# Matrix biorthogonal polynomials on the unit circle and non-abelian Ablowitz-Ladik hierarchy.

**Mattia Cafasso**

SISSA-UCLouvain

Workshop on Random Matrices, Related Topics and Applications.

CRM-Montreal

25th August-30th August 2008

# Plan of the talk.

# Plan of the talk.

- Review of the connection between biorthogonal polynomials on the unit circle and the Ablowitz-Ladik hierarchy (M.Adler-P.van Moerbeke '01,B.Simon I.Nenciu '05):

# Plan of the talk.

- Review of the connection between biorthogonal polynomials on the unit circle and the Ablowitz-Ladik hierarchy (M.Adler-P.van Moerbeke '01,B.Simon I.Nenciu '05):
  **Recursion coefficients for BOPUC satisfy semidiscrete zero-curvature equations for the Ablowitz-Ladik hierarchy.**

# Plan of the talk.

- Review of the connection between biorthogonal polynomials on the unit circle and the Ablowitz-Ladik hierarchy (M.Adler-P.van Moerbeke '01,B.Simon I.Nenciu '05):
  **Recursion coefficients for BOPUC satisfy semidiscrete zero-curvature equations for the Ablowitz-Ladik hierarchy.**
- Extension to the case of matrix biorthogonal polynomials on the unit circle:

# Plan of the talk.

- Review of the connection between biorthogonal polynomials on the unit circle and the Ablowitz-Ladik hierarchy (M.Adler-P.van Moerbeke ’01,B.Simon I.Nenciu ’05):
  **Recursion coefficients for BOPUC satisfy semidiscrete zero-curvature equations for the Ablowitz-Ladik hierarchy.**
- Extension to the case of matrix biorthogonal polynomials on the unit circle:
  **Recursion coefficients for matrix BOPUC satisfy semidiscrete zero-curvature equations for the non abelian Ablowitz-Ladik hierarchy.**

# Ablowitz-Ladik hierarchy; a discretization for AKNS (Ablowitz-Kaup-Newell-Segur).

# Ablowitz-Ladik hierarchy; a discretization for AKNS (Ablowitz-Kaup-Newell-Segur).

- Zakharov-Shabat spectral problem for AKNS:

$$\begin{cases} \partial_x \Psi = L\Psi \\ \partial_\tau \Psi = M\Psi \end{cases} \qquad L := \begin{pmatrix} z & q \\ r & -z \end{pmatrix}$$

$$\Longrightarrow \partial_\tau L := [M, L] + \partial_x M$$

# Ablowitz-Ladik hierarchy; a discretization for AKNS (Ablowitz-Kaup-Newell-Segur).

- Zakharov-Shabat spectral problem for AKNS:

$$\begin{cases} \partial_x \Psi = L\Psi \\ \partial_\tau \Psi = M\Psi \end{cases} \qquad L := \begin{pmatrix} z & q \\ r & -z \end{pmatrix}$$

$$\implies \partial_\tau L := [M, L] + \partial_x M$$

- Ablowitz-Ladik discretization:

$$\begin{cases} \Psi_{k+1} = L_k \Psi_k \\ \partial_\tau \Psi_k = M_k \Psi_k \end{cases} \qquad L_k := \begin{pmatrix} z & x_k \\ y_k & z^{-1} \end{pmatrix}$$

$$\implies \partial_\tau L_k = M_{k+1} L_k - L_k M_k$$

# An example: NLS and its discretization.

# An example: NLS and its discretization.

- Complexified non-linear Schrödinger:

$$L := \begin{pmatrix} z & q \\ r & -z \end{pmatrix} \quad M := \begin{pmatrix} 2z^2 - qr & 2zq + q_x \\ 2zr - r_x & -2z^2 + qr \end{pmatrix}$$

$$\Longrightarrow \begin{cases} \partial_\tau q = q_{xx} - 2q^2 r \\ \partial_\tau r = -r_{xx} + 2qr^2. \end{cases}$$

# An example: NLS and its discretization.

- Complexified non-linear Schrödinger:

$$L := \begin{pmatrix} z & q \\ r & -z \end{pmatrix} \quad M := \begin{pmatrix} 2z^2 - qr & 2zq + q_x \\ 2zr - r_x & -2z^2 + qr \end{pmatrix}$$

$$\Longrightarrow \begin{cases} \partial_\tau q = q_{xx} - 2q^2 r \\ \partial_\tau r = -r_{xx} + 2qr^2. \end{cases}$$

- Discrete complexified non-linear Schrödinger:

$$L_k := \begin{pmatrix} z & x_k \\ y_k & z^{-1} \end{pmatrix} \quad M_k := \begin{pmatrix} z^2 - 1 - x_k y_{k-1} & z x_k - z^{-1} x_{k-1} \\ z y_{k-1} - z^{-1} y_k & -z^{-2} + 1 + x_{k-1} y_k \end{pmatrix}$$

$$\Longrightarrow \begin{cases} \partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\ \partial_\tau y_k = -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}). \end{cases}$$

# Our goal, our strategy.

# Our goal, our strategy.

- The goal: Expressing the connection between Ablowitz-Ladik hierarchy and the theory of biorthogonal polynomials on the unit circle (M.Adler-P.van Moerbeke ’01,I.Nenciu ’05).

