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Mathematics of Dispersive Water Waves

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Abstract. A commuting hierarchy of dispersive water wave equations makes a three-Hamiltonian system which belongs to a general class of nonstandard integrable systems whose theory is developed. The modified water wave hierarchy is a bi-Hamiltonian system; its modification bifurcates. The water wave hierarchy, and the hierarchies of the Korteweg-de Vries and the modified Korteweg-de Vries equations, as well as the classical Miura map, are given new representations through various specializations of nonstandard systems.

1. Introduction

The current theory of integrable systems grew out of the analysis of the Korteweg-de Vries equation. Another, even more remarkable water wave equation, is the subject of this paper.

The classical dispersiveless long wave equations

$$u_t + uu_x + h_x = 0,\ h_t + (uh)_x = 0,\ u = u(x,t),\ h = h(x,t), \tag{1.1}$$

have a number of dispersive generalizations (see, e.g., review [1]). In this paper we shall be concerned with the following version

$$\begin{cases} u_t = (u^2/2 + h + \beta u_x)_x, \\ h_t = (uh + \alpha u_{xx} - \beta h_x)_x; \end{cases} \tag{1.2}$$

here α and β are arbitrary constants, and the sign of t is changed to make forthcoming formulae more natural. The invertible change of variables: $u = \bar{u}$, $h = \bar{h} + \gamma \bar{u}_x$, turns (1.2) into

$$\begin{cases} \bar{u}_t = (\bar{u}^2/2 + \bar{h} + \mu \bar{u}_x)_x, \\ \bar{h}_t = (\bar{u}\bar{h} - \mu \bar{h}_x)_x, \qquad \mu = \gamma + \beta = \pm\sqrt{\alpha + \beta^2}. \end{cases} \tag{1.3}$$

If μ does not vanish, it can be made into an arbitrary constant, say $\mu = \pm 1/2$, by rescaling x and t. For $\alpha = 1/3$, $\beta = 0$, the system (1.2) was derived by Broer [1] who called it "The oldest, simplest and most widely known set of equations...",

"...which are the Boussinesq equations proper." The same system (1.2), for $\beta=0$ and in terms of the potential $\varphi: u=\varphi_x$, was derived by Kaup [2] who also found its multi-soliton solutions and the following scattering problem: $\psi_{xx}+(\lambda^2+\lambda Q+R)\psi=0$, where Q and R are obtained by an invertible change of variables from u and h. Later Matveev and Yavor [3] found algebro-geometrically a large class of almost periodic solutions "...of some hydrodynamical equations derived recently by D. Kaup."

We shall see that the system (1.3) is connected with a large variety of ideas in the theory of integrable systems, which are important in their own right. Briefly, this system is the richest integrable system known to date. In this paper we establish the basic properties of the system (1.3) and the corresponding *hierarchy* by making use of the theory of nonstandard integrable systems [Eq. (2.3) below] developed in Sect. 2; the most important results are:

Theorem 2.0. *Consider the hierarchy*

$$L_t=[((P^\dagger)_{\geqq 1})^\dagger, L]=[-((P^\dagger)_{\leqq 0})^\dagger, L], \tag{1.4}$$

$$L=\xi+u+h\xi^{-1}, \tag{1.5}$$

$$P\in Z(L), \tag{1.6}$$

where $\xi=d/dx$; $Z(L)$ *is the centralizer of* L *in the ring of pseudo-differential operators* $C((\xi^{-1}))$ ([4, 5]), $C=C_{u,h}=\mathbb{C}[u^{(i)}, h^{(i)}]$ *is the corresponding differential algebra; "†" stands for the "adjoint"; for an element* $Q\in C((\xi^{-1}))$, $Q=\sum_{i<\infty} q_i\xi^i$, *we denote* $\operatorname{Res}Q=q_{-1}$, $Q_{\geqq k}=\sum_{i\geqq k} q_i\xi^i$, *and analogously for* $Q_{>k}$, $Q_{<k}$, $Q_{\leqq k}$. *Then:*

(i) *The hierarchy* (1.4) *has a common infinite set of conservation laws* $H_m=\frac{1}{m}\operatorname{Res}L^m$; (ii) *All these flows commute.*

Since the centralizer $Z(L)$ of L is a linear combination of $\{L^n | n\in\mathbb{Z}\}$, we can take $P=\frac{1}{2}L^2$ in (1.4).

The resulting system is readily found to be

$$\begin{cases} u_t=\partial(u^2+2h-u_x)/2, \\ h_t=\partial(2uh+h_x)/2, \end{cases} \qquad \partial=\partial/\partial x, \tag{1.7}$$

that is, (1.3) with $\mu=-1/2$.

Theorem 3.0. *The hierarchy* (1.4) *is a three-Hamiltonian system:* (i) *for* $P=L^m$ *it can be written in the form*

$$\begin{pmatrix} u \\ h \end{pmatrix}_t = B^{\mathrm{I}}\delta H_{m+1}=B^{\mathrm{II}}\delta H_m=B^{\mathrm{III}}\delta H_{m-1}, \tag{1.8}$$

where $\delta H=\begin{pmatrix} \delta H/\delta u \\ \delta H/\delta h \end{pmatrix}$ *is the vector of variational derivatives, and*

$$B^{\mathrm{I}}=\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \tag{1.9}$$

$$B^{\mathrm{II}}=\begin{pmatrix} 2\partial & \partial u-\partial^2 \\ u\partial+\partial^2 & h\partial+\partial h \end{pmatrix}, \tag{1.10}$$

$$B^{\mathrm{III}}=\begin{pmatrix} 2(u\partial+\partial u) & 2(h\partial+\partial h)+\partial(u-\partial)^2 \\ 2(h\partial+\partial h)+(u+\partial)^2\partial & (u+\partial)(h\partial+\partial h)+(h\partial+\partial h)(u-\partial)\end{pmatrix}. \tag{1.11}$$

(ii) *All matrices* B^{I}, B^{II}, B^{III} *are Hamiltonian.*

Theorem 4.0. (i) *The system* (1.4) *with* $P=L^m$, *and for* $-u$ *instead of* u, *has also the following Lax representation*

$$\mathscr{L}_t=[\mathscr{P}_{\geqq 0},\mathscr{L}]=[-\mathscr{P}_{<0},\mathscr{L}], \tag{1.12}$$

where $\mathscr{P}=\mathscr{L}^m$ *and*

$$\mathscr{L}=\xi+\sum_{i\geqq 0}\xi^{-i-1}hQ_i(u), \tag{1.13}$$

where

$$Q_i(u)=(\partial+u)^i(1). \tag{1.14}$$

(ii) *The conservation laws* $H_m=\frac{1}{m}\operatorname{Res}L^m$ *and* $\mathscr{H}_m=\frac{1}{m}\operatorname{Res}\mathscr{L}^m$ *coincide.*

Theorem 5.0. *The Korteweg-de Vries* (KdV) *hierarchy*

$$\ell_t=[\tilde{P}_{\geqq 0},\ell]=[-\tilde{P}_{<0},\ell],\ \ell=\xi^2+h,\ \tilde{P}=(\ell^{1/2})^{2m+1}, \qquad m\in\mathbb{Z}_+,$$

can be imbedded into the hierarchy (1.4) *when* u *vanishes and* $P=L^{2m+1}$. *In other words, set*

$$\bar{L}=\xi+h\xi^{-1} \tag{1.15}$$

and take $\bar{P}=\bar{L}^{2m+1}$. *Then:* (i) *The system* $\bar{L}_t=[((\bar{P}^{\dagger})_{\geqq 1})^{\dagger},\bar{L}]$ *is the* m^{th} KdV *flow;* (ii) *The Hamiltonian structures* B^{I} (1.9) *and* B^{III} (1.11) *can be properly restricted into the first and the second Hamiltonian structure, respectively, of the* KdV *hierarchy.*

Theorems 2.0–5.0 are proved below in Sects. 2–5 respectively, based on a theory of nonstandard integrable systems developed in Sect. 2. Details, explanations, and generalizations can be found in the course of the paper. Elementary facts about differential Lax equations are assumed to be known (see [4, 5]). Here I explain the method, devised in [6] for discrete Lax equations, used to prove Theorem 3.0. As an example, let us take the KdV hierarchy $\ell_t=[\ell,\tilde{P}_{<0}]$, with $\ell=\xi^2+h$, $\tilde{P}=(\ell^{1/2})^{2m+1}$. Set $H_m=m^{-1}\operatorname{Res}(\ell^{1/2})^m$, $(\ell^{1/2})^m=\sum_s p_s(m)\xi^s$. Then the m^{th} KdV equation becomes $h_t=2\partial p_{-1}(2m+1)$. On the other hand, recall that the basic formula of the Residue calculus in modules of differential forms over rings of pseudo-differential operators [5] is

$$\frac{1}{m+n}d\operatorname{Res}X^{m+n}\sim\frac{1}{n}\operatorname{Res}(X^m dX^n)\sim\frac{1}{n}\operatorname{Res}(dX^n\cdot X^m), \qquad m,n\in\mathbb{Z}, \tag{1.16}$$

for any pseudo-differential operator $X\in C((\xi^{-1}))$, where d is the differential, and $a\sim b$ means: $(a-b)\in\operatorname{Im}\partial$. Taking $X=\ell^{1/2}$, $n=2$, we obtain $\frac{1}{m+2}d\operatorname{Res}(\ell^{1/2})^{m+2}$ $=dH_{m+2}\sim\frac{1}{2}\operatorname{Res}(d\ell\circ(\ell^{1/2})^m)=\frac{1}{2}\operatorname{Res}(dh\sum p_s(m)\xi^s)=\frac{1}{2}p_{-1}(m)dh$. Therefore

$$p_{-1}(m)=2\frac{\delta H_{m+2}}{\delta h}, \tag{1.17}$$

and substituting this into $h_t = 2\partial p_{-1}(2m+1)$, we obtain the first Hamiltonian representation

$$h_t = 2\partial p_{-1}(2m+1) = 4\partial \frac{\delta H_{2m+3}}{\delta h}. \tag{1.18}$$

