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$\mathfrak{D}$-parallelism of normal and structure Jacobi operators for hypersurfaces in complex two-plane Grassmannians

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Abstract In this paper, we give non-existence theorems for Hopf hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel normal Jacobi operator $\bar{R}_N$ and $\mathfrak{D}$-parallel structure Jacobi operator R_ξ if the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the Reeb vector field is invariant by the shape operator, respectively.

Keywords Complex two-plane Grassmannians · $\mathfrak{D}$-parallel normal Jacobi operator · $\mathfrak{D}$-parallel structure Jacobi operator · Invariant Reeb vector · Hopf hypersurfaces

Mathematics Subject Classification (2010) Primary 53C40; Secondary 53C15

Introduction

The geometry of real hypersurfaces in complex projective space or in quaternionic projective space is one of the interesting parts in the field of differential geometry. Until now, there have been many characterizations for homogeneous hypersurfaces of type (A_1), (A_2), (B), (C), (D) and (E) in complex projective space $\mathbb{C}P^m$, of type (A_1),(A_2) and (B) in quaternionic projective space $\mathbb{H}P^m$ or of type (A) and (B) in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. Each corresponding geometric feature is classified and investigated by Berndt and Suh [2,3], Kimura [9] and Martinez and Pérez [10], respectively.

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Let $(\bar{M}, \bar{g})$ be a Riemannian manifold. A vector field U along a geodesic γ in a Riemannian manifold $\bar{M}$ is said to be a *Jacobi field* if it satisfies a differential equation

$$\bar{\nabla}^2_{\dot{\gamma}} U + \bar{R}(U(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0,$$

where $\bar{\nabla}_{\dot{\gamma}}$ and $\bar{R}$ respectively denote the covariant derivative of the vector field U along the curve γ in $\bar{M}$ and the curvature tensor of the Riemannian manifold $(\bar{M}, \bar{g})$. Then this equation is called the *Jacobi* equation.

The Jacobi operator $\bar{R}_X$ for any tangent vector field X at $x \in \bar{M}$, is defined by

$$(\bar{R}_X Y)(x) = (\bar{R}(Y, X)X)(x)$$

for any $Y \in T_x\bar{M}$, becomes a self adjoint endomorphism of the tangent bundle $T\bar{M}$ of $\bar{M}$. That is, the Jacobi operator satisfies $\bar{R}_X \in End(T_x\bar{M})$ and is symmetric in the sense of $\bar{g}(\bar{R}_X Y, Z) = \bar{g}(\bar{R}_X Z, Y)$ for any vector fields Y and Z on $\bar{M}$.

The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ denote a quaternionic Kähler structure of $\mathbb{H}P^m$ and N a unit normal field of M in $\mathbb{H}P^m$. In a quaternionic projective space $\mathbb{H}P^m$ Pérez and Suh [11] have classified real hypersurfaces in $\mathbb{H}P^m$ with $\mathfrak{D}^\perp$-parallel curvature tensor $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$, where R denotes the curvature tensor of M in $\mathbb{H}P^m$ and $\mathfrak{D}^\perp$ a distribution defined by $\mathfrak{D}^\perp = \text{Span}\,\{\xi_1, \xi_2, \xi_3\}$. In such a case, they are congruent to a tube of radius $\frac{\pi}{4}$ over a totally geodesic $\mathbb{H}P^k$ in $\mathbb{H}P^m$, $0 \leq k \leq m-1$.

Now let us consider such a parallelism related to the curvature tensor for hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ which consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The ambient space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It was known that the complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ (see Berndt and Suh [2]). Induced from such structures, some geometric characterizations for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ are investigated by Berndt and one of the present authors (see [2,3,17,18]).

As one of the examples Berndt and Suh [2] considered two natural geometric conditions for hypersurfaces in $G_2(\mathbb{C}^{m+2})$ that $[\xi] = \text{Span}\,\{\xi\}$ and $\mathfrak{D}^\perp = \text{Span}\,\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator. By using such conditions and the result in Alekseevskii [1], they have proved the following

Theorem A *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or
(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

The structure vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be a *Reeb* vector field. If the *Reeb* vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, M is said to be a *Hopf hypersurface*. In such a case, the integral curves of the *Reeb* vector field ξ are geodesics (see Berndt and Suh [3]).

In a paper due to Pérez et al. [4] we have introduced a notion of normal Jacobi operator $\bar{R}_N$ for hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ in such a way that

$$\bar{R}_N X = \bar{R}(X, N)N \in \text{End}\ T_x M, \quad x \in M,$$

for any tangent vector field X on M, where $\bar{R}$ denotes the curvature tensor of $G_2(\mathbb{C}^{m+2})$. The normal Jacobi operator $\bar{R}_N$ is *parallel* on the distribution $\mathfrak{D}$ of M in $G_2(\mathbb{C}^{m+2})$ means that the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are *parallel* along the distribution $\mathfrak{D}$ of M, where $\mathfrak{D}$ denotes the distribution orthogonal to the distribution $\mathfrak{D}^\perp$ such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$. Here the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are said to be *parallel* along the distribution $\mathfrak{D}$ if they are *invariant* with respect to any *parallel displacement* along the distribution $\mathfrak{D}$.

Related to such a normal curvature tensor $\bar{R}_N$, Jeong et al. [7] obtained a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator. Motivated by this fact, in such a paper, we consider more general notion of $\mathfrak{D}$-parallelism weaker than the notion of parallel normal Jacobi operator.

In Sect. 3, we consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-*parallel* normal Jacobi operator, that is, $\nabla_X \bar{R}_N = 0$, for all $X \in \mathfrak{D}$, where ∇, $\bar{R}$ and N respectively denote the induced Riemannian connection on M, the curvature tensor of the ambient space $G_2(\mathbb{C}^{m+2})$ and a unit normal vector of M in $G_2(\mathbb{C}^{m+2})$.

