# MATRIX INTERPRETATIONS AND APPLICATIONS OF THE CONTINUED FRACTION ALGORITHM 

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1. Introduction. This paper is concerned with certain aspects of the one to one correspondence between real sequences $\left\{c_{n}\right\}_{0}^{\infty}$, formal Laurent series $f(z)=\sum{ }_{0}^{\infty} c_{n} / z^{n+1}$ and infinite Hankel matrices $C=\left(c_{i+j}\right)_{i, j=0}^{\infty}$. The finite 'connected' submatrices of $C$ will be denoted by $C_{n}{ }^{(m)} \equiv\left(c_{m+i+j}\right)_{i, j=0}^{n-1}$, with $C_{n} \equiv C_{n}{ }^{(0)}$, and their determinants by ${c_{n}}^{(m)}=\operatorname{det} C_{n}{ }^{(m)}$.

Also associated with $\left\{c_{n}\right\}$ is the linear functional $c^{*}$ which acts on the vector space of real polynomials and is determined by $c^{*}\left(z^{n}\right)=$ $c_{n}, n \geqq 0$. With the ordinary (Cauchy) product of two polynomials $c^{*}(p q)$ becomes a (Cauchy) bilinear functional on the algebra of real polynomials. If $p(z)=\sum a_{i} z^{i}, q(z)=\sum b_{j} z^{j}$ and $a=\left(a_{0}, a_{1}, a_{2}, \cdots\right)^{T}$, $\boldsymbol{b}=\left(b_{0}, b_{1}, b_{2}, \cdots\right)^{T}$ are the column vectors of coefficients then $c^{*}(p q)=\sum a_{i} c_{i+j} b_{j}=a^{T} C b$.

The functional $c^{*}(p q)$ is an inner product if and only if $\left\{c_{n}\right\}, f$ and $C$ are positive definite, that is $c^{*}\left(p^{2}\right)>0$ if $p \neq 0$, or equivalently $c_{n}{ }^{(0)}>0$ for $n \geqq 0$. An alternative characterization is that $p \neq 0$ and $p(x) \geqq 0$ for $-\infty<x<+\infty$ imply $c^{*}(p)>0$. This involves the (unique) decomposition of such a (positive) polynomial $p$ as the sum of two squares of real polynomials whose zeros interlace (strictly) and gives rise to the geometric theory of moment spaces [20, 18, 19]. If the coefficients $c_{n}=\int_{-\infty}^{+\infty} t^{n} d \mu(t)$ are moments of a bounded nondecreasing function $\mu$ with infinitely many points of increase then all $c_{n}{ }^{(2 m)}>0$, since $c^{*}\left(t^{2 m} p^{2}\right)=\int_{-\infty}^{+\infty} t^{2 m} \quad[p(t)]^{2} d \mu(t)$. Conversely if all ${c_{n}}^{(0)}>0$ then the existence of such a $\mu$ follows by compactness arguments from the algebraic results to be given below $[12,34,1]$.
2. Lanczos polynomials. The algebraic aspects of the theory of orthogonal polynomials carry over to the case in which all $c_{n}{ }^{(0)} \neq 0$. Hence this will be assumed. The material of this section is readily adapted from [29, 32, 11] , for example.

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Theorem 1. There exists a unique sequence $\left\{q_{n}(z)\right\}_{0}^{\infty}=$ $\left\{\sum_{j=0}^{n} \ell_{n, j} z^{j}\right\}_{n=0}^{\infty}$ of monic polynomials for which $c^{*}\left(q_{m} q_{n}\right)=0$ when $m \neq n$. The determinant representations

$$
q_{n}(z)=\frac{1}{{c_{n}}^{(0)}} \operatorname{det}\left(\begin{array}{cccc}
c_{0} & c_{1} & \cdots & c_{n} \\
\vdots & \vdots & & \vdots \\
c_{n-1} & c_{n} & \cdots & c_{2 n-1} \\
1 & z & \cdots & z^{n}
\end{array}\right)
$$

