Mathematical Structures in Computer Science

Incomputability in Physics and Biology†

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Abstract

Computability has its origins in Logic within the framework formed along the original path laid down by the founding fathers of the modern foundational analysis for Mathematics (Frege and Hilbert). This theoretical itinerary, which was largely focused on Logic and Arithmetic, departed in principle from the renewed relations between Geometry and Physics occurring at the time. In particular, the key issue of physical measurement, as our only access to 'reality', played no part in its theoretical framework. This is in stark contrast to the position in Physics, where the role of measurement has been a core theoretical and epistemological issue since Poincaré, Planck and Einstein. Furthermore, measurement is intimately related to unpredictability, (in-)determinism and the relationship with physical space–time. Computability, despite having exact access to its own discrete data type, provides a unique tool for the investigation of 'unpredictability' in both Physics and Biology through its fine-grained analysis of undecidability – note that unpredictability coincides with physical randomness in both classical and quantum frames. Moreover, it now turns out that an understanding of randomness in Physics and Biology is a key component of the intelligibility of Nature. In this paper, we will discuss a few results following along this line of thought.

(Received October 01 2010)
(Revised July 01 2011)

Footnotes
† This paper is an extended and revised version of Longo (2010a), which was an Invited Lecture presented at the Computability in Europe conference held in the Azores in June 2010.

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Abstract. Computability originated from Logic within the frame of the original path proposed by the founding fathers of the modern foundational analysis in Mathematics (Frege and Hilbert). This theoretical itinerary, which largely focused in Logic and Arithmetic, departed in principle from the contemporary renewed relations between Geometry and Physics. In particular, the key issue of physical measurement, as our only access to "reality", is not part of its theoretical frame, in contrast to Physics. In this discipline, the role of measurement is a core theoretical and epistemological issue, since Poincaré, Planck and Einstein, and it is strictly related to unpredictability, (in-)determination and the relation to physical space-time. Computability, in spite of the exact access to its own discrete data type, provides a unique tool for the investigation of "unpredictability", both in Physics and Biology, by its fine analysis of undecidability. We note that unpredictability coincides with physical randomness, in classical and quantum frames. And today, in Physics and Biology, an understanding of randomness turns out to be a key component of intelligibility of Nature. In this paper, a few results along this line of thought will be discussed[1].

1   The issue of physical measurement

Before the crisis of the foundations in Mathematics, that is before the invention of non-euclidean Geometries, Mathematics was secured by the direct link between Euclid's geometry and physical space. Certainty was in the relation between our space of human action and Euclid's constructions, by ruler and compass, in particular once these were embedded by Descartes in abstract spaces and turned into an absolute by Newton. And a theorem proved at our scale could be transferred to the Stars or to Democritus atoms: the sum of internal angles of a triangle, say, would always be 180^0. But Riemann claimed that the relevant space manifolds, where "the cohesive forces among physical bodies could be related to the metrics" [Riemann 1854], were not closed under homotheties (this is Klein's version, to be precise). Thus, the invariance of scale (by homotheties), extending our proofs by ruler and compass to Stars and Atoms, was lost. Finally, with Einstein, Euclid's spaces turned out to be an irrelevant singularity, corresponding to curvature 0, or, at most, a local approximation in the novel geometrization

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[1] Extended and revised version of an Invited Lecture, Proceedings of **Computability in Europe**, Azores, Pt, June 2010, LNCS 6158 (Ferreira et al. eds.), Springer, 2010.

of Physics. In short, the physically crucial meaning of curved spaces demolished the presumed absolute correspondence between intuition, grounded on action in human space, and physical space-time (a survey of the epistemological challenges is in [Boi 1995]). This was a true "epistemological breakings", in Bachelard's terms, [Bachelard 1940] (or a "revolution", [Khun 1962]), as it stimulated a new foundational approach.

Frege's and Hilbert's response to this major crisis, though different as regards to meaning and existence, agreed in giving a central role to Arithmetics, with its "absolute laws", away from the loss of certitudes of the non-Euclidean turn as regards the intuitive relation to space. Logic or formal systems should categorically (for Frege, if we put it in modern terms) or completely (for Hilbert) allow mathematics to be reconstructed.

This "royal way out" in the foundation of Mathematics lead it to depart from the relation to physical space and, thus, to Physics. In particular, it proposed a foundational culture, in Mathematics, programmatically disregarding the constitutive interaction to physics, at the core of Riemann's geometry, and, thus, neglecting our forms of "access to the world", by measurement, in space and time. Mathematics had to be founded away from our forms of life, from action and space, on pure logic or on formal computations over meaningless strings of signs[2].

As a matter of fact, we "access" to physical processes only by measurement, from cognitive-perceptive measurements as relations to our environment to the very refined tools of Quantum measurement. And an entire arithmetizing community passed by a key aspect of the revolution that happened in physics at the turn of the XXth century: the novel relevance of physical measurement in understanding Nature. When turning back to natural phenomena, some project on them these arithmetic-computational views and their discrete structures of determination, as I will explain below.

Poincaré first understood, by his Three Body Theorem (1890), that the intrinsically approximated measurement of the initial conditions, jointly with the non-linearity of the mathematical description (and gravitational "resonances"), led to unpredictable, though deterministic, continuous dynamics. The Geometry of Dynamical Systems was born and classical randomness since then became understood as deterministic unpredictability (see below). Einstein's Relativity allowed knwoledge construction only when reference systems and their measure-invariant properties are explicitly given, and this following a revolutionary correlation between space and time measurements. Quantum Mechanics, in yet a different way, also brought the process of measurement to the limelight. It intro-

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[2] Mathematics had to be found independently of "wildest visions of delirium" proposed by the interpretations of non-Euclidean theories (in Frege's words: [Frege 1884], p. 20) and towards axiomatic systems for Geometry that had to contain no reference to intuition nor meaning in space (see [Hilbert 1899] and Poincaré's critique of it in [Poincaré 1903] as well as his critique of Frege's and Russell's approach, see [Goldfarb 1986]). Hermann Weyl similarly interpreted and criticized formalism and logicism, from a geometric perspective, [Weyl 1918], [Weyl 1927], [Weyl 1985].

duced an intrinsic randomness as soon as Schrödinger's deterministic dynamics of the state function, a probability density in Hilbert spaces, is made accessible (is measured in our space and time). And Planck's h gives a lower bound for the joint measurement of conjugated variables.