# Our goal, our strategy.

- The goal: Expressing the connection between Ablowitz-Ladik hierarchy and the theory of biorthogonal polynomials on the unit circle (M.Adler-P.van Moerbeke '01,I.Nenciu '05).
- The strategy: We slightly change the discrete Lax operator (P.Miller,N.Ercolani,I.Krichever, C.D.Levermore 1995.)

$$L_k = \begin{pmatrix} z & x_k \\ y_k & z^{-1} \end{pmatrix} \longrightarrow \mathfrak{L}_k := \begin{pmatrix} z & x_k \\ zy_k & 1 \end{pmatrix}$$

# Our goal, our strategy.

- The goal: Expressing the connection between Ablowitz-Ladik hierarchy and the theory of biorthogonal polynomials on the unit circle (M.Adler-P.van Moerbeke '01,I.Nenciu '05).
- The strategy: We slightly change the discrete Lax operator (P.Miller,N.Ercolani,I.Krichever, C.D.Levermore 1995.)

$$L_k = \begin{pmatrix} z & x_k \\ y_k & z^{-1} \end{pmatrix} \longrightarrow \mathcal{L}_k := \begin{pmatrix} z & x_k \\ zy_k & 1 \end{pmatrix}$$

- We write eigenvalues $\Phi_k$ such that

$$\Phi_{k+1} = \mathcal{L}_k \Phi_k$$

in terms of biorthogonal polynomials on the unit circle.

# Our goal, our strategy.

- The goal: Expressing the connection between Ablowitz-Ladik hierarchy and the theory of biorthogonal polynomials on the unit circle (M.Adler-P.van Moerbeke '01,I.Nenciu '05).
- The strategy: We slightly change the discrete Lax operator (P.Miller,N.Ercolani,I.Krichever, C.D.Levermore 1995.)

$$L_k = \begin{pmatrix} z & x_k \\ y_k & z^{-1} \end{pmatrix} \longrightarrow \mathcal{L}_k := \begin{pmatrix} z & x_k \\ zy_k & 1 \end{pmatrix}$$

- We write eigenvalues $\Phi_k$ such that

$$\Phi_{k+1} = \mathcal{L}_k \Phi_k$$

in terms of biorthogonal polynomials on the unit circle.
- We make them evolve using 2D-Toda flow.

# 2D-Toda I.

Definitions:

$$\Lambda := (\delta_{i,j-1})_{i,j\geq 0}, \quad \Lambda^{-1} := \Lambda^T \qquad \begin{cases} L_1 := \Lambda + \sum_{i\leq 0} a_i^{(1)}\Lambda^i \\ L_2 := a_{-1}^{(2)}\Lambda^{-1} + \sum_{i\geq 0} a_i^{(2)}\Lambda^i \end{cases}$$

# 2D-Toda I.

**Definitions:**

$$\Lambda := (\delta_{i,j-1})_{i,j\geq 0}, \quad \Lambda^{-1} := \Lambda^T \qquad \begin{cases} L_1 := \Lambda + \sum_{i\leq 0} a_i^{(1)} \Lambda^i \\ L_2 := a_{-1}^{(2)} \Lambda^{-1} + \sum_{i\geq 0} a_i^{(2)} \Lambda^i \end{cases}$$

**Zakharov-Shabat Spectral problem:**

$$\begin{cases} L_1 \Psi_1 = z \Psi_1 \\ L_2^T \Psi_2^* = z^{-1} \Psi_2^* \end{cases} \quad \begin{cases} \partial_{t_n} \Psi_1 = (L_1^n)_+ \Psi_1 \\ \partial_{t_n} \Psi_2^* = -(L_1^n)_+^T \Psi_2^* \end{cases} \quad \begin{cases} \partial_{s_n} \Psi_1 = (L_2^n)_- \Psi_1 \\ \partial_{s_n} \Psi_2^* = -(L_2^n)_-^T \Psi_2^* \end{cases}$$

