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VIRTUE THEORY OF MATHEMATICAL PRACTICES

Mathematizing as a virtuous practice: different narratives and their consequences for mathematics education and society

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Abstract
There are different narratives on mathematics as part of our world, some of which are more appropriate than others. Such narratives might be of the form 'Mathematics is useful', 'Mathematics is beautiful', or 'Mathematicians aim at theorem-credit'. These narratives play a crucial role in mathematics education and in society as they are influencing people's willingness to engage with the subject or the way they interpret mathematical results in relation to real-world questions; the latter yielding important normative considerations. Our strategy is to frame current narratives of mathematics from a virtue-theoretic perspective. We identify the practice of *mathematizing*, put forward by Freudenthal's 'Realistic mathematics education', as virtuous and use it to evaluate different narratives. We show that this can help to render the narratives more adequately, and to provide implications for societal organization.

Keywords Philosophy of mathematical practice · Narratives of mathematics · Images of mathematics · Values in mathematics · Mathematics for human flourishing · Mathematical beauty · Theorem credits · Mathematizing · Realistic mathematics education

> Probably no area of human activity is as afflicted as mathematics with a gap between the public perception of its nature and what its practitioners believe it to be.
>
> (Barbeau 1990, p. 42) as quoted in Picker and Berry (2000).

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1 Introduction

When the second author participated in an award ceremony for the local mathematics olympiads there was a greeting by the states minister of education. It started along the lines of "I've always been bad at mathematics, but I am happy that there are nerds out there who already know much more than most adults", followed by a pause for applause. An unusual theme to evoke there, but it is a repeating one throughout social events, may it be dinner parties or talk shows. A phrase such as "I've always been bad at mathematics" is a safe bet to gain sympathy from other participants, guests and the audience (as long as they are not mathematicians, which causes problems in the mentioned award ceremonies). Sometimes this narrative gets pierced by a much more positive narrative, something along the lines of "Look, mathematics is everywhere: in your smartphone, it brought us to the moon and made cars possible."[1]

We understand a narrative of mathematics as a story that tells us about the role of mathematics in our world and about the people doing mathematics, the professional mathematicians. A narrative aims at generalizing from the specific features, grasping the crucial ideas and putting them together in an easily imaginable picture. Because of this, narratives are hugely important for the ideas of mathematics which are entertained by the people in our society. If people use a certain narrative in their discourse about mathematics, then this can have an impact on society on a personal level (people might be discouraged to engage with mathematics since everyone tells them that it is too hard),[2] on the level of a group of people for whom it is important to understand a complex situation (a mathematical model of the situation might too quickly be interpreted as a correct description of the real situation), or on the level of reflecting about the nature of mathematical activity itself (mathematicians might be seen as serving the applied sciences such as physics or economics by providing mathematical tools).

In this paper, we want to analyze different narratives of mathematics and suggest that *mathematizing as a virtuous practice in its own right* is a better narrative of mathematics than, for example, extrinsic narratives which focus on the results of mathematical activity and the application of mathematics in non-mathematical contexts. By 'better' we mean that the mathematizing-narrative describes mathematical practice more adequately and that it allows for a shift in mathematics education that yields beneficial outcomes for our society. We argue that the *fundamental activity of doing mathematics,* or, more precisely, of *introducing, using, varying, applying … mathematical symbolism* is a virtuous practice—what we call *mathematizing*, drawing on Freudenthal's research in mathematics education.[3] Mathematizing means modelling a context in mathematical terms, which includes individual choices on the component factors of the model. We argue that mathematizing, parallel to virtues such as *art appreciation* or *art production*, is beneficial for personal flourishing as it opens up a

[1] For an in-depth discussion of such a narrative, see the work of Colli et al. (2014).

[2] Bishop (2012) investigates the importance of identity-narratives of mathematics students and finds that "the ways people talk and interact are powerful influences on who they are, and can become, with respect to mathematics" (p. 34).

[3] This paper is no text exegesis in the sense that we take the freedom to depart from Freudenthal's elaboration where this seems more adequate to us.

new aspect of reality—or at least a new perspective on it—that is not available without mathematizing. This virtue narrative focusing on mathematizing is better than other competing narratives that are currently more present in society. The latter often hide the arbitrary component factors of mathematical models which depart from the real-world context for reasons of reducing complexity or favoring simplicity of the mathematical tools for example. A mathematical model is often perceived as an objective and true representation of a societal context. If it, however, becomes clear that any normative conclusion, which is (partly) grounded in such a model, is directly connected to the choices made by building the model, then we can reduce the risk associated with the authority of formal tools in public debates.

The interaction between narratives and mathematics is considered on very different levels in the literature. There is formal narratology, or the question how mathematics is depicted in literature and its intersection with such.[4] Margolin (2012) analyzes how we could literally find narratives in proofs, where a proof is seen as a story and mathematical objects as actors. We want to focus on another point: the narratives we incorporate as a society concerning mathematics and its effects. As mentioned by Corfield (2012), the goals and aims of mathematicians are manifold. We here conceive of a narrative as the softly codified stereotypes and tropes we refer to in our day to day discourse, a broader nowhere directly told view or picture we have about a discipline. As most people in our society are not mathematicians by practice, this means especially narratives about mathematics or mathematicians in virtue of their goals, aims and practices.

In Sect. 2, we briefly introduce the notion of mathematizing, mostly focusing on Freudenthal's work. His approach is mainly used to understand education for primary school level children. We explain how it relates to mathematics at a research level by linking it to the mathematical activity of theory-building as introduced and popularized by Gowers (2000). We then give a case study about how we can understand mathematical logic as an instance of mathematizing. In Sect. 3, we discuss why this should be seen as an intellectually virtuous practice. In Sect. 4, we report and analyze narratives of mathematics that we encountered during our university studies as two trained mathematicians, a series of interview studies and research conferences in the fields of mathematics, philosophy, and mathematics education. In this sense, this analysis does not constitute a precise cartography of beliefs about the nature of mathematics but should be seen as contributing an exploratory exposition. We then evaluate their adequacy and the problems we encounter and see how the narratives change when we add the assumption that mathematizing is the main characteristic of mathematical activity.

2 Mathematizing

Starting with the words of the founder of the Realistic Mathematical Education ('Realistische mathematische Erziehung', in short RME):

[4] See for instance Gowers (2012) or Teissier (2012).

> In its first principles mathematics means mathematizing reality, and for most of its users this is the final aspect of mathematics, too. For a few ones this activity extends to mathematizing mathematics itself. The result can be a paper, a treatise, a textbook. A systematic textbook is a thing of beauty, a joy for its author, who knows the secret of its architecture and who has the right to be proud of it. […] What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics. (Freudenthal 1968, p. 7)

A detailed analysis of Freudenthal's notion of mathematizing can be found in Freudenthal (1968, 1973, 1983, 1991) as a primary resource or van den Heuvel-Panhuizen and Drijvers (2014) for an exposition. The term 'realistic' is not closely related to 'reality' as it is understood in English. Van den Heuvel-Panhuizen explains that the original Dutch term 'zich realizeren' rather means 'to imagine' (van den Heuvel-Panhuizen 2003, pp. 9–10). So, *realistic* mathematical education does not especially focus on a close relation to reality but on the process of mathematizing phenomena from the physical world as well as abstract structures. We note that this shifts also from "mathematics as a closed system" towards the action-based verb "mathematizing".

