MATHEMATICAL PROGRAMS WITH VANISHING CONSTRAINTS: OPTIMALITY CONDITIONS, SENSITIVITY, AND A RELAXATION METHOD*

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ABSTRACT

We consider a class of optimization problems with switch-off/switch-on constraints, which is a relatively new problem model. The specificity of this model is that it contains constraints that are being imposed (switched on) at some points of the feasible region, while being disregarded (switched off) at other points. This seems to be a potentially useful modeling paradigm, that has been shown to be helpful, for example, in optimal topology design. The fact that some constraints "vanish" from the problem at certain points, gave rise to the name of mathematical programs with vanishing constraints (MPVC). It turns out that such problems are usually degenerate at a solution, but are structurally different from the related class of mathematical programs with complementarity constraints (MPCC). In this paper, we first discuss some known first- and second-order necessary optimality conditions for MPVC, giving new very short and direct justifications. We then derive some new special second-order sufficient optimality conditions for these problems and show that, quite remarkably, these conditions are actually equivalent to the classical/standard second-order sufficient conditions in optimization. We also provide a sensitivity analysis for MPVC. Finally, a relaxation method is proposed. For this method, we analyze constraints regularity and boundedness of the Lagrange multipliers in the relaxed subproblems, derive a sufficient condition for local uniqueness of solutions of subproblems, and give some convergence estimates.

Key words: mathematical program with vanishing constraints, stationarity, second-order conditions, sensitivity, relaxation.

AMS subject classifications. 90C30, 90C33, 49M37, 65K10

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1 Introduction

We consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & H_i(x) \geq 0, \quad G_i(x)H_i(x) \leq 0,\, i = 1, \ldots, m, \end{array} \tag{1.1}$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is a smooth function and G, $H : \mathbf{R}^n \to \mathbf{R}^m$ are smooth mappings. Problems with this structure have been called *mathematical programs with vanishing constraints* (MPVC) [1, 9, 10, 8], because whenever $H_i(x) = 0$ for some index i, the second constraint indexed by the same i becomes redundant. On the other hand, for points $x \in \mathbf{R}^n$ for which $H_i(x) > 0$ for some index i, it must hold that $G_i(x) \leq 0$ for the same i. Problem (1.1) is therefore a prototype of the following problem with switch-on/switch-off constraints:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & G_i(x) \leq 0 \text{ if } x \in D_i,\ i = 1, \ldots, m, \end{array} \tag{1.2}$$

where $D_i \subset \mathbf{R}^n$, $i = 1, \ldots, m$. At points $x \in D_i$ the constraint $G_i(x) \leq 0$ is switched on, while at points $x \notin D_i$ it is switched off. When the sets D_i can be represented by the inequalities $H_i(x) > 0$, the general problem (1.2) is precisely (1.1). This seems to be a potentially useful modeling paradigm, as the situation described is not uncommon in practical considerations. For example, we may only care about a particular constraint when some variable is positive, and not care about it otherwise. One specific example of a practical problem where vanishing constraints arise naturally concerns optimal topology design, discussed in detail in [1, Section 2]. Very briefly, this application concerns computing cross-sectional areas for each potential bar in the given structure such that failure of the whole structure is prevented, the external load is supported by the structure, and some objective function is minimized (for example, the weight of the structure) [4]. The constraints of the problem are therefore on stress along each possible bar, and on internal force and buckling force along each possible bar. Obviously, those constraints make sense (need to be satisfied) only when the corresponding cross-sectional area is positive (i.e., the potential bar is actually present in the structure). An intuitively very natural view of this requirement is precisely that of a vanishing constraint in the sense specified above.

Before proceeding, we note that we could easily consider the more general problem with the additional "usual" equality and inequality constraints

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h(x) = 0, \quad g(x) \leq 0, \quad H_i(x) \geq 0, \quad G_i(x)H_i(x) \leq 0,\, i = 1, \ldots, m, \end{array}$$

where $h : \mathbf{R}^n \to \mathbf{R}^p$, $g : \mathbf{R}^n \to \mathbf{R}^q$. The usual constraints do not add any important structural differences, as all the essential difficulties in the analysis have to do with the switch-off/switch-on constraints. We consider the form of (1.1) to keep the notation simpler.

Because of the fact that constraints in (1.1) are likely to be degenerate at a solution, MPVC is a difficult class of problems requiring special treatment. To see why this happens, we consider the following model example, to be used to illustrate various properties throughout the paper.

Example 1.1 Let $n = 2$, $m = 1$, $f(x) = x_2 + ax_1^2/2$, $G(x) = x_1$, $H(x) = x_2$, where a is a real parameter.

The feasible set of problem (1.1) has the form

$$D = \{x \in \mathbf{R}^2 \mid x_1 \leq 0,\ x_2 \geq 0\} \cup \{x \in \mathbf{R}^2 \mid x_2 = 0\},$$

which is the union of an orthant and a ray (clearly, a rather special structure).

If $a \geq 0$ then $\bar{x} = 0$ is a solution of (1.1), and it is the unique solution if $a > 0$. Evidently, the point $\bar{x}$ does not satisfy the Mangasarian-Fromovitz constraint qualification (MFCQ), and hence also the linear independence constraint qualification (LICQ). And this situation is typical in some sense, as discussed next.

(We note, for future reference, that the two sets in the right-hand side of the above representation of D will be called branches of this set.)

Let $\bar{x} \in \mathbf{R}^n$ be a feasible point of problem (1.1). Following [1], define the index sets (associated to $\bar{x}$):

$$I_+ = I_+(\bar{x}) = \{i = 1, \ldots, m \mid H_i(\bar{x}) > 0\},$$

$$I_0 = I_0(\bar{x}) = \{i = 1, \ldots, m \mid H_i(\bar{x}) = 0\},$$

and further partitioning of I_+:

$$I_{+0} = I_{+0}(\bar{x}) = \{i \in I_+ \mid G_i(\bar{x}) = 0\},$$

$$I_{+-} = I_{+-}(\bar{x}) = \{i \in I_+ \mid G_i(\bar{x}) < 0\},$$

and of I_0:

$$I_{0+} = I_{0+}(\bar{x}) = \{i \in I_0 \mid G_i(\bar{x}) > 0\},$$

$$I_{00} = I_{00}(\bar{x}) = \{i \in I_0 \mid G_i(\bar{x}) = 0\},$$

$$I_{0-} = I_{0-}(\bar{x}) = \{i \in I_0 \mid G_i(\bar{x}) < 0\}.$$

Then, locally (near $\bar{x}$), the feasible set of (1.1) can be written in the form

$$D = \{x \in \mathbf{R}^n \mid H_{I_{0+}}(x) = 0,\ H_{I_{00} \cup I_{0-}}(x) \geq 0,\ G_{I_{+0}}(x) \leq 0,\ G_i(x)H_i(x) \leq 0,\ i \in I_{00}\}. \tag{1.3}$$

As is readily seen, the structure of D around $\bar{x}$ has a clear combinatorial aspect associated to the group of constraints indexed by I_{00} (if this set is nonempty), and this is precisely the cause of difficulties in dealing with MPVC. On the other hand, if $I_{00} = \emptyset$ then the combinatorial aspect disappears, and from (1.3) we obtain that

$$D = \{x \in \mathbf{R}^n \mid H_{I_{0+}}(x) = 0,\ H_{I_{0-}}(x) \geq 0,\ G_{I_{+0}}(x) \leq 0\}.$$

The latter can be locally treated as a feasible set of a usual inequality-constrained problem, and in this case MPVC can be tackled by standard tools.

Condition $I_{00} = \emptyset$ can be regarded as the *lower level strict complementarity condition* (LLSCC). However, LLSCC is rather restrictive, just as the corresponding (strict complementarity) conditions in the literature on complementarity problems [6] and on mathematical programs with complementarity constraints (MPCC) [13, 14].

In Example 1.1, LLSCC actually holds at any feasible point, except for $\bar{x} = 0$ (where $I_+ = \emptyset$, $I_0 = I_{00} = \{1\}$), while $\bar{x} = 0$ is the unique solution for all values of the parameter $a > 0$.

It is important to emphasize that if LLSCC does not hold, then the constraints of (1.1) are *inevitably* degenerate. Specifically, it was demonstrated in [1] that:

- If $I_0 \neq \emptyset$ then $\bar{x}$ violates LICQ.
- If $I_{00} \cup I_{0+} \neq \emptyset$ then $\bar{x}$ violates MFCQ.

Because of this inherent constraints degeneracy, much of classical theory does not apply and special analysis is needed.

In this respect, it should also be commented that MPVC can be reduced [1], in principle, to MPCC (for which a considerable amount of theory and practice is already available). Specifically, introducing a slack variable $u \in \mathbf{R}^m$, MPVC can be equivalently reformulated as MPCC

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & G(x) - u \leq 0, \quad H(x) \geq 0, \quad u \geq 0, \quad H_i(x)u_i = 0,\, i = 1,\, \ldots,\, m. \end{array} \tag{1.4}$$

However, this reformulation has the following serious drawback (apart form the increased dimension, which is undesirable by itself). Given a (local) solution $\bar{x}$ of MPVC (1.1) such that $I_0 \neq \emptyset$, the corresponding value of the slack variable is not uniquely defined (for any $u_{I_0} \geq G_{I_0}(\bar{x})$, $(\bar{x},\, u)$ is a (local) solution of (1.4)), and since u does not enter the objective function of (1.4), the corresponding local solutions of this problem will never be strict. This means that natural sufficient optimality conditions will never hold for MPCC (1.4). Consequently, various results based on sufficient optimality conditions (regarding convergence of numerical methods, sensitivity, etc.) will not be applicable to (1.4). For this reason, while some of the ideas from MPCC literature should certainly be useful to deal with MPVC, reformulation (1.4) is not helpful and independent analysis is needed. In fact, second-order conditions discussed in Section 3 show some important structural differences between MPCC and MPVC.

The rest of the paper is organized as follows. In Section 2, we discuss stationarity conditions for MPVC, giving also their short justification. Section 3 is devoted to second-order conditions. Among other things, we show that special second-order sufficient optimality conditions for MPVC are actually equivalent to standard second-order sufficient conditions (i.e., stated for MPVC as a usual optimization problem, without making use of the structure). Section 4 contains sensitivity analysis for MPVC, under different sets of assumptions (either existence of a regular branch or MPVC-LICQ). In Section 5, we propose a relaxation scheme for solving MPVC and analyze its properties.

2 First-Order Necessary Conditions

In this section, we discuss stationarity notions and special constraint qualifications relevant for MPVC. The main items of both Proposition 2.1 and Theorem 2.1 below have first been obtained in [1, 9, 8]. Nevertheless, as we have to introduce the terminology and state those results in any case, we prefer to give also their justification (in some cases different from the references above, while being very short and simple). By analogy with MPCC, we also introduce the notion of weak stationarity (and a tightened problem associated to MPVC), the notion of B-stationarity, and the piecewise MFCQ.

Define the standard Lagrangian of problem (1.1):

$$L(x,\, \lambda) = f(x) - \langle \lambda^H,\, H(x)\rangle + \sum_{i=1}^{m} \lambda_i^{GH} G_i(x) H_i(x), \tag{2.1}$$

where $x \in \mathbf{R}^n$ and $\lambda = (\lambda^G,\, \lambda^{GH}) \in \mathbf{R}^m \times \mathbf{R}^m$. Because of the lack of MFCQ (when LLSCC fails to hold), local optimality of $\bar{x}$ in (1.1) does not imply stationarity of this point in the classical sense, in general. This means that a multiplier $\lambda = (\lambda^G,\, \lambda^{GH}) \in \mathbf{R}^m \times \mathbf{R}^m$ satisfying the relations

$$\frac{\partial L}{\partial x}(\bar{x},\, \lambda) = 0, \tag{2.2}$$

$$\lambda^H_{I_0} \geq 0, \quad \lambda^H_{I_+} = 0, \quad \lambda^{GH}_{I_{+0} \cup I_0} \geq 0, \quad \lambda^{GH}_{I_{+-}} = 0, \tag{2.3}$$

may not exist. However, as will be explained below, this standard stationarity concept is still completely relevant for MPVC. In what follows, by $\Lambda = \Lambda(\bar{x})$ we denote the set of Lagrange multipliers associated to $\bar{x}$, i.e., those $\lambda = (\lambda^G,\, \lambda^{GH}) \in \mathbf{R}^m \times \mathbf{R}^m$ satisfying (2.2), (2.3).

To a given feasible point $\bar{x}$ of (1.1), we associate two auxiliary "usual" MP problems. The *relaxed nonlinear programming problem* (RNLP) has the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & H_{I_{0+}}(x) = 0, \quad H_{I_{00} \cup I_{0-}}(x) \geq 0, \quad G_{I_{+0}}(x) \leq 0, \end{array} \tag{2.4}$$

while the *tightened nonlinear programming problem* (TNLP) has the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & H_{I_{0+} \cup I_{00}}(x) = 0, \quad H_{I_{0-}}(x) \geq 0, \quad G_{I_{+0} \cup I_{00}}(x) \leq 0. \end{array} \tag{2.5}$$

The feasible sets of these problems will be denoted by $D_{\mathrm{RNLP}} = D_{\mathrm{RNLP}}(\bar{x})$ and $D_{\mathrm{TNLP}} = D_{\mathrm{TNLP}}(\bar{x})$, respectively. From (1.3) it evidently follows that locally (near $\bar{x}$) it holds that

$$D_{\mathrm{TNLP}} \subset D \subset D_{\mathrm{RNLP}}, \tag{2.6}$$

and the point $\bar{x}$ is feasible in all the three problems.