Shortly later, the arithmetizing culture in foundations, explicitly born against Riemann's and Poincaré's physicalization of Geometry (and geometrization of physics), produced remarkable formal-arithmetic machines. Computability was invented within Logic, in the '30s, for purely foundational purposes, by Herbrand, Gödel, Church, Kleene . . . . And Turing's Logical Computing Machine introduced a further – metaphysical – split from the world: the perfect (Cartesian) dualism of the distinction between hardware and software, based on a remarkable theoretical (and later practical) separation between the "logic" of a process and its physical realization. Turing's idea was the description of a human in the "least act of computing", or actually of thought. Wasn't Frege's and even Hilbert's project also an analysis of general human reasoning? Weren't neural spikes, the most visible traces of complex critical transitions of electrostatic potentials, quickly considered as some sort of $0, 1$ in the brain?

The physical realization of the Machine preserved Turing's fundamental split: its soft "soul" could act on discrete data types, with no problem as for access and independently from the hardware. No need of physically approximated measurement nor random information: its data bases are exactly given. Digital processes are exact, evolve away from the world, over a very artificial device, a most remarkable human alphanumeric invention: a discrete state physical process, capable of iterating identically whatever program it is given and, by this, reliable. Nothing like this existed before in the world. So rarely Nature iterates exactly (perhaps a few chemical processes, *in vitro* – we will go back to this).

I insist that the mathematical foundation of the Machine is in its exactness over discrete data types and, thus, in the reliable and identically iteratable interaction between hardware and software. It is given by the arithmetical certainty postulated by Frege and Hilbert within Logic, away from the fuzzy measurements of classical/relativistic dynamics, from the randomness of Quantum Mechanics. Networks of concurrent computers – distributed in physical space/time – are challenging today this original view.

2 Preliminaries: from equational determination to incompleteness.

In a short note of 2001, I suggested that Poincaré's three-body theorem may be considered an epistemological predecessor of Gödel's undecidability result[3], by understanding Hilbert's completeness conjecture as a meta-mathematical revival of Laplace's idea of the predictability of formally (equationally) determined systems. For Laplace, once the equations are given, one can completely derive the future states of affairs (with some – preserved – approximation). Or, more

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[3] This is more extensively discussed in [Longo 2009], [Longo 2010]

precisely, in "Le système du monde", he claims that the mathematical mechanics of moving particles, one by one, two by two, three by three ... compositionally and *completely* "covers", or makes understandable, the entire Universe. And, as for celestial bodies, by this progressive mathematical integration ... "We should be able to deduce all facts of astronomy", sayd he.

The challenge, for a closer comparison, is that Hilbert was speaking about completeness of formal systems as deducibility or *decidability* of purely mathematical *"yes or no"* questions, while unpredictability shows up in the relation between a physical system and a mathematical set of equations (or evolution function).

In order to consistently relate unpredictability to undecidability, one needs to effectivize the dynamical spaces and measure theory (typically, by a computable Lebesgue measure), the loci for dynamic randomness. This allows one to have a sound and purely mathematical treatment of the epistemological issue (and obtain a convincing correspondence between unpredictability and undecidability, see the next section).

Remark 1. On Proof Methods. Poincaré's and Gödel's theorems share also a methodological aspect: they both (and independently, of course) destroy the conjecture of predictability (Laplace) and decidability (Hilbert) *from inside.* Poincaré does not need to refer concretely to a physical process that would not be predictable, by measuring it "before and after". He shows, by a pure analysis of the equations, that the resulting bifurcations and "homoclinic intersections" (intersection points of stable and unstable manifolds or trajectories) lead to deterministic unpredictability (of course, the equations are derived in reference to three bodies in their gravitational fields, similarly as Peano Axioms are invented in reference to the ordered structure of numbers). Similarly, in the *statements* and *proofs* in his 1931 paper, Gödel formally constructs an undecidable sentence, by playing the purely syntactic game, with no reference whatsoever to "semantics", "truth" or suchlike, that is to the underlying mathematical structure.

Modern "concrete incompleteness" theorems (that is, Girard's normalization, Paris-Harrington or Friedman-Kruskal theorems, see [Longo 2002] and [Longo 2010] for references and discussions) resemble instead Laskar's results of the '90s [Laskar 1994], where "concrete unpredictability" is shown for the *Solar system*, as intended physical system, in reference to the *best possible astronomical measurements*. Similarly, concrete incompleteness was given by proving (unprovability and) truth over the *(standard) model*, thus comparing formal syntax and the intended mathematical structure.

Philosophically, the incompleteness of our formal (and equational) approaches to knowledge is a general and fundamental epistemological issue. It motivates our permanent need for new science: by inventing new *contentual* – i. e. meaningful – principles and conceptual constructions, we change directions, propose new intelligibilities and meaningfully grasp or organize new fragments of the world. There is no such thing as "the final solution to the foundational problem" in mathematics (as Hilbert dreamed – a true nightmare), nor in other sciences.

As a preliminary hint to biology, note that the "incompleteness" of the molecular theories for understanding life phenomena is a comparable issue. There is no way to completely understand/derive embryogenesis nor phylogenesis (Evolution) by looking only at the four letters of the bases of DNA (the formal language of molecular biology) – in spite of the claim of too many biologists (see [Monod 1973], [Maynard 1989] for two classics and [Fox Keller 2000] for a critical survey). The massive control and retro-action of the cell and the organism on DNA expression and subsequent molecular cascades is increasingly acknowledged also in molecular biology. Thus, the analysis of the global structure of the cell (and the organism) must parallel the absolutely crucial molecular analyses. The hard philosophical point to explain, to our colleagues in molecular biology, is that "incomplete" does not mean useless, but that we need also a (still missing) autonomous theory of the organism and further develop the Darwinian theory of Evolution.

By the way, randomness plays a key role in Evolution, but also in embryogenesis. But . . . what kind of randomness? Physics, classical/quantum, proposes two distinct notions of randomness . . . . Can logical undecidability (incomputability) help us in understanding this? We will focus on these questions at the end of this paper.