We now sketch how we understand mathematizing as an activity in general and in mathematics and give the case study of mathematizing metamathematics.

2.1 On Freudenthal's notion of mathematizing

Since we employ mathematization as the core part of our narrative of mathematics, let us start with the practice of counting which stands at the beginning of every mathematical career. This is because counting is closely related to the physical world for the trivial reason that there is nothing else to be mathematized: it can be connected to the steps of a stairway, one's fingers or anything which could be aligned in a row. This is reflected in a lot of theories in mathematics education, most prominently in the work of Tall (1980a, 1980b, 1991, 2013). He refers to the *embodied* world as the first layer of his model of mathematical development and sees this mathematical embodiment of the world as the first steps towards mathematizing the world.

Starting from this idea, RME evolved first in the Netherlands as opposed to the "New Maths" movement, which is based on a reduction of mathematics to set theory.[5] Treffers (1987) argues (and Freudenthal adapted his claims) that there are two kinds of mathematization. First *horizontal mathematization*: abstracting from observed phenomena and solving problems found in our physical world with adequate mathematical tools. For example, structuring an unstructured set in order to count its members is already horizontal mathematization. In contrast, *vertical mathematization* is what is called "mathematizing mathematics" in the quote of Freudenthal at the beginning of Sect. 2. This includes the organization of symbols and the study of these concepts, which were abstracted from the physical world.

[5] See Dieudonné (1961) for an early exposition of the *"New Math"* project. The main goal of this movement was to put mathematics education on a rigorous foundation. This often included starting with set theory and introducing new concepts in a more axiomatic and abstract way. A typical example for an idea of the New Math project is the introduction of groups in elementary schools.

The usage of symbolic notation is a key component of mathematizing. This might mean in the simplest cases, or at the beginning of an academic mathematical career, that we mathematize when we use numbers, sketches, and symbolisms to identify the crucial structures of a context. The usage of symbolic notation to encode mathematical content is also an important tool for the working mathematician. Kießwetter (2006) for instance stresses its role for overcoming limits of our working memory. It is an important feature of mathematics that these symbols and concepts themselves can become the object of mathematical study, i.e., that symbolizing itself can be iterated. For example, functions which map numbers to numbers might first be seen on one layer above the object level containing only numbers. But in higher mathematics, functions themselves are members of function-spaces that are studied by mathematical means.

A second fundamental feature of mathematizing is its individual nature. Mathematizing can be done in different ways, since it includes individual choices. For example, we can employ a continuous or discrete model to represent fine grained sand, we can see a doodle on a sheet of paper as an object in Euclidean space, or in a topological space. We could see aligned spoons as points in a grid, as rather complex shapes in Euclidean space, or even on a higher dimensional manifold. Therefore, one consequence of assuming that mathematizing is the main characteristic of mathematical activity is to emphasize the *individual choices* made in the mathematical formulation of a problem. In contrast, mathematics is nowadays often taught with the implicit assumption that there is a unique way of formulating and solving a given problem in a rather mechanical fashion. Mason and Watson speak for instance about their observation that "Mathematics is often seen by learners as a collection of concepts and techniques for solving problems assigned as homework" (Mason and Watson (2008) before putting their more variation focused approach forward. The authors' experiences also contrast with such a mechanical idea. After studying mathematics and attending mathematical research conferences as well as informally talking to mathematicians, we think that the situation is adequately described as follows: for professional mathematicians, it is part of their everyday life that there are different possibilities to formulate a problem and different methods are suitable to tackle a given problem. They often experience that some formulations are well suited for an easy solution, while others seem to require complicated constructions. We want to emphasize this gap between the practice of mathematics and its perceived objectivity in application: no mathematical way of formulating or solving a problem is the only correct way, there are always various ways and it is a challenge as well as it can be a joy to choose between those options or to come up with a new one. Our focus on mathematizing aims to highlight this aspect. If one is aware of this observation, it is obvious that the choices made in the mathematizing process should be made visible and can be discussed. We will come back to this aspect in more detail in Sects. 3.1 and 3.2.

2.2 Mathematizing in mathematics

In order to bring forward our idea of mathematizing as an integral part of mathematical research, we analyze the notion a bit further. Freudenthal indicates three main components of mathematizing:

> [T]he origin of the term mathematising [is] an analogue to axiomatising, formalising, schematising. … [I]t is not unusual, in particular in education, to restrict the term to one of its components. I myself insist on including in this one term the entire organising activity of the mathematician, whether it affects mathematical content and expression, or more naive, intuitive, say lived experience, expressed in everyday language. But let us not forget about the individual and the environmental dependence of "lived" and "everyday life" on expanding reality and progressing linguistic sophistication! (Freudenthal 1991, p. 31)

Firstly, we learn from Freudenthal that axiomatizing, formalizing and schematizing are the main components of the process of mathematizing.[6] Second, we learn that mathematizing is an organizing activity which can affect mathematical content and language as well as lived experience and everyday language. Hence, Freudenthal insists on the similarity of mathematical activity applied to everyday life contexts and mathematical activity in mathematical research itself. This is one of the key aspects of mathematics on which we follow Freudenthal. We take it to be inherent to mathematizing that it can be applied to various contexts, be it a phenomenon in the physical world or a mathematical research problem.

For further elucidation of the mathematizing process, we distinguish between content and form. Freudenthal points out that *the interplay of content and form* is one important characteristic of mathematical activity and emphasizes that form can become content. This aspect is significant to understand mathematizing in mathematics. Recall from above that mathematizing is mainly an organizing process:

> In mathematics … organising and reorganising is a continuing affair, and the newly acquired organisation forms may become content in the sense of subject matter to be examined as such. (Freudenthal 1991, p. 11)

The process of mathematizing usually starts in the physical world. By an emphasis on form, observed phenomena are structured[7] and people think about forms, which can be seen as (possibly tiny) mathematical models for the phenomena, for example, one could think about aligning dots when counting some candies lying disorganized on the table. Naturally, mathematicians are well trained in the process of mathematizing, and many mental objects and mental activities are part of their reality. Many forms that are acquired through mathematizing observed phenomena became content that can again be affected by mathematizing.

We observed above that the main components of mathematizing are schematizing, formalizing and axiomatizing. When mathematical content is organized in that way, mathematicians work towards the organization of a mathematical field; they try to find an adequate form, an explicit language and the fundamental laws governing the field.

[6] Mathematizing can be analyzed on different scales of granularity, in mathematics education we can find for instance the concrete study of involved processes, like *Superzeichenbildung* as discussed for instance by Nolte and Pamperien (2017) or Kießwetter (2006): when we solve a problem, we might assign a new sign to a structure we analyzed in detail beforehand.