In Example 1.1, $D_{\mathrm{RNLP}} = \{x \in \mathbf{R}^2 \mid x_2 \geq 0\}$, $D_{\mathrm{TNLP}} = \{x \in \mathbf{R}^2 \mid x_1 \leq 0,\ x_2 = 0\}$.

Define further the *MPVC-Lagrangian* of problem (1.1):

$$\mathcal{L}(x,\, \mu) = f(x) - \langle \mu^H,\, H(x)\rangle + \langle \mu^G,\, G(x)\rangle,$$

where $x \in \mathbf{R}^n$ and $\mu = (\mu^H,\, \mu^G) \in \mathbf{R}^m \times \mathbf{R}^m$. Evidently, this is the standard Lagrangian for TNLP (2.5) with additional constraints (inactive at $\bar{x}$)

$$H_{I_+}(x) \geq 0, \quad G_{I_{+-} \cup I_{0-}}(x) \leq 0, \quad G_{I_{0+}}(x) \geq 0. \tag{2.7}$$

MPVC-Lagrangian can also be regarded as the standard Lagrangian for RNLP (2.4) with additional constraints (2.7) and

$$G_{I_{00}}(x) \leq 0$$

(the latter constraint is active at $\bar{x}$).

Definition 2.1 A feasible point $\bar{x} \in \mathbf{R}^n$ of MPVC (1.1) is called a *strongly* (*weakly*) *stationary point* of this problem if it is stationary for RNLP (2.4) (TNLP (2.5)) in the classical sense.

Thus, weak stationarity of $\bar{x}$ means the existence of $\mu = (\mu^H, \mu^G) \in \mathbf{R}^m \times \mathbf{R}^m$ such that

$$\frac{\partial \mathcal{L}}{\partial x}(\bar{x}, \mu) = 0, \tag{2.8}$$

$$\mu^H_{I_{0-}} \geq 0, \quad \mu^H_{I_+} = 0, \quad \mu^G_{I_{+0} \cup I_{00}} \geq 0, \quad \mu^G_{I_{+-} \cup I_{0+} \cup I_{0-}} = 0, \tag{2.9}$$

while strong stationarity (the notion coined for MPVC in [8, Definition 2.1]) means that, additionally,

$$\mu^H_{I_{00}} \geq 0, \quad \mu^G_{I_{00}} = 0. \tag{2.10}$$

In the latter case, μ is referred to as *MPVC-multiplier* associated to the strongly stationary point $\bar{x}$. The set of MPVC-multipliers associated to $\bar{x}$ will be denoted by $\mathcal{M} = \mathcal{M}(\bar{x})$.

It was established in [1, Remark 1] that strong stationarity is in fact equivalent to usual stationarity for MPVC (1.1). More precisely, the following can be proved by a direct computation. (We note that the relation (2.14) below is new and it will be crucial later on, in the context of establishing the equivalence between various second-order sufficient optimality conditions.)

Proposition 2.1 *A feasible point $\bar{x}$ of problem* (1.1) *is a stationary point of this problem if, and only if, it is a strongly stationary point of this problem. Moreover, if $\lambda = (\lambda^H, \lambda^{GH})$ is a Lagrange multiplier associated with $\bar{x}$ then $\mu = (\mu^H, \mu^G)$ defined by*

$$\mu^H_i = \lambda^H_i = 0,\ i \in I_+, \quad \mu^H_i = \lambda^H_i - \lambda^{GH}_i G_i(\bar{x}),\ i \in I_{0+} \cup I_{0-}, \quad \mu^H_i = \lambda^H_i,\ i \in I_{00}, \tag{2.11}$$

$$\mu^G_i = \lambda^{GH}_i H_i(\bar{x}),\ i \in I_{+0}, \quad \mu^G_i = \lambda^{GH}_i H_i(\bar{x}) = 0,\ i \in I_{+-} \cup I_0, \tag{2.12}$$

is an MPVC-multiplier associated with $\bar{x}$. Conversely, if $\mu = (\mu^H, \mu^G)$ is an MPVC-multiplier associated with $\bar{x}$ then any $\lambda = (\lambda^H, \lambda^{GH})$ satisfying (2.11), (2.12) *and*

$$\lambda^{GH}_i \geq 0,\ i \in I_{00}, \quad \lambda^{GH}_i \geq \max\left\{0, -\frac{\mu^H_i}{G_i(\bar{x})}\right\},\ i \in I_{0+}, \quad 0 \leq \lambda^{GH}_i \leq -\frac{\mu^H_i}{G_i(\bar{x})},\ i \in I_{0-}, \tag{2.13}$$

is a Lagrange multiplier associated with $\bar{x}$.

Furthermore, for any $\xi \in \mathbf{R}^n$ and any $\lambda = (\lambda^G, \lambda^{GH}) \in \mathbf{R}^m \times \mathbf{R}^m$ and $\mu = (\mu^H, \mu^G) \in \mathbf{R}^m \times \mathbf{R}^m$ satisfying (2.11), (2.12)*, it holds that*

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \lambda)[\xi, \xi] = \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \mu)[\xi, \xi] + 2 \sum_{i \in I_{+0} \cup I_0} \lambda^{GH}_i \langle G'_i(\bar{x}), \xi \rangle \langle H'_i(\bar{x}), \xi \rangle. \tag{2.14}$$

Following [9, Definition 4.1], we say that *MPVC-linear independence constraint qualification* (MPVC-LICQ) holds at $\bar{x}$ if the gradients

$$H'_i(\bar{x}),\ i \in I_0, \quad G'_i(\bar{x}),\ i \in I_{+0} \cup I_{00} \quad \text{are linearly independent.} \tag{2.15}$$

Note that this is just the standard LICQ for TNLP (2.5) at $\bar{x}$. According to the left inclusion in (2.6), (local) optimality of $\bar{x}$ in MPVC implies its (local) optimality in TNLP. In particular, under MPVC-LICQ, weak stationarity is a necessary optimality condition for MPVC (1.1). A more subtle result (known from [9, Corollary 4.5], and justified below) consists of saying that under MPVC-LICQ, strong stationarity is also a necessary optimality condition.

Let $\mathcal{I} = \mathcal{I}(\bar{x})$ stand for the set of all partitions of I_{00}, i.e., pairs (I_1, I_2) of index sets such that $I_1 \cup I_2 = I_{00}$, $I_1 \cap I_2 = \emptyset$. Evidently, this set is finite, and $|\mathcal{I}| = 2^{|I_{00}|}$. For each $(I_1, I_2) \in \mathcal{I}$ define the *branch problem*

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & H_{I_{0+} \cup I_2}(x) = 0,\ H_{I_1 \cup I_{0-}}(x) \geq 0,\ G_{I_{+0} \cup I_1}(x) \leq 0, \end{array} \tag{2.16}$$

and denote by $D_{(I_1, I_2)} = D_{(I_1, I_2)}(\bar{x})$ its feasible set, which can be regarded as a *branch* (or *piece*) of the original feasible set D (see (1.3)). It can be easily seen that locally (near $\bar{x}$), the set D can be partitioned into such branches. In particular, the relations (2.6) can be made more precise: for any $(I_1, I_2) \in \mathcal{I}$ it holds that

$$\begin{aligned} D_{\mathrm{TNLP}} &= \bigcap_{(J_1, J_2) \in \mathcal{I}} D_{(J_1, J_2)} \\ &\subset D_{(I_1, I_2)} \\ &\subset D \\ &= \bigcup_{(J_1, J_2) \in \mathcal{I}} D_{(J_1, J_2)} \\ &\subset D_{\mathrm{RNLP}}. \end{aligned} \tag{2.17}$$

Moreover, $\bar{x}$ belongs to each branch, and each constraint defining each branch is active at $\bar{x}$.

Definition 2.2 A feasible point $\bar{x} \in \mathbf{R}^n$ of MPVC (1.1) is called a *B-stationary* (or *piecewise-stationary*) *point* of this problem if it is stationary in each of the branch problems (2.16), $(I_1, I_2) \in \mathcal{I}$.

Thus, B-stationarity means that for each $(I_1, I_2) \in \mathcal{I}$, there exist $\mu = (\mu^H, \mu^G) \in \mathbf{R}^m \times \mathbf{R}^m$ (depending on (I_1, I_2)) satisfying (2.8) and the relations

$$\mu^H_{I_1 \cup I_{0-}} \geq 0, \quad \mu^H_{I_+} = 0, \quad \mu^G_{I_{+0} \cup I_1} \geq 0, \quad \mu^G_{I_{+-} \cup I_{0+} \cup I_2 \cup I_{0-}} = 0. \tag{2.18}$$

Note that (2.18) is somewhat stronger than (2.9): it contains the additional condition $\mu^G_{I_2} = 0$.

We say that *piecewise MFCQ* holds at $\bar{x}$ if MFCQ holds at $\bar{x}$ for each branch problem (2.16), $(I_1, I_2) \in \mathcal{I}$. By the standard MP theory, and by the second inclusion in (2.17), it follows that, under piecewise MFCQ, B-stationarity is a necessary optimality condition for MPVC (1.1). Note that MPVC-LICQ evidently implies LICQ for each branch, and hence, piecewise MFCQ.

Note further that strong stationarity of $\bar{x}$ implies B-stationarity of this point, and moreover, any MPVC-multiplier is a Lagrange multiplier for each branch problem. On the other hand, B-stationarity of $\bar{x}$ evidently implies weak stationarity.

Suppose now that $\bar{x}$ is a B-stationary point of problem (1.1), satisfying MPVC-LICQ. According to the discussion above, this implies that for each partition $(I_1, I_2) \in \mathcal{I}$, there exists a unique multiplier $\mu = (\mu^H, \mu^G)$ satisfying (2.8) and (2.18). Moreover, as mentioned above, MPVC-LICQ is just LICQ for TNLP (2.5), which implies that the multiplier satisfying (2.8), (2.9) (and hence (2.8), (2.18)) must be unique. Therefore, the multipliers for all branch problems coincide with each other, and with the unique multiplier for TNLP. Moreover, by (2.18) we obtain that the relations $\mu^H_{I_1} \geq 0$ and $\mu^G_{I_2} = 0$ must hold for each $(I_1, I_2) \in \mathcal{I}$, which implies (2.10), and hence, strong stationarity of $\bar{x}$.

We thus proved the following result (we remind the reader that the most significant part of this result was originally established in [9, Corollary 4.5]). We believe that under MPVC-LICQ, strong stationarity of a local solution of MPVC can perhaps be obtained also using the framework of [19] (in particular, the linear independence assumption used throughout [19] becomes in our setting MPVC-LICQ (2.15)).

Theorem 2.1 *Assume that a feasible point $\bar{x} \in \mathbf{R}^n$ of problem* (1.1) *satisfies MPVC-LICQ* (2.15).

If $\bar{x}$ is a local solution of problem (1.1) *then $\bar{x}$ is a B-stationary point of this problem, and moreover, B-stationarity of this point is equivalent to its strong stationarity, and the associated MPVC-multiplier $\bar{\mu} = (\bar{\mu}^H, \bar{\mu}^G)$ is unique. Furthermore, weak stationarity condition, as well as stationarity condition for branch problem* (2.16) *for any partition $(I_1, I_2) \in \mathcal{I}$, do hold only with this multiplier.*

Summarizing, B-stationarity appears to be the most natural/proper stationarity concept for MPVC among those mentioned above, since it reflects the combinatorial/piecewise structure of the feasible set, and since it holds as a necessary optimality condition under the quite weak piecewise MFCQ. At the same time, combinatorial nature of B-stationarity is precisely what makes it difficult to use: verifying this condition requires verification of usual stationarity for $2^{|I_{00}|}$ "usual" MP problems.

At the same time, verifying weak or strong stationarity consists of verifying the usual stationarity for just one "usual" MP problem (TNLP and RNLP, respectively). Moreover, according to Theorem 2.1, under MPVC-LICQ strong stationarity is equivalent to B-stationarity.

In Example 1.1, MPVC-LICQ (2.15) holds at $\bar{x} = 0$, and this point is strongly stationary with the unique associated MPVC-multiplier $\bar{\mu} = (\bar{\mu}^H, \bar{\mu}^G) = (1, 0)$. Note that this point is also stationary in the usual sense, with the associated Lagrange multiplier $\lambda = (\bar{\mu}^H, \lambda^{GH})$ for an arbitrary $\lambda^{GH} \geq 0$.

3 Second-Order Optimality Conditions

We next discuss second-order necessary optimality conditions, first obtained in [8], and give a different very short justification. Then we proceed to establish some new results concerning sufficient optimality conditions.

Define the standard critical cone of problem (1.1) at $\bar{x}$:

$$C = C(\bar{x}) = \{\xi \in \mathbf{R}^n \mid H'_{I_{0+}}(\bar{x})\xi = 0,\ H'_{I_{00} \cup I_{0-}}(\bar{x})\xi \geq 0,\ G'_{I_{+0}}(\bar{x})\xi \leq 0,\ \langle f'(\bar{x}), \xi\rangle \leq 0\} \quad (3.1)$$

(see (1.3)). Note that this cone coincides with the critical cone for RNLP (2.4) at $\bar{x}$.