# 2D-Toda I.

## Definitions:

$$\Lambda := (\delta_{i,j-1})_{i,j\geq 0}, \quad \Lambda^{-1} := \Lambda^T \qquad \begin{cases} L_1 := \Lambda + \sum_{i\leq 0} a_i^{(1)} \Lambda^i \\ L_2 := a_{-1}^{(2)} \Lambda^{-1} + \sum_{i\geq 0} a_i^{(2)} \Lambda^i \end{cases}$$

## Zakharov-Shabat Spectral problem:

$$\begin{cases} L_1 \Psi_1 = z \Psi_1 \\ L_2^T \Psi_2^* = z^{-1} \Psi_2^* \end{cases} \quad \begin{cases} \partial_{t_n} \Psi_1 = (L_1^n)_+ \Psi_1 \\ \partial_{t_n} \Psi_2^* = -(L_1^n)_+^T \Psi_2^* \end{cases} \quad \begin{cases} \partial_{s_n} \Psi_1 = (L_2^n)_- \Psi_1 \\ \partial_{s_n} \Psi_2^* = -(L_2^n)_-^T \Psi_2^* \end{cases}$$

## Lax equations:

$$\partial_{t_n} L_i = \left[ (L_1^n)_+, L_i \right] \quad \partial_{s_n} L_i = \left[ (L_2^n)_-, L_i \right]$$

# 2D-Toda II (linearization).

## Theorem (Takasaki '84)

*Given any semi-infinite matrix* $M := S_1^{-1} S_2$ *with*

$$S_1 = \mathrm{I} + \sum_{i\geq 1} s_i^{(1)} \Lambda^{-i} \qquad S_2 = s_0^{(2)} + \sum_{i\geq 1} s_i^{(2)} \Lambda^{i}$$

*define its time evolution through*

$$M(t;s) := \exp(\xi(t,\Lambda)) M \exp(-\xi(s,\Lambda^{-1})) = S_1^{-1}(t;s) S_2(t;s).$$

*with* $\xi(t,\Lambda) = \sum_{i\geq 1} t_i \Lambda^i$.

# 2D-Toda II (linearization).

## Theorem (Takasaki '84)

*Given any semi-infinite matrix* $M := S_1^{-1} S_2$ *with*

$$S_1 = \mathrm{I} + \sum_{i \geq 1} s_i^{(1)} \Lambda^{-i} \qquad S_2 = s_0^{(2)} + \sum_{i \geq 1} s_i^{(2)} \Lambda^{i}$$

*define its time evolution through*

$$M(t;s) := \exp(\xi(t,\Lambda)) M \exp(-\xi(s,\Lambda^{-1})) = S_1^{-1}(t;s) S_2(t;s).$$

*with* $\xi(t,\Lambda) = \sum_{i \geq 1} t_i \Lambda^i$. *Then*

$$L_1 := S_1 \Lambda S_1^{-1} \quad L_2 := S_2 \Lambda^{-1} S_2^{-1}$$

$$\Psi_1 := \exp \xi(t,z) S_1 \chi(z) \quad \Psi_2^* := \exp(-\xi(s,z^{-1}))(S_2^{-1})^T \chi(z^{-1})$$

*with* $\chi(z) := (1, z, z^2, \ldots)^T$ *solve 2D-Toda.*

# Generalized biorthogonal polynomials

## Proposition

*Given a matrix* $M = S_1^{-1} S_2$ *define a bilinear pairing on the space of polynomials in* $z$ *imposing* $< z^i, z^j >_M = M_{i,j}$. *Then the polynomials*

$$q^{(1)} := (q_0^{(1)}, q_1^{(1)}, q_2^{(1)}, \ldots)^T = S_1 \chi(z)$$

$$q^{(2)} := (q_0^{(2)}, q_1^{(2)}, q_2^{(2)}, \ldots)^T = (S_2^{-1})^T \chi(z)$$

*are biorthonormal, i.e.* $< q_i^{(1)}, q_j^{(2)} >_M = \delta_{i,j}$

# The Toeplitz lattice I (Adler-van Moerbeke ’01).

# The Toeplitz lattice I (Adler-van Moerbeke '01).

Suppose that your initial datum $M$ is a Toeplitz matrix; i.e. $(M_{ij})_{i,j\geq 0} = (M_{j-i})_{i,j\geq 0}$. Then:

# The Toeplitz lattice I (Adler-van Moerbeke '01).

Suppose that your initial datum $M$ is a Toeplitz matrix; i.e. $(M_{ij})_{i,j\geq 0} = (M_{j-i})_{i,j\geq 0}$. Then:

- Toeplitz simmetry is conserved along 2D-Toda flow.