In Sect. 4, we prove a non-existence theorem for hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$- parallel normal Jacobi operator as follows:

Theorem 1 *There do not exist any connected Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel normal Jacobi operator if the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the Reeb vector field is invariant by the shape operator.*

On the other hand, Jeong et al. [5] obtained a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel structure Jacobi operator. Moreover, in a paper Pérez et al. [14] have given a classification of hypersurfaces in complex projective space $\mathbb{C}P^m$ with $\mathfrak{D}$-parallel structure Jacobi operator. So, in Sect. 4, we also consider hypersurfaces with $\mathfrak{D}$- parallel structure Jacobi operator, that is, $\nabla_X R_\xi = 0$ for all $X \in \mathfrak{D}$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$.

For any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$, we calculate the structure Jacobi operator R_ξ in such a way that

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= X - \eta(X)\xi - \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu \\ &\quad + 3g(\phi_\nu X,\xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\big\} + \alpha AX - \eta(AX)A\xi, \end{aligned}$$

where α denotes the function defined by $g(A\xi,\xi)$. The structure Jacobi operator R_ξ is *parallel* on the distribution $\mathfrak{D}$ of M in $G_2(\mathbb{C}^{m+2})$ means that the eigenspaces of the structure Jacobi operator R_ξ are *parallel* along the distribution $\mathfrak{D}$ of M. Here the eigenspaces of the structure Jacobi operator R_ξ are said to be *parallel* along the distribution $\mathfrak{D}$ if they are *invariant* with respect to any *parallel displacement* along the distribution $\mathfrak{D}$ (see [5,6]).

Then in Sect. 4, we prove another non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel structure Jacobi operator as follows:

Theorem 2 *There do not exist any connected Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel structure Jacobi operator if the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the Reeb vector field is invariant by the shape operator.*

1 Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section, we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to Berndt and Suh [2,3], and Suh et al. [16–18].

By $G_2(\mathbb{C}^{m+2})$, we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$.

We put $o = eK$ and identify $T_o G_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric $\bar{g}$ on $G_2(\mathbb{C}^{m+2})$.

In this way, $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons, we normalize $\bar{g}$ such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), \bar{g})$ is eight. When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$.

If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $tr(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), \bar{g})$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z = {} & \bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX \\ & -\bar{g}(JX,Z)JY - 2\bar{g}(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{\bar{g}(J_\nu Y,Z)J_\nu X - \bar{g}(J_\nu X,Z)J_\nu Y \\ & -2\bar{g}(J_\nu X,Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{\bar{g}(J_\nu JY,Z)J_\nu JX - \bar{g}(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{1.2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

2 Some fundamental formulas

In this section, we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [2,3,15–17]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression (1.2) for the curvature tensor $\bar{R}$, the Gauss and Codazzi equations are respectively given by

$$\begin{aligned}
R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y \\
& + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\
& + \sum_{\nu=1}^{3}\{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\} \\
& + \sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y, Z)\phi_\nu\phi X - g(\phi_\nu\phi X, Z)\phi_\nu\phi Y\} \\
& - \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\} \\
& - \sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y, Z) - \eta(Y)g(\phi_\nu\phi X, Z)\}\xi_\nu \\
& + g(AY, Z)AX - g(AX, Z)AY
\end{aligned}$$

and

$$\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
& + \sum_{\nu=1}^{3}\left\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\right\} \\
& + \sum_{\nu=1}^{3}\left\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\right\} \\
& + \sum_{\nu=1}^{3}\left\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\right\}\xi_\nu
\end{aligned}$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}
\phi_{\nu+1}\xi_\nu &= -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\
\phi\xi_\nu &= \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\
\phi_\nu\phi_{\nu+1}X &= \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\
\phi_{\nu+1}\phi_\nu X &= -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}.
\end{aligned} \tag{2.1}$$

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{2.2}$$

for any tangent vector X of M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1), we have that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX, \tag{2.3}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.4}$$

$$\begin{aligned}(\nabla_X \phi_\nu)Y = &-q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ &-g(AX, Y)\xi_\nu.\end{aligned} \tag{2.5}$$

Summing up these formulas, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu \xi) &= (\nabla_X \phi_\nu)\xi + \phi_\nu(\nabla_X \xi) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu \phi AX \\ &\quad -g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{2.6}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1, 2, 3$, it follows that

$$\phi\phi_\nu X = \phi_\nu \phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.7}$$

3 $\mathfrak{D}$-parallelism of the normal Jacobi operator

Now let us consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel normal Jacobi operator $\bar{R}_N$, that is, $\nabla_X \bar{R}_N = 0$ for any vector field $X \in \mathfrak{D}$.

Then first of all, we obtained the normal Jacobi operator $\bar{R}_N$, which is given by

$$\begin{aligned}\bar{R}_N(X) &= \bar{R}(X, N)N \\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad -\sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\Big\}.\end{aligned}$$

Of course, by (2.7) the normal Jacobi operator $\bar{R}_N$ is a symmetric endomorphism of T_xM, $x{\in}M$ (see [4]).

Now let us consider a covariant derivative of the normal Jacobi operator $\bar{R}_N$ along any direction X of T_xM, $x{\in}M$. Then it is given by

$$\begin{aligned}(\nabla_X \bar{R}_N)Y &= \nabla_X(\bar{R}_N Y) - \bar{R}_N(\nabla_X Y) \\ &= 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX \\ &\quad +3\sum_{\nu=1}^{3}\Big\{g(\phi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX\Big\} \\ &\quad -\sum_{\nu=1}^{3}\Big[2\eta_\nu(\phi AX)(\phi_\nu\phi Y - \eta(Y)\xi_\nu) - g(\phi_\nu AX, \phi Y)\phi_\nu\xi \\ &\qquad -\eta(Y)\eta_\nu(AX)\phi_\nu\xi - \eta_\nu(\phi Y)(\phi_\nu\phi AX - g(AX, \xi)\xi_\nu)\Big]\end{aligned} \tag{3.1}$$

for any tangent vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$(see [7]).