are valid. The sequence $\left\{q_{n}\right\}_{0}^{\infty}$ satisfies the three term recurrence relation

$$
\begin{align*}
q_{-1}(z) & =0, \quad q_{0}(z)=1 \\
q_{n+1}(z) & =\left(z-\alpha_{n+1}\right) q_{n}(z)-\beta_{n}^{2} q_{n-1}(z)  \tag{1}\\
n & =0,1,2, \cdots
\end{align*}
$$

in which

$$
\begin{aligned}
\alpha_{n+1} & =\pi_{2 n+1} / \pi_{2 n}, \quad \beta_{n}^{2}=\pi_{2 n} / \pi_{2 n-2} \quad\left(\beta_{0}^{2} \equiv c_{0}\right) \\
\pi_{2 n} & \equiv c^{*}\left(q_{n}^{2}\right)=c^{*}\left(z^{n} q_{n}\right)=c_{n+1}^{(0)} / c_{n}^{(0)} \\
\pi_{2 n+1} & \equiv c^{*}\left(z q_{n}^{2}\right)
\end{aligned}
$$

In accordance with their use in numerical linear algebra [16] the generalized orthogonal polynomials will be called the Lanczos polynomials of the first kind for $\left\{c_{n}\right\}$. They are related to the denominators of the $(n-1, n)$ Pade fractions for the formal power series $\sum_{0}^{\infty} c_{n} z^{n}$ by $q_{n}(z)=z^{n} q_{n-1, n}(1 / z)$. In this context (1) is a special Frobenius identity. Observe that $\left\{c_{n}\right\}$ is positive definite if and only if all $\pi_{2 n}>0$, or equivalently all ${\beta_{n}}^{2}>0$. In general $\beta_{n}{ }^{2}$ may be negative.

According to Wall [32], Chapter 11, the following algorithmic consequence of (1) dates back to Chebyshev [3]. It is more general than the quotient-difference algorithm [25, 13, 14, 17] which serves a similar purpose but also requires all $c_{n}{ }^{(m)} \neq 0$ (When $c_{n}=$ $\int_{0}^{\infty} t^{n} d \mu(t)$ one has all $c_{n}{ }^{(m)}>0$ and this is implied by $c_{n}{ }^{(0)}>0$ and $c_{n}{ }^{(1)}>0$ for $n \geqq 0$ [28].)

Theorem 2. The coefficients $\beta_{n-1}^{2}, \alpha_{n},\left\{\ell_{n, j}\right\}_{j=0}^{n}, n=1,2, \cdots, N$, may be computed recursively from the sequence $\left\{c_{n}\right\}_{0}{ }^{2 N-1}$ by the rational $0\left(N^{2}\right)$ process:

$$
\begin{aligned}
& \sigma_{-1}=1, \tau_{0}=0, \quad \ell_{0,0}=1, \\
& \text { for } n=0,1, \cdots, N-1 \\
& \sigma_{n}=\sum_{j=0}^{n} \ell_{n, j} c_{n+j}, \\
& \tau_{n+1}=\left(\sum_{j=1}^{n} \ell_{n, j} c_{n+j+1}\right) / \sigma_{n}, \\
& \beta_{n}{ }^{2}=\sigma_{n} / \sigma_{n-1}, \alpha_{n+1}=\tau_{n+1}-\tau_{n}, \\
& \ell_{n-1, n}=\ell_{n,-1}=0, \ell_{n+1, n+1}=1 \text {, } \\
& \text { for } j=0,1, \cdots, n \\
& \ell_{n+1, j}=\ell_{n, j-1}-\alpha_{n+1} \ell_{n, j}-\beta_{n}{ }^{2} \ell_{n-1, j} .
\end{aligned}
$$

Proof. Let

$$
\begin{aligned}
\sigma_{n} & \equiv \pi_{2 n}=c^{*}\left(q_{n}^{2}\right)=\beta_{0}^{2} \beta_{1}^{2} \cdots \beta_{n}^{2} \\
& =c^{*}\left(z^{n} q_{n}\right)=\sum_{j=0}^{n} \ell_{n, j} c_{n+j}
\end{aligned}
$$

and

$$
\sigma_{n} \tau_{n+1} \equiv c^{*}\left(z^{n+1} q_{n}\right)=\sum_{j=0}^{n} \ell_{n, j} c_{n+j+1}
$$