To obtain the second Hamiltonian form we need to re-express $\partial p_{-1}(2m+1)$ through $p_{-1}(2m-1)$. For this, we write

$$(\ell^{1/2})^{m+2} = (\ell^{1/2})^m \ell = \ell(\ell^{1/2})^m,$$

and pick out the ξ^{-1}- and ξ^{-2}-terms from both sides:

$$p_{-1}(m+2) = p_{-3}(m) + p_{-1}(m)h, \tag{1.19}$$

$$p_{-1}(m+2) = p_{-3}(m) + 2\partial p_{-2}(m) + \partial^2 p_{-1}(m) + hp_{-1}(m), \tag{1.20}$$

$$p_{-2}(m+2) = p_{-4}(m) + p_{-2}(m)h - p_{-1}(m)\partial h, \tag{1.21}$$

$$p_{-2}(m+2) = p_{-4}(m) + 2\partial p_{-3}(m) + \partial^2 p_{-2}(m) + hp_{-2}(m). \tag{1.22}$$

Comparing (1.19) with (1.20) we obtain

$$p_{-2}(m) = -\tfrac{1}{2}\partial p_{-1}(m) \tag{1.23}$$

[recall that $rkp_s(m) = m - s$ with $rkh^{(j)} = j + 2$]. Similarly, comparing (1.21) with (1.22), we obtain

$$\begin{aligned} 2\partial p_{-3}(m) &= -p_{-1}(m)\partial h - \partial^2 p_{-2}(m) = [\text{by } (1.23)] \\ &= -p_{-1}(m)\partial h + \tfrac{1}{2}\partial^3 p_{-1}(m). \end{aligned} \tag{1.24}$$

Substituting this into (1.19) we get

$$\begin{aligned} 2\partial p_{-1}(m+2) &= -p_{-1}(m)\partial h + \tfrac{1}{2}\partial^3 p_{-1}(m) + 2\partial(p_{-1}(m)h) \\ &= (\tfrac{1}{2}\partial^3 + h\partial + \partial h)p_{-1}(m), \end{aligned} \tag{1.25}$$

thus

$$h_t = 2\partial p_{-1}(2m+1) = 4\partial \frac{\partial H_{2m+3}}{\delta h} = [\partial^3 + 2(h\partial + \partial h)] \frac{\delta H_{2m+1}}{\delta h}, \tag{1.26}$$

and this is the second Hamiltonian form of the KdV hierarchy.

2. Nonstandard Integrable Systems

The strangely looking system (1.4) is a particular case of the following general set-up. For the rest of the paper, $k = 0$, 1 or 2. Let

$$L = \sum_{-k}^{n} u_i \xi^i \tag{2.1a}$$

or

$$L = \sum_{-\infty}^{n} u_i \xi^i, \tag{2.1b}$$

with $n \geqq 1$ and the normalization conditions

$$k=0: u_n = 1, u_{n-1} = 0; \quad k=1: u_n = 1. \tag{2.2}$$

Denote $C=C_u=\mathbb{C}[u_i^{(j)}]$ the differential algebra generated by u_i's (see [5]). The centralizer $Z(L)$ of L in $C((\xi^{-1}))$ is generated over $\mathbb{C}$ by $(L^{1/n})^m$, $m\in\mathbb{Z}$ (where we assume $u_n=\bar{u}_n^n$ for some fixed $\bar{u}_n$ when $k=2$). For $P\in Z(L)$, consider the following system,

$$L_t=[((P^\dagger)_{\geqq k})^\dagger, L]=[-((P^\dagger)_{<k})^\dagger, L]. \tag{2.3}$$

Since $((P^\dagger)_{<k})^\dagger$ is an operator of order $\leqq k-1$, the right-hand side of (2.3) has order $\leqq n+k-2$, which guarantees the preservation of the normalization conditions (2.2). For $k=0$ we recover the standard differential Lax equations, $L_t=[P_{\geqq 0}, L]=[-P_{<0}, L]$ ([4, 5]), since only for $k=0$ the projection $X\mapsto X_{<k}$, $X\in C((\xi^{-1}))$, and taking the adjoint, $X\mapsto X^\dagger$, commute. For the case (2.1a) of finite L and $k>0$, it remains to show that the right-hand side of (2.3) belongs to $C((\xi^{-1}))_{\geqq -k}$. Set $(P^\dagger)_{\geqq k}=Q\xi^k$ with some $Q\in C[\xi]=C((\xi^{-1}))_{\geqq 0}$. Then

$$[(Q\xi^k)^\dagger, L]_{<0}=[(Q\xi^k)^\dagger, L_{<0}]_{<0}=\left[(-\xi)^k Q^\dagger, \sum_{i=-k}^{-1} u_i\xi^i\right]_{<0}$$
$$=\left[\sum_{-k}^{-1}(-\xi)^k Q^\dagger u_i\xi^i-\sum_{-k}^{-1}u_i\xi^i(-\xi)^kQ^\dagger\right]_{<0}=\left[\sum_{-k}^{-1}(-\xi)^kQ^\dagger u_i\xi^i\right]_{<0}\in C((\xi^{-1}))_{\geqq -k}. \tag{2.4}$$

Theorem 2.5. *For fixed L, the evolution derivations of C given by* (2.3) *for various* $P\in Z(L)$, *commute between themselves and have an infinite common set of conservation laws* (=c.l.'s)

$$H_m=\frac{1}{m}\operatorname{Res}(L^{1/n})^m, \qquad m\in\mathbb{N}. \tag{2.6}$$

Proof. Let L be as in (2.1b) and denote by I_k the differential ideal in C generated by $\mathbb{C}[u_i^{(j)}]_{i<-k}$. Then from (2.3) and (2.4) we obtain

$$(L_{<-k})_t=[((P^\dagger)_{\geqq k})^\dagger, L]_{<-k}=[((P^\dagger)_{\geqq k})^\dagger, L_{<-k}]_{<-k},$$

so that $(I_k)_t\subset I_k$. Therefore, the case (2.1a) is a specialization C/I_k of the universal infinite case (2.1b), and we now restrict ourselves to this case only. Firstly, suppose $n=1$. Then from (2.3) we get $(L^m)_t=[((P^\dagger)_{\geqq k})^\dagger, L^m]$, and since $\operatorname{Res}([\ ,\ ])\sim 0$, we obtain an infinity of c.l.'s H_m. From the existence of the Hamiltonian forms of the system (2.3) (derived below) it follows that all these c.l.'s are in involution; thus, all the flows commute. If now $n>1$, we let $\bar{L}=L^{1/n}=\sum_{-\infty}^{1}\bar{u}_i\xi^i$. The map $L\to\bar{L}$ generates an invertible differential map (automorphism) $C_{\bar{u}}=\mathbb{C}[\bar{u}_i^{(j)}]\to C_u=\mathbb{C}[u_i^{(j)}]$ with respect to which $Z(L)$ and $Z(\bar{L})$ are "the same" too (isomorphic). Thus, each of Eq. (2.3) for L and $\bar{L}$ implies the other. In particular, the derivations of C_u commute since those of $C_{\bar{u}}$ do, as we found above.

Now we compute the (first) Hamiltonian form of the system (2.3). Set

$$P=L^m=\sum_s \xi^s p_s(m). \tag{2.7}$$

From (1.16) with $X=L$, we obtain

$$dH_{m+1}=\frac{1}{m+1}d\operatorname{Res}L^{m+1}\sim\operatorname{Res}(dL\circ L^m)=\operatorname{Res}\left(\sum_i du_i\xi^i\sum_s\xi^s p_s(m)\right)$$
$$=\sum du_i p_{-i-1}(m),$$

and thus

$$p_{-i-1}(m)=\frac{\delta H_{m+1}}{\delta u_i}, \ 1\geqq i>-\infty, \tag{2.8}$$

where we temporarily abandon the normalization conditions (2.2) and treat u_1 and u_0 as legitimate variables; the advantages of such course of action will be seen later on in the case $k\neq 0$. We consider separately three cases: $k=0$, 1, 2.

0) $k=0$. Set $v_i=u_{-i-1}$, so that

$$L=\xi+\sum_{i\geqq 0} v_i\xi^{-i-1}, \tag{2.9}$$

$$p_i(m)=\frac{\delta H_{m+1}}{\delta v_i}, \ i\geqq 0. \tag{2.10}$$

Denote

$$X_i=\frac{\delta H_{m+1}}{\delta v_i}.$$

Then

$$((P^\dagger)_{\geqq 0})^\dagger=P_{\geqq 0}=\sum_{s\geqq 0}\xi^s X_s,$$

and

$$\partial_t(v_i)=\xi^{-i-1}\text{-coef. in}$$

$$\left[\sum_{s\geqq 0}\xi^s X_s, \xi+\sum_{j\geqq 0} v_j\xi^{-j-1}\right]=\xi^{-i-1}\text{-coef. in}$$

$$\left[\sum_{s\geqq 0}\xi^s X_s, \sum_{j\geqq 0} v_j\xi^{-j-1}\right]=:\sum_j B_{ij}(X_j), \tag{2.11}$$

where B_{ij} is a differential operator depending linearly upon v's. (We do not compute the matrix elements B_{ij} at the moment since in a little while we will change our L.) For $B=(B_{ij})$ to be Hamiltonian, the corresponding algebra, say $\mathfrak{G}$, must be a Lie algebra [6–8]. To compute multiplication in $\mathfrak{G}$, we notice that, setting

$$X=\sum_{i\geqq 0}\xi^i X_i, \ Y=\sum_{j\geqq 0}\xi^j Y_j, \ v=\sum_{j\geqq 0} v_j\xi^{-j-1},$$

we obtain

$$\begin{aligned} \bar{Y}^t B\bar{X} &:= \sum_{i,j} B_{ij}(X_j)\cdot Y_i=\xi^{-1}\text{-coef. in } ([X,v]Y)=\operatorname{Res}([X,v]Y) \\ &\sim[\text{since } \operatorname{Res}(\ ,\ ) \text{ is an invariant form}]\sim\operatorname{Res}(v[Y,X]) \\ &=\sum_j v_j\,\{\text{left } \xi^j\text{-coef. in } [Y,X]\}=\sum_j v_j[Y,X]_j, \end{aligned} \tag{2.12}$$