Bearing in mind that $\phi\xi = 0$ and writing $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$, it follows:

$$\begin{aligned}
0 &= \phi\xi \\
&= \phi(\eta(X_0)X_0 + \eta(\xi_1)\xi_1) \\
&= \eta(X_0)\phi X_0 + \eta(\xi_1)\phi_1(\eta(X_0)X_0 + \eta(\xi_1)\xi_1) \\
&= \eta(X_0)\phi X_0 + \eta(X_0)\eta(\xi_1)\phi_1 X_0 \\
&= \eta(X_0)(\phi X_0 + \eta(\xi_1)\phi_1 X_0).
\end{aligned}$$

Lemma 3.1 *If we suppose $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ being $\eta(X_0)$ and $\eta(\xi_1)$ nonnull, then $\phi X_0 = -\eta(\xi_1)\phi_1 X_0$.*

Lemma 3.2 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel normal Jacobi operator. If the distribution $\mathfrak{D}$ or $\mathfrak{D}^{\perp}$ component of the Reeb vector field is invariant by the shape operator, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^{\perp}$.*

Proof When the function $\alpha = g(A\xi, \xi)$ identically vanishes, it can be verified directly by Pérez and Suh [12].

Now it remains to show the case when the function α is non-vanishing. Let us assume that $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$, non-zero functions $\eta(X_0)$ and $\eta(\xi_1)$. By putting $X = X_0$ and $Y = X_0$ in (3.1), we have

$$\begin{aligned}
0 &= (\nabla_{X_0}\bar{R}_N)X_0 \\
&= 3g(\phi AX_0, X_0)\xi + 3\eta(X_0)\phi AX_0 \\
&\quad +3\sum_{\nu=1}^{3}\Big\{g(\phi_\nu AX_0, X_0)\xi_\nu + \eta_\nu(X_0)\phi_\nu AX_0\Big\} \\
&\quad -\sum_{\nu=1}^{3}\Big[2\eta_\nu(\phi AX_0)(\phi_\nu\phi X_0 - \eta(X_0)\xi_\nu) - g(\phi_\nu AX_0, \phi X_0)\phi_\nu\xi \\
&\qquad -\eta(X_0)\eta_\nu(AX_0)\phi_\nu\xi - \eta_\nu(\phi X_0)(\phi_\nu\phi AX_0 - g(AX_0, \xi)\xi_\nu)\Big].
\end{aligned} \tag{3.2}$$

Since M is Hopf and the distributions $\mathfrak{D}$ or $\mathfrak{D}^{\perp}$ component of the Reeb vector field are invariant by the shape operator, we obtain

$$AX_0 = \alpha X_0, \quad A\xi_1 = \alpha\xi_1.$$

Substituting these formulas into (3.2) gives

$$0 = 3\alpha\eta(X_0)\phi X_0 + \sum_{\nu=1}^{3}\alpha g(\phi_\nu X_0, \phi X_0)\phi_\nu\xi.$$

Thus

$$
\begin{aligned}
& g((\nabla_{X_0}\bar{R}_N)X_0, \phi X_0) \\
&\quad = 3\alpha\eta(X_0){\eta_1}^2(\xi) + \alpha\sum_{\nu=1}^{3} g(\phi_\nu X_0, \phi X_0) g(\phi_\nu \xi, \phi X_0) \\
&\quad = 3\alpha\eta(X_0){\eta_1}^2(\xi) + \alpha\sum_{\nu=1}^{3} g(\phi_\nu X_0, \phi X_0)(-\eta_\nu(\xi)\eta(X_0)) \\
&\quad = 3\alpha\eta(X_0){\eta_1}^2(\xi) - \alpha g(\phi_1 X_0, \phi X_0)(\eta_1(\xi)\eta(X_0)) \\
&\quad = 3\alpha\eta(X_0)\eta^2(\xi_1) + \alpha\eta^2(\xi_1)\eta(X_0) \\
&\quad = 4\alpha\eta^2(\xi_1)\eta(X_0).
\end{aligned}
$$

From this, together with the assumption, it makes a contradiction. This means $\eta(X_0) = 0$ or $\eta(\xi_1) = 0$, that is, the Reeb vector ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$. □

Now we will prove our Theorem 1 in the introduction. That is, a non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel normal Jacobi operator $\bar{R}_N$ will be proved in this section. In order to do this, we need some lemmas as follows:

Lemma 3.3 *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof Assume that ξ is tangent to $\mathfrak{D}^\perp$. Then the unit normal N is a singular tangent vector of $G_2(\mathbb{C}^{m+2})$ of type $JN \in \mathfrak{J}N$. So there exists an almost Hermitian structure $J_1 \in \mathfrak{J}$ such that $JN = J_1 N$. Then we have

$$
\xi = \xi_1, \phi\xi_2 = -\xi_3, \phi\xi_3 = \xi_2, \phi\mathfrak{D} \subset \mathfrak{D}.
$$

Using (3.1), we consider $0 = (\nabla_X \bar{R}_N)\xi$ for any $X \in \mathfrak{D}$. Then we get

$$
\begin{aligned}
0 &= (\nabla_X \bar{R}_N)\xi \\
&= 3\phi AX + 3\sum_{\nu=1}^{3}\Big\{g(\phi_\nu AX, \xi)\xi_\nu + \eta_\nu(\xi)\phi_\nu AX\Big\} \\
&\quad -\sum_{\nu=1}^{3}\Big[2\eta_\nu(\phi AX)(-\xi_\nu) - \eta_\nu(AX)\phi_\nu\xi\Big] \\
&= 3\phi AX + 3g(\phi_2 AX, \xi_1)\xi_2 + 3g(\phi_3 AX, \xi_1)\xi_3 + 3\phi_1 AX \\
&\quad +2\eta_2(\phi AX)\xi_2 + 2\eta_3(\phi AX)\xi_3 + \eta_2(AX)\phi_2\xi + \eta_3(AX)\phi_3\xi \\
&= 3\phi AX + 3\eta_3(AX)\xi_2 - 3\eta_2(AX)\xi_3 + 3\phi_1 AX \\
&\quad +2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 - \eta_2(AX)\xi_3 + \eta_3(AX)\xi_2 \\
&= 3\phi AX + 6\eta_3(AX)\xi_2 - 6\eta_2(AX)\xi_3 + 3\phi_1 AX.
\end{aligned}
$$