3   Randomness vs. undecidability

As mentioned above, classical (physical) randomness is unpredictability of deterministic systems in finite time. For example, dice trajectories are theoretically determined: they follow the Hamiltonian, thus a unique geodetics; yet, they are so sensitive to initial and contour conditions that it is not worth writing the equations of movement. Now, algorithmic randomness, that is Martin-Löf's (and Chaitin's) number-theoretic randomness, is defined for infinite sequences, [Calude 2002]. How may this then yield a connection between Poincaré's unpredictability and Gödel's undecidability?

Classical physical randomness is deterministic unpredictability. Thus, it manifests itself at the interface "equations/process" and shows up at finite time (that is, after a finite number of iterations of the dynamics, if this is represented, as usual, by the technique of Poincaré's sections). Yet, also this physical randomness may be expressed as a limit or asymptotic notion and, by this, it may be soundly turned into a purely mathematical issue: this is Birkhoff's ergodicity (for any observable, limit time averages coincide with space averages: an equality of two infinite sums or integrals, see [Petersen 1983]). And this sense applies in (weakly chaotic) dynamical systems, within the frame of Poincaré's geometry of dynamical systems.

As for algorithmic randomness, Martin-Löf randomness is a "Gödelian" notion of randomness, since it is based on recursion theory and yields a strong form of undecidability for infinite 0-1 sequences (in short, a sequence is random if it passes all *effective statistical tests*; as a consequence, it contains no infinite r.e. subsequences). Recently, under Galatolo's and my supervision, M. Hoyrup

and C. Rojas proved that dynamic randomness (a la Poincaré, thus, but at the purely mathematical limit, in the ergodic sense), in suitable *effectively given* measurable dynamical systems, is equivalent to (a generalization of) Martin-Löf randomness (Schnorr's randomness). This is a non-obvious result, based also on a collaboration with P. Gacs, developed in those two parallel doctoral dissertations (defended in June 2008) [Gacs et al. 2009][4].

As for quantum randomness now, note that, because of entanglement, it mathematically differs from classical: if two classical dice interact and then separate, the probabilistic analysis of their values are independent. When two quanta interact and form a "system", they can no longer be separated: measurements on them give correlated probabilities of the results. Mathematically, they violate Bell's inequalities, see [Bailly et al. 2007] for a comparative introduction.

Algorithmic randomness provides a close analysis of classical randomness: how can this mathematics of discrete structures tell us more about randomness, in general?

4   Discrete vs. continua

One of the ideas extensively developed in a book, [Bailly et al. 2011], and in several papers with Francis Bailly[5] and Thierry Paul, two physicists (see our downloadable papers), is that the mathematical structures, constructed for the intelligibility of physical phenomena, according to their continuous (mostly in physics) or discrete nature (generally in computing), may propose different understandings of Nature.

In other words, in physics, the "causal relations", as mathematical determinations and structures of intelligibility (we "understand Nature" by them), are usually given by (differential) equations or evolution functions. Their physical meaning is dependent on the use of the continuum or the discrete and may deeply differ according to the choice of one of this mathematical frames. For example, in most non-linear systems, discrete approximations soon diverge from continuous evolutions and do not provide actual "models" of the intended physical processes. In a few cases, such as hyperbolic systems, "shadowing theorems" garanty at most that continuous evolutions approximate discrete ones (not the converse!), see [Pilyugin 1999]. In modern terms, continua or discrete underlying mathematical structures induce different symmetries and symmetry-breakings ([Bailly et al. 2011], chapters 4 and 5, [Longo et al. 2011]).

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[4] Undecidability over dynamical systems has been allready investigated and proved to yield a form of unpredictability – the undecidability, say, of a point to cross or not a given region of the phase space, see [Moore 1990], [da Costa 1991] among others. Here, we compare an independent notion of physical unpredictability, as classical randomness a la Birkhoff, to algorithmic randomness, as a strong form of gödelian undecidability.

[5] The enlightening collaboration with Francis, a physicist also interested in biology, has been fundamental for me. Francis recently passed away: a recorded Colloquium in his memory may be accessed from my web page, http://www.di.ens.fr/users/longo.

But what discrete (mathematical) structures are we referring to? There is one clear mathematical definition of "discrete": a structure is *discrete* when the discrete topology on it is "natural". Of course, this is not a formal definition, but in mathematics we all know what "natural w. r. to the intended purposes" means. For example, one can endow Cantor's real line with the discrete topology, but this is not "natural" (you do not do much with it), while, the integer numbers or a digital data base are naturally endowed with the discrete topology. Even though one may have good reasons to work with the latter also under a different structuring, in mathematical physics the interval topology on the reals has the "naturality" of deriving from the *interval* (approximated) nature of measurement. And this induces the discrete topology on the subset of integers or on any finite set of approximating numbers.

Church's thesis, introduced in the 1930s after the functional equivalence proofs of various formal systems for computability, concerns only computability over integers or discrete data types. As such, it is an extremely robust thesis: it ensures that any sufficiently expressive *finitistic formal system* over integers (a Hilbertian-type logico-formal system) computes exactly the recursive functions, as defined by Herbrand, Gödel, Kleene, Church, Turing . . . . This thesis therefore emerged within the context of mathematical logic. It is a thesis grounded on *languages* and *formal systems* for arithmetic computations, on discrete data types, programmatically invented by the founding fathers away from physics and its space-time.

The very first question to ask is the following: what happens if we extend *just* the formal framework? If we want to refer to continuous (differentiable) mathematical structures, the extension to consider is to the computable real numbers, see [Pour-El et al. 1989]. Are the various formalisms for computability over real numbers equivalent? An affirmative answer could suggest an extension of Church thesis to computability on "continua". Of course, the computable reals are countably many, but they are dense in the "natural" (interval) topology over Cantor's reals. As we shall see below, this yields a crucial difference.

Posing this question, we get closer to current physics, since it is within spatial and often also temporal continuity that we represent dynamical systems, that is, most mathematical models for (classical) physics. This does not imply that the World is continuous, but only that, since Newton and Leibniz, we better understand large parts of physics, such as space-time and movement in it, by continuous tools, as later very well specified by Cantor (but his continuum is not the only possible one: Lawvere and Bell, say, proposed a topos-theoretic continuum, without points, [Bell 1998]).