where $\sum_j \xi^j[Y,X]_j=[Y,X]$. Thus $\mathfrak{G}$ is the Lie algebra of differential operators, therefore the matrix $B=(B_{ij})$, being the natural Hamiltonian matrix on the dual space $\mathfrak{G}^*$, is indeed Hamiltonian. However, the coordinates we have ended up with on $\mathfrak{G}$ are not convenient, since the differential operators are written in the left form $\sum \xi^j X_j$ instead of the usual right form. Obviously, the root of the problem is L itself: as it is clear from the derivation of (2.8), we need to work with L in the left form in order for P to be naturally represented in the right form. The remedy, then, is clear: taking adjoint of (2.3) we obtain $-L_t^\dagger = [(P^\dagger)_{\geqq k}, L^\dagger]$, with $P^\dagger = (L^\dagger)^m$. Thus, we can discard the form (2.3) and instead use the following form:

$$L_t = [P_{\geqq k}, L] = [-P_{<k}, L], \tag{2.13}$$

$$L = \sum_{-\infty}^{n} \xi^i u_i, \qquad n=1, \tag{2.14}$$

with $P = L^m$, $m \in \mathbb{Z}_+$. Setting $H = \frac{1}{m}\operatorname{Res} L^m$, $P = L^m = \sum_s p_s(m)\xi^s$, we again obtain (2.8).

For $k=0$, repeating the derivation of (2.11) and (2.12), with $v_s = u_{-s-1}$ and $u_1 = 1$, $u_0 = 0$, we obtain

$$\partial_t(v_i) = \operatorname{Res}\left(\xi^i \left[\sum_{s\geqq 0} X_s \xi^s, \sum_{j\geqq 0} \xi^{-j-1} v_j\right]\right) = \sum_{j\geqq 0} B_{ij}^0(X_j), \tag{2.15}$$

and setting $X = \sum_{s\geqq 0} X_s \xi^s$, $Y = \sum_{i\geqq 0} Y_i \xi^i$, $v = \sum_{j\geqq 0} \xi^{-j-1} v_j$, we get

$$\bar{Y}^t B^0 \bar{X} = \sum_{i,j} Y_i B_{ij}^0(X_j) = \operatorname{Res}(Y[X,v]) \sim \operatorname{Res}([Y,X]v) = \sum_j v_j [Y,X]_j, \tag{2.16}$$

where $\sum_j [Y,X]_j \xi^j = [Y,X]$. Thus B^0 is the natural Hamiltonian matrix on the dual space to the Lie algebra of differential operators. In the proof of Theorem 4.0, we shall need the explicit form of the matrix elements of the matrix $B^r = (B_{ij}^r)$ associated to the dual space of the Lie algebra $\mathfrak{G}_{\geqq r}$ of (right) differential operators of order $\geqq r (\in \mathbb{Z}_+)$.

Let $X = \sum_{i\geqq 0} X_i \xi^{i+r}$, $Y = \sum_{j\geqq 0} Y_j \xi^{j+r}$. Then

$$XY = \sum X_i \binom{i+r}{s} Y_j^{(s)} \xi^{i+r-s} \xi^{j+r} = \sum X_i \binom{i+r}{s} Y_j^{(s)} \xi^{i+j+r-s} \xi^r,$$

so that

$$[X,Y] = \left\{\left[\sum \binom{i+r}{s} X_i Y_j^{(s)} - \sum Y_j \binom{j+r}{s} X_i^{(s)}\right] \xi^{i+j+r-s}\right\} \xi^r,$$

and if A_j is the coordinate on $(\mathfrak{G}_{\geqq r})^*$ dual to ξ^{j+r}, we obtain

$$\sum_m A_m [X,Y]_m = \sum A_{i+j+r-s}\left[\binom{i+r}{s} X_i Y_j^{(s)} - X_i^{(s)} \binom{j+r}{s} Y_j\right]$$

$$\sim \sum_{i,j} X_i \left[\sum_s A_{i+j+r-s} \binom{i+r}{s} \partial^s - \sum_s (-\partial)^s A_{i+j+r-s} \binom{j+r}{s}\right] Y_j .$$

Therefore

$$B_{ij}^r = \sum_s \binom{i+r}{s} A_{i+j+r-s} \partial^s - \sum_s (-\partial)^s A_{i+j+r-s} \binom{j+r}{s}. \tag{2.17}$$

1) Now let $k=1$, L being given by (2.14), $L^m=\sum_s p_s(m)\xi^s$, with p_s given by (2.8). We have firstly

$$L_{\geqq -1,t}=[L,P_{\leqq 0}]_{\geqq -1}=[L_{>0},P_{\leqq 0}]_{\geqq -1}=[\xi^1 u_1, p_0(m)+p_{-1}(m)\xi^{-1}+\dots]_{\geqq -1}$$
$$=u_1\partial p_0(m)+\partial(u_1 p_{-1}(m))\xi^{-1},$$

thus

$$u_{1,t}=0,\ u_{0,t}=u_1\partial p_0(m),\ u_{-1,t}=\partial u_1 p_{-1}(m), \tag{2.18}$$

and now we can as well put $u_1=1$. Using (2.8), we obtain

$$\begin{pmatrix} u_0 \\ u_{-1}\end{pmatrix}_t=\begin{pmatrix} 0 & \partial \\ \partial & 0\end{pmatrix}\begin{pmatrix} \delta/\delta u_0 \\ \delta/\delta u_{-1}\end{pmatrix}(H_{m+1}). \tag{2.19}$$

Now set $v_s=u_{-s-2}$, $s\geqq 0$, $P_{\geqq 1}=X=\sum_{s\geqq 0}X_s\xi^{s+1}$, $X_s=\dfrac{\delta H_{m+1}}{\delta v_s}$; $Y=\sum_{i\geqq 0}Y_i\xi^{i+1}$. Then

$$v_{i,t}=\mathrm{Res}(\xi^{i+1}[X,L])=\mathrm{Res}(\xi^{i+1}[X,v])=\sum_j B_{ij}(X_j),$$
$$\bar{Y}^t B\bar{X}=\sum Y_i B_{ij}(X_j)=\mathrm{Res}(Y[X,v])\sim\mathrm{Res}([Y,X]v)=\sum_j [Y,X]_j v_j,$$

where $\sum_j [Y,X]_j\xi^{j+1}=[Y,X]$. Thus $B=B^1$, and the (first) Hamiltonian form for the Eq. (2.13) with $k=1$ is

$$\left(\begin{array}{cc|c} 0 & \partial & \multirow{2}{*}{$\mathbf{0}$} \\ \partial & 0 & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & B^1 \end{array}\right) \tag{2.20}$$

2) Now let $k=2$. Picking out from the equation $L_t=[L,p_{<2}]$ the ξ^i-terms for $i\geqq -2$, we obtain

$$\begin{aligned} &u_{1,t}=(u_1\partial-u_1^{(1)})p_1;\ u_{0,t}=u_1\partial p_0-(u_0^{(1)}+u_1\partial^2)p_1;\ u_{-1,t}=\partial(u_1p_{-1}-u_{-1}p_1);\\ &u_{-2,t}=(\partial u_1+u_1^{(1)})p_{-2}+(u_0^{(1)}+\partial^2 u_1)p_{-1}-u_1\partial p_0-(u_2\partial+\partial u_2)p_1, \end{aligned} \tag{2.21}$$

where $p_i=p_i(m)$ and $u_i^{(j)}=\partial^j(u_i)$. Using (2.8) we put (2.21) into the form

$$\begin{pmatrix} u_1 \\ u_0 \\ u_{-1} \\ u_{-2}\end{pmatrix}_t=\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & u_1\partial-u_1^{(1)} \\ \hline 0 & 0 & u_1\partial & -u_1\partial^2-u_0^{(1)} \\ \hline 0 & \partial u_1 & 0 & -\partial u_{-1} \\ \hline \partial u_1+u_1^{(1)} & \partial^2 u_1+u_0^{(1)} & -u_{-1}\partial & -(u_{-2}\partial+\partial u_{-2}) \end{array}\right)\begin{pmatrix} \delta/\delta u_1 \\ \delta/\delta u_0 \\ \delta/\delta u_{-1} \\ \delta/\delta u_{-2}\end{pmatrix}$$
$$(H_{m+1}). \tag{2.22}$$

Denote b_2 the matrix in the right-hand side of (2.22). This matrix is Hamiltonian iff the corresponding algebra, say $\mathfrak{h}_2$, is a Lie algebra. Computing the commutator in $\mathfrak{h}_2$ we find

$$\left[\begin{pmatrix} X_{-2} \\ X_{-1} \\ X_0 \\ X_1\end{pmatrix},\begin{pmatrix} Y_{-2} \\ Y_{-1} \\ Y_0 \\ Y_1\end{pmatrix}\right]=\begin{pmatrix} X_{-2}Y_{-2}^{(1)}-Y_{-2}X_{-2}^{(1)} \\ X_{-2}Y_{-1}^{(1)}-Y_{-2}X_{-1}^{(1)} \\ \partial(X_{-2}Y_0-Y_{-2}X_0) \\ A\end{pmatrix}, \tag{2.23}$$

$$A:=(X_{-2}Y_1^{(1)}+2X_{-2}^{(1)}Y_1)-(Y_{-2}X_1^{(1)}+2Y_{-2}^{(1)}X_1)+(X_{-1}^{(1)}Y_0-Y_{-1}^{(1)}X_0)$$
$$+(Y_{-2}^{(2)}X_0-X_{-2}^{(2)}Y_0).$$