Taking its scalar product with ξ_2 and with ξ_3 respectively gives

$$
\begin{aligned}
0 &= 3g(\phi AX, \xi_2) + 6\eta_3(AX) + 3g(\phi_1 AX, \xi_2) \\
&= 3g(AX, \xi_3) + 6\eta_3(AX) - 3g(AX, \xi_3) \\
&= 6\eta_3(AX),
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= 3g(\phi AX, \xi_3) - 6\eta_2(AX) + 3g(\phi_1 AX, \xi_3) \\
&= -3g(AX, \xi_2) - 6\eta_2(AX) + 3g(AX, \xi_2) \\
&= -6\eta_2(AX).
\end{aligned}
$$

This gives a complete proof of our Lemma. □

Lemma 3.4 *Under the same assumptions as in Lemma 3.3, if* $\xi \in \mathfrak{D}$, *then* $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$.

Proof In this case, we have

$$
\begin{aligned}
0 &= (\nabla_X \bar{R}_N)\xi \\
&= 3\phi AX + 3\sum_{\nu=1}^{3} g(\phi_\nu AX, \xi)\xi_\nu + 2\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(AX)\phi_\nu\xi \\
&= 3\phi AX + 5\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(AX)\phi_\nu\xi.
\end{aligned}
$$

Taking its scalar product with $\phi_i\xi$, $i = 1, 2, 3$, we have

$$
\begin{aligned}
0 &= g((\nabla_X \bar{R}_N)\xi, \phi_i\xi) \\
&= 3g(\phi AX, \phi_i\xi) + 5\sum_{\nu=1}^{3}\eta_\nu(\phi AX)g(\xi_\nu, \phi_i\xi) + \sum_{\nu=1}^{3}\eta_\nu(AX)g(\phi_\nu\xi, \phi_i\xi) \\
&= 3g(\phi AX, \phi_i\xi) + \eta_i(AX) \\
&= -3g(AX, \phi\phi_i\xi) + \eta_i(AX) \\
&= 3g(AX, \xi_i) + \eta_i(AX) \\
&= 4\eta_i(AX).
\end{aligned}
$$

This gives a complete proof of our Lemma. □

Then by Lemmas 3.3 and 3.4, together with Theorem A, we know that any real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel normal Jacobi operator are locally congruent to real hypersurfaces of type (A) or type (B). Then in order to give a complete proof of Theorem 1, in Sect. 4, we will check whether the normal Jacobi operator $\bar{R}_N$ of real hypersurfaces of type (A) or type (B) in $G_2(\mathbb{C}^{m+2})$ is $\mathfrak{D}$-parallel or not.

4 $\mathfrak{D}$-parallelism of the structure Jacobi operator

In this section, we consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel structure Jacobi operator R_ξ, that is, $\nabla_X R_\xi = 0$ for any vector field $X \in \mathfrak{D}$.

Then first of all, we obtained the structure Jacobi operator R_ξ, which is given by

$$
\begin{aligned}
R_\xi(X) &= R(X, \xi)\xi \\
&= X - \eta(X)\xi - \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu \\
&\quad + 3g(\phi_\nu X, \xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\big\} + \alpha AX - \eta(AX)A\xi.
\end{aligned}
$$

Of course, the structure Jacobi operator R_ξ is a symmetric endomorphism of $T_x M$, $x \in M$ (see [5]).

Now let us consider a covariant derivative of the structure Jacobi operator R_ξ along any direction X of T_xM, $x \in M$. Then it is given by

$$
\begin{aligned}
(\nabla_X R_\xi)Y = & -g(\phi AX, Y)\xi - \eta(Y)\phi AX \\
& - \sum_{\nu=1}^{3} \Big[\, g(\phi_\nu AX, Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\
& +3\Big\{ g(\phi_\nu AX, \phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi + \eta_\nu(\phi Y)\big(\phi_\nu\phi AX \\
& -\eta(AX)\xi_\nu\big)\Big\} + 4\eta_\nu(\xi)\big(\eta_\nu(\phi Y)AX - g(AX, Y)\phi_\nu\xi\big) + 2\eta_\nu(\phi AX)\phi_\nu\phi Y \, \Big] \\
& +\eta\big((\nabla_X A)\,\xi\big)AY + 2\eta(A\phi AX)AY + \eta(A\xi)(\nabla_X A)Y - \eta\big((\nabla_X A)Y\big)A\xi \\
& -g(AY, \phi AX)A\xi - \eta(AY)(\nabla_X A)\xi - \eta(AY)A\phi AX
\end{aligned}
\tag{4.1}
$$

for any tangent vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$(see [5]).

The following Lemma 4.1 [3] will be used in the proof of our Lemmas.

Lemma 4.1 *If M is a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$ with geodesic Reeb flow, then*

$$
\begin{aligned}
& \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y) + 2g(\phi X, Y) \\
& \quad = 2\sum_{\nu=1}^{3} \Big\{ \eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi) \\
& \qquad\qquad -2\eta(X)\eta_\nu(\phi Y)\eta_\nu(\xi) + 2\eta(Y)\eta_\nu(\phi X)\eta_\nu(\xi) \Big\}.
\end{aligned}
$$

Lemma 4.2 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel structure Jacobi operator. If the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the Reeb vector field is invariant by the shape operator, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof When the function $\alpha = g(A\xi, \xi)$ identically vanishes, it can be verified directly by Pérez and Suh [12].