Now, as for this equivalence of formalisms, which is at the heart of Church's thesis, there remains very little when passing to computability over real numbers: the theories proposed are demonstrably different in terms of computational expressiveness (the classes of defined functions). The various systems (recursive analysis, first developed by Lacombe and Grezgorzcyk, in 1955-57; the Blum, Shub and Smale, BSS, system; the Moore-type recursive real functions; different forms of "analog" systems, such as threshold neurones, the GPAC . . .) yield

different classes of "continuous" computable functions. Some recent work established links, reductions between the various systems (more precisely: pairwise relations between subsystems and/or extensions). Yet, the full equivalence as in the discrete case is lost. Moreover, these systems have no "universal function" in Turing's sense. On the discrete, this function is constructed by a computable isomorphism between spaces of different dimension, that is $N^2 = N$. Instead, there is no continuous and computable isomorphism between the computable reals R_c and R_c^2, which would allow to transfer the notion of Universal Machine to computability over continua (see [Costa 2009] for more).

As a consequence of the $N^2 = N$ computable isomorphism, in computability on the discrete, the work spaces may be of any finite dimension: they are all effectively isomorphic or the "Cartesian dimension" does not matter! This is highly unsuitable for physics. First, dimensional analysis is a fundamental tool (one cannot confuse energy with force, nor with the square of energy ...). Second, dimension is a topological invariant, in all space manifolds for classical and relativistic physics. That is, take physical measurement, an interval, as a "natural" starting point for the metric (thus the interval or "real" topology), then you can *prove* that, if two such spaces have isomorphic open subsets, they have the same dimension. The topological invariance of dimension, on physically meaningful topologies, is a very simple, yet beautiful correspondence between mathematics and physics.

These facts weaken the computational approaches to the analysis of *physical* invariants, over continua, as two fundamental *computational* invariants are lost: equivalence (that is, the grounds for Church Thesis) and universality.

In summary, in discrete computability, a cloud of isolated points has no dimension, per se, and, for all theoretical purposes, one may encode them on a line. When you have dimension back, in computability over continua, where the trace of the interval topology maintains good physical properties, you lose the universal function and the equivalence of systems. Between the theoretical world of discrete computability and physico-mathematical continua there is a huge gap. While I believe that one should do better than Cantor as for continua, I would not give a penny for a physical theory whose dynamics take place only on discrete spaces, departing from physical measurement, dimensional analysis and the general relevance of dimensions in physics (again, from heat propagation to mean field theory, to relativity theory ... space dimension is crucial). The analysis over "computable continua" provides a more interesting frame for physics, adds relevant information, but loses two key invariants of computations over the discrete (typically, the Universal Function). We will discuss below the peculiar "discrete" nature of Quantum Mechanics.

Remark 2. On Physical Constants. By the way, are the main physical constants, G, c, h, *computable* (real numbers)? Of course, it depends on the choice of the reference system and the metrics. So, fix $h = 1$. Then, you have to renormalize all metrics and re-calculate, by equations, dimensional analyses *and* physical measurement, G and c. But physical measurement will always give an interval, as we said, or, in quantum frame, the probability of a value. If one interprets

the classical measurement interval as a Cantorian continuum (the best way, so far, to grasp fluctuations), then, for the believers in the absolute existence of the real numbers R, ... where are G and c? Computable reals form a dense subset of Lebesgues measure 0, with plenty of gaps and no jumps. Why should (the mathematical understanding of) fluctuations avoid falling in gaps and jump from computable real to computable real? Cristian Calude and I conjecture instead that random (thus highly incomputable) reals are a better structure of intelligibility for non-observable events.

Yet, the most striking mistake of many "computationalists" is to say: but, if "physics is not fully computable", then, some physical processes would super-compute (that is, "compute" non-computable functions)! No, this is not the point. Most physical processes, simply do not *define* a mathematical function. And the challenge, again, is due to our only form of access to the (physical) "world": *measurement.* In order to force a classical process to define a function, you have to fix a time for input, associate a (rational) number to the interval of measurement and ... let the process go. Then you wait for the output time and measure again. In order, for the process, to define the mathematical function $f(x) = y$, at the rational input x it must *always* associate a rational output y, over and over again, at each iteration of the process on input x. But restart now on x your physical double pendulum, say (a simple deterministic and chaotic process), that is restart it within the interval of measurement corresponding to x. Then a minor (thermal, typically) fluctuation, *below the interval defined by x*, will yield a different observable result y', even if you fixed a very short time of "computation". Of course, for processes that are modeled by non-linear dynamics, taking intervals as input and output does not solve the physical problem. Non-linear maps (or the solutions of non-linear equations, if any) may enlarge the interval exponentially (following the Lyapounov exponents, [Cencini 2010]) and are *mixing*, that is the extremes and the maxima/minima of the intervals are shuffled around. As numerical analysts know very well, this makes even fail the rarely applicable "shadowing theorems" (which at least guaranty that the discrete dymanics is approximated by the continuous one, but not conversely – as recalled above, see [Pilyugin 1999]).

So, a good question would be, instead: consider a physical process that does define a function, is this function computable?

The idea then is that the process should be sufficiently insensitive to initial conditions (some say: robust) as to actually define a function. But, then the question radically changes (and becomes trivial). Typically, one should be able to partition the World in little cubes of the smallest size, according to the best measurement as for insensitivity (fluctuations below that measurement do not affect the dynamics). *If* the Accessible World is considered finite[6], then one can make a finite list out of the input-output relation established by the given

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[6] But ... according to which *measure* it would be finite? What about Riemann's sphere as a model for the Relativistic Universe, and the squeezing to 0 little human moving with his/her meter towards the poles? For God, who holds it in His hand, that Universe is a finite, but it is infinite for his/her measurement.

process. *This is a "program"*. But then a good programming question would be: *is this program compressible?*

Remark 3. On Quanta and Discrete Space-Time. As for the relevance of the discrete, quantum mechanics started exactly by the discovery of a key (and unexpected) discretization of energy absorption or emission spectra of atoms. Then, a few dared to propose a discrete lower bound to measurement of *action*, that is of the product (energy × time). It is this physical dimension that bares a discrete structure. Clearly, one can then compute, by assuming the relativistic maximum for the speed of light, a Planck's length and time. But in no way are space and time thus discretized in small "quantum boxes or cubes". And this is the most striking and crucial feature of quantum mechanics: the "systemic" or entanglement effects, which yield non-separability of observables. In this context, no discrete space topology is natural as it would yield a separability of all (measurable) observables. That is, these quantum effects in space-time are the opposite of a discrete, thus well separated organization of space, while being at the core of its scientific originality. In particular, they motivate quantum computing (as well as our analysis of quantum randomness above). As a matter of fact, Thierry Paul and I claim that the belief in an absolutely separable topology of space continua is Einstein's mistake in EPR [Einstein et al. 1935], where entanglement is first examined and considered impossible (ongoing work, see [Longo 2010] for some further analysis).

