# Mathematics for Neuroscientists

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## Chapter 27.

**Exercise 2.** We compute

$$\begin{aligned}E(j+1)-E(j) &= -(\mathbf{s}^j)^T\mathbf{W}\mathbf{s}^{j+1}+(\mathbf{s}^{j-1})^T\mathbf{W}\mathbf{s}^j+\mathbf{b}^T(\mathbf{s}^{j+1}-\mathbf{s}^{j-1}) \\ &= -(\mathbf{s}^{j+1})^T\mathbf{W}^T\mathbf{s}^j+(\mathbf{s}^{j-1})^T\mathbf{W}\mathbf{s}^j+\mathbf{b}^T(\mathbf{s}^{j+1}-\mathbf{s}^{j-1}) \\ &= -(\mathbf{s}^{j+1}-\mathbf{s}^{j-1})^T(\mathbf{W}\mathbf{s}^j-\mathbf{b})\end{aligned}$$

as claimed. Now, if $\mathbf{s}^{j+1}\neq\mathbf{s}^{j-1}$ then $s_i^{j+1}=-s_i^{j-1}$ for some $i$. There are only two possibilities. If $s_i^{j+1}=1$ then $(\mathbf{W}\mathbf{s}^j)_i>b_i$ and so

$$(\mathbf{s}^{j+1}-\mathbf{s}^{j-1})_i(\mathbf{W}\mathbf{s}^j)_i=2(\mathbf{W}\mathbf{s}^j-\mathbf{b})_i>0,$$

while if $s_i^{j+1}=-1$ then $(\mathbf{W}\mathbf{s}^j)_i<b_i$ and so, again,

$$(\mathbf{s}^{j+1}-\mathbf{s}^{j-1})_i(\mathbf{W}\mathbf{s}^j)_i=-2(\mathbf{W}\mathbf{s}^j-\mathbf{b})_i>0.$$

Summing over $i$ we find that if $\mathbf{s}^{j+1}\neq\mathbf{s}^{j-1}$ then $\Delta E<0$. Hence, if there exists a period greater than 2, i.e., if $\mathbf{s}^{j-1}=\mathbf{s}^{j+1+k}\neq\mathbf{s}^{j+1}$ for some $k>0$ then $E(j)=E(j+2+k)$, is contradiction of $E(j+2+k)<E(j)$.

**Exercise 3.** If we integrate Eq. (27.2) on a small interval, say $(P-\varepsilon,P+\varepsilon)$, about the first input spike we find

$$\tau_E(g_{E,1}(P+\varepsilon)-g_{E,1}(P-\varepsilon))=-\int_{P-\varepsilon}^{P+\varepsilon}g_{E,1}(t)\,dt+w_{inp}. \tag{1}$$

Prior to the first spike we expect $g_{E,1}$ to remain at its initial value and so $g_{E,1}(P-\varepsilon))=0$ regardless of $\varepsilon$. Now, as $\varepsilon\to 0$ in Eq. (1) the integral vanishes and we arrive at $\tau_E g_{E,1}(P^+)=w_{inp}$ as claimed. This provides the initial data for $g_{E,1}$ from $t=P$ up to (but not including) $t=2P$. In that interval $g_{E,1}$ obeys $\tau_E g'_{E,1}(t)=-g_{E,1}(t)$ and so, proceeding as in §3.1 we find

$$g_{E,1}(t)=\exp((P-t)/\tau_E)w_{inp}/\tau_E \quad P\le t<2P \tag{2}$$

and so $g_{E,1}(2P^-)=\exp(-P/\tau_E)w_{inp}/\tau_E$. If we next integrate Eq. (27.2) across $t=2P$ we find $\tau_E(g_{E,1}(2P^+)-g_{E,1}(2P-))=w_{inp}$, and so conclude that $g_{E,1}(2P^+)=g_{E,1}(2P-)+w_{inp}/\tau_E=(1+\exp(-P/\tau_E))w_{inp}/\tau_E$ as claimed. Arguing as above we find