# The Toeplitz lattice I (Adler-van Moerbeke '01).

Suppose that your initial datum $M$ is a Toeplitz matrix; i.e. $(M_{ij})_{i,j\geq 0} = (M_{j-i})_{i,j\geq 0}$. Then:

- Toeplitz simmetry is conserved along 2D-Toda flow.
- The pairing $< .,. >_{M(t,s)}$ can be written as

$$< P(z), Q(z) >_M = \oint P(z)Q(z^{-1}) \exp(\xi(t,z))\gamma(z) \exp(-\xi(s,z^{-1}))\frac{dz}{2\pi iz}$$

with $\gamma(z) = \sum_k M_k z^k$ initial value.

# The Toeplitz lattice I (Adler-van Moerbeke '01).

Suppose that your initial datum $M$ is a Toeplitz matrix; i.e. $(M_{ij})_{i,j\geq 0} = (M_{j-i})_{i,j\geq 0}$. Then:

- Toeplitz simmetry is conserved along 2D-Toda flow.
- The pairing $< .,. >_{M(t,s)}$ can be written as
  $$< P(z), Q(z) >_M = \oint P(z)Q(z^{-1}) \exp(\xi(t,z))\gamma(z) \exp(-\xi(s,z^{-1})) \frac{dz}{2\pi iz}$$
  with $\gamma(z) = \sum_k M_k z^k$ initial value.
- Monic biorthogonal polynomials $\{p_i^{(1)}, p_j^{(2)}\}$ such that $< p_i^{(1)}(z), p_j^{(2)}(z) >_M = h_i \delta_{i,j}$ satisfy the recursion relation
  $$\begin{pmatrix} p_{n+1}^{(1)}(z) \\ \tilde{p}_{n+1}^{(2)}(z) \end{pmatrix} = \mathfrak{L}_n \begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} z & x_{n+1} \\ zy_{n+1} & 1 \end{pmatrix} \begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix}$$
  with $\tilde{p}_n^{(2)}(z) := z^n p_n^{(2)}(z^{-1})$

# The Toeplitz lattice II (Adler-van Moerbeke '01).

## Theorem

*Lax operators of the Toeplitz lattice are of the following form:*

$$h^{-1}L_1h = \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & \dots & \dots \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \dots \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$L_2(x,y) = (h^{-1}L_1h)^T(y,x).$$

# The Toeplitz lattice II (Adler-van Moerbeke '01).

**Theorem**

*Lax operators of the Toeplitz lattice are of the following form:*

$$h^{-1}L_1h = \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & \dots & \dots \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \dots \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$L_2(x,y) = (h^{-1}L_1h)^T(y,x).$$

Hence the Toeplitz lattice gives time evolution for recursion coefficients for BOPUC.

## Theorem

*Consider time-dipendent BOPUC* $\{p_n^{(1)}(z), p_n^{(2)}(z)\}$ *with respect to the pairing*

$$< P(z), Q(z) >:= \oint P(z)Q(z^{-1})\gamma(t,s;z)\frac{dz}{2\pi i z}$$

*with* $\gamma(t,s;z) := \exp(\xi(t,z))\gamma(z)\exp(-\xi(s,z^{-1}))$. *Then the related recursion operators* $\mathcal{L}_k$ *evolve according to semidiscrete zero-curvature equations for the Ablowitz-Ladik hierarchy*

$$\partial_{t_i/s_i}\mathcal{L}_k = \mathcal{M}_{t_i/s_i,k+1}\mathcal{L}_k - \mathcal{L}_k\mathcal{M}_{t_i/s_i,k}.$$

# Sketch of the proof.

# Sketch of the proof.

- $\Phi_k := (p_k^{(1)}(z), \tilde{p}_k^{(2)}(z))^T$ satisfy equation $\Phi_{k+1} = \mathcal{L}_k \Phi_k$. We have just to find $\mathcal{M}_{t_i/s_i,k}$ such that $\partial_{t_i/s_i} \Phi_k = \mathcal{M}_{t_i/s_i,k} \Phi_k$. Then semidiscrete zero-curvature equations will be given by compatibility conditions.

# Sketch of the proof.

- $\Phi_k := (p_k^{(1)}(z), \tilde{p}_k^{(2)}(z))^T$ satisfy equation $\Phi_{k+1} = \mathcal{L}_k \Phi_k$.
  We have just to find $\mathcal{M}_{t_i/s_i,k}$ such that $\partial_{t_i/s_i} \Phi_k = \mathcal{M}_{t_i/s_i,k} \Phi_k$.
  Then semidiscrete zero-curvature equations will be given by compatibility conditions.
- Using Zakharov-Shabat equations, for every $k$, $\partial_{t_i} p_k^{(1)}$ is a linear combination of $\{p_k^{(1)}, p_{k+1}^{(1)}, p_{k+2}^{(1)}, \dots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$.
  In the same way, for every $k$, $\partial_{t_i} \tilde{p}_k^{(2)}$ is a linear combination of $\{\tilde{p}_k^{(2)}, \tilde{p}_{k-1}^{(2)}, \tilde{p}_{k-2}^{(2)}, \dots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$.