Note finaly that loop gravitation and string theory do in fact assert that our world might be composed from (a very large number of) finite objects with discrete relations. However, these objects and their dynamical relations, crucially, are handled in abstract mathematical spaces, such as Hilbert's or Fock's spaces, possibly of infinite dimensions. In those spaces, far away from "ordinary" space-time, processes may even be represented by linear, thus fully computable, mathematical dynamics, such as given by Schrödinger equation. The problem then, and as usual, is to "bring back" these dynamics to our space-time, by measurement, as this is where intrinsic indetermination pops out.

In summary, continua, Cantorian or topos theoretic, take care rather well (but they are not an absolute) of the approximated nature of (classical) physical measurement, which is represented as an interval: the unknowable fluctuation is within the interval. In Quantum Physics the peculiar correlation of conjugated variables and intrinsic indetermination gives to measurement an even more important role. And (physical) measurement is our only form to access "reality". The arithmetizing foundation of mathematics went along another (and very fruitful) direction, based on perfectly accessible data types.

5   The originality of the Discrete State Machine

As I mentioned above, the Discrete State Alphanumeric Machine that compute is a remarkable and very original human invention, based on a long history. As hinted in [Longo 2008], this story begins with the invention of the alphabet,

probably the oldest experience of discretization. The continuous song of speech, instead of being captured by the design of concepts and ideas (by recalling "meaning", like in ideograms), is discretized by annotating phonetic pitches, an amazing idea (the people of Altham, in Mesopotamia, 3300 B.C.). Meaning is reconstructed by the sound, which acts as a compiler, either loud or in silence (but only after the IV century A.D. we learned to read "within the head"!).

The other key passage towards alphanumeric discretization is the invention of a discrete coding structure. This originated with Gödel-numbering, an obvious practice now, but another as remarkable as artificial idea. Turing's work followed: the Logical Computing Machine (LCM), as he first called it, is at the core of computing (right/left, $0, 1 \ldots$). Of course, between the alphabet and Turing, you also have Descartes "discretization' of thought (stepwise reasoning, along a discrete chain of intuitive certitudes . . .) and much more.

When, in the late '40s, Turing works again in physics, he changes the name to his LCM: in [Turing 1950] and [Turing 1952], he refers to it as *Discrete State* Machine (this is what matters as for its physical behavior). And twice in his 1950 paper (the "imitation game"), he calls it "Laplacian". Its evolution is theoretically predictable, even if there may be practical unpredictability (too long programs to be grasped, says he).

So, by using ideas from formal Logic, we invented and later physically realized an incredibly stable processor, which, by working on discrete data types, does what it is expected to do. And it iterates, very faithfully, I insist, this is its key feature. Primitive recursion and portability of software are forms of iterability: iteration and update of a register, do what you are supposed to do, respectively, even in slightly different contexts, over and over again. For example, program the evolution function of the most chaotic strange attractor you know. Push "restart": the digital evolution, by starting on the same initial digits, will follow exactly the same trajectory. This makes no physical sense, but it is very useful (also in meteorology: you may restart your turbulence, exactly, and try to better understand how it evolves . . .). Of course, one may imitate unpredictability by some pseudo-random generator or by . . . true physical randomness, added ad hoc. But this is cheating the observer, in the same way Turing's *imitation* of a woman's brain is meant to cheat the observer, not to "model" the brain. He says this explicitly, all the while working, in his 1952 paper, at a *model* of morphogenesis, as (non-)linear dynamics. The brain activity, says he, may depend on fluctuations below measurement, not his DSM (see [Longo 2008] for a closer analysis and, of course, Turing's two papers, which should always be read simultaneously). In contrast to imitation, a mathematical model tries to propose a "structure of determination" (for example, the equations for action, reaction, diffusion in the 1952 paper). Observe, finally, that our colleagues in networks and concurrency are so good that also programming in concurrent networks is reliable: programs do what they are supposed to do, they iterate and . . . give you the web page you want, identically, one thousands time, one million times. And this is hard, as physical space-time, which we better understand by continua and continuous approximations, steps in, yet still on discrete data types, and

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this allows iteration. Of course, identical iteration is the opposite of randomness (many define a process to be random when, iterated on the "same" – as for physical measurement – initial conditions, it follows a different evolution).

Those who claim that the Universe is a big digital computer, miss the originality of this machine of ours, its history, from the alphabet to Hilbert's formal systems, to the current work in concurrency in networks and its reliability as key objective. When we construct computers, we make a remarkable achievement, by producing a reliable, thus programmable, physical, but artificial device, far away from the natural world, iterating as we wish and any time we wish, even in networks. One should not miss the principles that guided this invention, as well as the principles by which we understand physical dynamics, since Poincaré. The very rich relations of computing to the (physical) world, to those dynamics in particular, is a non trivial issue, far away from the flat transfer of techniques or identification of models.

6 The relevance of negative results

Even without considering the Universe a big computer, some still claim that the "Laws of Physics" are computable. This is hard to (dis-)prove though, as I have never seen the Complete Table of the Laws of Physics, nor their enumeration algorithm.

