Applied Mathematical Sciences, Vol. 7, 2013, no. 142, 7093 - 7096
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2013.311658

Matrices Formula for Padovan and Perrin Sequences

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Abstract

The Padovan and Perrin numbers have the matrix formula,
$Q^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}$. The matrix product is a 3×2 matrix that when raised to the n^{th} power give a matrix product whose entries are Padovan and Perrin numbers. For which we established by mathematical induction such that,

$$Q^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} P_n & \mathcal{P}_n \\ P_{n+1} & \mathcal{P}_{n+1} \\ P_{n+2} & \mathcal{P}_{n+2} \end{bmatrix},$$

where P_n and $\mathcal{P}_n$ are the Padovan and Perrin sequences, respectively.

Mathematics Subject Classification: 11B39, 11B99

Keywords: Padovan sequences, Perrin sequences, Q-matrix

1 Introduction

The Padovan sequence is named after Richard Padovan who attributed its discovery to Dutch architect Hans van der Laan in his 1994 essay *Dom Hans van der Laan: Modern Primitive.*

In this paper, the Padovan sequence is the sequence of integers P_n defined by the initial values $P_0 = 0, P_1 = 0, P_2 = 1$ and the recurrence relation

$$P_n = P_{n-2} + P_{n-3}, \text{ for all } n \geq 3.$$

The first few values of P_n are 0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, ...

The Padovan numbers have the Q-matrix, $Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ such that $Q^n = \begin{bmatrix} P_{n-1} & P_{n+1} & P_n \\ P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \end{bmatrix}$, for all $n \geq 3$.

The Perrin sequence is the sequence of integers $\mathcal{P}_n$ defined by a recurrence relation, and is qualitatively similar to the Lucas sequence. The initial terms are $\mathcal{P}_0 = 3, \mathcal{P}_1 = 0, \mathcal{P}_2 = 2$ and subsequent terms are defined by

$$\mathcal{P}_n = \mathcal{P}_{n-2} + \mathcal{P}_{n-3}, \text{ for all } n \geq 3.$$

Here are the first few Perrin numbers: 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, ...

2 Main Results

In this study, we investigate the new property of Padovan and Perrin numbers in relation with the Padovan and Perrin matrices formula, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}$. More generally, we have $Q^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} P_n & \mathcal{P}_n \\ P_{n+1} & \mathcal{P}_{n+1} \\ P_{n+2} & \mathcal{P}_{n+2} \end{bmatrix}$. This strategy allow us to obtain the new relations for the Padovan and Perrin sequences.

Theorem 2.1. *For all $n \in \mathbb{N}$ we have,*

$$Q^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} P_n & \mathcal{P}_n \\ P_{n+1} & \mathcal{P}_{n+1} \\ P_{n+2} & \mathcal{P}_{n+2} \end{bmatrix}.$$

Proof. Let use the principle of mathematical induction on n. For $n = 1$, it is easy to see that

$$Q^1 \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^1 \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} P_1 & \mathcal{P}_1 \\ P_2 & \mathcal{P}_2 \\ P_3 & \mathcal{P}_3 \end{bmatrix} = \begin{bmatrix} P_1 & \mathcal{P}_1 \\ P_{1+1} & \mathcal{P}_{1+1} \\ P_{1+2} & \mathcal{P}_{1+2} \end{bmatrix}.$$

Assume that it is true for all positive integer $n = k$. That is,

$$Q^k \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^k \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} P_k & \mathcal{P}_k \\ P_{k+1} & \mathcal{P}_{k+1} \\ P_{k+2} & \mathcal{P}_{k+2} \end{bmatrix}.$$

Therefore, we have to show that it is true for $n = k + 1$. By the laws of associativity and exponents hold for the matrices such that their dimensions match. Consider,

$$\begin{aligned}
Q^{k+1}\begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} &= \left(QQ^k\right)\begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} \\
&= Q\left(Q^k\begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} P_k & \mathcal{P}_k \\ P_{k+1} & \mathcal{P}_{k+1} \\ P_{k+2} & \mathcal{P}_{k+2} \end{bmatrix} \\
&= \begin{bmatrix} P_{k+1} & \mathcal{P}_{k+1} \\ P_{k+2} & \mathcal{P}_{k+2} \\ P_k + P_{k+1} & \mathcal{P}_k + \mathcal{P}_{k+1} \end{bmatrix} \\
&= \begin{bmatrix} P_{k+1} & \mathcal{P}_{k+1} \\ P_{k+2} & \mathcal{P}_{k+2} \\ P_{k+3} & \mathcal{P}_{k+3} \end{bmatrix}.
\end{aligned}$$

Therefore, the result is true for every $n \geq 1$.

Let us generalize this observation using the Padovan and Perrin formula matrices.

Proposition 2.2. *For all integers m, n such that $3 \leq m < n$, we have the following relations:*

(a) $P_n = P_{m-1} \cdot P_{n-m} + P_{m+1} \cdot P_{n-m+1} + P_m \cdot P_{n-m+2}$,

(b) $\mathcal{P}_n = P_{m-1} \cdot \mathcal{P}_{n-m} + P_{m+1} \cdot \mathcal{P}_{n-m+1} + P_m \cdot \mathcal{P}_{n-m+2}$.

Proof. From the laws of exponent for the square matrices. So, we have

$$Q^n = Q^m Q^{n-m},$$

it follows that

$$Q^n\begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = Q^m\left(Q^{n-m}\begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}\right).$$

From the property of Padovan Q-matrix (see [2], page 2778) and Theorem 2.1 it follows that,

$$\begin{bmatrix} P_n & \mathcal{P}_n \\ P_{n+1} & \mathcal{P}_{n+1} \\ P_{n+2} & \mathcal{P}_{n+2} \end{bmatrix} = \begin{bmatrix} P_{m-1} & P_{m+1} & P_m \\ P_m & P_{m+2} & P_{m+1} \\ P_{m+1} & P_{m+3} & P_{m+2} \end{bmatrix}\begin{bmatrix} P_{n-m} & \mathcal{P}_{n-m} \\ P_{n-m+1} & \mathcal{P}_{n-m+1} \\ P_{n-m+2} & \mathcal{P}_{n-m+2} \end{bmatrix}$$

yielding, upon equating corresponding elements. That is,

$$P_n = P_{m-1} \cdot P_{n-m} + P_{m+1} \cdot P_{n-m+1} + P_m \cdot P_{n-m+2},$$

and

$$\mathcal{P}_n = P_{m-1} \cdot \mathcal{P}_{n-m} + P_{m+1} \cdot \mathcal{P}_{n-m+1} + P_m \cdot \mathcal{P}_{n-m+2}.$$

Completes the proof.

Remark 2.3. *In Proposition 2.2, if $m = 3$, then we have*

$$\begin{aligned} P_n &= P_2 \cdot P_{n-3} + P_4 \cdot P_{n-2} + P_3 \cdot P_{n-1}\,, \\ &= 1 \cdot P_{n-3} + 1 \cdot P_{n-2} + 0 \cdot P_{n-1}\,,\ (\textit{replaces } P_2 = P_4 = 1 \textit{ and } P_3 = 0) \\ &= P_{n-2} + P_{n-3}. \end{aligned}$$

Similary, we have $\mathcal{P}_n = \mathcal{P}_{n-2} + \mathcal{P}_{n-3}$.

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Received: November 25, 2013