# Sketch of the proof.

- $\Phi_k := (p_k^{(1)}(z), \tilde{p}_k^{(2)}(z))^T$ satisfy equation $\Phi_{k+1} = \mathfrak{L}_k \Phi_k$.
  We have just to find $\mathcal{M}_{t_i/s_i,k}$ such that $\partial_{t_i/s_i} \Phi_k = \mathcal{M}_{t_i/s_i,k} \Phi_k$.
  Then semidiscrete zero-curvature equations will be given by compatibility conditions.
- Using Zakharov-Shabat equations, for every $k$, $\partial_{t_i} p_k^{(1)}$ is a linear combination of $\{p_k^{(1)}, p_{k+1}^{(1)}, p_{k+2}^{(1)}, \dots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$.
  In the same way, for every $k$, $\partial_{t_i} \tilde{p}_k^{(2)}$ is a linear combination of $\{\tilde{p}_k^{(2)}, \tilde{p}_{k-1}^{(2)}, \tilde{p}_{k-2}^{(2)}, \dots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$.
- Using recursion relations you arrive to write $\mathcal{M}_{t_i,k}$ depending on $\{x_j, y_j, z\}$.

# Example:

# Example:

$$\partial_{t_1} p_k^{(1)} = -z p_k^{(1)} - x_{k+1} y_k p_k^{(1)} + p_{k+1}^{(1)} = -x_{k+1} y_k p_k^{(1)} + x_{k+1} \tilde{p}_k^{(2)}$$

$$\partial_{t_1} \tilde{p}_k^{(2)} = -z \frac{h_{k+1}}{h_k} \tilde{p}_{k-1}^{(2)} = z y_k p_k^{(1)} - z \tilde{p}_k^{(2)}$$

# Example:

$$\partial_{t_1} p_k^{(1)} = -zp_k^{(1)} - x_{k+1}y_k p_k^{(1)} + p_{k+1}^{(1)} = -x_{k+1}y_k p_k^{(1)} + x_{k+1}\tilde{p}_k^{(2)}$$

$$\partial_{t_1} \tilde{p}_k^{(2)} = -z\frac{h_{k+1}}{h_k}\tilde{p}_{k-1}^{(2)} = zy_k p_k^{(1)} - z\tilde{p}_k^{(2)}$$

$$\Longrightarrow \partial_{t_1} \begin{pmatrix} p_k^{(1)} \\ \tilde{p}_k^{(2)} \end{pmatrix} = \mathcal{M}_{t_1,k} \begin{pmatrix} p_k^{(1)} \\ \tilde{p}_k^{(2)} \end{pmatrix} = \begin{pmatrix} -x_{k+1}y_k & x_{k+1} \\ zy_k & -z \end{pmatrix} \begin{pmatrix} p_k^{(1)} \\ \tilde{p}_k^{(2)} \end{pmatrix}$$

# A natural question.

# A natural question.

- Time evolution for orthogonal polynomials on the real line leads to the Toda hierarchy.

# A natural question.

- Time evolution for orthogonal polynomials on the real line leads to the Toda hierarchy.
- Time evolution for biorthogonal polynomials on the unit circle leads to the Ablowitz-Ladik hierarchy.

# A natural question.

- Time evolution for orthogonal polynomials on the real line leads to the Toda hierarchy.
- Time evolution for biorthogonal polynomials on the unit circle leads to the Ablowitz-Ladik hierarchy.
- Time evolution for matrix orthogonal polynomials on the real line leads to the non-abelian Toda hierarchy.

# A natural question.

- Time evolution for orthogonal polynomials on the real line leads to the Toda hierarchy.
- Time evolution for biorthogonal polynomials on the unit circle leads to the Ablowitz-Ladik hierarchy.
- Time evolution for matrix orthogonal polynomials on the real line leads to the non-abelian Toda hierarchy.

**What about time evolution for matrix biorthogonal polynomials on the unit circle?**

# The setting I.

- Introduce a matrix-valued symbol $\gamma(z) = \sum_k M_k z^k$ with $\{M_k\}$ $(n \times n)$ matrices.