[7] Freudenthal tells us that "[s]tructuring, whether applied to products or to processes, means emphasising form. The first non-trivial structure as such, i.e., whole number as the product of the process of counting, begot rich process and product content which, organized by ever new structures, in turn begot new contents—a never ending cyclic process" (Freudenthal 1991, p. 10).

Important examples in this activity of a research mathematician include preparing a lecture or writing a textbook.

We have a look at each of the three activities in turn. Schematizing is a way of mathematizing that focuses on content: "schematising … is the counterpart to axiomatising and formalising insofar as contents rather than abstract form and language are concerned" (Freudenthal 1991, p. 31). By schematizing, the content is ordered. The result is a scheme that represents the context in a faithful way pointing out its crucial features but being closely connected to the context itself. An example for a scheme is the representation of the multiplication 5 times 8 by a rectangle with 5 rows each of 8 points.[8] Schematizing prepares the recognition of the structures that appropriately represent the problem, but it is a second step to isolate the forms of such a scheme since the scheme is always connected to reality, whereas the form is not. In Freudenthal's words, schematizing is the activity "to create schemes to fit reality" (Freudenthal 1991, p. 31).

Formalizing, in contrast, is a way of mathematizing that focuses on form. We search for appropriate symbols and formalisms to describe a scheme, a mental object, or an activity. Thereby, we get more and more sophisticated in our linguistic expression, which is necessary to precisely grasp the characteristics of the structures. We could start by describing a problem in natural language, and by trying to be exact in what we say, we start to use more symbolic language, looking for appropriate symbols to express our observation or thoughts.

Axiomatizing is the activity of identifying the rules which govern a context. Freudenthal gives the development of group theory as one example for an axiomatizing activity within mathematics:

> [The] first striking example [for the technique of axiomatizing] was groups. From the turn of the 18th century onwards, mathematicians were confronted with mappings of sets upon themselves, often singled out by invariance properties, and were led to compose such mappings. In this way they became acquainted with sets of transformations which, under composition, automatically satisfied the well-known postulates, required later on for groups. Cayley, in 1854, took the unifying step to define, by means of these postulates, the (finite) object he called a group; yet not before 1870 did this new conception become whole-heartedly accepted by leading creative mathematicians, and then also in infinite substrates. (Freudenthal 1991, pp. 30–31)[9]

Building on these first ideas of what mathematizing means for professional mathematicians, we relate the notion to further mathematical research activities. The priority of a mathematician whose main activities are schematizing, formalizing and axiomatizing is, we argue, *theory-building* rather than problem-solving. Gowers (2000) introduced this prominent distinction in his paper *Two Cultures of Mathematics*, where he emphasizes that "when I say that mathematicians can be classified into theory-builders and problem-solvers, I am talking about their *priorities*, rather than making the ridiculous

[8] See Freudenthal's example 7 (Freudenthal 1991, p. 43).

[9] Maddy also gives an informative summary of this development of group theory with an emphasis on the question when and why the group concept was accepted by mathematicians (Maddy 2011, p. 7 and pp. 134–135).

claim that they are exclusively devoted to only one sort of mathematical activity" (original emphasis, Gowers 2000, p. 66).

We understand theory-building and problem-solving as two typical research activities of a professional mathematician which are different in their goals. Some mathematicians might have a clear tendency to prefer one of those activities. Relating this distinction to the mathematizing process, we argue that problem-solving requires a rather short mathematizing process, and a prolonged and continued mathematizing process leads to theory-building—this process might even include metamathematics, as we elaborate in the next section. For example, problem-solving typically does not include axiomatizing, and, among other activities, often applying a given method in a structured context, which is not mathematizing in a narrow sense. As Freudenthal shows, mathematizing requires looking for essentials within a context. He provides a list of examples for what 'within a context' can mean, ranging from "within a situation and across situations" via "within a structure and across structures" to "within an axiomatic system and across axiomatic systems". He adds that looking across is important "[b]ecause discovering common features, similarities, analogies, isomorphisms is the way towards generalising" (Freudenthal 1991, all quotes on p. 35). The mathematizing process in problem-solving often consists in discovering similarities with known problems or with other mathematical structures in order to choose or develop an appropriate method that may be successfully applied to solve the problem. In contrast, the theory-building focuses on such similarities and common features as mathematical content and continues to mathematize those similarities and features to finally build a theory of them. Examples for theory-building in mathematics are the developments of group theory, set theory or homotopy type theory. The concepts of a *group*, a *set* and a *homotopy type* were each introduced as common structures which were isolated in a variety of mathematical contexts, since they adequately encode their similarities.

But, as Gowers (2000) himself observes, the boundaries between these notions are not sharp. For instance, the solution of some problems fueled the development of corresponding theories. The rather easily stated question of Fermat, whether there are any solutions for $m>2$ for $a^m + b^m = c^m$, was solved by Wiles and Taylor by translating it into a special form of cubic equations, namely elliptic curves. Important advancements in this theory made it possible to solve Fermat's Last Theorem. On the other hand: these advancements of the theory can be labelled in large parts with a corresponding theorem, namely the solved Taniyama-Shimura-Conjecture. Especially theory building must not be bigger than problem solving in the sense that there are problems solved over a long period of time with many contributors as well and there are of course more local examples of theory building in contrast to something like the emergence of group theory.

Although there is no sharp division between theory-building and problem-solving practices, we nevertheless argue that mathematizing is closer to theory-building since both activities can have fruitful interactions. Our focus on mathematizing thus emphasizes an essential part of mathematical activity that can be different from solving long standing open problems. Here, the reader should get a first impression of how mathematizing can help to render narratives more adequately.

2.3 Mathematizing metamathematics: the case of logic

In this section we briefly speak about the formalization of the notion of *proof* within mathematical logic and, in particular, within proof theory. There are several examples how mathematical fields emerged from the mathematization of informal notions. One may argue that topology or geometry formalizes our spatial intuitions or that arithmetic was born out of basic operation on collections of clearly distinguishable objects.[10] But there are two examples of mathematized notions which pertain to mathematical practice itself. Those are the notion of *algorithm* and the notion of *proof*. We will focus on the latter as the former is extensively discussed as the Turing-Church-thesis.[11]

The *Grundlagenkrise* in the early 20$^{\text{th}}$ century led to the field of *mathematical logic/metamathematics* and provided what is currently the most accepted framework for all mathematics, namely, first order logic and set theory. We cannot reconstruct the whole development of the proof concept in mathematical logic, and rather pick major contributions, which we think are relevant to understand the mathematizing that is going on in this debate. The Hilbert's school needed to find a formal counterpart to a real-world phenomenon, namely proofs, leading to the development of *proof theory*, whose well defined syntactical objects of inquiry are *derivations in formal systems*. The formal and informal concepts are very closely related, and we often use "proof" to refer to both concepts. For ease of reference we name the formal notion "derivation" or "formal proof" and the informal one "informal proof". Clearly, these two notions differ. The neglect of those differences was for example called out by Davis and Hersh, who address

> the error of identifying mathematics itself (what real mathematicians really do in real life) with its model or representation in metamathematics, or, if you prefer, first-order logic. (Davis and Hersh 1998, p. 354)

In (Leitgeb 2009, p. 268) we find a table listing differences of both notions: while derivations are given by a formal syntax, informal proofs are given informally, the former allow no gaps, while the latter is more lenient, etc. Derivations also hardly feature in mathematical practice, since most mathematical reasoning is conducted outside concrete formal systems.[12] But why do we then say things like, Gödel's Incompleteness Theorem showed that "provable" and "true" are not extensionally the same thing?

