module by the pointwise addition and scalar multiplication.

For an inclusion of simplexes $\sigma \subset \tau$ of $n(G^I)$, the ambient spaces of the charts of these simplexes are the same, i.e., the inclusion of local compositions $\cap\tau \subset \cap\sigma$ are in bijection with an inclusion of local compositions $K_\tau \subset K_\sigma \subset A@_R N$ for a specific module N. Since an affine function on $\cap\tau$ is the restriction of an affine morphism $A@h : A@_R N \to A@_R R$, f evidently extends to the restriction $A@h|_{K_\sigma}$, so the transition morphisms by restriction of affine functions are surjective. The corresponding complex of *affine functions* is denoted by $n\Gamma(G^I)$. The subcomplex $C = C_{G^I}$ of constant functions is defined by $C(\sigma) = \{f \in n\Gamma(G^I)(\sigma), f = \text{ constant on } \cap\sigma\}$. The set of global sections of the function complex is denoted by $\Gamma(G^I)$ and clearly identifies to the previous definition of $\Gamma(G^I)$.

Let $f^\iota : G^I \to H^J$ be a morphism of global compositions. Take a simplex σ in $n(G^I)$, and its image σ' under the associated simplicial map. Then, each restricted morphism

$$f|_{\cap\sigma} : \cap\sigma \to \cap\sigma'$$

gives rise to a map

$$f^\iota \star n\Gamma(H^J)(\sigma) = n\Gamma(H^J)(\sigma') \to n\Gamma(G^I)(\sigma)$$

by right composition with this restricted morphism. Moreover, the map is R-linear. Therefore, if M is any subcomplex of $n\Gamma(H^J)$, its induced complex $\overset{\iota}{f}/Id_A \star M$ is mapped R-linearly onto what is called the *retracted* module complex

$$M|\overset{\iota}{f}/Id_A \subset n\Gamma(G^I). \tag{13}$$

In particular, if $M = C_{H^J}$, we have $C_{H^J}|f^\iota \subset C_{G^I}$.

With these techniques in mind, the resolution functor res_A and its associated natural transformation δ_A give rise to a module complex of affine functions $\Delta n\Gamma(G^I) = n\Gamma(G^I)|res_{G^I}$ in $n\Gamma(\Delta_{G^I})$, for each global composition G^I. Call this complex *the resolution complex of composition* G^I. Moreover, this assignment commutes with the morphism of the resolution functor, i.e., for a morphism $f^\iota : G^I \to H^J$, we have a canonical inclusion

$$n\Gamma(\Delta_{H^J})|res_{f^\iota} \subset n\Gamma(\Delta_{G^I}) \tag{14}$$

of the the retracted resolution complex of H^J in the resolution complex of G^I. The next step deals with the reconstruction of G^I from $n\Gamma(\Delta_{G^I})$ and the related question of classification of global compositions by use of the resolution complex which is suggested by the functorial relation (14).

The generic situation from the preceding constructions is that we are given a module complex $M \subset n\Gamma(G^I)$, containing the constants C, and that we would like to construct a kind of "quotient" composition whose affine functions are those of M. We first look at the local situation.

Definition 3 *Let $S \subset A@_R U$ be a local composition in the R-module U. For a submodule $L \subset \Gamma(S)$ of affine functions on S, the evaluation map $\dot{} : S \to A@_R L^\star$ into the A-valued points of the dual module $L^\star = Hom_R(L, R)$ of L is defined by $\dot{s}(a)(l) = l(s)(a)$.*

The problem is that the evaluation is not a morphism of local compositions in general. But in the special case which is of interest, we have this guarantee: Let $S = {}_A\Delta_n \subset A@_R A^{\sqcup n}$. Then, the dotted points $\dot{\sigma_i} : A \to L^\star$ define the universal map $H_L : A^{\sqcup n} \to L^\star$, and we have interpreted $\dot{}: {}_A\Delta_n \to A@_R L^\star$ as a morphism of local compositions.

Next, suppose we are given two local compositiosns $S \subset A@_R U, T \subset A@_R V$ and a morphism

$$f : S \to T,$$

together with a module $L_T \subset \Gamma(T)$ whose retract $L_T|f$ is included in a module $L_S \subset \Gamma(S)$. We then have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \cdot\ } & A@_R L_S^\star \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle A@|f^\star} \\ T & \xrightarrow{\ \cdot\ } & A@_R L_T^\star \end{array}$$

where $|f^\star$ is the R-dual of the canonical linear map $|f : L_T \to L_S$.

This construction yields a morphism $\dot{f} : \dot{S} \to \dot{T}$ of local compositions in the ambient spaces $L_S^\star, L_T^\star$, respectively. With this technique we may associate a global composition with a module complex $N \subset n\Gamma({}_A\Delta_{n^\star})$ of affine functions in the standard composition ${}_A\Delta_{n^\star}$ of a standard covering $n^\star$.