# The setting I.

- Introduce a matrix-valued symbol $\gamma(z) = \sum_k M_k z^k$ with $\{M_k\}$ $(n \times n)$ matrices.
- Consider the two matrix-valued pairings

$$< P(z), Q(z) >_L = \oint P(z)\gamma(z)Q^*(z)\frac{dz}{2\pi iz}$$

$$< P(z), Q(z) >_R = \oint P^*(z)\gamma(z)Q(z)\frac{dz}{2\pi iz}$$

and corresponding matrix-valued biorthogonal polynomials

$$\{P_n^{(1)R}(z), P_n^{(1)L}(z), P_n^{(2)R}(z), P_n^{(2)L}(z)\}$$

such that

$$< P_k^{(2)R}, P_j^{(1)R} >_R = \delta_{kj} h_k^R \qquad < P_k^{(1)L}, P_j^{(2)L} >_L = \delta_{kj} h_k^L.$$

# The setting II.

- The following block recursion relations can be obtained:

$$\begin{pmatrix} P^{(1)L}_{N+1} \\ \tilde{P}^{(2)R}_{N+1} \end{pmatrix} = \mathcal{L}^L_N \begin{pmatrix} P^{(1)L}_{N} \\ \tilde{P}^{(2)R}_{N} \end{pmatrix} = \begin{pmatrix} z\mathrm{I} & x^L_{N+1} \\ zy^R_{N+1} & \mathrm{I} \end{pmatrix} \begin{pmatrix} P^{(1)L}_{N} \\ \tilde{P}^{(2)R}_{N} \end{pmatrix}$$

$$\begin{pmatrix} P^{(1)R}_{N+1} & \tilde{P}^{(2)L}_{N+1} \end{pmatrix} = \begin{pmatrix} P^{(1)R}_{N} & \tilde{P}^{(2)L}_{N} \end{pmatrix} \mathcal{L}^R_N = \begin{pmatrix} P^{(1)R}_{N} & \tilde{P}^{(2)L}_{N} \end{pmatrix} \begin{pmatrix} z\mathrm{I} & zy^L_{N+1} \\ x^R_{N+1} & \mathrm{I} \end{pmatrix}.$$

# The setting II.

- The following block recursion relations can be obtained:

$$\begin{pmatrix} P_{N+1}^{(1)L} \\ \tilde{P}_{N+1}^{(2)R} \end{pmatrix} = \mathcal{L}_N^L \begin{pmatrix} P_N^{(1)L} \\ \tilde{P}_N^{(2)R} \end{pmatrix} = \begin{pmatrix} z\mathrm{I} & x_{N+1}^L \\ zy_{N+1}^R & \mathrm{I} \end{pmatrix} \begin{pmatrix} P_N^{(1)L} \\ \tilde{P}_N^{(2)R} \end{pmatrix}$$

$$\begin{pmatrix} P_{N+1}^{(1)R} & \tilde{P}_{N+1}^{(2)L} \end{pmatrix} = \begin{pmatrix} P_N^{(1)R} & \tilde{P}_N^{(2)L} \end{pmatrix} \mathcal{L}_N^R = \begin{pmatrix} P_N^{(1)R} & \tilde{P}_N^{(2)L} \end{pmatrix} \begin{pmatrix} z\mathrm{I} & zy_{N+1}^L \\ x_{N+1}^R & \mathrm{I} \end{pmatrix}.$$

- Now introduce time evolution

$$\gamma(t,s;z) := \exp(\xi(t,z\mathrm{I}))\gamma(z)\exp(-\xi(s,z^{-1}\mathrm{I}))$$

and study time evolution for corresponding matrix biorthogonal polynomials.

# The setting III.

- Look for some matrices $\mathcal{M}^{L}_{t_i/s_i,n}, \mathcal{M}^{R}_{t_i/s_i,n}$ such that

$$\partial_{t_i/s_i} \begin{pmatrix} P_N^{(1)L}(z) \\ \tilde{P}_N^{(2)R}(z) \end{pmatrix} = \mathcal{M}^{L}_{t_i/s_i,n} \begin{pmatrix} P_N^{(1)L}(z) \\ \tilde{P}_N^{(2)R}(z) \end{pmatrix}$$

$$\partial_{t_i/s_i} \begin{pmatrix} P_N^{(1)R}(z) & \tilde{P}_N^{(2)L}(z) \end{pmatrix} = \begin{pmatrix} P_N^{(1)R}(z) & \tilde{P}_N^{(2)L}(z) \end{pmatrix} \mathcal{M}^{R}_{t_i/s_i,n}$$