We reason that "formal proof" abstracts from "informal proof" in a way that still keeps what we need to analyze for the purposes of mathematicians or in other words: the step from informal proofs to formal ones is simply an act of mathematization. Our judgement on the adequacy of this act of mathematization might change in the course of time. Before Einstein we would have thought that Euclidean geometry is the right model for space but now, we employ manifolds to represent certain spatial

[10] Numerous further examples can be found in Lakoff and Núñez (2000).

[11] Apparently, there is also literature on the Hilbert Thesis, see for instance Kahle and Oitavem (2018) or Hipolito and Kahle (2019). See below.

[12] Formal systems become more important due to recent developments in *formal mathematics*. This is a branch of mathematics using Automated Theorem Provers to generate derivations for non-trivial mathematical theorems, including the four-color theorem by Gonthier (2008) and the odd-order theorem by Gonthier et al. (2013).

features. The iterative refinement of scientific theories and the difference between the scientific model and their domain of discourse is analyzed in depth by philosophers of science.[13] In a similar fashion, we can have a debate whether formal proofs capture informal ones in the right way. Kahle (2019) mentions the first written occurrence of the affirmation of that question in a Handbook for Logic.

> [T]he informal notion of provable used in mathematics is made precise by the formal notion *provable in first-order logic*. Following a sug[g]estion of Martin Davis, we refer to this view as *Hilbert's Thesis*. (Barwise 1977, p. 41)

Here we will not focus too much on the "first order" part.[14] The exact interplay between these notions can be spelled out differently.

The adequacy of this act of mathematization can now be debated, which is mostly done in the debate on how closely such formal derivations would need to be related to the corresponding informal proofs. The most influential approach to this question is Azzouni's (2004) *Derivation-Indicator View* [DI-View], which suggests that there is an underlying derivation below an informal proof. He writes:

> [T]he day-to-day practice of mathematicians isn't to actually execute such derivations, but only [to] indicate, to themselves or to others in their profession, such derivations, it's clear why proof and not derivation must occupy centerstage in mathematical practice; and this despite the fact that, in a very clear sense, it's derivation which provides the skeleton for (the flesh of) proof. (Azzouni 2004, p. 95)

Azzouni notes himself that "indication" should not be overemphasized (Azzouni 2009, footnote 17). But it is unclear whether the informal proof is a kind of blueprint that can (maybe automatically) fill in details to complete it to a derivation as argued for instance by Carl and Koepke (2014). Tanswell raised the problem that there may be too many derivations fitting one proof (Tanswell 2015, 7, chapter 1). But those works would fit very well into the picture of mathematizing, where the connection between the object and the more abstract layer is not clear and up to modelling decisions.

In all these debates, we see that mathematizing mathematical activity itself requires a debate about the aspects that we want to take into account. Mathematical activity is a very rich context and by mathematizing it, we focus on certain aspects while neglecting others. Some views in these debates claim that modelling techniques which keep more qualitative information may be better suited. For instance, Lakatos (1976) stressed the dialectic nature of producing proofs. In his famous *Proofs and Refutations*, he investigates among other things the case of Euler characteristics. Mathematizing is certainly not a universal technique to model whatever context one considers. It will give a *mathematical* representation of a context. But, if one wishes for a mathematical representation of a context, then this is achieved by mathematizing it, which in turn requires one to make some simplifying assumptions and to make a choice on what will be included in the model. These choices and assumptions are made visible by

[13] See for example Bogen and Woodward (1988), Cartwright (1983), McMullin (1985), Weisberg (2007) or Woodward (2011). This does of course not mean that all disputes are settled.

[14] Berk (1982) analyzes in great detail whether first order logic is enough, or we need stronger logics.

mathematization and prevent one from (falsely) expecting *the* correct mathematical model.

2.4 Mathematizing as an intellectually virtuous practice

In this section, we present an argument for the claim that mathematizing is an intellectually virtuous practice. Our main thesis is that mathematizing reveals truth about reality and therefore leads to a substantially better understanding of it.

Typical intellectual virtues are humility, wisdom, or perseverance. King explains that "*[i]ntellectual* virtues are dispositions to think and act excellently as one carries out intellectual activities; that is, activities with intellectual ends" (King 2014, pp. 3782f) and intellectual ends or goods include "truth, knowledge, or understanding" (King 2014, p. 3792). Since mathematizing is a practice rather than a trait, we examine it here not as virtue but as a virtuous practice that is directed at the achievement of the intellectual good of understanding.

Mathematizing improves our understanding of societal problems and phenomena that we observe in the physical world. Nevertheless, we do not claim that we can understand these problems and phenomena completely by mathematizing them because mathematizing often includes abstraction and, therefore, eliminates some features of the problem or phenomenon under consideration that may be important. Mathematizing is one of many processes that lead to a better understanding. Other such processes could include the preparation and realization of experiments, literature research, or the formulation of good working hypotheses. However, mathematizing reveals an important aspect of the problem without which a full understanding cannot be achieved. Mathematizing does not just add a tiny improvement in understanding but an essential part of it.

In mathematics itself, mathematizing can lead to a full understanding of a mathematical context though it remains a matter of how far one wants to mathematize. We saw in Sect. 2.2 that problem-solving usually requires a shorter mathematizing process than theory-building. But theory-building stops at some point too and does not abstract any further. A continuation of the mathematizing process leads to a logical analysis of mathematical language and mathematical argumentation that is usually called metamathematics (see Sect. 2.3). Here we encounter again the aspect of preference, which is very fundamental to the mathematizing process. People have different preferences and they mathematize according to their preferences. Often, logic is too abstract for mathematicians. For the structures become poorer and poorer and, at some point, they are too poor to be still interesting for a mathematician, who prefers richer structures.[15]

[15] The notion of poor and rich structures is taken from (Freudenthal 1991, 1.2.1). As an example, he says: "As a geometric structure a tetrahedron is a richer structure than combinatorically. There is more one can say about it; within this structure one can, for instance, measure distances, edges, angles, surfaces and volume" (Freudenthal 1991, p. 20). Later, he explicitly describes first order logic structures as poor and mathematical structures as rich when he says that "Bourbaki's compass [shows how] to sail from the poor to the rich structures" (Freudenthal 1991, p. 131). He extends these notions to poor and rich contexts (Freudenthal 1991, 2.2.3), where rich contexts are better suited for education because pupils can ask more varied questions about a rich context than about a poor context.

Mathematicians improve their understanding of a mathematical context by mathematizing because they can understand *why* the method that applies to a problem works, and in which other contexts the method works. They do not lose the connection between the context and the method but focus on the mathematization of the context—to find appropriate schemes that fit the context, to become more sophisticated in their linguistic expression to speak about the context and to identify the laws that govern the context. In that way, the mathematizing mathematician understands more about the context than a non-mathematizing mathematician who applies a given method in a given context 'just because it works'.[16] For example, when we understand that some theorems about the real numbers are actually true due to the fact that we can see the reals as a topological space and not due to more specific properties of the real numbers. It can even trigger something like a paradigm shift and constitute whole new fields of studies.