Assumption 2 *In the following discussion of classification, we shall tacitly assume that all module complexes of affine functions have surjective transition morphisms. (We know that this is the case for retracted function modules from resolution morphisms!)*

If we apply the construction from diagram (4.3) to the situation where $S = {}_A\Delta_{n(\sigma)} \subset A@A^{\sqcup n(\sigma)}$, and $T = {}_A\Delta_{n(\tau)} \subset A@A^{\sqcup n(\tau)}$ for simplexes

$\tau \subset \sigma$ of ${}_A\Delta_{n^\star}$, and with $L_S = N(\sigma), L_T = N(\tau)$, then we have injective vertical arrows in the corresponding commutative diagram

$$\begin{array}{ccc} {}_A\Delta_{n(\sigma)} & \xrightarrow{\cdot} & A@_R N(\sigma)^\star \\ \text{inclusion}\big\downarrow & & \big\downarrow A@res^\star \\ {}_A\Delta_{n(\tau)} & \xrightarrow{\cdot} & A@_R N(\tau)^\star \end{array}$$

where res is the restriction map. We write $\cap\sigma = {}_A\dot{\Delta}_{n(\sigma)}$, and therefore get a surjective morphism $({}_A\Delta_{n(\sigma)} \to \cap\sigma)_{n^\star}$ of diagrams of local compositions over the nerve $n^\star$. Setting ${}_A\Delta_{n^\star}/N = colim_{n^\star} \cap \sigma$, we have a commutative diagram of sets

$$\begin{array}{ccc} {}_A\Delta_{n(\sigma)} & \xrightarrow{\cdot} & \cap\sigma \\ \big\downarrow & & \big\downarrow \\ {}_A\Delta_{n^\star} & \xrightarrow{/N = colim^{\cdot}} & {}_A\Delta_{n^\star}/N \end{array}$$

induced by the dot morphisms of local compositions. In order to complete the construction and to obtain a concise theorem, we shall now suppose the following:

Assumption 3 *We henceforth suppose that the global composition G^I* has a finitely generated projective atlas*, i.e., an atlas whose charts have finitely generated projective R-modules. We also suppose that G^I* has projective affine functions*, i.e., that the affine function modules on the zero-simplexes (the charts) are projective.*

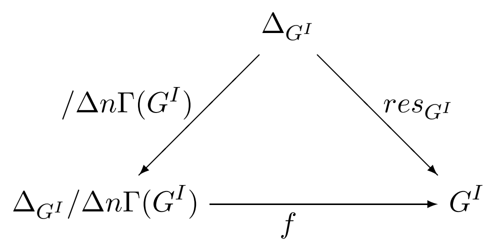
It is easily seen that under this assumption, the colimit diagram (4.3) has bijective horizontal arrows, and the images $\cap\sigma$ are injected into the limit ${}_A\Delta_{n^\star}/N$. So these images cover the limit and the images of the zero-dimensional simplexes build a canonical atlas of a global A-addressed composition, i.e., the diagram (4.3) becomes a bijective morphism of A-addressed global compositions.

So, we have constructed a canonical global composition and a bijective morphism from the free object to a global composition which is defined by the functions of N.

Definition 4 *We call this composition ${}_A\Delta_{n^\star}/N$ the N-quotient of ${}_A\Delta_{n^\star}$. The morphism ${}_A\Delta_{n^\star} \to {}_A\Delta_{n^\star}/N$ from diagram (4.3) is denoted by $/N$.*

In particular, if $res_{G^I} : \Delta_{G^I} \to G^I$ is the resolution of the composition G^I, we have the resolution complex $\Delta n\Gamma(G^I)$. Here is the crucial theorem [45]:

Theorem 6 *Under assumption 3 (for example, if R is semi-simple) we have a commutative triangle of morphisms of global compositions*



(15)

with an isomorphism f. All morphisms are isomorphisms of covering sets.

This means that we are able to reconstruct G^I from its retracted affine functions on the resolution. Moreover, in this case, the retracted module complex can also be recoverd from the quotient composition, i.e.,

$$\Delta n\Gamma(G^I) = n\Gamma({}_A\Delta_{G^I}/\Delta n\Gamma(G^I))|/\Delta n\Gamma(G^I) \tag{16}$$

so that we are now left with the question of charactrizing those module complexes of affine functions in ${}_A\Delta_{n^\star}$ which could give rise to compositions having this free object as their resolution.

Under assumption 3, we may proceed to the analysis of the following type of module complexes $N \subset n\Gamma({}_A\Delta_n)$: they are finitely generated projective (i.e., their zero-simplex modules are so), and contain the constant functions; call these complexes *representative*.