Then

$$
\begin{aligned}
\sigma_{n} \tau_{n+1} & =c^{*}\left(z^{n}\left(q_{n+1}+\alpha_{n+1} q_{n}+\beta_{n}^{2} q_{n-1}\right)\right) \\
& =\alpha_{n+1} \sigma_{n}+\beta_{n}^{2} \sigma_{n-1} \tau_{n} \\
& =\sigma_{n}\left(\tau_{n}+\alpha_{n+1}\right)
\end{aligned}
$$

so

$$
\tau_{n}=\alpha_{1}+\alpha_{2}+\cdots+\alpha_{n}\left(\tau_{0} \equiv 0\right)
$$

The rest follows by equating coefficients in (1).
It is known from practical experience with the positive definite case that $\beta_{n}{ }^{2}=\beta_{n}{ }^{2}\left(c_{0}, c_{1}, \cdots, c_{2 n}\right)$ and $\alpha_{n+1}=\alpha_{n+1}\left(c_{0}, c_{1}, \cdots, c_{2 n+1}\right)$ are ill conditioned functions of the moments $\left\{c_{n}\right\}$ although this has not been quantified precisely. Hence the algorithm can only be recommended if the $\left\{c_{n}\right\}$ are rational and rational arithmetic is used. How-
ever see [26] for a treatment of 'modified moments', [6,7] for related analysis and [33] for an application.
The $n$th reproducing kernel function,

$$
K_{n}(z, w) \equiv \sum_{j=0}^{n} \frac{q_{j}(z) q_{j}(w)}{\pi_{2 j}}=K_{n}(w, z)
$$

satisfies $p(z)=c_{t}^{*}\left(K_{n}(z, t) p(t)\right)$ when $\operatorname{deg} p \leqq n$.
Theorem 3. 1. (Christoffel-Darboux). The following relations hold:

$$
\begin{aligned}
K_{n}(z, w) & =\frac{q_{n+1}(z) q_{n}(w)-q_{n}(z) q_{n+1}(w)}{\pi_{2 n}(z-w)}, \\
& z \neq w, \\
& =\frac{q_{n+1}^{\prime}(z) q_{n}(z)-q_{n}^{\prime}(z) q_{n+1}(z)}{\pi_{2 n}}, \\
& z=w .
\end{aligned}
$$

2. The alternative representations

$$
\begin{aligned}
K_{n}(z, w) & =\frac{-1}{c_{n+1}^{(0)}} \operatorname{det}\left(\begin{array}{ccccc}
c_{0} & c_{1} & \cdots & c_{n} & 1 \\
c_{1} & c_{2} & \cdots & c_{n+1} & z \\
\vdots & \vdots & & \vdots & \vdots \\
c_{n} & c_{n+1} & \cdots & c_{2 n} & z^{n} \\
1 & w & \cdots & w^{n} & 0
\end{array}\right) \\
& =(-1)^{n} \frac{\operatorname{det}\left(c_{t}^{*}\left(t^{i+j}(t-z)(w-t)\right)_{i, j=0}^{n-1}\right.}{c_{n+1}^{(0)}}
\end{aligned}
$$

are valid.
3. (Christoffel). If $q_{n}(a) \neq 0, n \geqq 0$, then the Lanczos polynomials of the first kind associated with the linear functional $c_{a}{ }^{*}(p) \equiv$ $c^{*}((z-a) p)$ also exist and are given by

$$
q_{n}{ }^{a}(z)=\pi_{2 n} K_{n}(z, a) / q_{n}(a), \quad n \geqq 0 .
$$

The Lanczos polynomial of the second kind for $\left\{c_{n}\right\}$ are denoted by $\left\{p_{n}(z)\right\}_{0}^{\infty}$. They form a second linearly independent solution of (1):

$$
\begin{align*}
p_{-1}(z) & \equiv-1, p_{0}(z) \equiv 0 \\
p_{n+1}(z) & =\left(z-\alpha_{n+1}\right) p_{n}(z)-\beta_{n}{ }^{2} p_{n-1}(z), \quad n=0,1,2, \cdots \tag{2}
\end{align*}
$$

Note that $p_{n}(z)=c_{0} z^{n-1}+\cdots$ for $n \geqq 0$. The twin recurrence relations (1) and (2) lead to the following, via the elementary theory of continued fractions.