For our argument that mathematizing is an intellectually virtuous practice, it is important to stress that mathematizing is not necessary to solve problems. People could solve a problem by applying an imposed or given method. Consider Freudenthal's example of the commutativity of the addition of natural numbers, in which he distinguishes between horizontal and vertical mathematization: "Replacing 2 + 9 by 9 + 2 may be due to horizontal mathematising if 2 and 9 are visually or mentally combined as linearly structured sets and their combination is read backwards. It may be vertically interpreted as soon as the law of commutativity is generally applied" (Freudenthal 1991, pp. 42f). The horizontal mathematizing process is a clear example in which the understanding of addition of natural numbers is improved. When someone is clear about the fact that 2 + 9 is identical to 9 + 2, for example, because 2 dots aligned and then 9 dots attached gives the same mental object as 9 dots aligned and 2 attached, the person can easily apply the same scheme to other cases of addition. After several such examples, the children might get by induction to a general law that applies whenever natural numbers are added. After realizing this to be a law, formulating it by using symbols, and giving it the name 'the law of commutativity', it can be applied with understanding, which, Freudenthal tells us, is vertical mathematizing. In contrast, if a teacher says that one can always switch the numbers when adding, and pupils apply the law to calculate 9 + 2 instead of 2 + 9 because the first is easier, we would not claim that they understood more about the addition of natural numbers. Instead, we would say that they rather applied an imposed rule.

Another non-mathematizing way to deal with a problem is trial and error. To stick to our example, a child could also have heard that, actually, one can switch the numbers when adding them and tries it out with, say, about ten or twenty examples, and, then is convinced that this holds generally. Although we would claim that this child understood addition a bit better than the one who applied the imposed rule, it still misses the important part to recognize that it *has to* be the same in every case, not just the observation that it gave the same result in every concrete example.

We emphasized that a problem can be solved in other ways, but mathematizing a problem leads to a substantially better understanding. One could read this argu-

[16] Of course, the application of a method 'just because it works' can be a fruitful starting point to reflect further on the context, the problem and the method, for example by looking for essentials also across problems and methods.

mentation as a requirement that one should always mathematize everything to better understand it. Given the time and mental resources that are needed for a mathematizing process, this is not what we think is the right conclusion. Instead, division of labor is called for. If one wants to better understand a problem, then it is recommended that one mathematizes it. However, if one is fine with solving it quickly and going on to the next task, one can easily leave the understanding to somebody else, apply a given strategy for solving it, and continue with one's work. Mathematizing is opposed to cookbook like teaching which requires children to memorize the recipes and to apply them in suitable contexts. This unnecessarily limits the children and the RME tradition wants more. The hope is that

> [i]f the students experience the process of reinventing mathematics as expanding common sense, then they will experience no dichotomy between everyday life experience and mathematics. Both will be part of the same reality. (Gravemeijer and Doorman 1999, p. 127).

The facts that mathematizing improves our understanding of a problem and that mathematizing is not necessary to engage with it lead us to argue that mathematizing is an intellectually virtuous practice since the practice is directed at the achievement of an intellectual good: understanding. Mathematizing is in the sense we argued a particularly good practice to address a problem.

3 Narratives of mathematics

If we want to understand the narratives of mathematics as they are understood by a member of the broader public, that is, a layman, we can look into the literature. Paul Ernest puts the public image of mathematics as follows:

> A widespread public image of mathematics is that it is difficult, cold, abstract, theoretical, ultra-rational, but important and largely masculine. It also has the image of being remote and inaccessible to all but a few super-intelligent beings with „mathematical minds". (Ernest 1995, p. 1)

There are empirically informed approaches (closely related to Ernest's work) to the study of the image of mathematics (or mathematicians); see for instance Sam (1999), who handed questionnaires to 548 people and conducted semi-structured interviews with 62 people. From this, she distilled that there are five main views:

1. "Utilitarian view: mathematics is primarily viewed in terms of its utilitarian value." (Sam 1999, p. 253)
2. "Symbolic view: mathematics is perceived as a collection of numbers and symbols, or rules and procedures to be followed and memorised." (ibid, p. 254)
3. "Problem solving view: mathematics is related to a set of problems to be solved." (ibid, p. 256)
4. "Enigmatic view: mathematics is seen as mysterious but yet something to be explored and whose beauty is to be appreciated" (ibid, p. 257)
5. "Absolutist or dualistic view: mathematics is perceived as a set of absolute truths, or as a subject of which always has right or wrong answers." (ibid, p. 257)

As we are doing, she links these images to mathematics education. She observes, "[i]n particular, [that] many respondents' images of mathematics seem to be linked to their experiences of learning mathematics in school" (ibid, p. 343).

A crucial element for narratives of mathematics is also the depiction of mathematics in pop culture, for instance in films like "*A beautiful mind", "PI", "The Man Who Knew Infinity"* and many more. Here, apparently, historic contingency comes into play: the subject's history (as our society in general) was not as colorful as it could have been. This depiction was analyzed for example in Moreau et al. (2009, 2010), who stressed a white, male, middle-class, heterosexual picture of mathematicians, a theme closely related to the quote of Ernest above. It was also analyzed how this had consequences for mathematics learners, for instance by Picker and Berry (2000). A crucial finding of this work is that hardly any boy drew a female mathematician, when asked to draw a stereotypical mathematician. Even girls drew in only ~ 20% of the cases female mathematicians in Finland, Sweden, and Romania. The UK and US surpassed these percentages with ~ 57% and respectively ~ 30%.

In this section, we analyze the following narratives of mathematics:

1. Mathematics is useful
2. Mathematics is beautiful
3. Mathematicians aim at deep understanding
4. Mathematicians aim at theorem-credit

The first narrative of mathematics that we will investigate coincides with the *utilitarian view* identified by Sam. Its main characteristic lies in the assumption that research mathematics serves later applications in science which in turn serve our society in the form of technical progress: mathematics is useful for our everyday life.[17] The second narrative—mathematics is beautiful—takes a quite different view by emphasizing an intrinsic value of mathematics, independent of any application for science and society. This view has great similarities to Sam's *enigmatic view*. The narrative that mathematicians aim at deep understanding also has similarities to the enigmatic view. Here, the emphasized intrinsic value of mathematics is of an epistemological nature and not of an aesthetic nature as in the previous narrative. The last narrative we are concerned with is motivated by sociological aspects of the community of research mathematicians: the norms of the community seem to support that theorem credit is one of the main values for mathematicians.

We will have a look at each of these narratives in turn and investigate how taking into account the idea that mathematizing is the main characteristic of mathematics changes the respective narrative. Our aim is to show that the narratives are rendered more adequate by stressing the role of mathematization as a virtuous practice and that our society would be better off with these improved narratives.