A straightforward computation shows that the Jacobi identity is indeed satisfied in $\mathfrak{h}_2$. (See also Remark 2.27 below.) Thus, b_2 is Hamiltonian indeed. We now set $v_s = u_{-3-s}$, $s \geqq 0$,

$$v = \sum_{s \geqq 0} \xi^{-s-3} v_s, \; X = P_{\geqq 2} = \sum_{s \geqq 2} p_s(m)\xi^s = \sum_{s \geqq 0} X_s \xi^{s+2},$$

where $X_s = \delta H_{m+1}/\delta v_s$; $Y = \sum_{s \geqq 0} Y_s \xi^{s+2}$. Then

$$v_{i,t} = \mathrm{Res}(\xi^{i+2}[X, L]) = \mathrm{Res}(\xi^{i+2}[X, v]) = \sum_{j \geqq 0} B_{ij}(X_j),$$

$$\sum Y_i B_{ij}(X_j) = \mathrm{Res}(Y[X, v]) \sim \mathrm{Res}([Y, X]\, v) = \sum_j [Y, X]_j v_j,$$

where $\sum_j [Y, X]_j \xi^{j+2} = [Y, X]$. Thus, $B = B^2$. Therefore, Eqs. (2.13) are Hamiltonian for $k = 2$, with the Hamiltonian structure

$$\left(\begin{array}{c|c} b_2 & 0 \\ \hline 0 & B^2 \end{array}\right). \tag{2.24}$$

This concludes the proof of Theorems 2.5 and 2.1. □

Remark 2.25. For the standard case $k = 0$, a purely algebraic proof exists (bypassing the Hamiltonian formalism) of the commutativity of all the flows (2.3) [4], which uses the dressing operator $K: KLK^{-1} = u_n \xi^n$. A somewhat analogous proof can be given for the case $k = 1$, but not (I believe) for $k = 2$.

Remark 2.26. An analog of the case $k = 1$ for discrete integrable systems [6, 9] exists with $L = \sum_{-\infty}^{n} u_i \zeta^i$, where ζ is an automorphism.

Remark 2.27. For general $n \geqq 1$, the (first) Hamiltonian structure of the system (2.3) can easily be shown to be the direct sum of B^k (for the variables $v_s = u_{-s-k-1}$, $s \geqq 0$) and the natural Hamiltonian matrix on the dual space to the Lie factor-algebra $\mathfrak{G}_{<k}/\mathfrak{G}_{<-n-1}$ (for the variables $u_{-k}, \ldots, u_{-n}$) specialized by the normalization conditions (2.2).

Remark 2.28. Substituting the Poisson bracket $\{a, b\} = a_\xi b_x - a_x b_\xi$ instead of the commutator into the right-hand side of (2.3), we arrive at the dispersiveless integrable systems which can be considered as the quasiclassical limits of the full Eq. (2.3). The resulting flows all commute, have an infinity of common c.l.'s, and are Hamiltonian with the Hamiltonian structures which are the limits of the corresponding structures for the full equations.

Remark 2.29. It would be interesting to find out whether the second Hamiltonian form exists for the Eq. (2.3) when $k \neq 0$. For $k = 0$ it is not very difficult to show that it does exist (the quasi-classical limit of this structure, written down in [10], represents the second Hamiltonian form of the two-dimensional, free surface, long wave Eqs. [11–13]).

Remark 2.30. Any time a new integrable system, or class of systems, is introduced one has to tackle the (often overlooked) problem of triviality or nontriviality of flows and conservation laws. For complex systems the only known general avenue to analyze this problem is first solving the easier problem for the flows and then using the Hamiltonian formalism to analyze the conservation laws (see, e.g., [17, 18]). For the system (2.3) the answer is obvious: $\operatorname{Res} L^{m/n} \nsim 0$ for $m>0$ unless $k=0$ and $m \in \mathbb{N}n$, since the quasiclassical limit of $\operatorname{Res} L^{m/n}$ is simply a polynomial in u_i's with positive binomial coefficients, except for $k=0$ and $(L^{m/n})_{<0}=0$, that is, when $m \in \mathbb{N}n$. The nontriviality of flows follows by similar arguments: all the flows are nontrivial except for $k=0$ and $P=L^{m/n}$ with $m \in \mathbb{N}n$.

3. Hamiltonian Formalism for Dispersive Long Waves

In this section we prove Theorem 3.0. Retaining the notation $L=\xi+u+h\xi^{-1}$, $L^m=\sum_s \xi^s p_s(m)$, $H_m=m^{-1}\operatorname{Res} L^m$, we have by (2.8)

$$p_0(m)=\delta H_{m+1}/\delta h,\ p_{-1}(m)=\delta H_{m+1}/\delta u\,. \tag{3.1}$$

Picking out the ξ^0- and ξ^1-terms in the equality $L_t=[L,((P^\dagger)_{\leqq 0})^\dagger]$, we have

$$L_t=u_t+h_t\xi^{-1}=\left[\xi+u+h\xi^{-1}, \sum_{s\leqq 0} \xi^s p_s(m)\right]_{\geqq -1}=\partial p_0(m)+\partial p_{-1}(m)\xi^{-1},$$

so that

$$\begin{pmatrix} u \\ h \end{pmatrix}_t=\partial\begin{pmatrix} p_0(m) \\ p_{-1}(m)\end{pmatrix}=\begin{pmatrix} 0 & \partial \\ \partial & 0\end{pmatrix}\begin{pmatrix}\delta/\delta u \\ \delta/\delta h\end{pmatrix}(H_{m+1}), \tag{3.2}$$

which yields (1.9).

We now write down the identity $(L^\dagger)^{m+1}=(L^\dagger)^m L^\dagger=L^\dagger(L^\dagger)^m$:

$$\begin{aligned}\sum_s(-1)^s p_s(m+1)\xi^s &= \sum_s(-1)^s p_s(m)\xi^s(-\xi+u-\xi^{-1}h)\\ &=(-\xi+u-\xi^{-1}h)\sum_s(-1)^s p_s(m)\xi^s,\end{aligned} \tag{3.3}$$

and pick out the ξ^i-terms from it, for $i=0, -1, -2$:

$$p_0(m+1)=p_{-1}(m)-\partial p_0(m)+up_0(m)+\sum_{s\geqq 0}[hp_{s+1}(m)]^{(s)}, \tag{3.4}$$

$$-p_{-1}(m+1)=-p_{-2}(m)-p_{-1}(m)u-p_0(m)h, \tag{3.5}$$

$$-p_{-1}(m+1)=-p_{-2}(m)+\partial p_{-1}(m)-up_{-1}(m)-\sum_{s\geqq 0}[hp_s(m)]^{(s)}, \tag{3.6}$$

$$p_{-2}(m+1)=p_{-3}(m)+p_{-2}(m)u+p_{-1}(m)\partial u+p_{-1}(m)h+p_0(m)\partial h, \tag{3.7}$$

$$p_{-2}(m+1)=p_{-3}(m)-\partial p_{-2}(m)+up_{-2}(m)+\sum_{s\geqq -1}[hp_s(m)]^{(s+1)}. \tag{3.8}$$

Comparing (3.5) with (3.6) we obtain

$$p_{-1}(m)=\sum_{s\geqq 0}[hp_{s+1}(m)]^{(s)}, \tag{3.9}$$

and substituting this into (3.4) we get

$$p_0(m+1)=2p_{-1}(m)+(u-\partial)p_0(m)\,. \tag{3.10}$$

In particular,

$$\partial p_0(m+1)=2\partial p_{-1}(m)+\partial(u-\partial)p_0(m)\,. \tag{3.11}$$

Subtracting (3.8) from (3.7) we obtain

$$\partial p_{-2}(m)+p_{-1}(m)\partial u+p_0(m)\partial h=\sum_{s\geqq 0}[hp_s(m)]^{(s+1)}=\partial hp_0(m)$$
$$+\partial^2\sum_{s\geqq 0}[hp_{s+1}(m)]^{(s)}=[\text{by }(3.9)]=\partial hp_0(m)+\partial^2 p_{-1}(m). \quad (3.12)$$

Applying $(-\partial)$ to (3.5) and substituting (3.12) into the result, we arrive at

$$\partial p_{-1}(m+1)=(u\partial+\partial^2)p_{-1}(m)+(h\partial+\partial h)p_0(m). \quad (3.13)$$

Combining (3.11) and (3.13) together, we get

$$\begin{pmatrix} u \\ h \end{pmatrix}_t=\partial\begin{pmatrix} p_0(m) \\ p_{-1}(m)\end{pmatrix}=\begin{pmatrix} 2\partial & \partial(u-\partial) \\ (u+\partial)\partial & h\partial+\partial h\end{pmatrix}\begin{pmatrix} p_{-1}(m-1) \\ p_0(m-1)\end{pmatrix}, \quad (3.14)$$

which proves (1.10). Notice that only $\partial p_{-1}(m-1)$ is involved in the right-hand side of (3.14). We, thus, can iterate the procedure using (3.10) and (3.11):

$$\begin{aligned} u_t&=2\partial p_{-1}(m+1)+\partial(u-\partial)p_0(m-1)\\ &=2[(u\partial+\partial^2)p_{-1}(m-2)+(h\partial+\partial h)p_0(m-2)]\\ &\quad+\partial(u-\partial)[2p_{-1}(m-2)+(u-\partial)p_0(m-2)]\\ &=2(u\partial+\partial u)p_{-1}(m-2)+[2(h\partial+\partial h)+\partial(u-\partial)^2]p_0(m-2), \end{aligned} \quad (3.15a)$$

$$\begin{aligned} h_t&=(u\partial+\partial^2)p_{-1}(m-1)+(h\partial+\partial h)p_0(m-1)\\ &=(u+\partial)[(u+\partial)\partial p_{-1}(m-2)+(h\partial+\partial h)p_0(m-2)]\\ &\quad+(h\partial+\partial h)[2p_{-1}(m-2)+(u-\partial)p_0(m-2)]\\ &=[(u+\partial)^2\partial+2(h\partial+\partial h)]p_{-1}(m-2)\\ &\quad+[(u+\partial)(h\partial+\partial h)+(h\partial+\partial h)(u-\partial)]p_0(m-2), \end{aligned} \quad (3.15b)$$

which proves (1.11) for $m>1$: for $m=1$, $p_{-1}(1)$ and $p_0(-1)$ are not the variational derivatives of H_0 which is not defined. To fix this, we define $H_0=u/2$. Then (3.1) implies

$$p_{-1}(-1)=1/2,\ p_0(-1)=0. \quad (3.16)$$

Substituting this into (3.15) we obtain

$$u_t=\partial u,\ h_t=\partial h \quad (3.17)$$

which agrees with (3.2) for $m=1$ since $p_0(1)=u$ and $p_{-1}(1)=h$.