Theorem 4. There is a one to one correspondence between formal Laurent series $f(z)=\sum{ }_{0}^{\infty} c_{n} / z^{n+1}$ with all $c_{n}{ }^{(0)} \neq 0$ and formal (associated) continued fractions

$$
F(z)=\frac{\beta_{0}{ }^{2}}{z-\alpha_{1}}-\frac{\beta_{1}{ }^{2}}{z-\alpha_{2}}-\frac{\beta_{2}{ }^{2}}{z-\alpha_{3}}-\cdots
$$

with all $\boldsymbol{\beta}_{n}{ }^{2} \neq 0$. If

$$
w_{n}(z) \equiv p_{n}(z) / q_{n}(z) \quad(n \geqq 0)
$$

is the nth approximant of $F(z)$, then

$$
\begin{equation*}
p_{n}(z) q_{n-1}(z)-p_{n-1}(z) q_{n}(z) \equiv \pi_{2 n-2} \neq 0 \tag{3}
\end{equation*}
$$

and the Taylor expansion of $w_{n}(z)$ about $z=\infty$ coincides with $f(z)$ precisely through the term in $1 / z^{2 n}$ :

$$
\begin{equation*}
f(z)=w_{n}(z)+\pi_{2 n} / z^{2 n+1}+0\left(1 / z^{2 n+2}\right) . \tag{4}
\end{equation*}
$$

The following representations hold:

$$
\begin{aligned}
p_{n}(z) & =\frac{(-1)^{n}}{c_{n}{ }^{(0)}} \operatorname{det}\left(\begin{array}{ccccc}
0 & c_{0} & c_{0} z+c_{1} & \cdots & c_{0} z^{n-1}+\cdots+c_{n-1} \\
c_{0} & c_{1} & c_{2} & \cdots & c_{n} \\
\vdots & \vdots & \vdots & & \vdots \\
c_{n-1} & c_{n} & c_{n+1} & \cdots & c_{2 n-1}
\end{array}\right) \\
& =c_{t}^{*}\left(\frac{q_{n}(z)-q_{n}(t)}{z-t}\right) .
\end{aligned}
$$

The polynomials $\left\{p_{n}\right\}$ are related to the numerators of the ( $n-1, n$ ) Padé fractions for $\sum_{0}^{\infty} c_{n} z^{n}$ by $p_{n}(z)=z^{n-1} p_{n-1, n}(1 / z)$. In this context the use of bigradient determinants [17] provides a short proof of a generalization of (4).

It will be assumed that the zeros $\left\{z_{n, k}\right\}_{k=1}^{n}$ of $q_{n}$ are distinct for each $n \geqq 1$. One may then obtain a 'generalized' Gaussian quadrature formula. For the construction of such formulas in the positive definite case see $[5,8]$.

Theorem 5. The partial fraction expansion of $w_{n}(z)$ is

$$
w_{n}(z)=\sum_{k=1}^{n} \frac{w_{n, k}}{z-z_{n, k}} \text { with } w_{n, k}=\frac{1}{K_{n}\left(z_{n, k}, z_{n, k}\right)} .
$$

Consequently from (4),

$$
c^{*}(p)=\sum_{k=1}^{n} w_{n, k} p\left(z_{n, k}\right) \text { if } \operatorname{deg} p<2 n
$$

and

$$
c^{*}\left(z^{2 n}\right)-\sum_{k=1}^{n} w_{n, k} z_{n, k}^{2 n}=\pi_{2 n} \neq 0
$$

In particular $\left\{c_{n}\right\}$ is positive definite if and only if all $w_{n, k}>0$ and then the zeros $\left\{z_{n, k}\right\}_{k=1}^{n}$ are real, distinct and (strictly) separated by those of $p_{n}$ and $q_{n-1}$.
3. Matrix interpretations. The results of the previous section become more transparent when viewed in terms of matrices.