[17] This is also motivated by Ernest's quote above and the articles by Moreau et al. (2009, 2010). It is concerned with the possibly harmful stereotype of the typical mathematician persona. This narrative includes aspects of Sam's *symbolic view* (men are better in abstract thinking), *problem solving view* (you have to be a genius to solve a mathematical problem), and the *absolutist or dualistic view* (mathematicians tell you the one and only truth), but also a little bit of the enigmatic view (it is a great mystery what a genius mathematician can do).

3.1 Mathematics is useful

It is often reported that Galileo's dictum was that *the book of nature "is written in the language of mathematics*". Our society stresses that students should learn to apply mathematical tools as this is needed by natural sciences, engineering, and computer science. This is reason enough to include it into the manifold of STEM-initiatives with all the funding that comes along with them. Peterson (1991), as reported by Sam (1999), goes even further. She observes that:

> Even scientists and engineers whose jobs relate to mathematics 'often harbour an image of mathematics as a well-stocked warehouse from which to select ready-to-use formulae, theorems, and results to advance their own theories' (Peterson 1996). (Sam 1999, p. 14)[18]

In Germany, the *Kultusministerkonferenz* (2009) stresses that mathematical education is indeed a key component of the whole economic development in Germany. Here, we see also that this governmental institution is aware of another kind of beneficial outcome, namely that the individual profits. They stress that STEM abilities make it possible to participate in public discourse, adapt to technical improvements and become an active citizen. We can add that there might be other personality traits or virtues that might benefit from mathematical training. Among those are analytical thinking, frustration tolerance and patience or intellectual humility. This can be fostered due to the success of applied mathematics. Every engineering project or project in computer science will surely use some mathematical tools or knowledge.

This narrative is clearly very beneficial for mathematical research, especially as a justification. It seems that we strongly believe that even the most theoretic fields of study will find application in the long term. But this narrative is problematic in at least two senses:

1. It is an inadequate picture of mathematical practice.
2. It makes it harder to judge the value of mathematical results.

Concerning 1: The narrative overlooks that huge parts of mathematics (especially non-applied fields) might actually never aim for results that will be applied. Even worse, large chunks of mathematics are forgotten when the few specialists die or lose interest in the subject of their earlier studies.[19] The narrative focuses on the extrinsic values of mathematics. We do not do mathematics for its own sake in this narrative. This misses a large part of the practice: a lot of pure mathematicians would even stress that the abstractness and the lack of application is a motivating factor to work on their field of studies. We believe that the mathematizing narrative can incorporate the good part, it even explains the applicability or more precisely takes it as a fundamental notion because the first mathematical exercise necessarily originates

[18] Sam refers to Peterson (1996). Searching for new mathematics. (Articles on Public Understanding of Math). http://forum.swarthmore.edu/social/articles/ivars.html, but we did not find this source. Instead, Peterson (1991) contains the same quotation.

[19] As Gowers notes: "It is also true that many of the results proved by combinatorialists are somewhat isolated and will be completely forgotten (but this does not distinguish combinatorics from any other branch of mathematics)" (Gowers 2000, p. 69).

from the embodied world. It also stresses the process of doing mathematics in the sense of a process-oriented narrative, while also incorporating the role of the results in contrast to a narrative that solely stresses the results.

Concerning 2: Another important aspect connected to the "usefulness narrative" is that it borrows strongly from authority and not from an inherent interaction with mathematics. This authoritative aura has many consequences for the reception of mathematical results. Numbers often have authority in societal discussions. While mathematicians can profit from this and while the export of mathematics to other disciplines is a big part of the reason due to which mathematical departments are so big, we argue that the authority of models in social questions is not justified. Not everyone is aware that it is possible to tweak such results and are not able to interpret a precise technical statement on their own. An example is for instance that causation and correlation often get mixed up. And there are several studies showing our bad intuitions concerning conditionalized probabilities, see for example Díaz and de la Fuente (2007).

Here it would be useful to combine the narrative of usefulness with the narrative of mathematization. Mathematization is always partly dependent on choices and needs to miss out information about the object we abstract from. It is a virtue to do this rightly and also to *evaluate, understand* and *appreciate* such processes of mathematization. In mathematics education, we can see partly a fitting development in this direction: the one connected to the development of *modelling competencies*.[20] These developments stress the first step of mathematization from real-world phenomena but normally stop there and do not approach iterations as we would find them in applied mathematics. To frame this even stronger, it is very important to develop competencies to understand and analyze formal tools in the workplace and in public debates. A narrative of rationality and objectivity without reflection is highly problematic, since it might foster the risk of mathematics causing harm. This was problematized by Ernest (2009, 2016). We want to stress that the usefulness-narrative is crucial in this respect. Consultants trying to optimize several key-performance indicators of a company might feel obligated to forget other human and societal consequences of their work. This gets even more important when military-related research is disguised as optimization and engineering problems. We hope that the mathematizing narrative allows for a better implementation of a critical theory of the responsibilities which are connected to mathematical modelling. A missing understanding of the real mathematical practice might add to cases where the boundaries of a model are not rightfully taken into consideration, like with the Black–Scholes equations in financial modelling.

3.2 Mathematics is beautiful

The beauty of mathematics is often revealed both in the popularization of mathematics—for example, in newspaper articles like "Mathematics: Why the brain sees maths as beauty" by Gallagher (2014), "The beauty of mathematics: It can never lie to you" by Roberts (2017), or "The aesthetic beauty of math" by Olsson (2019)—as well as when mathematicians describe their own motivation for doing mathematics:

[20] See for instance Schukajlow et al. (2018), Kaiser (2010) or Kaiser and Schwarz (2010).

> For me, as a mathematician, [beauty] is hugely important. My enjoyment of the beauty of mathematics is part of what motivates me to study the subject. It is also a guide when I am working on a problem: if I think of a few strategies, I will choose the one that seems most elegant first. And if my solution seems clumsy then I will revisit it to try to make it more attractive. (Neale 2017)

The narrative of beauty as one of the main goals for mathematicians or one of the main characteristics of mathematics associates the abstract study of mathematical objects such as numbers, groups, or topological spaces with the creative activity of artists. In contrast to the previous narrative, it ascribes an intrinsic value to mathematics. The analogy between art and mathematics underscores the creative part of mathematical activity. The mathematician creates abstract forms and structures, explores those objects, and constructs extremely complex objects and relations. According to the narrative, this activity is driven intrinsically by the aesthetics of those objects and of their interplay.

A rather unsurprising fact that goes well with this analogy is that not every piece of art and not every piece of mathematics is beautiful. Some parts are more, or less, beautiful than others. Moreover, beauty can be connected to a feeling of depth: the mathematician found a deep fact or relation and the artist created a deep and inspiring piece. This goes with the expectation that one does not find beauty everywhere, that it rather takes time and reflection, expertise, and insight.