# The setting III.

- Look for some matrices $\mathcal{M}^L_{t_i/s_i,n}, \mathcal{M}^R_{t_i/s_i,n}$ such that

$$\partial_{t_i/s_i}\begin{pmatrix} P_N^{(1)L}(z) \\ \tilde{P}_N^{(2)R}(z)\end{pmatrix} = \mathcal{M}^L_{t_i/s_i,n}\begin{pmatrix} P_N^{(1)L}(z) \\ \tilde{P}_N^{(2)R}(z)\end{pmatrix}$$

$$\partial_{t_i/s_i}\begin{pmatrix} P_N^{(1)R}(z) & \tilde{P}_N^{(2)L}(z)\end{pmatrix} = \begin{pmatrix} P_N^{(1)R}(z) & \tilde{P}_N^{(2)L}(z)\end{pmatrix}\mathcal{M}^R_{t_i/s_i,n}$$

- Compatibility conditions will give non-abelian semidiscrete zero-curvature equations.

$$\partial_\tau \mathcal{L}_k^L = \mathcal{M}^L_{k+1}\mathcal{L}_k^L - \mathcal{L}_n^L\mathcal{M}_k^L$$
$$\partial_\tau \mathcal{L}_k^R = \mathcal{L}_k^R\mathcal{M}^R_{k+1} - \mathcal{M}_k^R\mathcal{L}_k^R.$$

# Non-abelian AL and MBOPUC

## Theorem (M.C.)

*Consider time-dipendent matrix BOPUC* $\{P_n^{(1)R}(z), P_n^{(1)L}(z)\}$ *and* $\{P_n^{(2)R}(z), P_n^{(2)L}(z)\}$ *with respect to the pairings*

$$< P, Q >_R := \oint P^*(z)\gamma(t, s; z)Q(z)\frac{dz}{2\pi iz}$$
$$< P, Q >_L := \oint P(z)\gamma(t, s; z)Q^*(z)\frac{dz}{2\pi iz}$$

*with* $\gamma(t, s; z) := \exp(\xi(t, z\mathrm{I}))\gamma(z)\exp(-\xi(s, z^{-1}\mathrm{I}))$. *Then the related block recursion operators* $\mathcal{L}^R$ *and* $\mathcal{L}^L$ *evolves according to the non-abelian AL equations:*

$$\partial_{t_i/s_i}\mathcal{L}_k^L = \mathcal{M}_{t_i/s_i,k+1}^L\mathcal{L}_k^L - \mathcal{L}_n^L\mathcal{M}_{t_i/s_i,k}^L$$
$$\partial_{t_i/s_i}\mathcal{L}_k^R = \mathcal{L}_k^R\mathcal{M}_{t_i/s_i,k+1}^R - \mathcal{M}_{t_i/s_i,k}^R\mathcal{L}_k^R.$$

# Discrete NLS and its non abelian version.

- Discrete NLS:

$$\begin{cases} \partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\ \partial_\tau y_k = -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}). \end{cases}$$

# Discrete NLS and its non abelian version.

- Discrete NLS:

$$\begin{cases} \partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\ \partial_\tau y_k = -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}). \end{cases}$$

- Non-abelian discrete NLS:

$$\begin{cases} \partial_\tau x_k^L = x_{k+1}^L - 2x_k^L + x_{k-1}^L - x_{k+1}^L y_k^R x_k^L - x_k^L y_k^R x_{k-1}^L \\ \\ \partial_\tau y_k^R = -y_{k+1}^R + 2y_k^R - y_{k-1}^R + y_{k+1}^R x_k^L y_k^R + y_k^R x_k^L y_{k-1}^R \end{cases}$$

$$\begin{cases} \partial_\tau x_k^R = x_{k+1}^R - 2x_k^R + x_{k-1}^R - x_{k-1}^R y_k^L x_k^R - x_k^R y_k^L x_{k+1}^R \\ \\ \partial_\tau y_k^L = -y_{k+1}^L + 2y_k^L - y_{k-1}^L + y_{k-1}^L x_k^R y_k^L + y_k^L x_k^R y_{k+1}^L \end{cases}$$

# Sketch of the proof I.

- Consider right and left block-Toeplitz matrices

$$T^L(t,s) := (M_{j-i})_{i,j\geq 0} \qquad T^R(t,s) := (M_{i-j})_{i,j\geq 0}$$

(they are moment matrices for $< .,. >_L$ and $< .,. >_R$.)

# Sketch of the proof I.

- Consider right and left block-Toeplitz matrices

$$T^L(t,s) := (M_{j-i})_{i,j\geq 0} \qquad T^R(t,s) := (M_{i-j})_{i,j\geq 0}$$

(they are moment matrices for $< .,. >_L$ and $< .,. >_R$.)

- Consider the two factorizations
$$T^L(t,s) = S_1(t,s)^{-1} S_2(t,s) \qquad T^R(t,s) = Z_2(t,s) Z_1^{-1}(t,s)$$

# Sketch of the proof I.

- Consider right and left block-Toeplitz matrices

$$T^L(t,s) := (M_{j-i})_{i,j\geq 0} \qquad T^R(t,s) := (M_{i-j})_{i,j\geq 0}$$

(they are moment matrices for $< .,. >_L$ and $< .,. >_R$.)