Let the unit left triangular matrices

$$
L \equiv\left(\ell_{i, j}\right)_{i, j=0}^{\infty} \quad \text { and } \quad L_{n} \equiv\left(\ell_{i, j} j_{i, j=0}^{n-1}\right.
$$

The 'orthogonality' of the polynomials $\left\{q_{n}\right\}$ is then equivalent with

$$
L C L^{T}=P \equiv \operatorname{diag}\left(\pi_{0}, \pi_{2}, \pi_{4}, \cdots\right) .
$$

and with

$$
\begin{equation*}
L_{n} C_{n} L_{n}^{T}=P_{n} \equiv \operatorname{diag}\left(\pi_{0}, \pi_{2}, \cdots, \pi_{2 n-2}\right) \tag{5}
\end{equation*}
$$

for $n \geqq 0$. In other words

$$
C=L^{-1} P\left(L^{-1}\right)^{T}
$$

is the Gauss-Banachiewicz LDR factorization of the symmetric Hankel matrix $C$. This factorization exists if and only if all $c_{n}{ }^{(0)} \neq 0$. The algorithm of theorem 2 effects the factorization (5) in $0\left(n^{2}\right)$ operations as compared with the usual $0\left(n^{3}\right)$ for an arbitrary (symmetric) $n \times n$ matrix.

The recurrence relations (1) and (2) are related to the tridiagonal matrix

$$
J \equiv \operatorname{tridiag}\left(\begin{array}{cccc}
1, & 1, & 1, & \cdots \\
\alpha_{1}, & \alpha_{2}, & \alpha_{3}, & \alpha_{4}, \\
\beta_{1}{ }^{2}, & \beta_{2}^{2}, & \beta_{3}^{2}, & \cdots
\end{array}\right),
$$

its $n$th leading principal submatrices $J_{n}$ and the submatrices $J_{n}{ }^{\prime}$ of $J_{n}$ in which the first row and column are deleted. One has

$$
\begin{aligned}
& p_{n}(z)=c_{0} \operatorname{det}\left(z I_{n-1}-J_{n}^{\prime}\right) \\
& q_{n}(z)=\operatorname{det}\left(z I_{n}-J_{n}\right)
\end{aligned}
$$

with $I_{n}$ the $n \times n$ identity matrix.
Let the translation matrix

$$
T \equiv\left(\delta_{i+1, j}\right)_{i, j=0}^{\infty}
$$

and the Frobenius matrices

$$
F_{n} \equiv\left(\begin{array}{cccccc}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
-\ell_{n, 0} & -\ell_{n, 1} & -\ell_{n, 2} & \cdots & -\ell_{n, n-2} & -\ell_{n, n-1}
\end{array}\right)
$$

The $F_{n}$ are the companion matrices of their characteristic polynomials $q_{n}$. From theorem 2 one then finds

$$
J L=L T \quad \text { and } \quad J_{n} L_{n}=L_{n} F_{n}
$$

The first of these relations was given by Stieltjes [27]; the second is its finite analogue. Finally, denote by

$$
V_{n} \equiv\left(z_{n, j}^{i-1}\right)_{i, j=1}^{n}
$$

the Vandermonde matrix of the zeros of $q_{n}$. Then

$$
\begin{aligned}
F_{n} V_{n} & =V_{n} D_{n}, D_{n} \equiv \operatorname{diag}\left(z_{n, 1}, z_{n, 2}, \cdots, z_{n, n}\right) \\
Q_{n} & \equiv L_{n} V_{n}=\left(q_{i-1}\left(z_{n, j}\right)\right)_{i, j=1}^{n}
\end{aligned}
$$