Now it remains to show the case when the function α is non-vanishing. Let us assume that $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$, non-zero functions $\eta(X_0)$ and $\eta(\xi_1)$. Taking $X = X_0$ in Lemma 4.1 and using $AX = \alpha X_0$ and $\phi X_0 = -\eta(\xi_1)\phi_1 X_0$ it follows

$$
\alpha A\phi X_0 + \alpha^2\phi X_0 - 2\alpha A\phi X_0 + 2\phi X_0 = -2\eta_1(\xi)\phi_1 X_0 + 4\eta^2(X_0)\eta_1(\xi)\phi_1 X_0
$$

which gives

$$
A\phi X_0 = \frac{\alpha^2 + 4\eta^2(X_0)}{\alpha}\phi X_0 \, .
$$

As $(\nabla_X R_\xi)Y = 0$ taking $X \in \mathfrak{D}$ for $X = X_0$ and $Y = \xi$ in (4.1), we get

$$\begin{aligned}
0 &= (\nabla_{X_0} R_\xi)\xi \\
&= -\phi A X_0 - \alpha A\phi A X_0 - \alpha\eta_1(\xi)\phi_1 X_0 + 4\alpha\eta_1(\xi)\eta(X_0)\phi_1\xi \\
&= -\alpha\phi X_0 - \alpha^2 A\phi X_0 - \alpha\eta_1(\xi)\phi_1 X_0 + 4\alpha\eta_1(\xi)\eta^2(X_0)\phi_1 X_0 \\
&= -\alpha\phi X_0 - \alpha^2 A\phi X_0 + \alpha\phi X_0 - 4\alpha\eta^2(X_0)\phi X_0 \\
&= -\alpha\{\alpha^2 + 4\eta^2(X_0)\}\phi X_0 - 4\alpha\eta^2(X_0)\phi X_0 \\
&= -\alpha^3\phi X_0 - 4\alpha\eta^2(X_0)\phi X_0 - 4\alpha\eta^2(X_0)\phi X_0 \\
&= (-\alpha^3 - 8\alpha\eta^2(X_0))\phi X_0.
\end{aligned}$$

From this, taking its scalar product with ϕX_0, we have

$$\begin{aligned}
0 &= g((\nabla_{X_0} R_\xi)\xi, \phi X_0) \\
&= (-\alpha^3 - 8\alpha\eta^2(X_0))g(\phi X_0, \phi X_0) \\
&= (-\alpha^3 - 8\alpha\eta^2(X_0))\eta_1^2(\xi),
\end{aligned}$$

where

$$g(\phi X_0, \phi X_0) = -g(\phi^2 X_0, X_0) = 1 - \eta^2(X_0) = \eta_1^2(\xi).$$

Thus

$$-\alpha^3 - 8\alpha\eta^2(X_0) = 0.$$

From this, we get

$$\eta^2(X_0) = \frac{-\alpha^2}{8}.$$

This makes a contradiction, so the result follows. □

Moreover, in order to prove our theorem, we need the following

Lemma 4.3 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel structure Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof First we get

$$\begin{aligned}
0 &= (\nabla_X R_\xi)\xi \\
&= -\phi A X - \alpha A\phi A X - 2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 - \phi_1 A X
\end{aligned} \tag{4.2}$$

if $X \in \mathfrak{D}$.

When the function $\alpha = g(A\xi, \xi)$ identically vanishes, it can be verified by (4.2). In fact, by taking its scalar product with ξ_2 and ξ_3 in (4.2), respectively, we have $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.

Now it remains to show the case when the function α is non-vanishing. Taking its scalar product with ξ_2

$$\alpha g(A\phi A X, \xi_2) + 2\eta_3(AX) = 0.$$

On the other hand, from Lemma 4.1

$$2g(A\phi A X, \xi_2) = \alpha g(A\phi X, \xi_2) + 2\eta_3(AX).$$

Therefore $0=\alpha^2 g(A\phi X,\xi_2)+(\alpha^2+4)\eta_3(AX)$. Thus

$$g(A\phi X,\xi_2)=-\frac{\alpha^2+4}{\alpha^2}\eta_3(AX). \tag{4.3}$$

If we change X by ϕX

$$-\alpha^2 g(AX,\xi_2)+(\alpha^2+4)\eta_3(A\phi X)=0 \tag{4.4}$$

and taking its scalar product with ξ_3 from (4.2) we have

$$\alpha g(A\phi AX,\xi_3)-2\eta_2(AX)=0.$$

From Lemma 4.1, we get

$$2g(A\phi AX,\xi_3)=\alpha g(A\phi X,\xi_3)-\alpha\eta_2(AX),$$

which yields

$$0=\alpha^2 g(A\phi X,\xi_3)-(\alpha^2+4)\eta_2(AX).$$

Thus

$$g(A\phi X,\xi_3)=\frac{\alpha^2+4}{\alpha^2}\eta_2(AX). \tag{4.5}$$

Now changing X by ϕX

$$\alpha^2 g(AX,\xi_3)+(\alpha^2+4)\eta_2(A\phi X)=0. \tag{4.6}$$

By applying (4.5) to (4.4), we have the following

$$0=\left(-\alpha^2+\frac{(\alpha^2+4)^2}{\alpha^2}\right)\eta_2(AX),$$

which gives $(8\alpha^2+16)\eta_2(AX)=0$, that is, $\eta_2(AX)=0$ for any $X\in\mathfrak{D}$.

Moreover, by applying (4.3) to (4.6), we have

$$0=\left(-\alpha^2+\frac{(\alpha^2+4)^2}{\alpha^2}\right)\eta_3(AX).$$

Similarly $\eta_3(AX)=0$ for any $X\in\mathfrak{D}$. Thus we have $g(A\mathfrak{D},\mathfrak{D}^\perp)=0$. □

Lemma 4.4 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m\geq 3$, with $\mathfrak{D}$-parallel structure Jacobi operator and $\xi\in\mathfrak{D}$. Then $g(A\mathfrak{D},\mathfrak{D}^\perp)=0$.*

Proof As now $0=g((\nabla_X R_\xi)Y,\xi)$ with $\xi\in\mathfrak{D}$ it follows

$$\begin{aligned}0=-g(\phi AX,Y)-\sum_{\nu=1}^{3}\Big[&\eta_\nu(Y)g(\phi_\nu AX,\xi)+3\eta_\nu(\phi Y)g(\phi_\nu\phi AX,\xi)\\&+2\eta_\nu(\phi AX)g(\phi_\nu\phi Y,\xi)\Big]-\alpha g(AY,\phi AX)\end{aligned} \tag{4.7}$$

for any tangent vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$.