We now prove that the matrices (1.9)–(1.11) are Hamiltonian.

The matrix B^{I} (1.9) is skew-symmetric constant-coefficient and is, thus, Hamiltonian [5].

Let K be a differential algebra with a derivation ∂; denote $D(K)$ K considered as a Lie algebra with the commutator $[X,Y]=X\partial Y-Y\partial X$. $D(K)$ acts on K by derivations: $(X,f)\mapsto X\partial f$. Denote the corresponding semidirect product Lie algebra by $\mathfrak{h}:\mathfrak{h}=D(K)\ominus K$. The commutator in $\mathfrak{h}$ is given by

$$\left[\begin{pmatrix} X \\ f\end{pmatrix},\begin{pmatrix} Y \\ g\end{pmatrix}\right]=\begin{pmatrix} X\partial Y-Y\partial X \\ X\partial g-Y\partial f\end{pmatrix}. \quad (3.18)$$

Let h and u denote coordinates on $\mathfrak{h}^*$ dual to $D(K)$ and K respectively. Then

$$h(X\partial Y - Y\partial X) + u(X\partial g - Y\partial f) \sim (f, X)\begin{pmatrix} 0 & \partial u \\ u\partial & h\partial + \partial h \end{pmatrix}\begin{pmatrix} g \\ Y \end{pmatrix}, \tag{3.19}$$

hence the linear part $B^{\mathrm{II}l}$ of B^{II} is the natural Hamiltonian matrix on $\mathfrak{h}^*$ [6–8]. Consider now the following skew-symmetric forms on $\mathfrak{h}$:

$$\omega_1\left(\begin{pmatrix} f \\ X \end{pmatrix}, \begin{pmatrix} g \\ Y \end{pmatrix}\right) = 2f\partial g = (f, X)b_1(g, Y)^t, b_1 = \begin{pmatrix} 2\partial & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.20}$$

$$\omega_2\left(\begin{pmatrix} f \\ X \end{pmatrix}, \begin{pmatrix} g \\ Y \end{pmatrix}\right) = X\partial^2 g - f\partial^2 Y = (f, X)b_2(g, Y)^t, b_2 = \begin{pmatrix} 0 & -\partial^2 \\ \partial^2 & 0 \end{pmatrix}. \tag{3.21}$$

It is easy to see that ω_1 and ω_2 are generalized two-cocycles on $\mathfrak{h}$, i.e., $\omega_i([A_1, A_2], A_3) + \text{c.p.} \sim 0$, $\forall A_1, A_2, A_3 \in \mathfrak{h}$, $i = 1, 2$, where "c.p." stands for "cyclic permutation." Thus [6, 7], the matrix $B^{\mathrm{II}} = B^{\mathrm{II}l} + b_1 + b_2$ is Hamiltonian.

The Hamiltonian property of the matrix B^{III} can be checked directly, by a tedious but straightforward computation, using methods of [13] (see, e.g., [14]). Alternatively, consider the following matrices over $C_2 = C_{U,v} = \mathbb{C}[U^{(i)}, v^{(j)}]$:

$$\bar{B}^{\mathrm{I}} = \begin{pmatrix} 2\partial & \partial \\ \partial & 0 \end{pmatrix}, \tag{3.22}$$

$$\bar{B}^{\mathrm{II}} = \begin{pmatrix} 2(U\partial + \partial U) & 2v\partial + \partial U - \partial^2 \\ 2\partial v + U\partial + \partial^2 & v\partial + \partial v \end{pmatrix}. \tag{3.23}$$

Let $\Phi: C_1 = C_{u,h} \to C_2$ be the differential homomorphism of differential algebras given on generators u, h by the formula

$$\Phi(u) = U, \Phi(h) = Uv - v^2 + v^{(1)}. \tag{3.24}$$

Let $J = J(\Phi)$ be the Fréchet Jacobian of Φ:

$$J = \begin{pmatrix} 1 & 0 \\ v & U - 2v + \partial \end{pmatrix}. \tag{3.25}$$

It is easy to check out that

$$J\bar{B}^i J^\dagger = \Phi(B^{i+\mathrm{I}}), \qquad i = \mathrm{I}, \mathrm{II}. \tag{3.26}$$

In addition, the map Φ (3.24) is obviously injective (this also follows from a more general injectivity result from [15]). This implies that the matrix $B^{i+\mathrm{I}}$ is Hamiltonian provided $\bar{B}^i$ is (since $B^{i+\mathrm{I}}$ is just the reduction of $\bar{B}^i$ on the image $\Phi(C_1) \subset C_2$, see [15]).

The matrix $\bar{B}^{\mathrm{I}}$ is skew-symmetric constant-coefficient, and thus is Hamiltonian. It is easy to associate a generalized 2-cocycle and a Lie algebra to the matrix $\bar{B}^{\mathrm{II}}$ (3.23). Instead, let us consider two more differential algebras $C_3 = C_{\varphi,V}$ and $C_4 = C_{w,\psi}$, together with the following matrices b_3 and b_4 over C_3 and C_4 respectively:

$$b_4 = \begin{pmatrix} 2(w\partial + \partial w) & -2\psi^{(1)} + \partial \\ 2\psi^{(1)} + \partial & 0 \end{pmatrix}, \tag{3.27}$$

$$b_3 = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \end{pmatrix}. \tag{3.28}$$

Let $\varphi_3 : C_2 \to C_3$ and $\varphi_4 : C_2 \to C_4$ be the differential homomorphisms given by

$$\varphi_3(U) = \varphi^{(1)} + 2V, \ \varphi_3(v) = V, \tag{3.29}$$

$$\varphi_4(U) = w, \ \varphi_4(v) = \psi^{(1)} + \tfrac{1}{2}w. \tag{3.30}$$

The maps φ_3, φ_4 are evidently injective. If J_j is the Fréchet Jacobian of the map φ_j, $j = 3, 4$, one can easily check out that

$$J_j b_j J_j^{\dagger} = \varphi_j(\bar{B}^{\mathrm{II}}). \tag{3.31}$$

Since b_3 is obviously Hamiltonian, it follows that $\bar{B}^{\mathrm{II}}$ is Hamiltonian too, and, thus, B^{III} is Hamiltonian as well. Theorem 3.0 is proved.

The matrices $\bar{B}^{\mathrm{I}}$, $\bar{B}^{\mathrm{II}}$, b_3, and b_4 have the following interpretation. U and v are the variables of the modified long wave hierarchy

$$\begin{pmatrix} U \\ v \end{pmatrix}_t = \bar{B}^{\mathrm{I}} \delta(\Phi H_m) = \bar{B}^{\mathrm{II}} \delta(\Phi H_{m-1}) \tag{3.32}$$

which is, thus, integrable and bi-Hamiltonian. The canonical map Φ is an analog of the Miura map. In addition, we have two *different Hamiltonian* modified-modified systems

$$\begin{pmatrix} \varphi \\ V \end{pmatrix}_t = b_3 \delta(\varphi_3 \Phi H_{m-1}), \ \begin{pmatrix} w \\ \psi \end{pmatrix}_t = b_4 \delta(\varphi_4 \Phi H_{m-1}). \tag{3.33}$$

This phenomenon is absent in the standard theory of integrable systems ($k = 0$); neither three-Hamiltonian systems are present when $k = 0$.

It remains to show that b_4 is Hamiltonian. Let $\mathfrak{h}_1$ be the semidirect product Lie algebra $D(K) \circledS K$, where $D(K)$ acts now on K via the rule $(X, f) \mapsto \partial(Xf)$. Thus, the commutator in $\mathfrak{h}_1$ is

$$\left[\begin{pmatrix} X \\ f \end{pmatrix}, \begin{pmatrix} Y \\ g \end{pmatrix}\right] = \begin{pmatrix} X\partial Y - Y\partial X \\ \partial(Xg - Yf) \end{pmatrix}. \tag{3.34}$$

Let w and ψ be the corresponding coordinates on $\mathfrak{h}_1^*$. Then

$$w(X\partial Y - Y\partial X) + \psi\partial(Xg - Yf) \sim (X, f) \begin{pmatrix} w\partial + \partial w & -\psi_x \\ \psi_x & 0 \end{pmatrix} \begin{pmatrix} Y \\ g \end{pmatrix}, \tag{3.35}$$

and hence the linear part b_4^l of b_4 is twice the natural Hamiltonian matrix on $\mathfrak{h}_1^*$. Consider the following skew-symmetric form on $\mathfrak{h}_1$:

$$\omega\left(\begin{pmatrix} X \\ f \end{pmatrix}, \begin{pmatrix} Y \\ g \end{pmatrix}\right) = X\partial g + f\partial Y = (X, f) \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} Y \\ g \end{pmatrix}. \tag{3.36}$$

This form is clearly seen to be a generalized two-cocycle on $\mathfrak{h}_1$. Hence b_4 is indeed Hamiltonian.

Remark 3.37. Not only matrices B^{I}, B^{II}, and B^{III} are individually Hamiltonian, but an arbitrary linear combination $\sum_{\mathrm{I}}^{\mathrm{III}} \lambda_i B^i$ is Hamiltonian as well. This follows by the same line of reasoning as above, when combined with the obvious formula

$$B^{\mathrm{III}}(u+\lambda)=B^{\mathrm{III}}(u)+2\lambda B^{\mathrm{II}}(u)+\lambda^2 B^{\mathrm{I}},\ \lambda_i,\ \lambda\in\mathbb{C}.$$

Remark 3.38. For arbitrary α and β in (1.2), all the results in this section apply after a slight change is made in matrices B^i, $\bar{B}^i$, b_3, b_4 and the maps Φ, φ_3, φ_4. For example, for $\beta=-1/2$, the corresponding matrix B^{III} (1.11) adds on $2\alpha\begin{pmatrix}0 & 2\partial^3\\ 2\partial^3 & u\partial^3+\partial^3 u\end{pmatrix}$.