Another similarity is that people judge differently. There can be agreement about the beauty of particular pieces of art or proofs. For example, people may agree on the beauty of da Vinci's *Mona Lisa* or van Gogh's *The Starry Night*, and mathematicians may agree on the beauty of some of the *Proofs from THE BOOK* by Aigner and Ziegler (2004); however, on other pieces they could disagree a little bit, or even strongly. Inglis and Aberdein (2016) report that the mathematician's proof appraisal is not intersubjective. Presented with the same proof, there were mathematicians judging it as beautiful while others indicated that 'beautiful' is not an accurate description of the proof.[21]

A valuable aspect of the narrative of mathematics striving for beauty is that it takes a strong stance against those narratives that claim mathematics to be cold and unapproachable. Philosophers argued strongly for a focus on the aesthetic aspect of mathematics (for example Dreyfus and Eisenberg (1986), Tymoczko (1993), and Sinclair (2001)). But, of course, it is not enough to declare that proofs can be beautiful, that mathematicians enjoy the beauty of mathematics and that mathematics is, therefore, "super-exciting". For, the main problem of the beauty narrative is: how can anyone

[21] The framing of the proof (for instance if the mathematicians were told that it's a proof from the Book) played also a role in the judgements on beauty. Importantly, the authors also checked on a correlation between the research area, respectively the career stage, and the appraisals and found that they were not influenced by that. They write in the discussion section: "We found a remarkable level of disagreement between our participants' ratings of the proof. For each of the four dimensions of proof appraisal there were participants who thought the proof should score high on that dimension, and there were participants who thought the proof should score low on that dimension. Furthermore, neither research area nor career stage seemed to be predictive of mathematicians' appraisals on any of the four dimensions." (Inglis and Aberdein 2016, p. 173).

understand such a claim who never experienced an appreciation of beauty connected to mathematics?

Tymoczko, when presenting the aesthetics of a proof step, accepts that some people may lack an ability: "if they say 'I could never have thought of that in a million years', we'd question their ability to appreciate mathematics" (Tymoczko 1993, p. 75). This supports the view that one needs mathematical skills to appreciate its beauty. Dreyfus and Eisenberg (1986) take a similar stance when they show that students were not able to appreciate mathematical beauty and suggest educating them in their aesthetic abilities. In contrast, Sinclair argues that "students can and do behave aesthetically in the mathematics classroom" (Sinclair 2003, p. 204) and that their aesthetic judgments are just different from those of expert mathematicians. In a small study with four students, she discusses with them the best way to construct a square with a given digital tool. The students try out their ways of constructing a square and, subsequently, evaluate their different approaches in discussion with each other. This is a very good example for a mathematization process in the classroom: the students are presented with a rich tool and it becomes very clear to them that there are various ways to construct a square, which they can compare according to their preferences.

We think that mathematizing can be very helpful in one's attempts to understand what beauty in mathematics can be. This is due to two aspects of mathematizing: its structuring activity and its acknowledgment of subjective components. Sinclair reveals an account of beauty that "interprets aesthetic response as a cognisance of fit, of structure or order, perceived part as being intuitive and recognized at an emotional level as being pleasurable" (Sinclair 2001, p. 25). We think this is a plausible account of beauty in mathematics that does not separate between students and expert mathematicians. Since such a cognisance of fit can be perceived by students when mathematizing at school, they can experience the related pleasure in the same way that expert mathematicians experience pleasure when they identify a clear structure in a mathematical context of their interest.

Beauty and mathematizing are both reflected in subjective judgments. When a problem is mathematized, there are different ways of doing so: different ways of formulating the problem and different ways of solving it. Those choices are made according to the person's preferences, so, also according to the person's aesthetic judgments. One could find one way of mathematizing a problem particularly insightful, striking, or appealing in comparison to another way that would also work but is seen as less insightful, striking, or appealing as the first one. Since those adjectives correspond to the aesthetics dimension of the mathematician's proof appraisal,[22] such a judgment is comparable to the judgment of a mathematician who finds a proof beautiful. People may feel the beauty of mathematics when they choose a particular way of mathematizing according to their own preferences.

[22] Inglis and Aberdein found out by an explorative factor analysis that 'beautiful' loaded highly on what they called the aesthetics-factor, which also includes adjectives such as 'deep', 'ingenious' or 'cute' as well as 'insightful', 'striking', or 'appealing' (Inglis and Aberdein 2015). This empirical investigation can give an idea on the meaning of beauty in mathematics, but the meaning remains an open question and there is much controversy about it.

3.3 Mathematicians aim at deep understanding

Our next narrative may sound trivial at first but becomes difficult to defend against the circumstances of mathematical practice. Mathematizing can, again, shed light on the narrative and emphasize its valuable aspects.

In the narrative of deep understanding, mathematics is the activity of advancing our human understanding of mathematical objects; mathematicians aim to understand mathematical structures and relations and the final goal of mathematics is to delve deeper and deeper in that understanding. In mathematics, understanding comes second to truth; that is, mathematicians first find a proof of a new theorem, but they can still be puzzled by the theorem because they do not understand *why* it holds.

The narrative of a deep understanding could be suggested for all sciences and humanities: the respective scholars aim at a deep understanding of a specific subject matter. In this sense, the narrative is rather natural and general.[23]

According to this narrative, mathematicians should value explanations, which give reasons why theorems hold, higher than proofs without any explanatory power. However, it is hard to find evidence for that in mathematical practice. Mathematicians are generally satisfied with good proofs of important theorems. On the other hand, a mathematician could view the accumulation of important theorems about a specific object as exactly the way to go to understand the object. In that view, from outside the mathematical community, it looks like only theorems count, but from the inside, the theorems are only valued if they lead to a better understanding.[24]

Deep understanding involves acquiring knowledge about a mathematical structure or an established fact from different perspectives, learning all one can learn about it, knowing its difficulties, its advantages, its nice features and how it is connected to other close mathematical objects or facts. We imagine that if a mathematician has a deep understanding of a mathematical object, then everything that one can say about that object would not be surprising, but clear and expected; even more, we would imagine that the mathematician can easily offer various explanations.[25] In other words, the mathematician has a very good idea of the bigger picture in this specific context and can easily talk about it and present it to others. This might very well include using suitable and telling metaphors, which illustrate the connections and objects.

It is harder to give an example of a mathematical object of which one could claim mathematicians have deeply understood it since conceivably there may be surprising new facts even about the numbers from 1 to 10. For example, there is a result in set theory, that people often quote as an unexpected theorem: Shelah proved that $2^{\aleph_\omega} < \aleph_{\omega_4}$ if $\aleph_\omega$ is a strong limit.[26] People are surprised about the number 4 appearing

[23] In mathematics, it comes with the difficulty to specify the subject matter. We speak here roughly about mathematical objects and facts without entering the debate about the existence of such objects.

[24] See Sect. 3.4 below.

[25] It seems again that mathematicians differ in their judgments whether a proof is explanatory or not, as problematized by Inglis and Aberdein (2016). There is much debate about explanations in mathematics. However, we do not want to elaborate on the concept of explanation in mathematics, but on the idea that mathematical activity may be driven by the desire for deep understanding.

[26] See (Jech 2006, p. 476) for a proof. The theorem gives an upper bound for the cardinal number $2^{\aleph_\omega}$, where $\aleph_\omega$ is the ω-th cardinal number, and $\aleph_{\omega_4}$ is the ω_4-th cardinal number.

in the inequality and there could be something about the number 4 that we learn by that fact. Shelah himself asks: “Why the hell is it four? Can we replace it by one? Is 4 an artifact of the proof or the best possible bound?” (Shelah 2002, p. 560).