- Consider the two factorizations
$T^L(t,s) = S_1(t,s)^{-1}S_2(t,s) \qquad T^R(t,s) = Z_2(t,s)Z_1^{-1}(t,s)$

- Define the following block Lax matrices and block wave vectors:

$$L_1 := S_1\Lambda S_1^{-1} \quad L_2 := S_2\Lambda^{-1}S_2^{-1}$$

$$R_1 := Z_1^{-1}\Lambda^{-1}Z_1 \quad R_2 := Z_2^{-1}\Lambda Z_2$$

$$\Psi_1(z) := \exp(\xi(t,z\mathrm{I}))S_1\chi(z) \quad \Psi_2^*(z) := \exp(-\xi(s,z^{-1}\mathrm{I}))(S_2^{-1})^T\chi(z^{-1})$$

$$\Phi_1(z) := \exp(\xi(t,z\mathrm{I}))\left[\chi(z)\right]^T Z_1 \quad \Phi_2^*(z) := \exp(-\xi(s,z^{-1}\mathrm{I}))\chi(z^{-1})^T(Z_2^{-1})^T.$$

# Sketch of the proof II.

## Theorem

*The following equations hold true:*

$$
\begin{aligned}
L_1\Psi_1(z) &= z\Psi_1(z) & \Phi_1(z)R_1 &= z\Phi_1(z) \\
L_2^T\Psi_2^*(z) &= z^{-1}\Psi_2^*(z) & \Phi_2^*(z)R_2^T &= z^{-1}\Phi_2^*(z) \\
\partial_{t_n}\Psi_1 &= (L_1^n)_+\Psi_1 & \partial_{t_n}\Phi_1 &= \Phi_1(R_1^n)_- \\
\partial_{s_n}\Psi_1 &= (L_2^n)_-\Psi_1 & \partial_{s_n}\Phi_1 &= \Phi_1(R_2^n)_+ \\
\partial_{t_n}\Psi_2^* &= -(L_{1+}^n)^T\Psi_2^* & \partial_{t_n}\Phi_2^* &= -\Phi_2^*(R_{1-}^n)^T \\
\partial_{s_n}\Psi_2^* &= -(L_{2-}^n)^T\Psi_2^* & \partial_{s_n}\Phi_2^* &= -\Phi_2^*(R_{2+}^n)^T.
\end{aligned}
$$

## Theorem (A.Sinap-W.van Assche '96,M.C.)

*Lax operators are given in terms of recursion coefficients by*

$$(L_1)_{N,M+1} = -x^l_{N+1}\Big(\prod_{j=N+2}^{M-}(\mathrm{I}-y^r_j x^l_j)\Big)y^r_{M+1} \quad \forall\, N > M \geq -1$$

$$(R_2)_{N,M+1} = -y^r_{N+1}\Big(\prod_{j=N+2}^{M-}(\mathrm{I}-x^l_j y^r_j)\Big)x^l_{M+1} \quad \forall\, N > M \geq -1$$

$$(L_2)_{M+1,N} = -h^{-l}_{M+1}x^r_{M+1}\Big(\prod_{j=N+2}^{M+}(\mathrm{I}-y^l_j x^r_j)\Big)y^l_{N+1}h^l_N \quad \forall\, N > M \geq -1$$

$$(R_1)_{M+1,N} = -h^{-r}_{M+1}y^l_{M+1}\Big(\prod_{j=N+2}^{M+}(\mathrm{I}-x^r_j y^l_j)\Big)x^r_{N+1}h^r_N \quad \forall\, N > M \geq -1$$

$$(L_1)_{N,N+1} = (R_2)_{N,N+1} = \mathrm{I}$$

$$(L_2)_{N+1,N} = h^l_{N+1}h^{-l}_N \quad (R_1)_{N+1,N} = h^r_{N+1}h^{-r}_N.$$

# Conclusions and Perspectives

# Conclusions and Perspectives

- The link between BOPUC and AL hierarchy is extended to the matrix case considering the non-abelian version of AL.

# Conclusions and Perspectives

- The link between BOPUC and AL hierarchy is extended to the matrix case considering the non-abelian version of AL.
- In the scalar case AL equations combined with string equations coming from the unitary matrix models lead to Painleve' and discrete Painleve' equations. Is it possible to generalize these results to the matrix case?
- Any applications in combinatorics for the matrix case?

(...Gessel,Baik,Deift,Johansson,Its,Widom,Tracy,Borodin, Okounkov,Adler,van Moerbeke,Hisakado...)