and

$$
J_{n} Q_{n}=Q_{n} D_{n}
$$

showing explicitly the similarities among the matrices $J_{n}, F_{n}$ and $D_{n}$.
Further consequences arise from theorem 3. First of all the Chris-toffel-Darboux formula shows that the matrices

$$
\left(K_{n}\left(z_{n, i}, z_{n, j}\right)\right)=Q_{n}^{T} P_{n}^{-1} Q_{n}
$$

are diagonal. There follows

$$
\begin{equation*}
Q_{n} W_{n} Q_{n}^{T}=P_{n}, \quad W_{n} \equiv \operatorname{diag}\left(w_{n, 1}, w_{n, 2}, \cdots, w_{n, n}\right) \tag{6}
\end{equation*}
$$

and with (5),

$$
\begin{equation*}
C_{n}=V_{n} W_{n} V_{n}^{T} \tag{7}
\end{equation*}
$$

exhibiting the explicit congruences among $C_{n}, P_{n}$ and $W_{n}$. Equation (6) shows the 'orthogonality' of the polynomials $\left\{q_{k}\right\}_{0}^{n-1}$ with respect to 'weighted' summation over the zeros of $q_{n}$, and (7) essentially contains the 'quadrature formula'. From theorem $3.2 K_{n-1}(z, w)=$ $\sum b_{i, j} w^{i-1} z^{j-1}$ is the generating function for the elements of $B_{n} \equiv$ $C_{n}{ }^{-1}$. Theorem 2 and the Christoffel-Darboux formula thus provide an $0\left(n^{2}\right)$ algorithm for the inversion of $n \times n$ Hankel matrices:

$$
\begin{aligned}
& \text { for } i=0,1, \cdots, n-1 \\
& \qquad \begin{array}{l}
b_{i, n+1}=0 \\
\text { for } j=i+1, i+2, \cdots, n \\
\text { if } i=0 \text { then } b_{0, j}=0 \\
b_{i+1, j}=b_{j, i+1}=b_{i, j+1}+\left(\ell_{n-1, i} \ell_{n, j}-\ell_{n, i} \ell_{n-1, j}\right) / \sigma_{n-1} .
\end{array}
\end{aligned}
$$

See also [30, 31].
Let the $n$th resolvent

$$
\begin{aligned}
R_{n}(z) & \equiv\left(r_{i, j}^{(n)}(z)\right)_{i, j=0}^{n-1} \equiv\left(z I_{n}-J_{n}\right)^{-1} \\
& =\sum_{m=0}^{\infty} J_{n}^{m} / z^{m+1} \equiv \sum_{k=1}^{n} \frac{R_{n, k}}{z-z_{n, k}}
\end{aligned}
$$

Then the residues $R_{n, k}$ satisfy

$$
I_{n}=\sum_{k=1}^{n} R_{n, k} \text { and } J_{n}=\sum_{k=1}^{n} z_{n, k} R_{n, k}
$$

Moreover from the above

$$
R_{n}(z)=Q_{n}\left(z I_{n}-P_{n}\right)^{-1} W_{n} Q_{n}^{T} P_{n}^{-1}
$$

giving

$$
\pi_{2 j} r_{i, j}^{(n)}(z)=\sum_{k=1}^{n} \frac{w_{n, k} q_{i}\left(z_{n, k}\right) q_{j}\left(z_{n, k}\right)}{z-z_{n, k}}
$$

and

$$
R_{n, k}=w_{n, k}\left(q_{i}\left(z_{n, k}\right) q_{j}\left(z_{n, k}\right) / \pi_{2 j}\right)_{i, j=0}^{n-1}
$$

