Mathematics for Computer Graphics

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Recommended Reading

- John Vince 2005. Mathematics for Computer Graphics SpringerÂVerlag London
- Comninos, P. 2005. Mathematical and Computer Programming Techniques for Computer Graphics (Hardcover). Springer.
- Harris J.W. and Stocker H. 1998 Handbook of Mathematics and Computational Science. Springer.

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- Harris J.W. and Stocker H. 1998 Handbook of Mathematics and Computational Science. Springer.
- Lipschutz, S. 1982. Schaum's Outline of Essential Computer Mathematics (Paperback).'s Outline Series.

Syllabus

- Algebra
- Number Theory
- Boolean Algebra
- Circuit Diagrams
- Thrigonometry
- Vectors
- Matrices
- Analytic Geometry
- 2D Transformations
- Set Theory

Algebra

- One fundamental concept of algebra is the idea of giving a name to an unknown quantity.
- Rene Descartes (1596-1650) formalized the idea of using letters from the beginning of the alphabet ($a, b, c,$ etc.) to represent arbitrary numbers, and letters at the end of the alphabet ($p, q, r, s, t, ..x, y, z$) to identify variables representing quantities such as pressure (p), temperature (t), and coordinates (x, y, z).

Algebra

- With the aid of the arithmetic operators we can develop expressions that describe the behavior of a specific computation.
- For example $ax + by - d = 0$ represents a straight line.
- The variables x and y are the coordinates of any point on the line and the values of a, b, d determine the position and orientation of the line.
- There is an implied multiplication between ax and by which would be expressed as $a * x$ and $b * y$ using a programming language.
- The $=$ sign permits the line equation to be expressed as a self-evident statement
- Such a statement implies that the expressions on the left-hand and right-hand sides of the $=$ sign are **equal** or **balanced**.

Algebraic Expressions

- Algebraic expressions also contain a wide variety of other notation, such as
  $\sqrt{x}$ square root of x;
  x^n x to the power on n
  $\sin\alpha$ sine of α
- For example $ax + by - d = 0$ represents a straight line.
- Parentheses are used to isolate part of an expression in order to select a sub-expression that is manipulated in a particular way. For example, the parentheses in $c(a + b) + d$ ensure that the variables a and b are added together before being multiplied by c and finally added to d.

Algebraic Laws

- There are three basic laws for manipulating algebraic expressions:
    - associative
    - commutative
    - distributive
- In the following descriptions, the term binary operation represents the arithmetic operations $+$, $-$ or $\times$, which are always associated with a pair of numbers or variables.

Associative Law

- The $associative$ law in algebra states that when three or more elements are linked together through a binary operation, the result is independent of how each pair of elements is grouped.

- Addition:
  $a + (b + c) = (a + b) + c$ $\quad\quad 5 + (2 + 7) = (5 + 2) + 7$

- Multiplication:
  $a \times (b \times c) = (a \times b) \times c$ $\quad\quad 5 \times (2 \times 7) = (5 \times 2) \times 7$

- Subtration is not associative:
  $a - (b - c) \neq (a - b) - c$ $\quad\quad 9 - (7 - 3) \neq (9 - 7) - 3$

- What about the division, is it associative?

Commutative Law

- The $commutative$ law in algebra states that when two elements are linked through some binary operation, the result is independent of the order of the elements.
- Addition:
  $a + b = b + a$ $\qquad$ $(2 + 7) = (7 + 2)$
- Multiplication:
  $a \times b = b \times a$ $\qquad$ $5 \times 9 = 9 \times 5$
- Subtraction is not commutative:
  $a - -b- \neq -b - a$ $\qquad$ $15 - 6 \neq 6 - 15$
- What about the division, is it commutative?

Distributive Law

- The $distributive$ law in algebra describes an operation which when performed on a combination of elements is the same as performing the operation on the individual elements.

- Multiplication over addition holds:
  $a \times (b + c) = (a \times b) + (a \times c)$ $\quad$ $5 \times (2 + 7) = (5 \times 2) + 5 \times 7$

- Multiplication over subtraction holds:
  $a \times (b - c) = (a \times b) - (a \times c)$ $\quad$ $5 \times (7 - 4) = (5 \times 7) - 5 \times 4$

- Addition over multiip;ication does not hold:
  $a + (b \times c) \neq (a + b) \times (a + c)$ $\quad$ $5 + (8 \times 2) \neq (5 + 8) \times (5 + 2)$

Powers(1)

- Indices (or powers, or exponents) are very useful in mathematics. Indices are a convenient way of writing multiplications that have many repeated terms.

- Example 7^3 means *multiply 7 by itself 3 times*
  $7^3 = 7 \times 7 \times 7$
  $$x^n = \underbrace{x \times x \times ... \times x}_{\text{n times}}$$

- Example 6^{-4} means *multiply* $\frac{1}{6}$ *by itself 4 times*
  $$6^{-3} = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$$
  $$x^{-n} = \underbrace{\frac{1}{x} \times \frac{1}{x} \times ... \times \frac{1}{x}}_{\text{n times}}$$

Powers(2)

- Any number a, $a \neq 0$ raised to the power of 1 is a
  $a^1 = a$

- Any number a, $a \neq 0$ raised to the power of -1 is $\frac{1}{a}$
  $a^{-1} = \frac{1}{a}$

- Multiplying numbers with the same Base: $x^m \times x^n = x^{m+n}$

- Dividing numbers with the same Base: $\frac{x^m}{x^n} = x^{m-n}$

Powers(2)

- Raising a power expression to a power:

  $a^{mn} = a^{m \times n} = a^{mn}$

- Raising a power expression to a negative power:

  $a^{-m} = \dfrac{1}{a^m}$

- Raising a product to a power: $(a \times b)^m = a^m \times b^m$

- Raising a fraction to a power: $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

- The is no formula for adding variables raised to power: $a^m + b^n$

Powers(3)

- Square root is equivalent to raising a number to power $\frac{1}{2}$

  $\sqrt{x} = x^{\frac{1}{2}}$ $\qquad$ $\sqrt{49} = 49^{\frac{1}{2}} = 7$

- Cubic root:

  $\sqrt[3]{x} = x^{\frac{1}{3}}$ $\qquad$ $\sqrt[3]{216} = 216^{\frac{1}{3}} = 6$

- n-root is equivalent to raising a number to power $\frac{1}{n}$

  $\sqrt[n]{a} = a^{\frac{1}{n}}$

- Proot properties
  - $\sqrt[n]{a \times b} = a^{\frac{1}{n}} \times b^{\frac{1}{n}}$
  - $\sqrt[n]{a + b} \neq a^{\frac{1}{n}} + b^{\frac{1}{n}}$