Put the subset $\mathfrak{D}_0$ of $\mathfrak{D}$ as $\mathfrak{D}_0=\{X\in\mathfrak{D}|\ X\perp\xi,\ \phi_1\xi,\ \phi_2\xi,\ \phi_3\xi\ \}$. Then the tangent vector space T_xM for any point $x\in M$ is decomposed as

$$T_xM=\mathfrak{D}\oplus\mathfrak{D}^\perp=[\xi]\oplus[\phi_1\xi,\ \phi_2\xi,\ \phi_3\xi]\oplus\mathfrak{D}_0\oplus\mathfrak{D}^\perp,$$

where $[\phi_1\xi,\ \phi_2\xi,\ \phi_3\xi]$ denotes a subspace of the distribution $\mathfrak{D}$ spanned by the vectors $\{\phi_1\xi,\ \phi_2\xi,\ \phi_3\xi\}$.

In order to show that $g(AX, \xi_\mu) = 0$ for any $X \in \mathfrak{D}$ and $\mu = 1, 2, 3$, first we consider for $X = \xi$. Then we have $g(A\xi, \xi_\mu) = \alpha g(\xi, \xi_\mu) = 0$ for any $\mu = 1, 2, 3$.

Next, we consider the case that $X \in [\phi_1\xi, \phi_2\xi, \phi_3\xi]$. Put $X = \phi_\nu\xi$, $\nu = 1, 2, 3$. Since $\eta(\xi_\nu) = 0$ for any $\nu = 1, 2, 3$, we see that $g(\nabla_{\xi_\mu}\xi, \xi_\nu) = -g(\xi, \nabla_{\xi_\mu}\xi_\nu)$ for any $\mu = 1, 2, 3$. Thus we have

$$\begin{aligned} g(A\phi_\nu\xi, \xi_\mu) &= g(\phi\xi_\nu, A\xi_\mu) \\ &= -g(\xi_\nu, \phi A\xi_\mu) \\ &= -g(\xi_\nu, \nabla_{\xi_\mu}\xi) \\ &= g(\nabla_{\xi_\mu}\xi_\nu, \xi) \\ &= g(q_{\nu+2}(\xi_\mu)\xi_{\nu+1} - q_{\nu+1}(\xi_\mu)\xi_{\nu+2} + \phi_\nu A\xi_\mu, \xi) \\ &= g(\phi_\nu A\xi_\mu, \xi) \\ &= -g(A\phi_\nu\xi, \xi_\mu)\,. \end{aligned}$$

Consequently we have

$$g(A\phi_\nu\xi, \xi_\mu) = 0$$

for $\mu,\ \nu = 1, 2, 3$.

Finally, we consider the case that any $X \in \mathfrak{D}_0$. To avoid confusion, we put $X = X_0 \in \mathfrak{D}_0$, where the distribution $\mathfrak{D}_0$ is defined by

$$\mathfrak{D}_0 = \{X \in \mathfrak{D} |\ X \perp \xi,\ \phi_1\xi,\ \phi_2\xi,\ \phi_3\xi\}.$$

By putting $X = X_0 \in \mathfrak{D}_0$ and $Y = \xi_i$, $i = 1, 2, 3$ in (4.6), we have

$$\begin{aligned} 0 &= -g(\phi AX_0, \xi_i) - \sum_{\nu=1}^{3}\Big[\ \eta_\nu(\xi_i)g(\phi_\nu AX_0, \xi) + 3\eta_\nu(\phi\xi_i)g(\phi_\nu\phi AX_0, \xi) \\ &\qquad +2\eta_\nu(\phi AX_0)g(\phi_\nu\phi\xi_i, \xi)\Big] - \alpha g(A\xi_i, \phi AX_0) \\ &= -\alpha g(A\xi_i, \phi AX_0), \end{aligned}$$

where in the second equality, we have used $\eta_\nu(\phi\xi_i) = 0$ and

$$g(\phi_\nu\phi, \xi) = -g(\phi\xi_i, \phi_\nu\xi) = g(\phi^2\xi_i, \xi_\nu) = -\delta_{i\nu}.$$

So we consider the two cases, that is, $\alpha = 0$ or $g(A\phi AX_0, \xi_i) = 0$.

CASE 1. $\alpha = 0$.

By putting $X = X_0 \in \mathfrak{D}_0$ and $Y = \phi\xi_i$, $i = 1, 2, 3$ in (4.7), we have

$$\begin{aligned} 0 = -g(\phi AX_0, \phi\xi_i) - \sum_{\nu=1}^{3}\Big[&\eta_\nu(\phi\xi_i)g(\phi_\nu AX_0, \xi) + 3\eta_\nu(\phi\phi\xi_i)g(\phi_\nu\phi AX_0, \xi) \\ &+2\eta_\nu(\phi AX_0)g(\phi_\nu\phi\phi\xi_i, \xi)\Big]. \end{aligned}$$

From this, we obtain the following

$$0 = -g(AX_0, \xi_i) + 3g(\phi_i\phi AX_0, \xi) \tag{4.8}$$

because

$$g(\phi AX_0, \phi\xi_i) = g(AX_0, \xi_i),$$
$$\eta_\nu(\phi\xi_i) = 0,$$
$$\eta_\nu(\phi\phi\xi_i) = -\eta_\nu(\xi_i),$$

and

$$g(\phi_\nu\phi\phi\xi_i, \xi) = 0.$$

From (4.8), by using $g(\phi_i\phi AX_0, \xi) = -g(AX_0, \xi_i)$ we have

$$0 = -4g(AX_0, \xi_i).$$

So this yields

$$g(AX_0, \xi_i) = 0, i = 1, 2, 3$$

for any $X_0 \in \mathfrak{D}_0$.