Remark 3.39. Under zero-dispersion limit the dispersive system (1.7) goes into the classical long wave system (1.1), while the limit of the matrices B^{I}, B^{II}, and B^{III} produces the corresponding three-Hamiltonian form of this system ([16]).

4. Canonical Maps

Suppose B_1 and B_2 are Hamiltonian matrices over differential rings C_1 and C_2 respectively and let $\Phi: C_1\to C_2$ be a differential homomorphism. Φ is called canonical if $\Phi X_H = X_{\Phi H}\Phi$, for any $H\in C_1$; here X_G is the Hamiltonian evolution derivation whose Hamiltonian is G. If $J=J(\Phi)$ is the Fréchet Jacobian of Φ, then the condition on Φ to be canonical can conveniently be written in the form (see [6, 15])

$$JB_2J^{\dagger}=\Phi(B_1). \tag{4.1}$$

We fix $r\in\mathbb{Z}_+$. Let $C_2=C_{u,h}$, and define B_2 as

$$B_2=B_2(r)=\begin{pmatrix}0 & \partial(\partial+u)^r\\ (u-\partial)^r\partial & h(\partial+u)^r-(u-\partial)^r h\end{pmatrix}. \tag{4.2}$$

Let $B_1=B^r$ be the natural Hamiltonian matrix (2.17) on the dual space $(\mathfrak{G}_{\geqq r})^*$ to the Lie algebra of differential operators of order $\geqq r$. For each $i\in\mathbb{Z}_+$ define $Q_i=Q_i(u)\in C_u$ by

$$Q_i=(\partial+u)^i(1). \tag{4.3}$$

Theorem 4.4. *The map* $\Phi: C_1=C_A=\mathbb{C}[A_i^{(j)}]\to C_2=C_{u,h}$ *given by*

$$\Phi(A_i)=hQ_i(u) \tag{4.5}$$

is canonical between $B_2(r)$ (4.2) *and* B^r (2.17).

Remark 4.6. Let $\varphi: C_2\to C_1$ be the following rational map: $\varphi(h)=A_0$, $\varphi(u)=A_1/A_0$. Then $\Phi\varphi=id$. Thus φ is an epimorphism. Since B^r is Hamiltonian, from Theorem 4.4 it would follow that $B_2(r)$ is Hamiltonian too.

Before proving Theorem 4.4, we record a few properties of the polynomials Q_i (4.3). Notice, that by (4.3),

$$Q_{i+1}=(\partial+u)(Q_i),\ Q_0=1. \tag{4.7}$$

Lemma 4.8. *For* $m\in\mathbb{Z}$,

$$(\xi+u)^m=\sum_{\alpha\geqq 0}\binom{m}{\alpha}Q_\alpha\xi^{m-\alpha}. \tag{4.9}$$

Proof. We use induction on $|m|$. Let first $m \geqq 0$. For $m=0$ (4.9) is obviously true. If it is true for $0 \leqq m \leqq M$, then

$$\begin{aligned}(\xi+u)^{M+1} &= (\xi+u)(\xi+u)^M = (\xi+u)\sum\binom{M}{\alpha}Q_\alpha \xi^{M-\alpha} \\ &= \sum\binom{M}{\alpha}[(\partial+u)(Q_\alpha)+Q_\alpha\xi]\xi^{M-\alpha} \\ &= \sum\binom{M}{\alpha}Q_{\alpha+1}\xi^{M-\alpha}+\sum\binom{M}{\alpha}Q_\alpha\xi^{M+1-\alpha} \\ &= \sum\left[\binom{M}{\alpha-1}+\binom{M}{\alpha}\right]Q_\alpha\xi^{M+1-\alpha} = \sum\binom{M+1}{\alpha}Q_\alpha\xi^{M+1-\alpha},\end{aligned}$$

and the induction step is made.

Now let $m=-1$. Since $\binom{-1}{\alpha}=(-1)^\alpha$, (4.9) becomes

$$(\xi+u)^{-1} = \sum_{\alpha \geqq 0}(-1)^\alpha Q_\alpha \xi^{-\alpha-1}. \tag{4.10}$$

To check (4.10) we apply $\xi+u$ to the right-hand side of (4.10):

$$(\xi+u)\sum(-1)^\alpha Q_\alpha\xi^{-\alpha-1} = \sum(-1)^\alpha[Q_{\alpha+1}\xi^{-\alpha-1}+Q_\alpha\xi^{-\alpha}] = Q_0\xi^{-0}=1,$$

which proves (4.10). Now suppose (4.9) is true for $0>m\geqq -M$, and let

$$(\xi+u)^{-M-1} = \sum_{\alpha\geqq 0}\binom{-M-1}{\alpha}F_\alpha\xi^{-M-1-\alpha} \tag{4.11}$$

with some F_α. To prove that $F_\alpha = Q_\alpha$, we apply $\xi+u$ to (4.11) and get

$$(\xi+u)^{-M} = \sum_{\alpha\geqq 0}\binom{-M-1}{\alpha}[(\partial+u)(F_\alpha)\xi^{-M-1-\alpha}+F_\alpha\xi^{-M-\alpha}]. \tag{4.12}$$

But

$$(\xi+u)^{-M} = \sum_{\alpha\geqq 0}\binom{-M}{\alpha}Q_\alpha\xi^{-M-\alpha} \tag{4.13}$$

by induction assumption. Equating $\xi^{-M-\alpha}$-terms in (4.12) and (4.13) we obtain $F_0 = Q_0$ and

$$\binom{-M-1}{\alpha+1}F_{\alpha+1} = \binom{-M}{1+\alpha}Q_{\alpha+1} - \binom{-M-1}{\alpha}(\partial+u)(F_\alpha). \tag{4.14}$$

Since

$$\binom{-M-1}{1+\alpha}+\binom{-M-1}{\alpha}=\binom{-M}{\alpha+1},$$

the induction on α for (4.14) shows that $F_\alpha = Q_\alpha$. □

Corollary 4.15.

$$(\xi-u)^{-1} = \sum_{i\geqq 0}\xi^{-i-1}Q_i. \tag{4.16}$$

Proof. Take adjoint of (4.10). □

Lemma 4.17. *For* $m, r \in \mathbb{Z}_+$,

$$\sum_{\alpha \geqq 0} \binom{m}{\alpha} Q_{m+r-\alpha} \xi^\alpha = \sum_{\alpha \geqq 0} \binom{m}{\alpha} Q_{m-\alpha} \xi^\alpha Q_r . \tag{4.18}$$

Proof. We use induction on m. For $m=0$, (4.18) is obviously true. Assume it is true for $0 \leqq m \leqq M$. Noticing that the right-hand side of (4.18) can be written as $(\xi+u)^m Q_r$ by (4.9), we have

$$\begin{aligned}(\xi+u)^{M+1} Q_r &= (\xi+u)(\xi+u)^M Q_r = (\xi+u)^M (\xi+u) Q_r \\ &= (\xi+u)^M (Q_{r+1} + Q_r \xi) = [\text{by (4.18) with } m=M] \\ &= \sum \binom{M}{\alpha} Q_{M+r+1-\alpha} \xi^\alpha + \sum \binom{M}{\alpha} Q_{M+r-\alpha} \xi^{\alpha+1} \\ &= \sum \left[\binom{M}{\alpha} + \binom{M}{\alpha-1} \right] Q_{M+r+1-\alpha} \xi^\alpha = \sum \binom{M+1}{\alpha} Q_{M+1+r-\alpha} \xi^\alpha ,\end{aligned}$$

which finishes the induction step. □

Denote $D(Q_m) = D_u(Q_m) = \sum_{i \geqq 0} \frac{\partial Q_m}{\partial u^{(i)}} \xi^i$ the Fréchet derivative of Q_m with respect to u.

Lemma 4.19.

$$D(Q_m)\xi + Q_m = (\xi+u)^m . \tag{4.20}$$

Proof. Use induction on m, (4.20) being true for $m=0$. Suppose (4.20) is true for $0 \leqq m \leqq M$. Recall that the operator D is a derivation: $D(AB) = AD(B) + BD(A)$. Applying D to (4.7), we obtain

$$D(Q_{m+1}) = (\xi+u) D(Q_m) + Q_m ,$$

therefore

$$\begin{aligned}D(Q_{M+1})\xi + Q_{M+1} &= (\xi+u) D(Q_M)\xi + Q_M \xi + Q_{M+1} \\ &= [\text{by induction assumption}] \\ &= (\xi+u)[(\xi+u)^M - Q_M] + Q_M \xi + u Q_M + \partial(Q_M) = (\xi+u)^{M+1} . \quad \square\end{aligned}$$

Proof of Theorem 4.4. From (4.5) we get, denoting $\varphi_i = \Phi(A_i)$:

$$D_u(\varphi_i) = h D(Q_i),\ D_h(\varphi_i) = Q_i .$$

Taking (ij)-entry of the left-hand side of (4.1) we obtain, using (4.2) and (2.17)

$$\begin{aligned}&h[D(Q_i)\xi + Q_i](\xi+u)^r Q_j - Q_i (u-\xi)^r [Q_j - \xi D(Q_j)^\dagger] h \\ &= [\text{by (4.20) and its adjoint}] \\ &= h(\xi+u)^{r+i} Q_j - Q_i (u-\xi)^{r+j} h = [\text{by (4.18) and its adjoint}] \\ &= h \sum_\alpha \binom{r+i}{\alpha} Q_{i+r+j-\alpha} \xi^\alpha - \sum_\alpha (-\xi)^\alpha \binom{r+j}{\alpha} Q_{r+j+i-\alpha} h \\ &= [\text{by (2.17), (4.5)}] = \Phi(B^r_{ij}) . \quad \square\end{aligned}$$

Proof of Theorem 4.0. The change of u into $-u$ amounts to considering $(\xi+u+h\xi^{-1})^\dagger$ instead of $\xi+u+h\xi^{-1}$. Thus, put $L=\xi+u+\xi^{-1}h$. Notice that for any pseudo-differential operator $X=\sum a_i\xi^i$, $a_i\in$ some C, and any $v\in C$, $\operatorname{Res}X=\operatorname{Res}\sum a_i(\xi-v)^i$. Thus,

$$\begin{aligned}\operatorname{Res}L^m &= \operatorname{Res}[\xi+(\xi-u)^{-1}h]^m = [\text{by (4.16)}]\\ &= \operatorname{Res}(\xi+\textstyle\sum\xi^{-i-1}Q_ih)^m = \Phi\operatorname{Res}(\xi+\textstyle\sum\xi^{-i-1}A_i)^m,\end{aligned}\tag{4.21}$$

which proves (ii).