Furthermore, for each partition $(I_1,\, I_2) \in \mathcal{I}$ define the critical cone of the corresponding branch problem (2.16) at $\bar{x}$:

$$C_{(I_1,\, I_2)} = C_{(I_1,\, I_2)}(\bar{x}) = \left\{ \xi \in \mathbf{R}^n \;\middle|\; \begin{array}{l} H'_{I_{0+}\cup I_2}(\bar{x})\xi = 0,\; H'_{I_{0-}\cup I_1}(\bar{x})\xi \geq 0,\; G'_{I_{+0}\cup I_1}(\bar{x})\xi \leq 0, \\ \langle f'(\bar{x}),\, \xi\rangle \leq 0 \end{array} \right\}.$$

Set

$$\begin{aligned} C_2 &= C_2(\bar{x}) \\ &= \bigcup_{(I_1,\, I_2)\in\mathcal{I}} C_{(I_1,\, I_2)} \\ &= \left\{ \xi \in \mathbf{R}^n \;\middle|\; \begin{array}{l} H'_{I_{0+}}(\bar{x})\xi = 0,\; H'_{I_{00}\cup I_{0-}}(\bar{x})\xi \geq 0,\; G'_{I_{+0}}(\bar{x})\xi \leq 0, \\ \langle G'_i(\bar{x}),\, \xi\rangle\langle H'_i(\bar{x}),\, \xi\rangle \leq 0,\; i \in I_{00},\; \langle f'(\bar{x}),\, \xi\rangle \leq 0 \end{array} \right\}, \end{aligned} \tag{3.2}$$

where the subscript "2" indicates that, unlike C, this cone takes into account the second-order information about the last constraint in (1.1). By direct comparison of (3.1) and (3.2), we have that

$$C_2 = \{\xi \in C \mid \langle G'_i(\bar{x}),\, \xi\rangle\langle H'_i(\bar{x}),\, \xi\rangle \leq 0,\; i \in I_{00}\},$$

and in particular, $C_2 \subset C$.

Assume that $\bar{x}$ is a local solution of MPVC (1.1) satisfying MPVC-LICQ. Recall that the latter implies LICQ for each branch problem (2.16) at $\bar{x}$. From the second equality in (2.17) it thus follows that for each $(I_1,\, I_2) \in \mathcal{I}$, the standard second-order necessary condition holds at $\bar{x}$ for problem (2.16). Taking into account Theorem 2.1 and the second equality in (3.2), we thus justified the following result (first established in [8, Theorem 4.3]).

Theorem 3.1 *Assume that a feasible point $\bar{x} \in \mathbf{R}^n$ of problem* (1.1) *satisfies MPVC-LICQ* (2.15).

If $\bar{x}$ is a local solution of problem (1.1) *then for the unique MPVC-multiplier $\bar{\mu} = (\bar{\mu}^H,\, \bar{\mu}^G)$ associated with $\bar{x}$, it holds that*

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x},\, \bar{\mu})[\xi,\, \xi] \geq 0 \quad \forall\, \xi \in C_2.$$

By standard MP theory (using the stationarity conditions and standard arguments to eliminate the dependence on f in the original definition of the critical cone), for any $\lambda \in \Lambda$ we can write

$$C = \left\{ \xi \in \mathbf{R}^n \;\middle|\; \begin{array}{l} H'_{I_{0+}}(\bar{x})\xi = 0,\; H'_{I_{00}\cup I_{0-}}(\bar{x})\xi \geq 0,\; G'_{I_{+0}}(\bar{x})\xi \leq 0, \\ \lambda_i^H\langle H'_i(\bar{x}),\, \xi\rangle = 0,\; i \in I_{00}\cup I_{0-},\; \lambda_i^{GH}\langle G'_i(\bar{x}),\, \xi\rangle = 0,\; i \in I_{+0} \end{array} \right\}. \tag{3.3}$$

Similarly, for any $\mu \in \mathcal{M}$ and for each $(I_1,\, I_2) \in \mathcal{I}$, taking also into account the equality in (2.10), we can write

$$C_{(I_1,\, I_2)} = \left\{ \xi \in \mathbf{R}^n \;\middle|\; \begin{array}{l} H'_{I_{0+}\cup I_2}(\bar{x})\xi = 0,\; H'_{I_{0-}\cup I_1}(\bar{x})\xi \geq 0,\; G'_{I_{+0}\cup I_1}(\bar{x})\xi \leq 0, \\ \mu_i^H\langle H'_i(\bar{x}),\, \xi\rangle = 0,\; i \in I_1\cup I_{0-},\; \mu_i^G\langle G'_i(\bar{x}),\, \xi\rangle = 0,\; i \in I_{+0} \end{array} \right\}, \tag{3.4}$$

and hence, by the second equality in (3.2),

$$C_2 = \left\{ \xi \in \mathbf{R}^n \;\middle|\; \begin{array}{l} H'_{I_{0+}}(\bar{x})\xi = 0,\ H'_{I_{00}\cup I_{0-}}(\bar{x})\xi \geq 0,\ G'_{I_{+0}}(\bar{x})\xi \leq 0, \\ \mu_i^H \langle H'_i(\bar{x}),\, \xi\rangle = 0,\ i \in I_{00} \cup I_{0-},\ \mu_i^G \langle G'_i(\bar{x}),\, \xi\rangle = 0,\ i \in I_{+0}, \\ \langle G'_i(\bar{x}),\, \xi\rangle \langle H'_i(\bar{x}),\, \xi\rangle \leq 0,\ i \in I_{00} \end{array} \right\} \tag{3.5}$$

(cf. [8, (21)]).

We proceed with sufficient optimality conditions, that are all either new or stronger than previous results in the literature.

Theorem 3.2 *Let $\bar{x} \in \mathbf{R}^n$ be a feasible point of problem* (1.1)*. If*

$$C_2 = \{0\} \tag{3.6}$$

then the linear growth condition holds at $\bar{x}$ for problem (1.1)*, and in particular, $\bar{x}$ is a strict local solution of this problem.*

Proof. According to the second equality in (3.2), condition (3.6) means that the equality

$$C_{(I_1,\, I_2)} = \{0\}$$

holds for each $(I_1,\, I_2) \in \mathcal{I}$. By standard MP theory, the latter condition evidently implies the linear growth condition at $\bar{x}$ for each branch problem (2.16), and the needed assertion follows now from the second equality in (2.17). ∎

Theorem 3.2 can be regarded as the second-order-relaxed first-order sufficient condition. We next present the second-order sufficient condition.

Theorem 3.3 *Let $\bar{x} \in \mathbf{R}^n$ be a feasible point of problem* (1.1)*.*

If $\bar{x}$ is a strongly stationary point of problem (1.1)*, and*

$$\forall\, \xi \in C_2 \setminus \{0\}\ \ \exists\, \mu \in \mathcal{M} \text{ such that } \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x},\, \mu)[\xi,\, \xi] > 0, \tag{3.7}$$

then the quadratic growth condition holds at $\bar{x}$ for problem (1.1)*, and in particular, $\bar{x}$ is a strict local solution of this problem.*

Proof. According to the second equality in (3.2), condition (3.7) implies that for each $(I_1,\, I_2) \in \mathcal{I}$, the standard second-order sufficient optimality condition holds at $\bar{x}$ for branch problem (2.16). By standard MP theory, the latter implies the quadratic growth condition at $\bar{x}$ for each branch problem, and the needed assertion follows again from the second equality in (2.17). ∎

Note that condition (3.7) is weaker than the sufficient optimality condition from [8, Theorem 4.4]: the latter employs a "universal" MPVC-multiplier (the same for all $\xi \in C_2 \setminus \{0\}$).

In Example 1.1, $C_2 = C = \{\xi \in \mathbf{R}^2 \mid \xi_2 = 0\}$, and (3.7) holds with the unique associated MPVC-multiplier $\mu = \bar{\mu} = (1, 0)$, for any $a > 0$.

The following example shows that, in general, condition (3.7) does not imply the standard second-order sufficient condition for RNLP (2.4) even under MPVC-LICQ (this issue can be relevant for justification of numerical methods and for sensitivity analysis, as will be seen in the sequel).

Example 3.1 Consider the model Example 1.1, but with the objective function replaced by $f(x) = (x_1 - x_2)^2$.

The point $\bar{x} = 0$ is the unique solution of MPVC (1.1), and MPVC-LICQ (2.15) holds at this point. It can be easily checked that $\mathcal{M} = \{0\}$, $C = \{\xi \in \mathbf{R}^2 \mid \xi_2 \geq 0\}$, $C_2 = \{\xi \in \mathbf{R}^2 \mid \xi_2 \geq 0,\, \xi_1\xi_2 \leq 0\}$, and that (3.7) holds at $\bar{x}$, while the second-order sufficient condition for RNLP (2.4) does not (moreover, after adding some third-order terms to the objective function, $\bar{x}$ will no longer be a local solution of (2.4)).

However, quite remarkably, condition (3.7) turns out to be equivalent to the standard second-order sufficient optimality condition for the original problem (1.1), which has the form

$$\forall\, \xi \in C \setminus \{0\} \;\; \exists\, \lambda \in \Lambda \text{ such that } \frac{\partial^2 L}{\partial x^2}(\bar{x},\, \lambda)[\xi,\, \xi] > 0. \tag{3.8}$$

Specifically, the following holds.

Theorem 3.4 *Let $\bar{x} \in \mathbf{R}^n$ be a strongly stationary point of problem* (1.1).

Then for any $\mu \in \mathcal{M}$ it holds that

$$\frac{\partial^2 L}{\partial x^2}(\bar{x},\, \lambda)[\xi,\, \xi] = \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x},\, \mu)[\xi,\, \xi] \quad \forall\, \xi \in C_2 \tag{3.9}$$

for all $\lambda = (\lambda^H,\, \lambda^{GH}) \in \mathbf{R}^m \times \mathbf{R}^m$ satisfying (2.11), (2.12) *and*

$$\lambda^{GH}_{I_{00} \cup I_{0-}} = 0. \tag{3.10}$$

Moreover, condition (3.7) *is equivalent to the condition*

$$\forall\, \xi \in C_2 \setminus \{0\} \;\; \exists\, \lambda \in \Lambda \textit{ such that } \frac{\partial^2 L}{\partial x^2}(\bar{x},\, \lambda)[\xi,\, \xi] > 0, \tag{3.11}$$

which is further equivalent to the condition (3.8).

Proof. By formula (2.14) in Proposition 2.1, and by (3.2), we obtain that for any $\xi \in C_2$, any $\mu \in \mathcal{M}$, and any λ satisfying (2.11), (2.12) and (3.10), it holds that

$$\begin{aligned}
\frac{\partial^2 L}{\partial x^2}(\bar{x},\, \lambda)[\xi,\, \xi] &= \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x},\, \mu)[\xi,\, \xi] + 2 \sum_{i \in I_{+0}} \lambda_i^{GH} \langle G_i'(\bar{x}),\, \xi\rangle \langle H_i'(\bar{x}),\, \xi\rangle \\
&= \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x},\, \mu)[\xi,\, \xi] + 2 \sum_{i \in I_{+0}} \frac{\mu_i^G}{H_i(\bar{x})} \langle G_i'(\bar{x}),\, \xi\rangle \langle H_i'(\bar{x}),\, \xi\rangle \\
&= \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x},\, \mu)[\xi,\, \xi],
\end{aligned}$$

where the last equality follows from the representation (3.5) of C_2. We thus proved (3.9).

According to Proposition 2.1, for any $\mu \in \mathcal{M}$ and any λ satisfying (2.11), (2.12), (3.10) and the second group of conditions in (2.13), it holds that $\lambda \in \Lambda$. If for a given $\xi \in C_2$ the inequality in (3.7) holds with some $\mu \in \mathcal{M}$, we can take any λ satisfying these conditions so that, by (3.9), the inequality in (3.11) would hold. Conversely, suppose that the inequality in (3.11) holds with some λ. Define μ according to (2.11), (2.12). Then, by Proposition 2.1, $\mu \in \mathcal{M}$. Furthermore, by formula (2.14),

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \mu)[\xi, \xi] &= \frac{\partial^2 L}{\partial x^2}(\bar{x}, \lambda)[\xi, \xi] - 2 \sum_{i \in I_{+0} \cup I_0} \lambda_i^{GH} \langle G_i'(\bar{x}), \xi \rangle \langle H_i'(\bar{x}), \xi \rangle \\
&= \frac{\partial^2 L}{\partial x^2}(\bar{x}, \lambda)[\xi, \xi] - 2 \sum_{i \in I_{00} \cup I_{0-}} \lambda_i^{GH} \langle G_i'(\bar{x}), \xi \rangle \langle H_i'(\bar{x}), \xi \rangle \\
&= \frac{\partial^2 L}{\partial x^2}(\bar{x}, \lambda)[\xi, \xi] - 2 \sum_{i \in I_{00}} \lambda_i^{GH} \langle G_i'(\bar{x}), \xi \rangle \langle H_i'(\bar{x}), \xi \rangle \\
&\quad -2 \sum_{i \in I_{0-}} \frac{\lambda_i^H - \mu_i^H}{G_i(\bar{x})} \langle G_i'(\bar{x}), \xi \rangle \langle H_i'(\bar{x}), \xi \rangle \\
&= \frac{\partial^2 L}{\partial x^2}(\bar{x}, \lambda)[\xi, \xi] - 2 \sum_{i \in I_{00}} \lambda_i^{GH} \langle G_i'(\bar{x}), \xi \rangle \langle H_i'(\bar{x}), \xi \rangle \\
&\geq \frac{\partial^2 L}{\partial x^2}(\bar{x}, \lambda)[\xi, \xi] \\
&> 0,
\end{aligned}$$

where the second equality is by (3.3) and by the inclusion $C_2 \subset C$, the third is by (2.11), the fourth is by (3.3) and (3.5), and the first inequality is by (3.5) and by the nonnegativity condition $\lambda^{GH} \geq 0$. We thus proved that the conditions (3.7) and (3.11) are equivalent.