CASE 2. $g(A\phi AX_0, \xi_i) = 0$.

Taking $X = X_0 \in \mathfrak{D}_0$ and $Y = \xi_i$, $i = 1, 2, 3$ in Lemma 4.1 and using $\xi \in \mathfrak{D}$ it follows

$$0 = \alpha g(A\phi X_0, \xi_i) + \alpha g(\phi AX_0, \xi_i) - 2g(A\phi AX_0, \xi_i)$$

for any $X_0 \in \mathfrak{D}_0$.

And we knew that $g(\phi AX_0, \xi_i) = 0$ and $g(A\phi AX_0, \xi_i) = 0$.

So we have

$$\alpha g(A\phi X_0, \xi_i) = 0.$$

Thus we consider the two cases that $\alpha = 0$ or $g(A\phi X_0, \xi_i) = 0$.

SUBCASE 2-1. $\alpha = 0$.

By the result of CASE 1, we have

$$g(AX_0, \xi_i) = 0, i = 1, 2, 3$$

for any $X_0 \in \mathfrak{D}_0$.

SUBCASE 2-2. $g(A\phi X_0, \xi_i) = 0$.

If $X_0 \in \mathfrak{D}_0$, then $\phi X_0 \in \mathfrak{D}_0$. So let us replace X_0 by ϕX_0 in this case. Then we have

$$\begin{aligned}0 &= g(A\phi^2 X_0, \xi_i)\\ &= g(A(-X_0 + \eta(X_0)\xi), \xi_i)\\ &= -g(AX_0, \xi_i)\end{aligned}$$

for any $X_0 \in \mathfrak{D}_0$ and $i = 1, 2, 3$.

This gives a complete proof of our Lemma. □

Then by Lemmas 4.3 and 4.4, together with Theorem A, we know that any real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel structure Jacobi operator are locally congruent to real hypersurfaces of type (A) or type (B). Then in order to give a complete proof of Theorem 2, in this section we will check whether the normal Jacobi operator R_ξ of real hypersurfaces of type (A) or type (B) in $G_2(\mathbb{C}^{m+2})$ is $\mathfrak{D}$-parallel or not.

Now including the result in Sect. 3 related to the normal Jacobi operator $\bar{R}_N$, we want to check whether real hypersurfaces of type (A) or of type (B) mentioned in Theorem A could satisfy $\nabla_{\mathfrak{D}}\bar{R}_N = 0$ or $\nabla_{\mathfrak{D}}R_\xi = 0$.

In order to do this, we introduce a proposition due to Berndt and Suh [2] as follows:

Proposition A *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r)\ ,\ \beta = \sqrt{2}\cot(\sqrt{2}r)\ ,\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 2\ ,\ m(\lambda) = 2m-2 = m(\mu),$$

and the corresponding eigenspaces we have

$$\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1,\\
T_\beta &= \mathbb{C}^{\perp}\xi = \mathbb{C}^{\perp}N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3,\\
T_\lambda &= \{X | X \perp \mathbb{H}\xi,\ JX = J_1X\},\\
T_\mu &= \{X | X \perp \mathbb{H}\xi,\ JX = -J_1X\},
\end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector ξ and $\mathbb{C}^{\perp}\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

First, let us check $\nabla_{\mathfrak{D}}\bar{R}_N = 0$ on the distribution $\mathfrak{D}$ for $\xi \in \mathfrak{D}^{\perp}$. By taking $\xi \in \mathfrak{D}^{\perp}$ and $X \in \mathfrak{D}$, we get

$$(\nabla_X \bar{R}_N)\xi = 3\phi AX + 6\eta_3(AX)\xi_2 - 6\eta_2(AX)\xi_3 + 3\phi_1 AX.$$

From Proposition A if $X_i \in T_\lambda$ we have

$$(\nabla_{X_i}\bar{R}_N)\xi = 3\phi AX_i + 3\phi_1 AX_i = 3\lambda\phi X_i + 3\lambda\phi_1 X_i = 6\lambda\phi X_i.$$

If $(\nabla_{X_i}\bar{R}_N)\xi = 0$, then $\lambda = 0$ but $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ for some $r \in (0, \pi/\sqrt{8})$. Thus, in this case, no real hypersurface satisfies our condition.

Next let us consider $\mathfrak{D}$-parallesm for the structure Jacobi operator R_ξ for $\xi \in \mathfrak{D}^{\perp}$ mentioned in the introduction. That is, let us assume that $\nabla_{\mathfrak{D}} R_\xi = 0$ for $\xi \in \mathfrak{D}^{\perp}$. This gives

$$(\nabla_X R_\xi)\xi = -\phi AX - \alpha A\phi AX - 2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 - \phi_1 AX$$

for $X \in \mathfrak{D}$. Bearing in mind Proposition A, taking $X_i \in T_\lambda$ we have

$$\begin{aligned}
(\nabla_{X_i} R_\xi)\xi &= -\phi AX_i - \alpha A\phi AX_i - 2\eta_3(AX_i)\xi_2 + 2\eta_2(AX_i)\xi_3 - \phi_1 AX_i\\
&= -\lambda\phi X_i - \alpha\lambda A\phi X_i - \lambda\phi_1 X_i\\
&= -2\lambda\phi X_i - \alpha\lambda A\phi X_i.
\end{aligned}$$

Let us see that if $X_i \in T_\lambda$ then $\phi X_i \in T_\lambda$.