4.4 Orbit Spaces and Classifying Schemes

By the universal property of the standard compositions, the automorphism group $S_{A,n^\star}$ of the standard composition ${}_A\Delta_{n^\star}$ identifies to a subgroup of the symmetric group $\mathfrak{S}_{m+1}$ of permutations of ${}_A\Delta_{n^\star}$ if the standard covering is defined on the integer interval $[0, m]$ as discussed above. By retraction, this group acts from the right on the set $Rep_{A,n}$ of representative module complexes on ${}_A\Delta_{n^\star}$

$$ret : Rep_{A,n^\star} \times S_{A,n^\star} \to Rep_{A,n^\star} : (N, g) \mapsto N|g. \tag{17}$$

The orbit space of this action has this role [45]:

Theorem 7 *The orbit space $Rep_{A,n^\star}/S_{A,n^\star}$ is in bijection with the set of isomorphism classes of A-addressed global compositions with projective functions and finitely generated projective atlases which have a covering complex isomorphic to $n^\star$. This bijection is induced by the retraction of the function module complex to the resolution ${}_A\Delta_{n^\star}$, in one direction, and by the quotient composition on a given representative module complex on ${}_A\Delta_{n^\star}$, in the other.*

In particular, this classification result is valid for the global compositions having as their address a module A over a semi-simple commutative ring R.

A more in-depth discussion of the action of the automorphism group of the standard composition on module complexes yields this geometric classification spaces [45]:

Theorem 8 *For an addresse A which is locally free of rank m over the commutative ring R, there is a subscheme $J^{n^\star}$ of a projective $Spec(R)$-scheme of finite type such that its S-valued points $J^{n^\star}(Spec(S))$ for a R-algebra S are in bijection with the classifying orbits of module complexes N in ${}_{S\otimes_R A}\Delta_{n^\star}$ which are locally free of defined co-ranks on the zero-simplexes of $n^\star$.*

In particular, if the ground ring R is semi-simple, this theorem gives the classification of any global composition which are addressed in a finitely generated R-module A.

What could now be, after all, the program of classification? Its core value is that it deals with understanding musical works. And we should stress that our concept of a musical work is not the narrow one which restricts to those individual opera which—at least in Europe—started to emanate in the Renaissance. It includes as well general musical corpora such as scales, systems, everything that can be represented by means of global compositions.

From the precise parametric description of a work and of its ambiguities, this work appears as a point configuration in a more or less complex space. However this configuration is already a determinate perspective which shows a multitude of relations among its ingredients. It is the composer's perspective (now including an abstract 'composer' or creator of a general musical structure like a scale). For example, the choice of tonality, instrumentation, tempo, etc., are points of view which may or may not pertain to the composition, this is a question of the epoch of creation. But their character can undoubtedly be subject to variation. Among others, here we do address the question of historical instrumentation for early music.

In order to understand the relations among different parts of a composition, and even to simply recognize them, a change of the given perspective is mandatory. If a never seen object must be inspected, what should we do? You walk around it. This is the most common version of Yoneda's lemma. The analogy to cartography is straightforward: The natural perspective of the landscape in which we live does not coincide with the perspective which meets our need for orientation best. To reach this goal, we preferrably build maps which show the landscape from an infinitely high point.

The same happens to music. You play a piece in slow motion 'from very near', in a zoomed optics, a complex chord is arpeggiated, i.e., viewed from a skew angle, and so forth. This idea of variation of the perspective has in fact been integrated in the compositional thinking of the 20th century, perhaps most prominently by Edgar Varèse, especially in his comments on the composition "Intégrales" [61, p.67]. There, he invokes a geometric analogon of a machine which is able to project a mobile spatial object from variable space angles onto a luminous surface. This latter idea is astonishingly akin to the resolution projection from points in general position to points in special position.

5 Towards Grand Unification

In this section, we shall shortly illustrate on a concrete musicological situation: harmony and counterpoint, why some of the above general concepts have been introduced, and how they create perspectives of unification.

5.1 An Isomorphism Between Instances of Harmony and Counterpoint

Classically, mathematical music theory worked on the pitch class space $PiMod_{12}$ introduced above. In what follows, we shall slightly adjust it by the "fifth circle" automorphism $.7 : \mathbb{Z}_{12} \xrightarrow{\sim} \mathbb{Z}_{12}$, i.e., we consider the synonymous form

$$FiPiMod_{12} \underset{@.7}{\longrightarrow} \mathbf{Syn}(PiMod_{12})$$

which means that pitch denotators are now thought in terms of multiples of fifths, a common point of view in harmony. On this pitch space, two extensions are necessary: extension to intervals and extension to chords. The first one will be realized by a new form space

$$IntMod_{12} \longrightarrow \mathbf{Simple}(\mathbb{Z}_{12}[\epsilon])$$

with the module $\mathbb{Z}_{12}[\epsilon]$ of dual numbers over the pitch module $\mathbb{Z}_{12}$. We have the evident form embedding