$$\begin{aligned}g_{E,1}(t) &= g_{E,1}(2P^+)\exp((2P-t)/\tau_E) \\ &= \exp((P-t)/\tau_E)(1+\exp(P/\tau_E))w_{inp}/\tau_E, \quad 2P\le t<3P\end{aligned} \tag{3}$$

as claimed. Upon comparing Eqs. (2) and (3) we deduce that

$$g_{E,1}(t)=\exp((P-t)/\tau_E)(w_{inp}/\tau_E)\sum_{m=0}^{n-2}\exp(P/\tau_E)^m, \quad (n-1)P\le t<nP. \tag{4}$$

We recognize this sum as a finite geometric series and find

$$\sum_{m=0}^{n-2}\exp(P/\tau_E)^m=\frac{1-\exp((n-1)P/\tau_E)}{1-\exp(P/\tau_E)}.$$

On inserting this back into Eq. (4) we arrive at the desired Eq. (27.41).

**Exercise 8.** If we define $V(\theta) \equiv W(\theta) \star f(\theta) - U(\theta)$ then the convolution theorem and linearity yield $\hat{V}_n = \hat{W}_n \hat{f}_n - \hat{U}_n$. This, together with Parseval's equality yields

$$E(W) = \sum_{n=-\infty}^{\infty} |\hat{V}_n|^2 + \lambda |\hat{W}_n|^2 = \sum_{n=-\infty}^{\infty} |\hat{W}_n \hat{f}_n - \hat{U}_n|^2 + \lambda |\hat{W}_n|^2, \tag{5}$$

as claimed. Now, for $n$ fixed, we choose $\hat{W}_n$ as the minimizer of

$$E_n(W) \equiv |W \hat{f}_n - \hat{U}_n|^2 + \lambda |W|^2.$$

We write $W = a + ib$ and expand the squared magnitudes and find

$$E_n(W) = E_n(a,b) = (|\hat{f}_n|^2 + \lambda)(a^2 + b^2) - (a + ib)\hat{U}_n^* \hat{f}_n - (a - ib)\hat{U}_n \hat{f}_n^*.$$

It remains to determine its critical point. We compute

$$\partial_a E_n = 2a(|\hat{f}_n|^2 + \lambda) - (\hat{U}_n^* \hat{f}_n + \hat{U}_n \hat{f}_n^*) \quad \text{and} \quad \partial_b E_n = 2b(|\hat{f}_n|^2 + \lambda) - i(\hat{U}_n^* \hat{f}_n - \hat{U}_n \hat{f}_n^*).$$

These vanish when

$$a = \frac{\hat{U}_n^* \hat{f}_n + \hat{U}_n \hat{f}_n^*}{2(|\hat{f}_n|^2 + \lambda)} = \frac{\Re(\hat{f}_n^* \hat{U}_n)}{|\hat{f}_n|^2 + \lambda} \quad \text{and} \quad b = \frac{\hat{U}_n^* \hat{f}_n - \hat{U}_n \hat{f}_n^*}{2(|\hat{f}_n|^2 + \lambda)} = \frac{\Im(\hat{f}_n^* \hat{U}_n)}{|\hat{f}_n|^2 + \lambda}.$$

On combining these as $W = a + ib$ we recover Eq. (27.33).

**Exercise 10.** (i) We proceed from the definition

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(-x) e^{-inx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} \, dx = \hat{f}_{-n}.$$

(ii) We note that $U(-\theta) = \sigma^{-1}(f(-\theta)) = \sigma^{-1}(f(\theta)) = U(\theta)$.
(iii) The result follows from $\hat{f}_n^* = \hat{f}_{-n}$.
(iv) We compute

$$\begin{aligned} W'(\theta) &= \lim_{h \to 0} \frac{W(\theta + h) - W(\theta)}{h} \\ &= \lim_{h \to 0} \frac{W(-\theta - h) - W(-\theta)}{h} \\ &= -\lim_{h \to 0} \frac{W(-\theta - h) - W(-\theta)}{-h} = -W'(-\theta). \end{aligned}$$