Firstly, if $X_i \in T_\lambda$ then $X_i \perp \mathbb{H}\xi = \{\xi, N, \xi_2, \xi_3\}$. Thus $\phi X_i \perp \mathbb{H}\xi$. And if $X_i \in T_\lambda$ then $\phi(\phi X_i) = \phi_1(\phi X_i)$. In fact,

$$\phi_1\phi X_i = \phi\phi_1 X_i - \eta_1(X_i)\xi + \eta(X_i)\xi_1 = \phi\phi_1 X_i = \phi(\phi X_i).$$

So, as $\phi X_i \in T_\lambda$ we have

$$\begin{aligned}
(\nabla_{X_i} R_\xi)\xi &= -2\lambda\phi X_i - \alpha\lambda A\phi X_i\\
&= -2\lambda\phi X_i - \alpha\lambda^2\phi X_i\\
&= (-2\lambda - \alpha\lambda^2)\phi X_i.
\end{aligned}$$

If $(\nabla_{X_i} R_\xi)\xi = 0$ then $2\lambda + \alpha\lambda^2 = 0$. This means either $\lambda = 0$ or $2 + \alpha\lambda = 0$ for some $r \in (0, \pi/\sqrt{8})$. In the first case, $\lambda = 0$ for some $r \in (0, \pi/\sqrt{8})$, this gives a contradiction.

Now let us consider the latter case, $\alpha\lambda + 2 = 0$, we obtain

$$\begin{aligned} 0 &= \Big[\left(\sqrt{8}\cot\left(\sqrt{8}r\right)\right)\left(-\sqrt{2}\tan\left(\sqrt{2}r\right)\right) \Big] + 2 \\ &= -4\cot\left(\sqrt{8}r\right)\tan\left(\sqrt{2}r\right) + 2 \\ &= 2\tan^2\left(\sqrt{2}r\right). \end{aligned}$$

Thus $\tan(\sqrt{2}r) = 0$ for some $r \in (0, \pi/\sqrt{8})$, which gives a contradiction.

On the other hand, we recall a proposition due to Berndt and Suh [2] as follows:

Proposition B *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r)\ ,\ \beta = 2\cot(2r)\ ,\ \gamma = 0\ ,\ \lambda = \cot(r)\ ,\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 3 = m(\gamma)\ ,\ m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi\ ,\ T_\beta = \mathfrak{J}J\xi\ ,\ T_\gamma = \mathfrak{J}\xi\ ,\ T_\lambda\ ,\ T_\mu\ ,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp\ ,\ \mathfrak{J}T_\lambda = T_\lambda\ ,\ \mathfrak{J}T_\mu = T_\mu\ ,\ JT_\lambda = T_\mu.$$

First let us consider $\mathfrak{D}$-parallelism for the normal Jacobi operator $\bar{R}_N$ when the Reeb vector field $\xi \in \mathfrak{D}$. By taking $\xi \in \mathfrak{D}$ and any $X \in \mathfrak{D}$, we get

$$(\nabla_X \bar{R}_N)\xi = 3\phi AX + 5\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(AX)\phi_\nu\xi.$$

From Proposition B, taking $X_i \in T_\lambda$, we have

$$\begin{aligned} (\nabla_{X_i}\bar{R}_N)\xi &= 3\phi AX_i + 5\sum_{\nu=1}^{3}\eta_\nu(\phi AX_i)\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(AX_i)\phi_\nu\xi \\ &= 3\phi AX_i + 5\sum_{\nu=1}^{3}\eta_\nu(\phi AX_i)\xi_\nu \\ &= 3\lambda\phi X_i + 5\lambda\sum_{\nu=1}^{3}\eta_\nu(\phi X_i)\xi_\nu \\ &= 3\lambda\phi X_i. \end{aligned}$$

If now $(\nabla_{X_i}\bar{R}_N)\xi = 0$ for $X_i \in T_\lambda$, then $\lambda = 0$ for some $r \in (0, \pi/4)$ which gives a contradiction.

Next, let us consider $\mathfrak{D}$-parallelism for the structure Jacobi operator R_ξ when the Reeb vector field $\xi \in \mathfrak{D}$. By taking $\xi \in \mathfrak{D}$ and any $X \in \mathfrak{D}$, we have

$$(\nabla_X R_\xi)\xi = -\phi AX - \alpha A\phi AX - \sum_{\nu=1}^{3}\Big\{-\eta_\nu(\phi AX)\xi_\nu + 3\eta_\nu(AX)\phi_\nu\xi\Big\}.$$

Bearing in mind Proposition B for $X_i \in T_\lambda$ it follows

$$\begin{aligned}(\nabla_{X_i} R_\xi)\xi &= -\phi AX_i - \alpha A\phi AX_i - \sum_{\nu=1}^{3}\Big\{-\eta_\nu(\phi AX_i)\xi_\nu + 3\eta_\nu(AX_i)\phi_\nu\xi\Big\}\\ &= -\lambda\phi X_i - \alpha\lambda A\phi X_i.\end{aligned}$$

If $(\nabla_{X_i} R_\xi)\xi = 0$, then we get

$$\begin{aligned}0 &= \lambda\phi X_i + \alpha\lambda A\phi X_i\\ &= \lambda\phi X_i + \alpha\lambda\mu\phi X_i\\ &= (\lambda + \alpha\lambda\mu)\phi X_i.\end{aligned}$$

Thus we have $\lambda(1+\alpha\mu) = 0$. For the case where $\lambda = 0$, we can make a contradiction, because $\lambda = \cot(r)$ never vanishing for some $r \in (0, \pi/4)$. This yields

$$1 + \alpha\mu = 1 + (-2\tan(2r))(-\tan(r)) = 1 + 2\tan(2r)\tan(r) = 0$$

for some $r \in (0, \pi/4)$, which gives $2\Big[\frac{2\tan(r)}{1-\tan^2(r)}\Big]\tan(r) + 1 = 0$. From this, we have

$$4\tan^2(r) = -(1-\tan^2(r)) = -1 + \tan^2(r).$$

Then it follows

$$3\tan^2(r) = -1.$$

This gives a contradiction.

Summing up all the formulas mentioned in Sects. 3 and 4, we know that both normal Jacobi operator $\bar{R}_N$ and structure Jacobi operator R_ξ for any hypersurfaces of type (A) or type (B) in Theorem A cannot satisfy $\mathfrak{D}$-parallelism. From this, we complete the proof of our Theorems 1 and 2 in the introduction.

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