$$\otimes 1 : FiPiMod_{12} \rightarrowtail IntMod_{12} : x \mapsto x \otimes 1$$

of this extension, where we should pay attention to the interpretation of a zero-addressed interval denotator

$$D : 0 \rightsquigarrow IntMod_{12}(a + \epsilon.b).$$

It means that D has cantus firmus pitch a and interval quantity b in terms of multiples of fifths. For example, the interval coordinate $1 + \epsilon.5$ denotes the pitch of fifth from the basic pitch (say 'g' if zero corresponds to pitch 'c'), together with the interval of $7.5 = 11$, i.e., the major seventh ('b' in our setup). The set K_ϵ of consonant intervals in counterpoint are then given by the zero-addressed denotators with coordinate $a + \epsilon.k, k \in K = \{0, 1, 3, 4, 8, 9, \}$. The set D_ϵ of dissonant intervals are the remaining denotators $a + \epsilon.d, d \in D = \mathbb{Z}_{12} - K$.

The counterpoint model of mathematical music theory [34] which yields an excellent coincidence of counterpoint rules between this model and Fux' traditional rules [23] is deduced from a unique affine automorphism, the autocomplementary involution $AC = e^2.5$ on the pitch space: we have $AC(K) = D, AC(D) = K$. It can be shown [34, 50] that this unique involution and the fact that K is a multiplicative monoid uniquely characterize the consonance-dissonance dichotomy among all 924 mathematically possible 6-6-dichotomies. This model's involution has also been recognized by neurophysiological investigations in human depth EEG [40]. Consider the consonance stabilizer $Trans(K_\epsilon, K_\epsilon) \subset \mathbb{Z}_{12}[\epsilon]@\mathbb{Z}_{12}[\epsilon]$. This one is canonically related to Riemann harmony in the following sense.

In his PhD thesis, Noll succeeded in reconstructing Riemann harmony on the basis of "self-addressed chords". This means that pitch denotators

$$D : \mathbb{Z}_{12} \rightsquigarrow FiPiMod_{12}(e^y.x)$$

are considered instead of usual zero-address pitch denotators which here appear as those which factor through the zero address change $\alpha : \mathbb{Z}_{12} \to 0$, i.e., the constant pitches. A *self-addressed chord* is defined to be a local composition with ambient space $FiPiMod_{12}$, and Noll's point was to replace zero-addressed chords by self-addressed ones.

In Riemann's spirit [54, 55, 56], the harmonic "consonance perspective" between the constant dominant triad $Dominant : 0 \rightsquigarrow FiPiMod_{12}(1, 5, 2)$

and the constant tonic triad $Tonic : 0 \rightsquigarrow FiPiMod_{12}(0,4,1)$ is defined by the monoid $Trans(Dominant, Tonic) \subset \mathbb{Z}_{12}@\mathbb{Z}_{12}$, a self-addressed chord generated by the transporter set of all morphisms $u : Dominant \to Tonic$.

This self-addressed chord is related to the above stabilizer as follows: Consider the tensor multiplication embedding

$$\otimes\epsilon : \mathbb{Z}_{12}@\mathbb{Z}_{12} \rightarrowtail \mathbb{Z}_{12}[\epsilon]@\mathbb{Z}_{12}[\epsilon] : e^u.v \mapsto e^{(u+\epsilon.0)}.(v \otimes \mathbb{Z}_{12}[\epsilon]).$$

Then we have a "grand unification" theorem ([50], see also [51] for more details):

Theorem 9 *With the above notations, we have*

$$Trans(Dominant, Tonic) = \otimes\epsilon^{-1} Trans(K_\epsilon, K_\epsilon).$$

This means that the Fux and Riemann theories are intimately related by this denotator-theoretic connections. At present, it is not known to what extent this structural relation has been involved in the historical development from contrapuntal polyphony to harmonic homophony.

5.2 Conclusion and Preview

If we review the overall power of mathematics in the description, analysis and performance of music, it turns out that it has a unique unifying character: Seemingly disparate subjects become related and comparable only through the universal language and methods of modern mathematics. Moreover, the operationalization of the abstract theories on the technical level of computers and software is an immediate and very important empirical and theoretical consequence of mathematization. For the first time, models and experimental setups can be applied in a scientific, i.e., precise and objective framework. Finally, the embedding of the historically grown existing theories in the mathematical concept framework preconizes a natural extension of facticity to fictitious variants, thereby opening the way to the comprehension of the crucial question of musicology: Why do we have this music and no other?

Of course, there will be other musics. But mathematical methods and associated technologial tools will undoubtedly play a dominant role it their discovery and exploration, be it on the level of instrumental realization, be it on the very concept space which transcends pure intuition and catalyzes fantasy to an unprecedented degree.

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