Now notice that the change $u\to -u$ does not change the matrix B^{I} (1.9). By Theorem 4.4 with $r=0$, the map $\Phi:\Phi(A_i)=hQ_i$ is canonical between the Hamiltonian structure B^{I} (1.9) of Eqs. (1.4) and the matrix B^0 (2.16) representing the Hamiltonian structure of Eq. (2.13) with $k=0$. Therefore, the corresponding Hamiltonian derivations in $C_{u,h}$ and C_A are compatible for Φ-connected Hamiltonians, and (4.21) shows that the Hamiltonians are compatible indeed. This proves (i). □

5. Specializations

In this section we prove Theorem 5.0 and place it into a general context of specializations of the system (2.3).

Using the notation of Sect. 3, we rewrite (3.15a) using (3.2):

$$\partial p_0(m)=2(u\partial+\partial u)p_{-1}(m-2)+[2(h\partial+\partial h)+\partial(u-\partial)^2]p_0(m-2). \tag{5.1}$$

Corollary 5.2. *If $u=0$, that is,*

$$L=\xi+h\xi^{-1}, \tag{5.3}$$

then

$$p_0(2m+1)=0, \qquad m\in\mathbb{Z}_+. \tag{5.4}$$

Proof. Since $p_0(1)=0$, induction on m with the help of (5.1) yields $\{p_0(2m-1)=0 \Rightarrow \partial p_0(2m+1)=0\}$ which implies $p_0(2m+1)=0$, since $rk\,p_s(m)=m-s$ with $rk\,h^{(j)}=j+2$, $rk\,\mathbb{C}=0$. □

Corollary 5.5. *The equation*

$$L_t=[((P^\dagger)_{\geq 1})^\dagger, L],\ P=L^{2m+1},\ L=\xi+h\xi^{-1}, \tag{5.6}$$

is meaningful, that is, it preserves the subring $C_h\subset C_{u,h}$.

Proof. By (3.2), $u_t=\partial p_0(2m+1)$ for the full system (1.4), and we have just seen that $p_0(2m+1)=0$. □

For the variable h, we use (3.2) and (3.15b):

$$h_t=\partial p_{-1}(2m+1)=[\partial^3+2(h\partial+\partial h)]p_{-1}(2m+1). \tag{5.7}$$

On the other hand, $2p_{-1}(2m-1)=p_0(2m)$ by (3.10), and $p_0(2m)=\delta H_{2m+1}/\delta h$ by (3.1), since letting u vanish does not interfere with taking the variational derivative with respect to h. Thus,

$$h_t=\tfrac{1}{2}\partial\frac{\delta H_{2m+3}}{\delta h}=[\tfrac{1}{2}\partial^3+h\partial+\partial h]\frac{\delta H_{2m+1}}{\delta h}, \tag{5.8}$$

which proves the (ii) part of Theorem 5.0. Notice that $H_{2m} \sim 0$ by (3.1), since $p_0(2m-1)=0$ by (5.4). On the other hand, $H_{2m+1} \nsim 0$, since the quasiclassical limit of H_{2m+1} is $\frac{1}{2m+1}\binom{2m+1}{m} h^{m+1} \nsim 0$. Now, $H_3 \sim h^2$, so (5.8) yields

$$h_t = h_{xxx} + 6hh_x, \tag{5.9}$$

which is the KdV equation. Comparing (5.7) with (1.25) we see that each of Eqs. (5.6) differs by a constant multiple from the corresponding member of the KdV hierarchy. This proves the (i) part of Theorem 5.0.

The possibility of the specialization $u=0$ for the operator $\xi+u+h\xi^{-1}$ is a particular instance of the following general picture.

Theorem 5.10. *Let L be given as*

$$L = \sum^{n} \xi^i u_i. \tag{5.11}$$

Then the equations

$$L_t = [P_{\geqq k}, L],\ P = L^{m/n} \tag{5.12}$$

preserve the relation

$$L^\dagger = (-1)^n \xi^k L \xi^{-k} \tag{5.13}$$

for

$$m = 2M+1, \qquad M \in \mathbb{Z}_+ \tag{5.14}$$

(recall that $k=0, 1$ or 2).

Proof. Since (5.13) is equivalent to $\mathscr{L}^\dagger = -\xi^k \mathscr{L} \xi^{-k}$ for $\mathscr{L} = L^{1/n}$, we can restrict ourselves to the case $n=1$ only.

Lemma 5.15. *Suppose*

$$L = \sum^{1} \xi^i u_i;\ P = L^m = \sum_s p_s(m)\xi^s, \qquad m \in \mathbb{Z}_+; \tag{5.16}$$

$$L^\dagger = -\xi^r L \xi^{-r}, \qquad r \in \mathbb{Z}. \tag{5.17}$$

Then

$$(P\xi^{-r})^\dagger = (-1)^{m+r} P\xi^{-r}, \tag{5.18}$$

$$(P_{\geqq r}\xi^{-r})^\dagger = (-1)^{m+r} P_{\geqq r}\xi^{-r}. \tag{5.19}$$

Proof. $P^\dagger = (L^m)^\dagger = (L^\dagger)^m = [\text{by } (5.17)] = (-1)^m \xi^r L^m \xi^{-r} = (-1)^m \xi^r P \xi^{-r}$, hence $(P\xi^{-r})^\dagger = (-1)^r \xi^{-r} P^\dagger = (-1)^{m+r} P\xi^{-r}$. Using $P = P_{\geqq r} + P_{<r}$, we rewrite (5.18) in the form

$$(P_{\geqq r}\xi^{-r})^\dagger + (P_{<r}\xi^{-r})^\dagger = (-1)^{m+r}[P_{\geqq r}\xi^{-r} + P_{<r}\xi^{-r}], \tag{5.20}$$

and picking out the differential part of (5.20) yields (5.19). □

We have to show that the relations $L^\dagger = -\xi^k L \xi^{-k}$ and $L_t = [P_{\geqq k}, L]$ result in

$$L_t^\dagger = -\xi^k L_t \xi^{-k}. \tag{5.21}$$

The left-hand side of (5.21) is $([P_{\geq k}, L])^{\dagger} = [L^{\dagger}, (P_{\geq k})^{\dagger}] =$ [by (5.19) with $r=k$] $= [-\xi^k L \xi^{-k}, \ -\xi^k P_{\geq k} \xi^{-k}] = \xi^k [L, P_{\geq k}] \xi^{-k} = -\xi^k [P_{\geq k}, L] \xi^{-k}$, which is the right-hand side of 5.21. □

Remark 5.22. The KdV hierarchy is associated with the Lax operator $L = \xi^2 + h$. This operator can be viewed as a member of two different series: 1) general scalar Lax operators $L = \sum_{i=0}^{n} u_i \xi^i$, $u_n = 1$, $u_{n-1} = 0$; 2) specialized Lax operators $L = \sum_{i=0}^{n} u_i \xi^i$, $u_n = 1$, $u_{n-1} = 0$, satisfying the condition $L^{\dagger} = (-1)^n L$. The second point of view, which we took in Theorem 5.10, leads to a rather nontrivial theory in the standard case $k=0$ ([19, 20]).

Remark 5.23. The flows (5.12)–(5.14) are all nontrivial except when $k=0$ and $m \in \mathbb{N}n$, which can be seen by going to the quasiclassical limit and changing the commutator in the right-hand side (5.12) into the Poisson bracket. The same type of reasoning shows that the c.l.'s H_{2M+1} remain nontrivial after specialization (5.13) is imposed. It seems very likely that the c.l.'s H_{2M} become trivial but I couldn't prove this in general.

We conclude this section by discussing other specializations of the general operator

$$L = \sum_{-k}^{n} u_i \xi^i \tag{5.24}$$

for $k = 1, 2$. Firstly, since

$$L_t = [((P^{\dagger})_{\geq k})^{\dagger}, L], \tag{5.25}$$

it follows that $\partial_t(L_{<0}) = [((P^{\dagger})_{\geq k})^{\dagger}, L_{<0}]_{<0}$, and hence

$$\{u_i = 0 \mid -k \leq i < 0\} \tag{5.26}$$

is an invariant submanifold (speaking geometrically) of (5.25). Now assume that we have already reduced (5.24) on (5.26) and let us write the resulting L in the *left* form:

$$L = \sum_{i=0}^{n} \xi^i v_i \,. \tag{5.27}$$

If $P = \sum_s \xi^s p_s$, then

$$L_t = \left[\sum_{s \geq k} \xi^s p_s, \sum_{i \geq 0} \xi^i v_i \right]. \tag{5.28}$$

It follows that (5.28) has the invariant submanifold $J_r^k := \{v_i = 0 \mid 0 \leq i \leq r\}$ for each $r < k$. Thus, we have three possibilities:

$$J_0^1 = \{v_0 = 0\}, \ k = 1\,, \tag{5.29}$$

$$J_0^2 = \{v_0 = 0\}, \ k = 2\,, \tag{5.30}$$

$$J_1^2 = \{v_0 = v_1 = 0\}, \ k = 2\,. \tag{5.31}$$

Consider for example the first nontrivial case: $n=2$ on J_0^1 (5.29). Here

$$L = \xi^2 + \xi v\,, \tag{5.32}$$

and for $P=\xi$ we obtain $v_t=\partial v$. The next nontrivial flow is provided by $P=L^{3/2}$: since $L^{1/2}=\xi+\frac{1}{2}v+a\xi^{-1}+b\xi^{-2}+\ldots$, where $a=\frac{1}{4}v_x-\frac{1}{8}v^2$, $b=-\frac{1}{2}(\partial+v)(a)$, we have $(((L^{3/2})^\dagger)_{\leqq 0})^\dagger=-b+\ldots$. Hence $L_t=\xi v_t=[\xi^2+\xi v,\ -b+\ldots]=\xi 2\partial(-b)$ $=\xi\partial(\partial+v)(a)$, so that

$$v_t=\partial(v_{xx}/4-v^3/8)=(2v_{xxx}-3v^2v_x)/8, \tag{5.33}$$

which is just the modified KdV (m-KdV) equation.