It remains to show that (3.8) holds automatically for $\xi \in C \setminus C_2$. (This is actually a particular instance of a more general fact from optimization theory, see [2, Lemma 3], but in this case it is easier to give a short proof instead of referring the reader elsewhere.) According to (3.1) and (3.2), for any such ξ there exists $i \in I_{00}$ such that

$$\langle G_i'(\bar{x}), \xi \rangle \langle H_i'(\bar{x}), \xi \rangle > 0. \tag{3.12}$$

Take any $\lambda \in \Lambda$ (it exists according to strong stationarity of $\bar{x}$ and Proposition 2.1). By Proposition 2.1, increasing λ_i^{GH} while keeping the rest of the components of λ fixed preserves the inclusion $\lambda \in \Lambda$ ((2.11)–(2.13) remain valid), while by formula (2.14), and by (3.12), the needed inequality in (3.8) will hold automatically for all λ_i^{GH} large enough. ■

We finish our discussion of second-order optimality conditions with the following conjecture, indicating some important structural differencies between MPCC and MPVC. We believe that in the case of MPVC (unlike for MPCC; see [12, Proposition 2]), the condition

$$\exists\, \bar{\mu} \in \mathcal{M} \text{ such that } \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x}, \bar{\mu})[\xi, \xi] > 0 \quad \forall\, \xi \in C_2 \setminus \{0\}$$

(existence of a "universal" MPVC-multiplier) generally does not imply the condition

$$\exists\, \bar{\lambda} \in \Lambda \text{ such that } \frac{\partial^2 L}{\partial x^2}(\bar{x},\, \bar{\lambda})[\xi,\, \xi] > 0 \quad \forall\, \xi \in C \setminus \{0\}$$

(existence of a "universal" Lagrange multiplier), even under MPVC-LICQ.

4 Sensitivity Analysis of MPVC

Let now f, G and H (and hence, all the objects defined by them) depend on a parameter $\sigma \in \mathbf{R}^s$ characterizing perturbations. Thus, now we have a smooth function $f : \mathbf{R}^s \times \mathbf{R}^n \to \mathbf{R}$ and smooth mappings $G,\, H : \mathbf{R}^s \times \mathbf{R}^n \to \mathbf{R}^m$. It is important to stress that here we deal with perturbations of G and H, preserving the MPVC structure, rather than with arbitrary perturbations of the constraints of (1.1).

Let $\bar{x}$ be a local solution of the problem

$$\begin{array}{ll} \text{minimize} & f(\sigma,\, x) \\ \text{subject to} & H_i(\sigma,\, x) \geq 0, \quad G_i(\sigma,\, x)H_i(\sigma,\, x) \leq 0,\, i = 1,\, \ldots,\, m, \end{array} \tag{4.1}$$

for $\sigma = \bar{\sigma}$, where $\bar{\sigma} \in \mathbf{R}^s$ is the base parameter value. Furthermore, let the index sets I_+, I_0, I_{+0}, I_{+-}, I_{0+}, I_{00}, and I_{0-}, the function $\mathcal{L}$, and the sets $\mathcal{M}$, C and C_2 be defined for the base parameter value. Fix a closed ball B centered at $\bar{x}$ such that $\bar{x}$ is a global solution of the problem

$$\begin{array}{ll} \text{minimize} & f(\sigma,\, x) \\ \text{subject to} & H_i(\sigma,\, x) \geq 0, \quad G_i(\sigma,\, x)H_i(\sigma,\, x) \leq 0,\, i = 1,\, \ldots,\, m, \quad x \in B, \end{array} \tag{4.2}$$

for $\sigma = \bar{\sigma}$. If $\bar{x}$ is a *strict* local solution of (4.1) for $\sigma = \bar{\sigma}$, we always assume that B is chosen small enough, so that $\bar{x}$ is the *unique* global solution of (4.2) for $\sigma = \bar{\sigma}$. For each $\sigma \in \mathbf{R}^s$, let $S(\sigma)$ stand for the (global) solution set of problem (4.2), and define the local optimal value function $v : \mathbf{R}^s \to \mathbf{R}$ of problem (4.1) as the optimal value function of problem (4.2):

$$v(\sigma) = \inf_{x \in D(\sigma) \cap B} f(\sigma,\, x),$$

where $D(\sigma)$ stands for the feasible set of (4.1).

4.1 Sensitivity in the Case of Existence of a Regular Branch

Throughout this subsection, we assume that there exists at least one partition $(I_1,\, I_2) \in \mathcal{I}$ such that MFCQ holds at $\bar{x}$ for the corresponding branch problem

$$\begin{array}{ll} \text{minimize} & f(\sigma,\, x) \\ \text{subject to} & H_{I_{0+} \cup I_2}(\sigma,\, x) = 0, \quad H_{I_1 \cup I_{0-}}(\sigma,\, x) \geq 0, \quad G_{I_{+0} \cup I_1}(\sigma,\, x) \leq 0, \end{array} \tag{4.3}$$

for $\sigma = \bar{\sigma}$, i.e.,

$$\operatorname{rank} \frac{\partial H_{I_{0+} \cup I_2}}{\partial x}(\bar{\sigma},\, \bar{x}) = |I_{0+}| + |I_2|, \tag{4.4}$$

and there exists $\bar{\xi} \in \mathbf{R}^n$ such that

$$\frac{\partial H_{I_{0+} \cup I_2}}{\partial x}(\bar{\sigma},\, \bar{x})\bar{\xi} = 0, \quad \frac{\partial H_{I_1 \cup I_{0-}}}{\partial x}(\bar{\sigma},\, \bar{x})\bar{\xi} > 0, \quad \frac{\partial G_{I_{+0} \cup I_1}}{\partial x}(\bar{\sigma},\, \bar{x})\bar{\xi} < 0. \tag{4.5}$$

Then, by Robinson's stability theorem [16],

$$\operatorname{dist}(\bar{x},\, D_{(I_1,\, I_2)}(\sigma)) = O(\|\sigma - \bar{\sigma}\|), \tag{4.6}$$

where for each $\sigma \in \mathbf{R}^s$, we denote by $D_{(I_1,\, I_2)}(\sigma)$ the feasible set of problem (4.3). At the same time, it is easy to check that the second equality in (2.17) is preserved under perturbations of the given class: for each $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$, locally (near $\bar{x}$) it holds that

$$D(\sigma) = \bigcup_{(I_1,\, I_2) \in \mathcal{I}} D_{(I_1,\, I_2)}(\sigma), \tag{4.7}$$

and hence, by (4.6),

$$\operatorname{dist}(\bar{x},\, D(\sigma)) = O(\|\sigma - \bar{\sigma}\|). \tag{4.8}$$

According to [3, Theorem 3.1], an estimate of the distance from $\bar{x}$ to the perturbed feasible set guarantees solution stability and a certain upper bound for the optimal value of the perturbed problem. Specifically, combining estimate (4.8) with [3, Theorem 3.1] results in the following.

Theorem 4.1 *Let $\bar{x}$ be a local solution of problem* (4.1) *for $\sigma = \bar{\sigma}$. Let there exist a partition $(I_1,\, I_2) \in \mathcal{I}$ such that MFCQ holds at $\bar{x}$ for the corresponding branch problem* (4.3) *for $\sigma = \bar{\sigma}$.*

Then v is continuous at $\bar{\sigma}$, and for $\sigma \in \mathbf{R}^s$ it holds that

$$v(\sigma) \leq v(\bar{\sigma}) + O(\|\sigma - \bar{\sigma}\|).$$

Moreover, $S(\sigma) \neq \emptyset$ for all $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$, and

$$\sup_{x \in S(\sigma)} \operatorname{dist}(x,\, S(\bar{\sigma})) \to 0 \text{ as } \sigma \to \bar{\sigma}.$$

In particular, if $\bar{x}$ is a strict local solution of problem (4.1) *for $\sigma = \bar{\sigma}$ then*

$$\sup_{x \in S(\sigma)} \|x - \bar{x}\| \to 0 \text{ as } \sigma \to \bar{\sigma}. \tag{4.9}$$

Furthermore, according to [3, Theorems 3.2, 3.3], an upper bound on the optimal value function combined with certain sufficient optimality conditions results in solution estimates and lower bounds for the optimal value of the perturbed optimization problem. Employing the upper estimate from Theorem 4.1, we thus obtain the next two theorems.

Theorem 4.2 *Under the assumptions of Theorem 4.1, suppose that* (3.6) *holds.*

Then for $\sigma \in \mathbf{R}^s$ it holds that

$$\sup_{x \in S(\sigma)} \|x - \bar{x}\| = O(\|\sigma - \bar{\sigma}\|), \tag{4.10}$$

$$|v(\sigma) - v(\bar{\sigma})| = O(\|\sigma - \bar{\sigma}\|).$$

Proof. The constraints defining the feasible set of (4.1) can be locally written in the form

$$F(\sigma, x) \in K,$$

where $F : \mathbf{R}^s \times \mathbf{R}^n \to \mathbf{R}^{|I_{0+}|} \times \mathbf{R}^{|I_{00}|+|I_{0-}|} \times \mathbf{R}^{|I_{+0}|} \times \mathbf{R}^{|I_{00}|}$,

$$F(\sigma, x) = (H_{I_{0+}}(\sigma, x),\, H_{I_{00}\cup I_{0-}}(\sigma, x),\, G_{I_{+0}}(\sigma, x),\, y(\sigma, x)),$$

$$y_i(\sigma, x) = G_i(\sigma, x)H_i(\sigma, x),\ i \in I_{00},$$

$$K = \{0\} \times \mathbf{R}_+^{|I_{00}|+|I_{0-}|} \times \mathbf{R}_-^{|I_{+0}|} \times \mathbf{R}_-^{|I_{00}|}.$$

(cf. (1.3)). Note that $F(\bar{\sigma}, \bar{x}) = 0$, that K is a polyhedral cone, and that by the definition of I_{00}, for all $i \in I_{00}$, all $d \in \mathbf{R}^s$, and all $\xi \in \mathbf{R}^n$, it holds that

$$\left\langle \frac{\partial y_i}{\partial \sigma}(\bar{\sigma}, \bar{x}),\, d \right\rangle = G_i(\bar{\sigma}, \bar{x}) \left\langle \frac{\partial H_i}{\partial \sigma}(\bar{\sigma}, \bar{x}),\, d \right\rangle + H_i(\bar{\sigma}, \bar{x}) \left\langle \frac{\partial G_i}{\partial \sigma}(\bar{\sigma}, \bar{x}),\, d \right\rangle = 0, \tag{4.11}$$

and similarly,

$$\left\langle \frac{\partial y_i}{\partial x}(\bar{\sigma}, \bar{x}),\, \xi \right\rangle = G_i(\bar{\sigma}, \bar{x}) \left\langle \frac{\partial H_i}{\partial x}(\bar{\sigma}, \bar{x}),\, \xi \right\rangle + H_i(\bar{\sigma}, \bar{x}) \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}),\, \xi \right\rangle = 0. \tag{4.12}$$

Now, in order to apply [3, Theorem 3.2 (iii)], it suffices to show that the following two conditions hold:

$$\left\{ \xi \in \mathbf{R}^n \;\middle|\; \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi \in K,\ \frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \in K + \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}),\ \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}),\, \xi \right\rangle \leq 0 \right\} = \{0\}, \tag{4.13}$$

and for each $\sigma \in \mathbf{R}^s$

$$-\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})(\sigma - \bar{\sigma}) \in K + \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}). \tag{4.14}$$

Using (3.2), it is an easy exercise to check that the set in the left-hand side of (4.13) is contained in C_2 (the reader can also check that the two sets coincide under MPVC-LICQ). Thus, condition (3.6) implies (4.13).

Furthermore, taking into account (4.11) and (4.12), condition (4.14) can be stated as follows: there exists $x \in \mathbf{R}^n$ such that

$$\frac{\partial H_{I_{0+}}}{\partial x}(\bar{\sigma}, \bar{x})x - \frac{\partial H_{I_{0+}}}{\partial \sigma}(\bar{\sigma}, \bar{x})(\sigma - \bar{\sigma}) = 0, \tag{4.15}$$

$$\frac{\partial H_{I_{00}\cup I_{0-}}}{\partial x}(\bar{\sigma}, \bar{x})x - \frac{\partial H_{I_{00}\cup I_{0-}}}{\partial \sigma}(\bar{\sigma}, \bar{x})(\sigma - \bar{\sigma}) \geq 0, \tag{4.16}$$

$$\frac{\partial G_{I_{+0}}}{\partial x}(\bar{\sigma}, \bar{x})x - \frac{\partial G_{I_{+0}}}{\partial \sigma}(\bar{\sigma}, \bar{x})(\sigma - \bar{\sigma}) \leq 0. \tag{4.17}$$

By (4.4), we obtain the existence of $\tilde{x} \in \mathbf{R}^n$ such that

$$\frac{\partial H_{I_{0+}\cup I_2}}{\partial x}(\bar{\sigma}, \bar{x})\tilde{x} = \frac{\partial H_{I_{0+}\cup I_2}}{\partial \sigma}(\bar{\sigma}, \bar{x})(\sigma - \bar{\sigma}).$$

Then from the equality in (4.5) it follows that for each t, the vector $x = \tilde{x} + t\bar{\xi}$ satisfies (4.15) and (as equalities) those inequalities in (4.16) that correspond to $i \in I_2$. Finally, the rest of the inequalities in (4.16), (4.17) will hold for $t > 0$ large enough, according to the inequalities in (4.5). We thus proved (4.14).