What should be analyzed is the effectiveness of our *mathematical writing* of physical invariants. Of course, equations, evolution functions . . . are given by sums, products, exponents, derivations, integrations . . . all computable operations. No one is so crazy as to put an incomputable real as a coefficient or exponent in an equation (even if h could be so . . .). This gives us remarkable approximations and, most often, qualitative information: Poincaré's geometry of dynamical systems or Hadamard's analysis of the geodetic flow on hyperbolic surfaces, do not give predictions, but very relevant global information (by attractors, for example, or regularities in flows . . . that we beautifully see today, as never before, by fantastic approximations, "shadowed" on our computers' screens). Yet, we can write two equations that "model", in the best way we know of, the non-linear dynamics of a double pendulum and even compute a computable solution. Too bad though that that solution does not follow the actual physical dynamics: you measure its initial conditions and a fluctuation below your best measurement let the pendulum go along a completely different trajectory than the one you rush to compute.

Moreover, Pour-El, Richards, Baverman, Yampolsky . . . were able to find unique (or only) non computable solutions of effective equational writing, [Pour-El et al. 1989], [Braverman et al. 2006], [Weihrauch 2002]. At the end, these relevant results reduce the question to the halting problem, that is to the *diagonal writing* of sequences of digits. Computing is "a man provided with paper, pencil, and rubber . . . " says Turing. Computing is not in the World. It is an alpha-numeric extraordinary invention of ours, based on written language, a form of re-writing made possible by the very abstract and dualistic nature (no

meaning in signs) of the alphabet (classical Chinese computers are geometric devices). Both the alphabet and computing are, respectively, a very rough approximation of the expressivity of the continuous song of language and of the physical world.

As I said, it is too hard for us to isolate Absolute Laws of Nature: our human insight is provided by the constructive theorizing on the phenomenal veil, at the interface between us and the World. These active constructions are effective (we understand a lot and transform the World, not always for the best) and mostly computable (we use the alphabet, computable operations and codings, I insist).

Yet, *predictable processes are not many in Nature*: you can compute the date of a few forthcoming Eclipses, at human time scale, but the Solar System is chaotic in astronomical times, as Poincaré proved and Laskar quantified by computing an upper bound to predictability.

Note that unpredictable processes are the mathematical and computational challenge, as a computable physical process is, by definition, deterministic and predictable. In order to predict (pre-dicere, "to say in advance" in Latin), just "say" or write the corresponding program and compute in advance. Thus, the results mentioned above, by showing the equivalence of unpredictability and (strong) undecidability, ML-randomness, prove this equivalence, by logical duality. Unpredictability may pop-out in computers' networks and this because of *physical* space-time, as observed above: we then make them computable and predictable-reliable by forcing semaphores, handling interleaving . . . In Nature, many (most, fortunately) processes escape predictions, thus our computations. Fortunately, otherwise there would be no change, nor life in particular: randomness is crucial. And when we compute unpredictable evolutions, we just approximate their initial paths, as I said, or give qualitative information, both very relevant tasks. Thus the fine analysis of unpredictability and randomness is an essential component of scientific knowledge. Moreover, by "saying no" to strong programs (Laplace, Hilbert), unpredictability and undecidability started new science: modern Dynamics and Mathematical Logic.

Computability is as artificial (and, thus, as useful) as the alphabet, as mentioned above. By formalizing what we can effectively say (compute/predicere), it gives some *detached* symbolic images of the World and, in particular, the best way (the only one?) we have to describe what we *cannot* say (in advance). As already recalled, computability was invented by proving incomputability. This is why I insist that a peculiar and relevant role, in the relation between physics (and biology, see later) and mathematics, as a form of meaningful "writing" about the World, must be given to Incomputability (and randomness).

As a matter of fact, the only mathematical way I know to define randomness, in classical dynamics, is Birkhoff's ergodicity. But it is very specific (certain dynamics). Otherwise, randomness is given in terms of probability measure. But this is unsatisfactory, as probability gives a *measure* of randomness, not a definition. It is the theory of algorithms, that, asymptotically, gave a fully general, mathematical, notion of randomness, as a strong form of incomputability, independently of probability theory. Again, physical (classical) randomness

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is deterministic unpredictability and, by the bridging results above and more in the literature, the role of computational randomness further comes to the limelight. In particular, it provides a very flexible theory of randomness: you can adjust the class of effective randomness tests (Martin-Löf, Schnorr ... and many more). A wild conjecture is that this may help to better grasp, for example, the mathematical difference between classical and quantum randomness.

7 Randomness, Entropy and Anti-entropy in Biology

7.1 Embryogenesis and ontogenesis.

In biology, randomness is even more relevant than in physics. About 50% of conceptions in mammals fail (do not reach birth): a very bad performance for the DNA as a program. This is due to the fact that, in *no reasonable sense*, the DNA is a "program". While iterability, as reliability, is at the core of software (and hardware) design, the key principle for understanding life – at the phenotypic level – is *variability*, a form of non-iterability, joint to "structural stability" along unexpected change, a very different matter. In Evolution as well as in ontogenesis, a cell is never identical to the mother cell and this is crucial to life. So, the principles of intelligibility are the exact opposite: variability and failures correspond to the crucial *possibility* that a random mutant better fits a changing environment.

Of course, some molecular processes iterate, in particular in vitro, but there is an increasing tendency to analyses molecular cascades in terms of stochastic phenomena, in particular in eukaryotic cells, see [Kaern et al. 2005], [Raj et al. 2008] (and this is where also good computational approaches may help to understand, by stochastic networks interactions, see for example [Krivine et al. 2008]). It has been even said that "the DNA is a random generator of proteins, regulated by the cell, the organism and the environment", [Kupiec 2009]. This is an extreme, yet empirically motivated reaction to the too prolonged domination of the "one gene - one protein" hypothesis and Crick's 1958 "central dogma" (on the unidirectional, linear determination from DNA to RNA to proteins and, then, to the phenotype). These imposed until early this decade a laplacian frame for molecular biology (the DNA is a program paradigm).

I consider that the complexity of life processes is also in the blend of conceptually opposite aspects. The central dogma is almost always false, but a few molecular cascades may actually follow Crick's dogma: the colors in plants, apparently, are predictably and uniquely determined by a fragment of DNA, a gene in the classical sense – a very rare fact. And some laplacian molecular cascades can be reproduced in vitro or observed in bacteria (very rarely in eukaryote cells). Yet, large parts of DNA or RNA statistically interact in a non linear way with and in a turbulent context, in the presence of quasi-chaotic enthalpic oscillations of huge molecules, in particular in the cytoplasm of eukaryotes.