Remark 5.34. The Lax representation $(\xi^2+\xi v)_t=[K,\ \xi^2+\xi v]$ with $K=\xi^3+\frac{3}{2}v\xi^2+(\frac{3}{4}v^2+\frac{9}{4}v_x)\,\xi+\frac{3}{4}(v^2+v_x)_x\ [=\xi^3+\frac{3}{2}\xi^2 v+\xi\frac{3}{4}(v^2-v_x)=(((L^{3/2})^\dagger)_{\geqq 1})^\dagger]$ for the m-KdV Eq. (5.33) was found by Knörrer [21].

Let us show that the higher flows

$$L_t=[((P^\dagger)_{\geqq 1})^\dagger, L],\ L=\xi^2+\xi v, P=L^{m/2}, m\equiv 1\ (\mathrm{mod}\,2) \tag{5.35}$$

are exactly the higher m-KdV equations (for even m, $P=((P^\dagger)_{\geqq 1})^\dagger$ and $L_t=[((P^\dagger)_{\geqq 1})^\dagger, L]$ is a trivial equation). First, the quasiclassical arguments show that Eqs. (5.35) are all nontrivial, and c.l.'s $H_{2m+1}=\dfrac{2}{2m+1}\operatorname{Res}L^{(2m+1)/2}$ are nontrivial as well. They are also homogeneous (of weight $2m+2$) in the grading $rkv^{(j)}=j+1$. Now, for $L^{m/2}=\sum\xi^s p_s(m)$ we have

$$L_t=\xi v_t=\Big[-\sum_{s\leqq 0}\xi^s p_s(2m+1), \xi^2+\xi v\Big]_{\geqq 1}=\xi 2\partial p_0(2m+1),$$

so that

$$v_t=2\partial p_0(2m+1). \tag{5.36}$$

Also, since $dH_m=\dfrac{2}{m}d\operatorname{Res}L^{m/2}\sim\operatorname{Res}(dL\circ L^{(m-2)/2})=\operatorname{Res}[\xi dv\sum\xi^s p_s(m-2)]$ $\sim\operatorname{Res}[dv\sum\xi^s p_s(m-2)\xi]=dv[p_{-2}(m-2)-\partial p_{-1}(m-2)]$ by (1.16), we obtain

$$p_{-2}(m)-\partial p_{-1}(m)=\frac{\delta H_{m+2}}{\delta v}. \tag{5.37}$$

It's not exactly what we need, which is $p_0(m)$, as (5.36) shows. We continue as follows. From the double identity

$$\sum\xi^s p_s(m+2)=(\xi^2+\xi v)\sum\xi^s p_s(m)=\sum\xi^s p_s(m)(\xi^2+\xi v), \tag{5.38}$$

we find

$$p_{-1}(m+2)=p_{-3}(m)+vp_{-2}(m)+v^{(1)}p_{-1}(m) \tag{5.39}$$

$$=p_{-3}(m)-2\partial p_{-2}(m)+\partial^2 p_{-1}(m)+p_{-2}(m)v-v\partial p_{-1}(m), \tag{5.40}$$

$$p_0(m+2)=p_{-2}(m)+vp_{-1}(m). \tag{5.41}$$

Comparing (5.39) and (5.40) we find

$$vp_{-1}(m)=-2p_{-2}(m)+\partial p_{-1}(m), \tag{5.42}$$

and substituting this into (5.41) we obtain

$$p_0(m+2)=-p_{-2}(m)+\partial p_{-1}(m). \tag{5.43}$$

Together with (5.37) this results in

$$p_0(m) = -\frac{\delta H_m}{\delta v}, \tag{5.44}$$

and (5.36) finally yields

$$v_t = -\partial \frac{\delta H_{2m+1}}{\delta v}, \tag{5.45}$$

which is the standard Hamiltonian form of the m-KdV hierarchy. Since $rkH_{2m+1} = 2m+2$ and $H_{2m+1} \nsim 0$, the uniqueness of the conservation laws of the m-KdV equation [22] implies that H_{2m+1}, up to a constant multiple, is equivalent to the standard c.l. $\#(2m+1)$ of the m-KdV hierarchy. Since the Hamiltonian form of this hierarchy is the same as (5.45), the flows too differ at most by a constant multiple.

In most theories of integrable systems the centerpiece is a (generalized) Miura map. Since we have realized the classical modified KdV hierarchy inside the $k=1$-case of the general system (2.3), the reader will undoubtedly wonder about the realization of the classical Miura map in our general context. Here is the answer. For $k=1$ and $L = \sum^{n} \xi^i u_i$, $P = L^{m/n} = \sum_s p_s \xi^s$, with $L_t = [P_{\geqq 1}, L] = [-P_{\leqq 0}, L]$, we have $u_{n-1,t} = n\partial p_0$. Therefore, if we introduce a new variable w: $\partial w = -\frac{1}{n} u_{n-1}$, we can lift the system $L_t = [P_{\geqq 1}, L]$ from the ring $C_1 = C_{u_{n-1}, u_{n-2}, \ldots}$ into the ring $C_2 = C_{w, u_{n-2}, \ldots}$. Now consider the following conjugation:

$$\mathscr{L} = e^{-w} L e^{w}, \mathscr{P} = e^{-w} P e^{w} = \mathscr{L}^{m/n}. \tag{5.46}$$

Then

$$\begin{aligned} \mathscr{L}_t &= e^{-w} L_t e^{w} + [\mathscr{L}, w_t] = [\mathscr{L}, e^{-w} P_{\leqq 0} e^{-w} - p_0] \\ &= [\mathscr{L}, -e^{-w} P_{\geqq 1} e^{w} - p_0] = [\mathscr{L}, \mathscr{F}_{<0}] = [\mathscr{L}, -\mathscr{H}_{\geqq 0}], \end{aligned}$$

where $\mathscr{F} = \mathscr{F}_{<0} = e^{-w} P_{\leqq 0} e^{w} - p_0$, $\mathscr{H} = \mathscr{H}_{\geqq 0} = e^{-w} P_{\geqq 1} e^{w} + p_0$. Since $\mathscr{F} + \mathscr{H} = \mathscr{P}$, we see that $\mathscr{F} = \mathscr{P}_{<0}$, $\mathscr{H} = \mathscr{P}_{\geqq 0}$, and we arrive at the standard $k=0$-case

$$\mathscr{L}_t = [\mathscr{P}_{\geqq 0}, \mathscr{L}] = [-\mathscr{P}_{<0}, \mathscr{L}]. \tag{5.47}$$

The map $L \to \mathscr{L} = e^{-w} L e^{w}$ is the desired Miura map. Since $e^{-w} \xi^i e^{w} = (\xi + w^{(1)})^i = \left(\xi - \frac{1}{n} u_{n-1}\right)^i$, we see that the Miura map amounts to changing ξ into $\xi - \frac{1}{n} u_{n-1}$:

$$\sum^{n} \xi^j U_j = \sum^{n} \left(\xi - \frac{1}{n} u_{n-1}\right)^i u_i, U_n = 1, U_{n-1} = 0, u_n = 1. \tag{5.48}$$

In particular, for $L = \xi^2 + \xi v$ (5.32), $\xi^2 + u = (\xi - \frac{1}{2}v)^2 + (\xi - \frac{1}{2}v)v = \xi^2 + \frac{1}{2}v_x - \frac{1}{4}v^2$, hence $u = \frac{v_x}{2} - \frac{v^2}{4}$, and this is the usual Miura map. For $n=1$, we have from (5.48):

$$\sum_{-\infty}^{1}(\xi-u_0)^i u_i=\xi-u_0+u_0+\sum_{i\geqq 0}(\xi-u_0)^{-i-1}u_{-i-1}$$
$$=[\text{by the adjoint of (4.9)}]$$
$$=\xi+\sum_{i\geqq 0}\sum_{\alpha\geqq 0}\xi^{-i-1-\alpha}(-1)^\alpha\binom{-i-1}{\alpha}Q_\alpha(u_0)u_{-i-1},$$

so that

$$U_{-k-1}=\sum_{i+\alpha=k}(-1)^\alpha\binom{-i-1}{\alpha}Q_\alpha(u_0)u_{-i-1}. \tag{5.49}$$

Suppose now, that $L=\xi+u+\xi^{-1}h$. Then

$$\mathscr{L}=\xi+(\xi-u)^{-1}h=[\text{by (4.16)}]=\xi+\sum_{i\geqq 0}\xi^{-i-1}Q_i(u)h=\xi+\sum_{i\geqq 0}\xi^{-i-1}A_i, \tag{5.50}$$

and Theorem 4.4 (for $r=0$) says that the Miura map (5.50) is canonical, as it should be in general. It is very likely that the full Miura map (5.49) is also canonical between the Hamiltonian structures B^0 (2.17) of the $k=0$-case and (2.20) of the $k=1$-case respectively; the quasi-classical limit of this map is indeed canonical between the quasiclassical limits of the corresponding Hamiltonian structures (this is another generalization, different from the $r=0$-case of Theorem 4.4, of the compatibility Theorem 0.7 in [5]).

Remark 5.48. In physical language, the interpretation of the Miura map as a conjugation was proposed by J. Gibbons [23].

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