The needed result now follows from [3, Theorem 3.2 (iii)] and Theorem 4.1. ■

It is obvious that the estimates obtained in Theorems 4.1 and 4.2 cannot be improved.

Theorem 4.3 *Under the assumptions of Theorem 4.1, let $\bar{x}$ be a strongly stationary point of problem* (4.1) *for $\sigma = \bar{\sigma}$, and suppose that* (3.7) *holds.*

Then for $\sigma \in \mathbf{R}^s$ we have that

$$\sup_{x \in S(\sigma)} \|x - \bar{x}\| = O(\|\sigma - \bar{\sigma}\|^{1/2}),$$

$$|v(\sigma) - v(\bar{\sigma})| = O(\|\sigma - \bar{\sigma}\|).$$

Proof. This follows immediately from [3, Theorems 3.3] and Theorems 3.4, 4.1. ■

The next example (cf. [5, Example 4.23], [3, Example 6.1]) demonstrates that the estimates obtained in Theorem 4.3 cannot be improved, even when perturbations are performed along a given direction in the space of parameter values.

Example 4.1 Let $s = 1$, $n = 4$, $m = 2$, $f(\sigma, x) = -x_2 + (x_3^2 + x_4^2)/2$, $G(\sigma, x) = (x_3, x_4)$, $H(\sigma, x) = (-x_2 - x_1^2 + \sigma, -x_2 + x_1^2)$.

It can be easily seen that $\bar{x} = 0$ is the unique solution of MPVC (4.1) for $\sigma = \bar{\sigma} = 0$, $I_+ = \emptyset$, $I_0 = I_{00} = \{1, 2\}$, and for $(I_1, I_2) = (\{1, 2\}, \emptyset)$ (and only for this partition!), MFCQ holds at $\bar{x}$ for branch problem (4.3) for $\sigma = \bar{\sigma}$. Furthermore,

$$\mathcal{M} = \{\mu = (\mu^H, 0) \in \mathbf{R}^2 \times \mathbf{R}^2 \mid \mu_1^H + \mu_2^H = 1,\ \mu_1^H \geq 0,\ \mu_2^H \geq 0\},$$

$$C_2 = C = \{\xi \in \mathbf{R}^4 \mid \xi_2 = 0\},$$

and (3.7) holds even with a "universal" multiplier, say $\mu = ((1, 0), 0))$.

The reader can easily check that for any $\sigma \geq 0$ small enough

$$v(\sigma) = -\sigma/2, \quad S(\sigma) = \{(\pm(\sigma/2)^{1/2}, \sigma/2, 0, 0)\}.$$

4.2 Sensitivity under MPVC-LICQ

Let now a local solution $\bar{x}$ of the problem (4.1) for $\sigma = \bar{\sigma}$ satisfy MPVC-LICQ. In this case, the sensitivity results obtained in Section 4.1 can be quantitatively sharpened and completed.

Recall that, according to Theorem 2.1, under the stated assumptions there exists the unique $\mu = \bar{\mu} = (\bar{\mu}^H, \bar{\mu}^G) \in \mathbf{R}^m \times \mathbf{R}^m$ satisfying (2.8), (2.9). And, moreover, this multiplier satisfies (2.10) as well, i.e., $\bar{\mu}$ is the MPVC-multiplier associated to the strongly stationary point $\bar{x}$. Moreover, for each partition $(I_1, I_2) \in \mathcal{I}$, the point $\bar{x}$ is a local solution of the corresponding branch problem (4.3) for $\sigma = \bar{\sigma}$, satisfying LICQ, and the sensitivity theory presented in [5, Chapters 4, 5] is completely applicable to each of the branch problems. Note finally that $\bar{\mu}$ is the unique Lagrange multiplier associated to a stationary point $\bar{x}$ of each branch problem. These are the key observations for the approach adopted in this section.

For each partition $(I_1, I_2) \in \mathcal{I}$, and each $\sigma \in \mathbf{R}^s$, let $S_{(I_1, I_2)}(\sigma)$ stand for the (global) solution set of problem

$$\begin{array}{ll} \text{minimize} & f(\sigma, x) \\ \text{subject to} & H_{I_{0+}\cup I_2}(\sigma, x) = 0, \quad H_{I_1 \cup I_{0-}}(\sigma, x) \geq 0, \quad G_{I_{+0}\cup I_1}(\sigma, x) \leq 0, \quad x \in B, \end{array} \tag{4.18}$$

and define the local optimal value function $v_{(I_1, I_2)} : \mathbf{R}^s \to \mathbf{R}$ of problem (4.3) as the optimal value function of problem (4.18):

$$v_{(I_1, I_2)}(\sigma) = \inf_{x \in D_{(I_1, I_2)}(\sigma) \cap B} f(\sigma, x).$$

From this point on, we will need the explicit dependence of MPVC-Lagrangian on the parameter: let

$$\mathcal{L}(\sigma, x, \mu) = f(\sigma, x) - \langle \mu^H, H(\sigma, x) \rangle + \langle \mu^G, G(\sigma, x) \rangle,$$

where $\sigma \in \mathbf{R}^s$, $x \in \mathbf{R}^n$ and $\mu = (\mu^H, \mu^G) \in \mathbf{R}^m \times \mathbf{R}^m$.

We first deal with perturbations along a given direction $d \in \mathbf{R}^s$. Since LICQ implies directional regularity (Gollan's condition, to be discussed in more detail in Section 5) in any direction (see [5, Proposition 5.50 (v)]), by [5, Proposition 5.50 (ii), Theorem 5.51 (i)] we obtain that for any partition $(I_1, I_2) \in \mathcal{I}$, and any mapping $\rho : \mathbf{R}_+ \to \mathbf{R}^s$ such that $\rho(t) = o(t)$, for $t \geq 0$ it holds that

$$v_{(I_1, I_2)}(\bar{\sigma} + td + \rho(t)) \leq v_{(I_1, I_2)}(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\mu}), d \right\rangle t + o(t), \tag{4.19}$$

where $\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\mu}), d \rangle$ coincides with the optimal value of the LP problem

$$\begin{array}{ll} \text{minimize} & \left\langle \frac{\partial f}{\partial \sigma}(\bar{\sigma}, \bar{x}), d \right\rangle + \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \xi \right\rangle \\ \text{subject to} & \frac{\partial H_{I_{0+}\cup I_2}}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{\partial H_{I_{0+}\cup I_2}}{\partial x}(\bar{\sigma}, \bar{x})\xi = 0, \\ & \frac{\partial H_{I_1\cup I_{0-}}}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{\partial H_{I_1\cup I_{0-}}}{\partial x}(\bar{\sigma}, \bar{x})\xi \geq 0, \\ & \frac{\partial G_{I_{+0}\cup I_1}}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{\partial G_{I_{+0}\cup I_1}}{\partial x}(\bar{\sigma}, \bar{x})\xi \leq 0. \end{array} \tag{4.20}$$

In particular, the optimal value of the latter problem does not depend on $(I_1, I_2) \in \mathcal{I}$.

Furthermore, let $\bar{x}$ be a strict local solution of MPVC (4.1) (and hence, of any branch problem) for $\sigma = \bar{\sigma}$. Then by [5, Theorem 4.26], and by the uniqueness of the multiplier $\bar{\mu}$, estimate (4.19) is exact (holds as an equality), which implies that $v_{(I_1, I_2)}$ is differentiable at $\bar{\sigma}$, and the derivative

$$v'_{(I_1, I_2)}(\bar{\sigma}) = \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\mu})$$

does not depend on a partition $(I_1, I_2) \in \mathcal{I}$.

In order to transmit these results onto the original problem (4.1), let us employ the equality (4.7), which immediately yields that for $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$ we have that

$$v(\sigma) = \min_{(I_1,\, I_2)\in\mathcal{I}} v_{(I_1,\, I_2)}(\sigma), \tag{4.21}$$

$$S(\sigma) = \bigcup_{\substack{(I_1,\, I_2)\in\mathcal{I},\\ v_{(I_1,\, I_2)}(\sigma)=v(\sigma)}} S_{(I_1,\, I_2)}(\sigma). \tag{4.22}$$

(In order to avoid the discussion of a local nature of (4.7), we can assume from now on that B is so small that (4.7) holds for each $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$, with the left-hand side and the right-hand side intersected with B.) By (4.21) and the discussion above, we immediately obtain the following result.

Theorem 4.4 *Assume that a local solution $\bar{x}$ of problem* (4.1) *for $\sigma = \bar{\sigma}$ satisfies MPVC-LICQ* (2.15), *and let $\bar{\mu}$ be the unique MPVC-multiplier associated with $\bar{x}$.*

Then for any direction $d \in \mathbf{R}^s$ and for $t \geq 0$, it holds that

$$v(\bar{\sigma} + td + \rho(t)) \leq v(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\mu}),\, d \right\rangle t + o(t). \tag{4.23}$$

Moreover, if $\bar{x}$ is a strict local solution of problem (4.1) *for $\sigma = \bar{\sigma}$ then this estimate is exact (holds as an equality), which implies that v is differentiable at $\bar{\sigma}$, and its gradient is given by*

$$v'(\bar{\sigma}) = \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\mu}). \tag{4.24}$$

For MPCC, the equality similar to (4.24) was obtained in [11]. It is interesting to note that under MPVC-LICQ, the first derivative of the optimal value function does not reflect the combinatorial structure of the feasible set of MPVC. However, the combinatorial aspect inevitably appears in the higher-order analysis.

Indeed, for each partition $(I_1,\, I_2) \in \mathcal{I}$, and each $d \in \mathbf{R}^s$, denote by $S_{(I_1,\, I_2)}(d) = S_{(I_1,\, I_2)}(\bar{\sigma},\, \bar{x};\, d)$ the solution set of LP problem (4.20), and consider another auxiliary problem:

$$\begin{array}{ll} \text{minimize} & \frac{\partial^2 \mathcal{L}}{\partial(\sigma, x)^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[(d,\, \xi),\, (d,\, \xi)] \\ \text{subject to} & \xi \in S_{(I_1,\, I_2)}(d) \end{array} \tag{4.25}$$

(this is actually a QP problem). By [5, Theorem 5.51 (ii)], employing the optimal value of problem (4.25), one can sharpen the estimate (4.19) as follows: for $t \geq 0$ it holds that

$$\begin{aligned} v_{(I_1,\, I_2)}(\bar{\sigma} + td) \;\; \leq \;\; & v_{(I_1,\, I_2)}(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\mu}),\, d \right\rangle t \\ & + \frac{1}{2} \inf_{\xi \in S_{(I_1,\, I_2)}(d)} \frac{\partial^2 \mathcal{L}}{\partial(\sigma,\, x)^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[(d,\, \xi),\, (d,\, \xi)]t^2 + o(t^2). \end{aligned} \tag{4.26}$$

Now, by (4.21), we obtain the following sharpened version of estimate (4.23).

Theorem 4.5 *Assume that a local solution $\bar{x}$ of problem* (4.1) *for $\sigma = \bar{\sigma}$ satisfies MPVC-LICQ* (2.15)*, and let $\bar{\mu}$ be the unique MPVC-multiplier associated with $\bar{x}$.*

Then for any direction $d \in \mathbf{R}^s$ and for $t \geq 0$, it holds that

$$\begin{aligned} v(\bar{\sigma} + td) \quad &\leq \quad v(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\mu}),\, d \right\rangle t \\ &\quad + \frac{1}{2} \min_{(I_1,\, I_2) \in \mathcal{I}} \inf_{\xi \in S_{(I_1,\, I_2)}(d)} \frac{\partial^2 \mathcal{L}}{\partial (\sigma,\, x)^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[(d,\, \xi),\, (d,\, \xi)] t^2 + o(t^2). \end{aligned} \tag{4.27}$$

Note that the combinatorial aspect is explicit in the estimate (4.27).

Moreover, since $\mathcal{I}$ is finite, according to (4.22), (4.27), [5, Theorem 4.95], and relation (4.9) in Theorem 4.1, we obtain the following result.