But we may argue that there are mathematical facts that are deeply understood in the sense that mathematicians could give various different explanations of the fact. For example, the facts that there are more real numbers than there are natural numbers, or that every polynomial of degree n has n complex roots (where the same root can appear more than once), or that every vector space has a basis. Those are important standard results, which are given in different formulations and with different proofs in textbooks depending on how the author thinks it is best explained.

Mathematizing has a very close connection to the deep understanding narrative. We argued above that mathematizing is a virtuous practice since it helps to improve our understanding of the real world and of mathematics itself. Emphasizing mathematizing as a main characteristic of mathematical activity means arguing that mathematical activity is about extending our understanding. This directly implies our narrative of mathematics driven by the desire for a deep understanding.

Since we argued in 2.1 that those mathematicians who prioritize theory-building mathematize more than others, we relativize the implication and conclude from the mathematizing narrative that some mathematicians aim for a deep understanding, and others may have different goals. This is, we believe, correct regarding contemporary practices in research mathematics. However, taking into account that we also argued that mathematizing is virtuous, the deep understanding narrative describes, in contrast to the other narratives, a kind of virtuous mathematical activity.

3.4 Mathematicians aim at theorem-credit

There are different examples of mathematicians who became very famous by proving a conjecture which was open for many, many years, and there is a narrative which tells us that mathematics can offer the opportunity to find a place in history by proving an earth-shaking theorem; one will be admired for finding such a proof and mathematics can be the way to go to achieve an honorary place. Exemplary cases are Grigori Perelman’s proof of the Poincaré conjecture or Andrew Wiles’ proof of Fermat’s Last Theorem. In this narrative, mathematical activity is driven by the search for the solution of important conjectures.

Various aspects of mathematical practice suggest that research mathematicians are recognized to be good and valued in the community if they produce virtuous proofs, where a virtuous proof is particularly beautiful, deep, innovative, or is judged to have similar such properties.[27] One main argument for that narrative is the observation that mathematicians are given much credit when they prove an important theorem, and that there is sometimes even emotional debate on who deserves the credit for a proof and

[27] A proof can have different qualities, some of which are judged virtuous and others not. For example, a proof that is judged elementary is probably not seen to be virtuous since it uses standard methods of the respective research area, and is, hence, not original, or innovative. Qualities of mathematical proofs have recently been investigated by Inglis and Aberdein (2015, 2016) (their work was mentioned above in Sect. 3.2 on beauty in mathematics and 3.3 on deep understanding).

who does not (see for instance the debate between Perelman and Shing-Tung).[28] If someone proves a theorem and the proof or theorem is judged to be particularly virtuous by the community, the theorem is named after the mathematician who found the proof. For example, in set theory, fundamental theorems are König's lemma, Cantor's theorem or the Martin-Steel theorem and so on. In such cases, the respective mathematicians are recognized by the community, for they made a valuable contribution to the joint endeavor of research mathematics, and the more virtuous the proof or theorem is, the higher the theorem credit.

There are several arguments in favor of the described theorem-credit narrative, for example the publishing practices in the mathematical community, which support the narrative that finding virtuous proofs is part and parcel of being a good mathematician today. To succeed as a research mathematician in a practical sense, that is, to get a permanent position as a research mathematician, publications in high-ranked journals are essential. Those journals claim to publish original and innovative work. Thus, one could argue, the work of a research mathematician is mainly driven by his attempt to find such theorems and proofs.[29]

Although seemingly adequately representing aspects of mathematical practice, this narrative reduces mathematical activity to the production of theorems. The focus of that narrative lies on the products of mathematical activity instead of on the process and the activity itself. This, we think, is a negative aspect of this narrative. The theorem-credit narrative quantifies mathematical activity in a harmful way. Of course, it is included in the narrative that not every theorem is a desired product, but rather particularly virtuous proofs are desired. Thus, it is not a complete quantification, but it neglects the process of finding a proof and its key ideas, where for example other people could have contributed a large part by mathematizing a relevant context. Presented with an amazing proof, one can find oneself puzzled about the question of how that person was able to come up with such a wonderful proof. Putting mathematizing in the equation, it is not puzzling at all. The proof does not represent the process of finding the proof. The proof was rather found by a particularly sophisticated way of mathematizing. Therefore, the theorem-credit narrative hides a main characteristic of mathematics and is not adequate in this sense. It creates a distance between the public and the final products of mathematics that is not a priori, since people who are used to mathematizing everyday problems can easily imagine that mathematizing further in other contexts and possibly in a more sophisticated way, can indeed lead to the discovery of amazing proofs. But it is the preceding process of mathematizing that is virtuous, and we should admire a mathematician for that rather than for the final product.

[28] See an article in the New Yorker: Nasar and Gruber (2006).

[29] We already mentioned in Sect. 3.4 on deep understanding that one may argue that the accumulation of theorems is not a goal in itself, but that it is rather aimed at the final goal of deepening our understanding of mathematics. This is what someone preferring the narrative of deep understanding would object to the reasons from mathematical practice which rather speak in favor of the theorem credit narrative. However, here we want to take the narrative as an emphasis on the products of mathematics—proofs and theorems—and consider the idea that mathematicians are mainly concerned with finding good proofs and new interesting theorems that they can publish in major journals of their respective discipline.

4 Conclusion

We argued that mathematizing is a virtuous practice, because it offers tools to grasp a larger part of reality and is beneficial for epistemically valuable ends such as understanding. Our main point is that *mathematizing as a virtuous practice* is a narrative which captures integral parts of mathematical practice, including the activity of research mathematicians, mathematics learners or applied scientists. It accounts for the important role of theory-building processes for mathematics and how essential it is to organize and systematize mathematical knowledge, for example in textbooks. We argued that it thus constitutes a more precise and adequate narrative of mathematical practice than its competing alternatives. Furthermore, we suggested that a narrative of mathematics stressing mathematizing provides beneficial outcomes for society; for example, importantly, by relativizing the authority of formal arguments in public discourse due to the emphasis on the modelling component that comes with any real world application of mathematical tools.

One limitation of our paper is that we do not provide in-depth investigations of specific narratives, be it their description or their importance. This lies beyond the scope of this article because our intention was to overview a larger topic. We compared what we found in relevant discourses about mathematics and elaborated ways to integrate the activity of doing mathematics, mathematizing, in those narratives. This also shows that narratives are flexible and can be improved. It is subject to future research to try to answer some of the following questions: Can mathematizing contribute to individual human flourishing? Which are the benefits of bringing mathematical laymen and research mathematicians closer together in their way of understanding and valuing mathematical activity? Which other narratives of mathematics are important in our societies, do they reflect on mathematizing? Another important issue is gender-related narratives.

We see our contribution as an impulse directed towards a continued development and reflection on these matters which, we believe, can result in a valuable change of the role of mathematics for our society—initiated by choosing a more adequate and better narrative of mathematical activity.

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