Whatever is the validity or level of the blend that I suggest, these new views on randomness open the way to an increasing role of epigenetics and, thus, to

the relevance of downwards regulating effects, from the cell and the organism to DNA expression.

Random effects persist during life. In particular, the recent Darwinian perspective in cancerogenesis proposes *growth* as "default state" of all cells [Sonnenschein et al. 1999]. This largely random proliferation is usually controlled by the various regulating activities of the organism on cells: cancer would then be mostly due to the failure of this control and/or of the exchanges between cells in a tissue, generally in presence of external carcinogenic factors.

7.2 Evolution.

By a remarkable analysis spanning many articles and two books [Gould 1989], [Gould 1998], S. J. Gould stressed the role of randomness in Evolution. In particular, we – primates – are a random complexification in a bacterial world, along a contingent diffusive path. The expansion of life, "punctuated" by sudden explosions and massive extinctions of species, preserved a few invariants while constantly changing organisms and their ecosystems. In order to set these remarks and the associated paleontological evidence on mathematical grounds, we proposed the notion of anti-entropy, as formalized biological complexity (a qualitative evaluation of cellular, functional and phenotypical differentiation), see [Bailly et al. 2009]. I briefly survey some aspects of that technical paper, as an application of the role of randomness in Evolution.

Anti-entropy formally opposes entropy as a new observable (it has the opposite sign), in the same way that anti-matter opposes matter as proper to new (observable) particles in Quantum Mechanics. Anti-entropy is a form of (increasing) information in embryogenesis and Evolution. Organism become more "complex", from bacteria to eukaryotes to multicellular organisms, and this is the result of an asymmetric diffusion of biomass over anti-entropy, along random paths. Anti-entropy locally opposes (increasing) entropy proper to all irreversible processes, by the (increasing) structuring of organisms, both in embryogenesis and Evolution.

In order to compare shortly anti-entropy to information, observe that, traditionally, Shannon's information is considered as negentropy (Brillouin). Then, by definition:
- the sum of a quantity of information (negentropy) and an equal quantity of entropy gives 0;
- information (Shannon, but also Kolmogorov) is "insensitive to coding" (it is also an analysis of coding, but one can "encrypt" and "decrypt" as much as one wishes and the information content will not be lost/gained, in principle).

Note that by this, we passed now from Turing's *elaboration* of information (Computability Theory), also insensitive to coding and dimension, to the analysis of information (Shannon, Brillouin . . .). This notion, of which the applications are of an immense relevance, for machines' elaboration and transmission, is not sufficient for an investigation of the living state of matter. DNA, usually considered as digital information, is the most important component of the cell, as

I said, but it is also necessary to analyze the *organization* of the living objects, as an observable specific to biological theorization. Without a proper theory of the organism and its genesis, comparable to the remarkable one we have of Evolution, we may get stuck in the current situation, where there is no general theoretical frame relating genotype to phenotype (only very long lists of, mosty differential, correlations and only a few direct, positive ones, from the wild gene to the phenotype (see [Fox Keller 2000], [Longo et al. 2007)]). In short, modulo a few exceptions, we have no idea on how the discrete chemical structures of the DNA and of other active macromolecules contribute to the construction of biological "forms", in general. For sure, randomness and constraints (including deformations, torsions, relative geometric positions . . .), regulation and integration as well as timing . . . have a radically different role from the one, if any, they may have in programs which generate forms from digits, in machines.

In our attempts, anti-entropy, as biological complexity, may be understood as "information specific to the form", including the intertwining and enwrapping of levels of organization, at the core of the autonomy of life. Anti-entropy yields a *strict* extension, in a logical sense, of the thermodynamics of entropy (it extends some balance equations, see [Bailly et al. 2009]). It is compatible with information as negentropy, but it differs from it. First, the production of entropy and that of anti-entropy are summed in an "extended critical singularity" ([Bailly et al. 2008], [Longo et al. 2011]), an organism, never to 0, in contrast to Brillouin's and others' negentropy. Second, as it is linked to spatial forms, anti-entropy is "*sensitive to coding*", contrarily to digital information (it depends on the dimensions of embedding manifolds, on folds, on fractal structures, on singularities . . .).

Its use in metabolic balance equations has produced a certain number of results. We have, in particular, proposed a diffusion equation of biomass over anti-entropy, following Schrödinger's "operational method" in Quantum Mechanics. This has enabled to operate a mathematical reconstruction of this diffusion, which fits Gould's curve describing phenotypic complexity along the evolution of species. As mentioned above, we could then mathematically describe the random complexification during Evolution, evidenced by Gould, by an asymmetric diffusion equation. The original asymmetry (Gould's "*left* wall" of least complexity, that is the formation of bacteria, a critical transition from the inert to the living state of matter) propagates as *right* average bias along *random* evolutive paths. And by purely local effects, as in any asymmetric random diffusion, biological complexity, qualitatively described by anti-entropy, grows along Evolution with no goal-directedness nor program whatsoever.

7.3 Computability in bio-chemical cycles

In [Rosen 1991], a stimulating investigation of incomputability at the molecular level is hinted. By an abstract analysis of metabolic cycles and a refined distinction between mechanical simulation and modeling, the claim is made that some auto-referential steps would lead to incomputability (or non mechanizability). Unfortunately, the technical sections on this matter are flawed by notational

ambiguities and (crucial) misprints. By proposing a *possible* interpretation of Rosen's equations, in [Mossio et al. 2009], we gave a λ-calculus, thus computable, fixed point solution to these equations. Other interpretations and, of course, other, possibly incomputable, solutions can be given. Yet, a computable (possibly optimal in Scott's domains) may be obtained. Unfortunately, wrong ideas for too long spammed a lively "rosenian" debate. Some (see [Mossio et al. 2009] for references) claimed that "life is not computable", *because* Rosen's equations are:

1. leading to divergence (while computable functions should always be total);
2. circular (self-referential);
3. impredicative;
4. set-theoretically non-well-founded.