Proposition 4.1 *Assume that a strict local solution $\bar{x}$ of problem* (4.1) *for $\sigma = \bar{\sigma}$ satisfies MPVC-LICQ* (2.15)*, and let $\bar{\mu}$ be the unique MPVC-multiplier associated with $\bar{x}$.*

Then for any sequences $\{t_k\} \subset \mathbf{R}_+ \setminus \{0\}$ and $\{x^k\} \subset \mathbf{R}^n$ such that $\{t_k\} \to 0$, and $x^k \in S(\bar{\sigma} + t_k d)\ \forall\, k$, any limit point of the sequence $\{(x^k - \bar{x})/t_k\}$ is a solution of problem (4.25) *for some partition $(I_1,\, I_2) \in \mathcal{I}$ such that*

$$\begin{aligned} &\liminf_{k \to \infty} \frac{v_{(I_1, I_2)}(\bar{\sigma} + t_k d) - v(\bar{\sigma}) - \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\mu}),\, d \right\rangle t_k}{t_k^2} \\ &\leq \min_{(J_1,\, J_2) \in \mathcal{I}} \inf_{\xi \in S_{(J_1,\, J_2)}(d)} \frac{\partial^2 \mathcal{L}}{\partial (\sigma,\, x)^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[(d,\, \xi),\, (d,\, \xi)]. \end{aligned}$$

Further analysis must employ sufficient optimality conditions. For example, for a given partition $(I_1,\, I_2) \in \mathcal{I}$, let the corresponding unperturbed branch problem satisfy the standard second-order sufficient condition at $\bar{x}$:

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[\xi,\, \xi] > 0 \quad \forall\, \xi \in C_{(I_1,\, I_2)} \setminus \{0\},$$

where $C_{(I_1,\, I_2)}$ is defined for the base parameter value. By [5, Theorem 5.9], we then obtain the Lipschitzian solution estimate: for $\sigma \in \mathbf{R}^s$

$$\sup_{x \in S_{(I_1,\, I_2)}(\sigma)} \|x - \bar{x}\| = O(\|\sigma - \bar{\sigma}\|). \tag{4.28}$$

Moreover, by [5, Theorem 5.53 (b), (d)], for any fixed $d \in \mathbf{R}^s$, the estimate (4.26) is exact, i.e., for $t \geq 0$

$$\begin{aligned} v_{(I_1, I_2)}(\bar{\sigma} + td) \quad &= \quad v_{(I_1, I_2)}(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\mu}),\, d \right\rangle t \\ &\quad + \frac{1}{2} \inf_{\xi \in S_{(I_1,\, I_2)}(d)} \frac{\partial^2 \mathcal{L}}{\partial (\sigma,\, x)^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[(d,\, \xi),\, (d,\, \xi)] t^2 + o(t^2), \end{aligned} \tag{4.29}$$

and for any solution $\bar{\xi}$ of problem (4.25), branch problem (4.3) for $\sigma = \bar{\sigma} + td$ has an $o(t^2)$-solution of the form $\bar{x} + t\bar{\xi} + O(t^2)$. Employing (4.21), (4.22), (4.28), and (4.29), we now obtain the following.

Theorem 4.6 *Assume that a local solution $\bar{x}$ of problem* (4.1) *for $\sigma = \bar{\sigma}$ satisfies MPVC-LICQ* (2.15)*, and suppose that*

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[\xi,\, \xi] > 0 \quad \forall\, \xi \in C_2 \setminus \{0\}, \tag{4.30}$$

where $\bar{\mu}$ is the unique MPVC-multiplier associated with $\bar{x}$.

Then for $\sigma \in \mathbf{R}^s$, (4.10) *holds. Moreover, for any direction $d \in \mathbf{R}^s$, the estimate* (4.27) *is exact, i.e., for $t \geq 0$*

$$\begin{aligned} v(\bar{\sigma} + td) &= v(\bar{\sigma}) + \left\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\mu}),\, d \right\rangle t \\ &\quad + \frac{1}{2} \min_{(I_1,\, I_2) \in \mathcal{I}} \inf_{\xi \in S_{(I_1,\, I_2)}(d)} \frac{\partial^2 \mathcal{L}}{\partial (\sigma,\, x)^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[(d,\, \xi),\, (d,\, \xi)]t^2 + o(t^2), \end{aligned} \tag{4.31}$$

and for any solution $\bar{\xi}$ of problem (4.25) *for any partition $(I_1,\, I_2) \in \mathcal{I}$ such that*

$$\begin{aligned} &\inf_{\xi \in S_{(I_1,\, I_2)}(d)} \frac{\partial^2 \mathcal{L}}{\partial (\sigma,\, x)^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[(d,\, \xi),\, (d,\, \xi)] \\ &= \min_{(J_1,\, J_2) \in \mathcal{I}} \inf_{\xi \in S_{(J_1,\, J_2)}(d)} \frac{\partial^2 \mathcal{L}}{\partial (\sigma,\, x)^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[(d,\, \xi),\, (d,\, \xi)], \end{aligned}$$

MPVC (4.1) *for $\sigma = \bar{\sigma} + td$ has an $o(t^2)$-solution of the form $\bar{x} + t\bar{\xi} + O(t^2)$.*

For MPCC, an estimate similar to (4.31) was derived in [11].

Finally, let us discuss the case when, in addition to MPVC-LICQ, for a given partition $(I_1,\, I_2) \in \mathcal{I}$ the so-called strong second-order sufficient condition holds at $\bar{x}$ for the corresponding unperturbed branch problem:

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[\xi,\, \xi] > 0 \quad \forall\, \xi \in C^+_{(I_1,\, I_2)} \setminus \{0\}, \tag{4.32}$$

where

$$C^+_{(I_1,\, I_2)} = C^+_{(I_1,\, I_2)}(\bar{\sigma},\, \bar{x},\, \bar{\mu}) = \left\{ \xi \in \mathbf{R}^n \,\middle|\, \begin{array}{l} H'_{I_{0+} \cup I_2}(\bar{x})\xi = 0, \\ \bar{\mu}_i^H \langle H_i'(\bar{x}),\, \xi \rangle = 0,\, i \in I_1 \cup I_{0-}, \\ \bar{\mu}_i^G \langle G_i'(\bar{x}),\, \xi \rangle = 0,\, i \in I_{+0} \end{array} \right\} \tag{4.33}$$

(by (3.4), $C_{(I_1, I_2)} \subset C^+_{(I_1, I_2)}$). Then according to [5, Proposition 5.38], $(\bar{x},\, \bar{\mu})$ is a strongly regular solution of the KKT system of problem (4.3) for $\sigma = \bar{\sigma}$, and it now follows from [5, Theorem 5.13] that for each $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$, problem (4.3) has near $\bar{x}$ the unique local solution and the unique stationary point $x(\sigma)$, with the unique associated Lagrange multiplier $\mu(\sigma) = (\mu^H(\sigma),\, \mu^G(\sigma))$. Moreover, the mapping $(x(\cdot),\, \mu(\cdot))$ is Lipschitz-continuous near $\bar{\sigma}$. Then by the above-mentioned result on differentiability of $v_{(I_1,\, I_2)}$ at $\bar{\sigma}$, it can be easily derived that this function is actually continuously differentiable near $\bar{\sigma}$ (just apply the differentiability result with $\bar{\sigma}$ replaced by σ and $\bar{x}$ replaced by $x(\sigma)$). Furthermore, assuming that (4.32) holds for each partition $(I_1, I_2) \in \mathcal{I}$, by (4.21) we immediately obtain that v is piecewise smooth, and, according to (4.22), the perturbed MPVC (4.1) for σ close to $\bar{\sigma}$ can have near $\bar{x}$ only a finite number (not more than $2^{|I_{00}|}$) of different solutions.

We say that the *upper-level strict complementarity condition* (ULSCC) holds at $\bar{x}$ for the associated MPVC-multiplier $\bar{\mu}$ if

$$\bar{\mu}^H_{I_{00}\cup I_{0-}} > 0, \quad \bar{\mu}^G_{I_{+0}} > 0. \tag{4.34}$$

If this condition holds then, by (4.33), for each partition $(I_1, I_2) \in \mathcal{I}$ we have that

$$C^+_{(I_1,\, I_2)} = \{\xi \in \mathbf{R}^n \mid H'_{I_0}(\bar{x})\xi = 0,\ G'_{I_{+0}}(\bar{x})\xi = 0\},$$

and condition (4.32) takes the form

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \bar{\mu})[\xi,\, \xi] > 0 \quad \forall\, \xi \in \mathbf{R}^n \setminus \{0\} \text{ such that } H'_{I_0}(\bar{x})\xi = 0,\ G'_{I_{+0}}(\bar{x})\xi = 0 \tag{4.35}$$

(independent of (I_1, I_2)).

For MPCC, the following argument goes through:

1. Under ULSCC, strong second order sufficient condition for each branch is just strong second order sufficient condition for TNLP. Combined with MPCC-LICQ, this implies strong regularity for TNLP, and hence, perturbed TNLP has the unique local solution with the unique associated multiplier.

2. Under ULSCC, a solution of the KKT system for any branch problem is a solution of the Lagrange system for TNLP.

Thus, under these assumptions, the solutions of all perturbed branch MPCC problems coincide with the unique solution of perturbed TNLP, and hence, for each $\sigma \in \mathbf{R}^s$ close to $\bar{\sigma}$, the perturbed MPCC has near $\bar{x}$ the unique local solution $x(\sigma)$, with the unique associated MPCC-multiplier $\mu(\sigma)$. The mapping $(x(\sigma),\, \mu(\sigma))$ is Lipschitz-continuous near $\bar{\sigma}$, while v is continuously differentiable near $\bar{\sigma}$ (the fact first established in [17, Theorem 11]).

For MPVC, item 1 is valid, but not item 2: the KKT system for TNLP contains the condition $G_{I_2}(\sigma,\, x) \leq 0$, which is missing in the KKT system for the corresponding branch problem. And vice versa: the KKT system for the branch problem subsumes the condition $\mu_{I_2} = 0$, which does not follow from the KKT system for TNLP.

The next example demonstrates that item 2 is indeed not valid for MPVC.

Example 4.2 Let $s = 4$, $n = 2$, $m = 1$, $f(\sigma,\, x) = x_2 + a x_1^2/2 + \sigma_1 x_1 + \sigma_2 x_2$, $G(\sigma,\, x) = x_1 - \sigma_3$, $H(\sigma,\, x) = x_2 - \sigma_4$, where $a > 0$. This is just a parameterized version of model Example 1.1, with the linear perturbation of the objective function, and the simplest perturbations of G and H (right-hand side perturbations).

The point $\bar{x} = 0$ is the unique solution of the unperturbed MPVC corresponding to $\sigma = \bar{\sigma} = 0$, satisfying MPVC-LICQ (2.15) and the second-order sufficient condition (4.30) with the unique associated MPVC-multiplier $\bar{\mu} = (\bar{\mu}^H,\, \bar{\mu}^G) = (1,\, 0)$. Moreover, ULSCC

(4.34) holds at $\bar{x}$ for this multiplier, and condition (4.35) is satisfied too (actually, for each of the two partitions $(I_1, I_2) \in \mathcal{I}$, it holds that $C^+_{(I_1, I_2)} = C_{(I_1, I_2)} = C = C_2$).

Furthermore, for each $d \in \mathbf{R}^4$, it holds that $\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\mu}), d\rangle = d_4$, and for each of the two partitions $(I_1, I_2) \in \mathcal{I}$, the equality $S_{(I_1, I_2)}(d) = \{\xi \in \mathbf{R}^2 \mid \xi_2 = d_4\}$ holds, and problem (4.25) takes the form

$$\begin{array}{ll} \text{minimize} & a\xi_1^2/2 + d_1\xi_1 + d_2\xi_2 \\ \text{subject to} & \xi_2 = d_4. \end{array}$$

The unique solution of this problem is $\bar{\xi} = (-d_1/a, d_2)$, and the optimal value is equal to $-d_1^2/(2a) + d_2 d_4$.

It can be easily checked that for each $\sigma \in \mathbf{R}^2$ close enough to 0, it holds that

$$v(\sigma) = \sigma_4 - \frac{1}{2a}\sigma_1^2 + \sigma_2\sigma_4, \quad S(\sigma) = \left\{ \left(-\frac{1}{a}\sigma_1, \sigma_4 \right) \right\},$$

which completely agrees with the result of Theorem 4.6.

Note, however, that for each $\sigma \in \mathbf{R}^2$ close enough to 0,

$$S_{(\emptyset, \{1\})}(\sigma) = \{(-\sigma_1/a, \sigma_4)\},$$

while

$$S_{(\{1\}, \emptyset)}(\sigma) = \begin{cases} \{(-\sigma_1/a, \sigma_4)\} & \text{if } \sigma_1 \geq -a\sigma_3, \\ \{(\sigma_3, \sigma_2)\} & \text{if } \sigma_1 < -a\sigma_3, \end{cases}$$

the latter being also the solution set of the perturbed TNLP. Thus, the solutions of TNLP (and of the second branch problem) do not coincide with the solution of the first branch problem when $\sigma_1 < -a\sigma_3$.

However, in this example the solution of the perturbed MPVC is still unique, and this is not accidental. This is clarified by the next theorem, where we refer to the branch problem (4.3) corresponding to $I_1 = \emptyset$, $I_2 = I_{00}$ as the *minimal branch problem.* (One should not take this terminology literally: the feasible set of this branch problem is not necessarily contained in the feasible sets of the other branches, and in particular, the feasible set of this problem is not necessarily contained in D_{TNLP}.)

Theorem 4.7 *Assume that a local solution $\bar{x}$ of problem* (4.1) *for $\sigma = \bar{\sigma}$ satisfies MPVC-LICQ* (2.15)*, that ULSCC* (4.34) *holds at $\bar{x}$ for the unique associated MPVC-multiplier $\bar{\mu}$, and that* (4.35) *holds.*

Then for $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$, problem (4.1) *has near $\bar{x}$ the unique local solution and the unique strongly stationary point $x(\sigma)$. Moreover, this solution coincides with the unique solution of the minimal branch problem, the associated MPVC-multiplier $\mu(\sigma) = (\mu^H(\sigma), \mu^G(\sigma))$ is unique, and the mapping $(x(\sigma), \mu(\sigma))$ is Lipschitz-continuous near $\bar{\sigma}$, while v is continuously differentiable near $\bar{\sigma}$.*

Proof. According to (4.7), for any $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$, any local solution $x(\sigma)$ of

MPVC (4.1) near $\bar{x}$ must be a local solution of the branch problem (4.3) for some partition $(I_1,\, I_2) \in \mathcal{I}$. According to the discussion above, there exists the unique associated Lagrange multiplier $\mu(\sigma)$, and $\mu(\sigma) \to \bar{\mu}$ as $\sigma \to \bar{\sigma}$. Substituting $(x(\sigma),\, \mu(\sigma))$ into the KKT system of the branch problem, and employing ULSCC (4.34), we obtain the relations

$$H_{I_0}(\sigma,\, x(\sigma)) = 0, \quad G_{I_{0+}}(\sigma,\, x(\sigma)) = 0,$$

assuring that $x(\sigma)$ belongs to the feasible set of the minimal branch problem. But then, again according to (4.7), $x(\sigma)$ is a local solution of the minimal branch problem. The rest follows from the discussion above applied to the minimal branch. ■

We finally provide an example demonstrating that the perturbed MPVC may have infinitely many solutions if the strong second-order sufficient condition (4.32) does not hold for some partition $(I_1,\, I_2) \in \mathcal{I}$.

Example 4.3 Let $s = n = 2$, $m = 1$, $f(\sigma,\, x) = (x_1 - x_2)^2$, $G(\sigma,\, x) = x_1 - \sigma_1$, $H(\sigma,\, x) = x_2 - \sigma_2$. This is just the parameterized version of Example 3.1, with the right-hand side perturbations of G and H.

The point $\bar{x} = 0$ is the unique solution of the unperturbed MPVC corresponding to $\sigma = \bar{\sigma} = 0$, satisfying MPVC-LICQ (2.15) and the second-order sufficient condition (4.30) with the unique associated MPVC-multiplier $\bar{\mu} = 0$. The strong second-order sufficient condition (4.32) holds for $(I_1,\, I_2) = (\emptyset,\, \{1\})$, but it does not hold for the second possible partition $(I_1,\, I_2) = (\{1\},\, \emptyset)$.

Furthermore, for each $d \in \mathbf{R}^2$, it holds that $\langle \frac{\partial \mathcal{L}}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\mu}),\, d\rangle = 0$, and for each of the two partitions $(I_1,\, I_2) \in \mathcal{I}$, (4.20) is just a feasibility problem, giving $S_{(\emptyset,\, \{1\})}(d) = \{\xi \in \mathbf{R}^2 \mid \xi_2 = d_2\}$ and $S_{(\{1\},\, \emptyset)}(d) = \{\xi \in \mathbf{R}^2 \mid \xi_2 \geq d_2,\ \xi_1 \leq d_1\}$. In the first case, problem (4.25) takes the form

$$\begin{array}{ll} \text{minimize} & 2(\xi_1 - \xi_2)^2 \\ \text{subject to} & \xi_2 = d_2, \end{array}$$

and its unique solution is $\bar{\xi} = (d_2,\, d_2)$. In the second case, problem (4.25) takes the form

$$\begin{array}{ll} \text{minimize} & 2(\xi_1 - \xi_2)^2 \\ \text{subject to} & \xi_2 \geq d_2, \quad \xi_1 \leq d_1, \end{array}$$

and its solution set is $[(d_2,\, d_2),\, (d_1,\, d_1)]$ if $d_1 > d_2$, and $\{(d_2,\, d_2)\}$ if $d_1 \leq d_2$. In both cases, the optimal value of problem problem (4.25) equals 0.

It can be easily checked that for each $\sigma \in \mathbf{R}^2$ it holds that

$$v(\sigma) = 0, \quad S(\sigma) = \begin{cases} [(\sigma_2,\, \sigma_2),\, (\sigma_1,\, \sigma_1)] & \text{if } \sigma_1 > \sigma_2, \\ (\sigma_2,\, \sigma_2) & \text{if } \sigma_1 \leq \sigma_2, \end{cases}$$

which again completely agrees with the result of Theorem 4.6.

5 A Relaxation Method

One of the approaches to solving MPCC consists in embedding the original problem into a family of perturbed problems, where the “troublesome” complementarity constraint is relaxed [18, 15]. Applying this idea to MPVC, we consider problems of the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & H_i(x) \geq 0, \quad G_i(x)H_i(x) \leq t, \ i = 1, \ldots, m, \end{array} \tag{5.1}$$

where $t \geq 0$ is the relaxation parameter. Note that, unlike in MPVC (1.1), for $t > 0$ the constraints in problem (5.1) can be expected to be regular at its solution, which makes the problem more tractable. In Example 1.1, this relaxation is actually “exact”, in the sense that if $a > 0$ then the unique solution $\bar{x} = 0$ of (1.1) is also the unique solution of (5.1) for any $t > 0$. Moreover, LICQ holds for (5.1) at $\bar{x} = 0$.

In general, of course, exactness cannot be expected and the behaviour of (5.1) as $t \to 0+$ should be investigated. Let D and $D(t)$ be the feasible sets of (1.1) and (5.1), respectively. Evidently, $D(0) = D$ and

$$D \subset D(t) \quad \forall\, t \geq 0. \tag{5.2}$$

Let $\bar{x}$ be a local solution of (1.1) and let B be a closed ball centered at $\bar{x}$, defined for the unperturbed problem (corresponding to the base parameter value $t = 0$) the same way as in Section 4. At issue is the behaviour of the relaxed problem (5.1), localized to the ball B around $\bar{x}$. To this end, for each $t \geq 0$, let $S(t)$ and $v(t)$ denote the solution set and the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & H_i(x) \geq 0, \quad G_i(x)H_i(x) \leq t, \ i = 1, \ldots, m, \quad x \in B. \end{array}$$

We are interested in estimates of how close is the solution $\bar{x}$ of the original problem (1.1) to the set $S(t)$, and how close is the optimal value $f(\bar{x})$ of (1.1) to $v(t)$. Note that since $\bar{x} \in D(t)$, it holds that

$$v(t) \leq f(\bar{x}) \quad \forall\, t \geq 0. \tag{5.3}$$

Some facts follow immediately from observations above and sensitivity results of [3, Theorems 3.1, 3.2] (we note that these facts are actually valid for any relaxation schemes). In particular, it holds that v is continuous at 0 from the right, and

$$\sup_{x \in S(t)} \operatorname{dist}(x, S(0)) \to 0 \text{ as } t \to 0+\,.$$

Moreover, if the first-order sufficient condition

$$C = \{0\}$$

is satisfied, then

$$\sup_{x \in S(t)} \|x - \bar{x}\| = O(t), \tag{5.4}$$

$$v(t) \geq f(\bar{x}) + O(t). \tag{5.5}$$

It is obvious that those (Lipschitzian) estimates cannot be improved.

In the case of MPVC specifically, further analysis gives the following.

Theorem 5.1 *Let $\bar{x}$ be a strongly stationary point of problem* (1.1), *and suppose that* (3.7) *holds.*

Then for $t \geq 0$ it holds that

$$\sup_{x \in S(t)} \|x - \bar{x}\| = O(t^{1/2}), \tag{5.6}$$

and the estimate (5.5) *is satisfied.*

Proof. By Theorem 3.4, (3.7) is equivalent to the standard second order sufficient condition (3.8). The assertions now follow from [3, Theorem 3.3], taking into account (5.3). ∎

The estimate (5.6) is sharp, as can be seen from the following example.

Example 5.1 Consider the problem from Example 3.1. Recall that $\bar{x} = 0$ is the unique solution of MPVC (1.1), MPVC-LICQ (2.15) holds at this point, $\mathcal{M} = \{0\}$, $C_2 = \{\xi \in \mathbf{R}^2 \mid \xi_2 \geq 0,\ \xi_1\xi_2 \leq 0\}$, and that (3.7) holds at $\bar{x}$.

As is readily seen, for $t \geq 0$ it holds that

$$v(t) = 0, \quad S(t) = \{x \in \mathbf{R}^2 \mid 0 \leq x_1 = x_2 \leq t^{1/2}\},$$

and hence,

$$\sup_{x \in S(t)} \|x - \bar{x}\| = t^{1/2}.$$

Note that Theorem 5.1 does not employ any CQ-type conditions. Estimate (5.6) can be improved under the assumptions of MPVC-LICQ and the second-order sufficient condition for RNLP (2.4) (recall that the latter is violated in the example above). By Theorem 2.1, in the case of MPVC-LICQ the MPVC-multiplier $\bar{\mu}$ is unique, and since the critical cones for MPVC (1.1) and for RNLP (2.4) coincide, the second-order sufficient condition for RNLP takes the form

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x},\, \bar{\mu})[\xi,\, \xi] > 0 \quad \forall\, \xi \in C \setminus \{0\}. \tag{5.7}$$

Theorem 5.2 *Assume that a local solution $\bar{x}$ of problem* (1.1) *satisfies MPVC-LICQ* (2.15), *and suppose that the second-order sufficient condition* (5.7) *is satisfied, where $\bar{\mu} = (\bar{\mu}^H,\, \bar{\mu}^G)$ is the unique MPVC-multiplier associated with $\bar{x}$.*

Then for $t \geq 0$ the estimate (5.4) *is valid.*

Proof. The assertion would follow from [5, Theorem 4.55], once we show that problem (5.1) satisfies at $\bar{x}$, for $t = 0$, the directional regularity condition of Gollan [7] in the direction $d = 1$, as well as the associated (directional) second-order condition (see [5, (4.139)]).

We start with the latter. To this end, for $t \geq 0$, $x \in \mathbf{R}^n$, $\lambda = (\lambda^G,\, \lambda^{GH}) \in \mathbf{R}^m \times \mathbf{R}^m$, let

$$L(t,\, x,\, \lambda) = f(x) - \langle \lambda^H,\, H(x) \rangle + \sum_{i=1}^{m} \lambda_i^{GH}(G_i(x)H_i(x) - t),$$

be the Lagrangian of (5.1). Note that, for $t = 0$, this function coincides with the Lagrangian of the original problem (1.1), as defined in (2.1). Hence, by Theorem 2.1 and Proposition 2.1, the set Λ of Lagrange multipliers for problem (5.1) with $t = 0$ consists precisely of $\lambda = (\lambda^G,\, \lambda^{GH}) \in \mathbf{R}^m \times \mathbf{R}^m$ that satisfy (2.11)–(2.13) for $\mu = \bar{\mu}$. Observe that

$$\frac{\partial L}{\partial t}(0,\, \bar{x},\, \lambda) = -\sum_{i=1}^{m} \lambda_i^{GH}. \tag{5.8}$$

From (2.11)–(2.13) it is evident that the maximum of (5.8) over Λ is attained at the unique point $\bar{\lambda} = (\bar{\lambda}^G,\, \bar{\lambda}^{GH}) \in \mathbf{R}^m \times \mathbf{R}^m$ corresponding to

$$\bar{\lambda}_i^{GH} = 0,\ i \in I_{00} \cup I_{0-}, \quad \bar{\lambda}_i^{GH} = \max\left\{0,\, -\frac{\bar{\mu}_i^H}{G_i(\bar{x})}\right\},\ i \in I_{0+}. \tag{5.9}$$

(The other components of $\bar{\lambda}$ are all fixed according to (2.11), (2.12).)

By Proposition 2.1 (see (2.14)), for any $\xi \in C$ it holds that

$$\begin{aligned}
\frac{\partial^2 L}{\partial x^2}(0, \bar{x},\, \bar{\lambda})[\xi,\, \xi] &= \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x},\, \bar{\mu})[\xi,\, \xi] + 2\sum_{i \in I_{+0} \cup I_0} \bar{\lambda}_i^{GH} \langle G_i'(\bar{x}),\, \xi\rangle \langle H_i'(\bar{x}),\, \xi\rangle \\
&= \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x},\, \bar{\mu})[\xi,\, \xi] + 2\sum_{i \in I_{+0} \cup I_{0+}} \bar{\lambda}_i^{GH} \langle G_i'(\bar{x}),\, \xi\rangle \langle H_i'(\bar{x}),\, \xi\rangle \\
&= \frac{\partial^2 \mathcal{L}}{\partial x^2}(\bar{x},\, \bar{\mu})[\xi,\, \xi],
\end{aligned} \tag{5.10}$$

where the second equality follows from (5.9), and the last follows from (3.3). The assumption (5.7) and the relation (5.10) now imply that

$$\frac{\partial^2 L}{\partial x^2}(0,\, \bar{x},\, \bar{\lambda})[\xi,\, \xi] > 0 \quad \forall\, \xi \in C \setminus \{0\},$$

which is the needed second-order condition from [5, (4.139)].

It remains to show that problem (5.1) satisfies for $t = 0$ the directional regularity condition in the direction $d = 1$. In this context, this means that there should exist $\bar{\xi} \in \mathbf{R}^n$ such that

$$H_{I_0}'(\bar{x})\bar{\xi} > 0, \quad G_i(\bar{x})\langle H_i'(\bar{x}),\, \bar{\xi}\rangle < 1,\ i \in I_{0+} \cup I_{0-}, \quad H_i(\bar{x})\langle G_i'(\bar{x}),\, \bar{\xi}\rangle < 1,\ i \in I_{+0}.$$

By MPVC-LICQ, one can always find $\bar{\xi}$ satisfying the first inequality above, while the other inequalities will be satisfied after the appropriate re-scaling of this $\bar{\xi}$. This completes the proof. ■

One expects the suggested relaxation approach to be useful if for $t > 0$ the constraints of problem (5.1) will be regular at its solution $x(t)$. We next establish this property, as well as boundedness of Lagrange multipliers associated with $x(t)$ as $t \to 0+$, under the assumptions of Theorem 5.2.