Now, in the computablity community, we know very well that:

1. This branch of mathematics is born by proving the existence of "(relevant) *partial* computable functions that cannot be extended to total ones" (they are intrinsically diverging: Gödel's and Turing's work on incompleteness and halting, respectively).
2. *Circularity* is at the core of recursion; type-free λ-calculus, in particular, is based on self-reference (such as $f(f) = f$ equations: an incomputable miracle, for some rosenian ... with a one-line computable solution in λ-calculus). Reflexive domain equations ... and λ-calculus fixed point constructions, both rich forms of circularity, may give interesting (and useful) non-normalizing terms. And this may correspond to the correct intuition that formal (computable) metabolic "cycles" are not supposed to stop.
3. *Impredicativity* is an integral part of Girard's Type Theory, which computes exactly the recursive functions that are provably total in II order Arithmetic (see [Asperti et al. 1991] for models of λ-calculus and impredicative Type Theory).
4. Models of *anti-foundation* axioms (non-well-founded sets), can be (relatively) interpreted in constructive models of type-free λ-calculus (and conversely – see [Mossio et al. 2009] for a discussion and references).

(In-)computability in biology is a delicate issue, also related to computer simulation and Artificial Life. Equational writings and their solutions are generally computable, as discussed above, with very hard ("construed") counterexamples. These contrexamples are always proved by reducing the problem to Turing's halting, a pure "diagonal" game of signs. Once more, computability and its opposite are a (very relevant) alpha-numeric linguistic construction, they are not in the World. As for incomputability, I would say, by a metaphor, that it stays to Natural Sciences in the same way as it relates to Cantor's real numbers: these are "almost all" non-computable (they are actually all ML-random, except a set of measure 0), yet it is extremely hard, *by our mathematical writing* to describe one: Turing's example and Chaitin's random number Ω are the only examples given so far – with infinitely many variants, of course. In Nature, we can *informally* point out many incomputable (or unpredictable) processes, but it has

been very difficult to single them out formally. Poincaré had to give meaning to the *absence* of analytic solutions of certain equations – an unusual step in mathematical physics. The computational difficulty is that, not only our writing is effective, but also, I insist, that we invented computability and its machines as alpha-numeric *(re-)writing systems*, of which λ-calculus is a paradigm. That is, when we write (and re-write) we obtain computable structures. And in order to depart from them and formally provide an example of incomputable object, we can only, so far, diagonalize or reduce to some diagonalization process. The first examples were invented by Gödel and Turing and the reduction to the second of them was applied by a few (from Chaitin to Pour-El, Braverman and collaborators).

Yet, as a form of unpredictability, incomputability in biology should be analyzed, well beyond classical randomness, that is besides the strong incomputability of deterministic unpredictability or quantum randomness. Unpredictability and incomputability may certainly depend on non-linear and network interactions, but also on an understanding of organisms as *specific* (contingent) autonomous systems in a changing ecosystem, as proposed by many, including Rosen. Equationally determined objects are instead *generic* (mathematically invariant), a crucial difference, see [Bailly et al. 2011]. Moreover, "resonance" effects may take place between different levels of organization, an analysis which does not belong, so far, to the mathematical analysis inherited from physics (since Poincaré we understood the importance of gravitational resonance, a critical non-linear interaction, as for deterministic unpredictability at one level of organization: a planetary system, say). An organism contains networks of cells in tissues as part of organs subject to morphogenesis, integrated and regulated within an organism by hormones, neural and immunitary system .... Thus, proper biological randomness is a further mathematical challenge, yet to be explored, a component of the difficult issue of the "mathematization" of biology, see [Buiatti et al. 2012].

8 Conclusion

In this paper, we have synthetically compared physical (dynamic) randomness to algorithmic randomness (at the center of *algorithmic theories of information*). This was a way to discuss incomputability in physics. While hinting to randomness in Evolution, anti-entropy has been mentioned (a "*geometrical* extension" of the notion of information) as an observable that we added in thermodynamic balance equations. This allows to mathematize the globally random complexification of life (any diffusion is grounded on random paths). A further, ongoing, analysis of some aspects of the stable/unstable, far from equilibrium, dissipative state of living matter is based on a theory of "extended criticality", a notion that mathematically extends the point-wise critical transitions in physics.

The scientific finality of this work, whose conceptual frame is presented in [Bailly et al. 2011], may also entail some epistemological consequences.

First, by science, we grasp, at most and approximatively, some relevant, but changing fragments of the World; relevant for us, from our perspective of randomly complexified bacteria, up to the neotenic monkeys we are, with free hands and a relatively too big brains, in constant historical evolution. The understanding that physical and biological processes do not coincide with formal computations, and that our symbolic writings cannot even represent them faithfully and completey, constitutes a constant request towards new forms of knowledge. Our (mathematical) descriptions are not absolute and they are "reasonably effective" exactly because they derive from a concrete friction between our evolutive (and historical) being and the World.

The close analysis of incomputability or of unpredictability, by negative results, in search for qualitative analysis, for an understanding of limits and, thus, for new perspectives, is part of this quest. If, one day, molecular biology will *prove from inside*, like Poincaré's and Gödel's analyses did in mathematics and Logic, that the sole analysis of the DNA doesn't allow to predict ... the shape of the ear, then I would say that something theoretically very original and relevant has happened. Instead, some still claim to have found the gene of marital fidelity (Young et al., *Nature*, 400, 766-788, 1999).

Second, this view should participate to the epistemological debate regarding the notion of information, the updating of its theoretical principles, as part of the many existing interactions with physics and biology. Information, both as a concept and as a Theory, in the current sense (of negentropy, say), seems largely inadequate to express biological dynamics. And a possible outcome of these interactions, concerning the notion of information, could be to start thinking to ... the next machine, along another path w. r. to the one explored by Quantum Computing. I bet that this nice DSM of ours is not the "final machine" that Mankind has invented, as computationalists seem to claim when considering the World identical to it or a faithful image of it or its logic.

As for biology in particular, "structural stability", as non-identical iteration of a morphogenetic process, as well as the role of contingency in phylogenesis and ontogenesis radically depart from these views. And randomness, at the core of life's contingency, seems to depend also on quantum effects, which are increasingly witnessed in cells, and on non-linear interactions, amplified by (metabolic) circularities, but also on "resonance" effects between different levels of organization, both within the cytoplasm and the organism's integration-regulation system. But this is ongoing work, see [Buiatti et al. 2012].

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