Theorem 5.3 *Assume that a feasible point $\bar{x}$ of problem* (1.1) *satisfies MPVC-LICQ* (2.15). *Let the sequences $\{t_k\} \subset \mathbf{R}_+ \setminus \{0\}$ and $\{x^k\} \subset \mathbf{R}^n$ be such that $\{t_k\} \to 0$,*

$$\|x^k - \bar{x}\| = o(t_k^{1/2}), \tag{5.11}$$

where for each k, x^k is a feasible point of problem (5.1) *with $t = t_k$.*

Then for each k large enough, the constraints of (5.1) *satisfy LICQ at x^k, and if x^k is a stationary point of problem* (5.1) *with $t = t_k$, then the sequence of associated uniquely defined Lagrange multipliers $\lambda^k = ((\lambda^H)^k,\, (\lambda^{GH})^k)$ is bounded, and*

$$(\lambda^H_{I_+})^k = 0, \quad (\lambda^{GH}_{I_{+-} \cup I_{00} \cup I_{0-}})^k = 0, \quad (\lambda^H_i)^k (\lambda^{GH}_i)^k = 0 \; \forall\, i \in I_{0+}. \tag{5.12}$$

Moreover, if $\bar{x}$ is a strongly stationary point of problem (1.1), *with the unique associated MPVC multiplier $\bar{\mu} = (\bar{\mu}^H,\, \bar{\mu}^G)$, then $\{\lambda^k\}$ has no more than $2^{|I_{0+}|}$ different accumulation points, and*

$$(\lambda^H_i)^k \to \bar{\mu}^H_i \; \forall\, i \in I_{00} \cup I_{0-}, \quad (\lambda^{GH}_i)^k H_i(x^k) \to \bar{\mu}^G_i \; \forall\, i \in I_{+0},$$
$$(\lambda^H_i)^k - (\lambda^{GH}_i)^k G_i(x^k) \to \bar{\mu}^H_i \; \forall\, i \in I_{0+}.$$

Proof. By the definition of the index sets involved, there exists a neighborhood of $\bar{x}$ such that for any $t > 0$, the constraints $H_{I_+}(x) \geq 0$ and $G_i(x)H_i(x) \leq t$, $i \in I_{+-} \cup I_{0-}$, cannot be active at any feasible point of problem (5.1) in this neighborhood. Furthermore, according to (5.11), for $i \in I_{00}$ it holds that

$$\begin{aligned} G_i(x^k)H_i(x^k) &= O(\|x^k - \bar{x}\|^2) \\ &= o(t_k) \\ &< t_k \end{aligned}$$

for all k large enough. Hence, the constraints $G_i(x)H_i(x) \leq t_k$, $i \in I_{00}$, are also inactive at x^k for such k. Thus, the only constraints of problem (5.1) that can be active at x^k are the following: $H_{I_0}(x) \geq 0$, $G_i(x)H_i(x) \leq t_k$, $i \in I_{+0} \cup I_{0+}$. Moreover, for $i \in I_{0+}$, the constraints $H_i(x) \geq 0$ and $G_i(x)H_i(x) \leq t_k$ cannot be active simultaneously.

Set $I_H = \{i \in I_0 \mid H_i(x^k) = 0\}$. Generally, I_H depends on k, but we may omit this dependence without loss of generality (it suffices to split the sequence $\{x^k\}$ into a finite number of subsequences, each corresponding to some fixed I_H, and to consider these subsequences separately). Now the only constraints of (5.1) that can be active at x^k are the following: $H_{I_H}(x) \geq 0$ and $G_i(x)H_i(x) \leq t_k$, $i \in I_{+0} \cup (I_{0+} \setminus I_H)$. The corresponding gradients have the form:

$$H_i'(x^k), \; i \in I_H,$$
$$G_i(x^k)H_i'(x^k) + H_i(x^k)G_i'(x^k), \; i \in I_{+0}, \quad G_i(x^k)H_i'(x^k) + H_i(x^k)G_i'(x^k), \; i \in I_{0+} \setminus I_H.$$

Again taking into account the definition of the index sets involved, we obtain that the gradients in the second group tend to $H_i(\bar{x})G_i'(\bar{x})$ as $k \to \infty$, while the gradients in the third group tend to $G_i(\bar{x})H_i'(\bar{x})$. Thus, the entire collection of gradients tends to

$$H_i'(\bar{x}), \; i \in I_H, \quad H_i(\bar{x})G_i'(\bar{x}), \; i \in I_{+0}, \quad G_i(\bar{x})H_i'(\bar{x}), \; i \in I_{0+} \setminus I_H,$$

which, after the appropriate re-scaling, is a subset of the linearly independent system in (2.15). This shows asymptotic regularity of constraints in (5.1). Boundedness of the sequence of multipliers associated to stationary points of the subproblems follows by a standard argument, while (5.12) is a consequence of inactivity of the subset of the constraints along the sequence, as established above.

Furthermore, according to the last group of equalities in (5.12), there exists an index set $I \subset I_{0+}$ such that for all k large enough

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x}(t_k,\, x^k,\, \lambda^k) \\
&= f'(x^k) - \sum_{i\in I_0} (\lambda_i^H)^k H_i'(x^k) + \sum_{i\in I_{+0}\cup I_{0+}} (\lambda_i^{GH})^k (G_i(x^k)H_i'(x^k) + H_i(x^k)G_i'(x^k)) \\
&= f'(x^k) - \sum_{i\in I_{00}\cup I_{0-}} (\lambda_i^H)^k H_i'(x^k) + \sum_{i\in I_{+0}} (\lambda_i^{GH})^k (H_i(x^k)G_i'(x^k) + G_i(x^k)H_i'(x^k)) \\
&\quad - \sum_{i\in I_{0+}} ((\lambda_i^H)^k H_i'(x^k) - (\lambda_i^{GH})^k (G_i(x^k)H_i'(x^k) + H_i(x^k)G_i'(x^k))) \\
&= f'(x^k) - \sum_{i\in I_{00}\cup I_{0-}} (\lambda_i^H)^k H_i'(x^k) + \sum_{i\in I_{+0}} (\lambda_i^{GH})^k (H_i(x^k)G_i'(x^k) + G_i(x^k)H_i'(x^k)) \\
&\quad - \sum_{i\in I} (\lambda_i^H)^k H_i'(x^k) + \sum_{i\in I_{0+}\setminus I} (\lambda_i^{GH})^k (G_i(x^k)H_i'(x^k) + H_i(x^k)G_i'(x^k)),
\end{aligned} \tag{5.13}$$

$$(\lambda^H_{I_{0+}\setminus I})^k = 0, \quad (\lambda_I^{GH})^k = 0. \tag{5.14}$$

We next split the sequence $\{\lambda^k\}$ into (no more than $2^{|I_{0+}|}$) subsequences corresponding to different I (and we split $\{t_k\}$ and $\{x^k\}$ accordingly). In the rest of the proof, we deal with one of these subsequences, without using subindices.

Similarly to the argument above, we observe that

$$H_i'(x^k) \to H_i'(\bar{x})\ \forall\, i \in I \cup I_{00} \cup I_{0-},$$

$$H_i(x^k)G_i'(x^k) + G_i(x^k)H_i'(x^k) \to H_i(\bar{x})G_i'(\bar{x})\ \forall\, i \in I_{+0},$$
$$G_i(x^k)H_i'(x^k) + H_i(x^k)G_i'(x^k) \to G_i(\bar{x})H_i'(\bar{x})\ i \in I_{0+} \setminus I,$$

where the limiting vectors are linearly independent. It then follows from the first two groups of equalities in (5.12), and from (5.13) and (5.14), that the sequence $\{\lambda^k\}$ converges to some $\bar{\lambda} = (\bar{\lambda}^H,\, \bar{\lambda}^{GH})$, and

$$f'(\bar{x}) - \sum_{i\in I_{00}\cup I_{0-}} \tilde{\mu}_i^H H_i'(\bar{x}) + \sum_{i\in I_{+0}} \tilde{\mu}_i^G G_i'(\bar{x}) - \sum_{i\in I} \tilde{\mu}_i^H H_i'(\bar{x}) - \sum_{i\in I_{0+}\setminus I} \tilde{\mu}_i^H H_i'(\bar{x}) = 0,$$

where $\tilde{\mu}^H_{I\cup I_{00}\cup I_{0-}} = \bar{\lambda}^H_{I_{00}\cup I_{0-}}$, $\tilde{\mu}_i^G = \bar{\lambda}_i^{GH} H_i(\bar{x})$, $i \in I_{+0}$, $\tilde{\mu}_i^H = -\bar{\lambda}_i^{GH} G_i(\bar{x})$, $i \in I_{0+} \setminus I$, while the rest of the components of $\tilde{\mu} = (\tilde{\mu}^H,\, \tilde{\mu}^G)$ are zero. Then $\tilde{\mu}$ satisfies (2.8) and the equalities in (2.9), (2.10). Observe finally that for all k, it holds that $\lambda^k \geq 0$ (since it is a Lagrange multiplier of an inequality-constrained problem), and hence, $\bar{\lambda} \geq 0$. This implies the nonnegativity conditions in (2.9), (2.10). Thus, $\tilde{\mu}$ is an MPVC-multiplier associated with

$\bar{x}$. And since $\bar{\mu}$ is the unique MPVC-multiplier associated with $\bar{x}$, we have that $\tilde{\mu} = \bar{\mu}$. The needed assertion is now evident. ■

We complete this section with conditions that guarantee that the solution set $S(t)$ is a singleton for all $t \geq 0$ small enough.

Theorem 5.4 *Assume that a local solution $\bar{x}$ of problem* (1.1) *satisfies MPVC-LICQ* (2.15)*, that ULSCC* (4.34) *holds at $\bar{x}$ for the unique associated MPVC-multiplier $\bar{\mu}$, and that the second-order condition* (4.35) *holds. Assume further that $I_{0+} = \emptyset$.*

Then for $t \geq 0$ small enough, $S(t)$ consists of a single element $x(t)$, which is a stationary point of problem (5.1) *with the unique associated Lagrange multiplier $\lambda(t) = (\lambda^H(t),\, \lambda^{GH}(t))$, and the mapping $(x(\cdot),\, \lambda(\cdot))$ is Lipschitz-continuous near* 0.

Proof. Under the assumptions of this theorem, Theorem 5.2 is obviously applicable and implies the estimate (5.4). Then, by Theorem 5.3, for any $t > 0$ small enough, any $x(t) \in S(t)$ must be a stationary point of problem (5.1), with the unique associated Lagrange multiplier $\lambda(t) = (\lambda^H(t),\, \lambda^{GH}(t))$, and $\lambda^H_{I_+}(t) = 0$, $\lambda^{GH}_{I_{+-} \cup I_0}(t) = 0$ (recall the assumption $I_{0+} = \emptyset$). Thus, the only constraints of problem (5.1) that can be active at $x(t)$ are the following: $H_{I_{00} \cup I_{0-}}(x) \geq 0$, $G_i(x)H_i(x) \leq t$, $i \in I_{+0}$. Hence, $x(t)$ must be a stationary point of the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & H_{I_{00} \cup I_{0-}}(x) \geq 0, \quad G_i(x)H_i(x) \leq t,\ i \in I_{+0}, \end{array} \tag{5.15}$$

with the associated Lagrange multiplier $(\lambda^H_{I_{00} \cup I_{0-}}(t),\, \lambda^{GH}_{I_{+0}}(t))$.

Furthermore, it follows from (2.8)–(2.10) and from the assumption $I_{0+} = \emptyset$ that $\bar{x}$ is a stationary point of problem (5.15) for $t = 0$, with the associated Lagrange multiplier $(\bar{\mu}^H_{I_{00} \cup I_{0-}},\, \bar{\lambda}^{GH}_{I_{+0}})$, where $\bar{\lambda}^{GH}_i = \bar{\mu}^G_i / H_i(\bar{x})$, $i \in I_{+0}$. Moreover, MPVC-LICQ (2.15) implies the usual LICQ for the constraints of problem (5.15) at $\bar{x}$, while ULSCC (4.34) ensures that (4.35) is the usual strong second-order sufficient condition for problem (5.15) at $\bar{x}$. Then according to [5, Proposition 5.38], $(\bar{x},\, ((\bar{\mu}^H_{I_{00} \cup I_{0-}},\, \bar{\lambda}^{GH}_{I_{+0}})))$ is a strongly regular solution of the KKT system of problem (5.15) for $t = 0$, and according to [5, Theorem 5.13], for each $t > 0$ small enough, $x(t)$ is the unique stationary point of problem (5.15) near $\bar{x}$, while $(\lambda^H_{I_{00} \cup I_{0-}}(t),\, \lambda^{GH}_{I_{+0}}(t))$ is the unique Lagrange multiplier associated with $x(t)$. The same result ensures that the mapping $(x(\cdot),\, (\lambda^H_{I_{00} \cup I_{0-}}(\cdot),\, \lambda^{GH}_{I_{+0}}(\cdot)))$ is Lipschitz-continuous near 0. ■

Employing the last statement of Theorem 5.3, the following extension of Theorem 5.4 can be proved. Let the assumptions of the latter hold, but with the condition $I_{0+} = \emptyset$ replaced by $\bar{\mu}^H_i \neq 0\ \forall\, i \in I_{0+}$. Then for $t \geq 0$ small enough, $S(t)$ consists of no more than $2^{|I_{0+}|}$ elements, each of which is a stationary point of problem (5.1) with the unique associated Lagrange multiplier.

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