In particular the $n$th approximant of the continued fraction $F(z)$ is

$$
w_{n}(z)=p_{n}(z) / q_{n}(z)=c_{0} r_{0,0}^{(n)}(z)
$$

and likewise

$$
r_{n-1, n-1}^{(n)}(z)=q_{n-1}(z) / q_{n}(z)
$$

The Christoffel-Darboux formula shows that the residues are projectors: $R_{n, k}^{2}=R_{n, k}$. When $\left\{c_{n}\right\}$ is positive definite the polynomials $\left\{\tilde{q}_{n}\right\} \equiv\left\{q_{n} / \pi_{2 n}^{1 / 2}\right\}$ are orthonormal, $\tilde{Q}_{n} \equiv P_{n}^{-1 / 2} Q_{n} W_{n}^{1 / 2}$ is an orthogonal matrix, $\tilde{J}_{n} \equiv P_{n}{ }^{-1 / 2} J_{n} P_{n}^{1 / 2}$ is symmetric and $\tilde{J}_{n} \tilde{Q}_{n}=\tilde{Q}_{n} D_{n}$. The residues $\tilde{R}_{n, k}=P_{n}{ }^{-1 / 2} R_{n, k} P_{n}{ }^{1 / 2}$ of the resolvent of $\tilde{J}_{n}$ are then also orthogonal projectors: $\tilde{R}_{n, k}=\tilde{R}_{n, k}^{T}$.

Let $A$ be a real $N \times N$ matrix and consider the Krylov sequences

$$
x_{n} \equiv A x_{n-1}=A^{n} x_{0}, y_{n} \equiv A^{T} y_{n-1}=\left(A^{T}\right)^{n} y_{0}
$$

of $A$ and $A^{T}$ with respect to initial vectors $x_{0}, y_{0}$ for which $y_{0}{ }^{T} x_{0} \neq 0$. For $n \geqq 1$ put

$$
X_{n} \equiv\left(x_{0}, x_{1}, \cdots, x_{n-1}\right), Y_{n} \equiv\left(y_{0}, y_{1}, \cdots, y_{n-1}\right)
$$

and

$$
C^{n} \equiv Y_{n}^{T} X_{n}=\left(y_{i}^{T} x_{j}\right)=\left(y_{0}^{T} A^{i+j} \boldsymbol{x}_{0}\right)=\left(c_{i+j}\right)
$$

The matrices $C_{n}$ are ultimately singular, at least for $n>N$. Let $\nu$ be such that $c_{1}{ }^{(0)} c_{2}{ }^{(0)} \cdots c_{\nu}{ }^{(0)} \neq 0$ but $c_{\nu+1}^{(0)}=0$. Then the sequence $\left\{q_{n}\right\}$ 'terminates' with $q_{\nu}$ but all previous relations, as well as the following, hold for $n \leqq \nu$. If

$$
\begin{gather*}
\hat{x}_{n} \equiv q_{n}(A) x_{0}=\sum_{k=0}^{n} \ell_{n, k} x_{k},  \tag{8}\\
\hat{y}_{n} \equiv q_{n}\left(A^{T}\right) y_{0}=\sum_{k=0}^{n} \ell_{n, k} y_{k}
\end{gather*}
$$

then

$$
\begin{gathered}
\hat{X}_{n} \equiv\left(\hat{x}_{0}, \hat{x}_{1}, \cdots, \hat{x}_{n-1}\right)=X_{n} L_{n}^{T} \\
\hat{Y}_{n} \equiv\left(\hat{y}_{0}, \hat{y}_{1}, \cdots, \hat{y}_{m-1}\right)=Y_{n} L_{n}^{T}
\end{gathered}
$$

and (5) shows that

$$
\begin{equation*}
\hat{Y}_{n}^{T} \hat{X}_{n}=P_{n} \tag{9}
\end{equation*}
$$

That is $\left\{\hat{x}_{n}\right\}_{0}$ and $\left\{\hat{y}_{n}\right\}_{0}$ are biorthogonal. From (8)

$$
A X_{n}=\left(x_{1}, x_{2}, \cdots, x_{n}\right)=X_{n} F_{n}^{T}+\left(0, \cdots, 0, \hat{x}_{n}\right)
$$

and the relation $J_{n} L_{n}=L_{n} F_{n}$ gives

$$
A \hat{X}_{n}=\hat{X}_{n} J_{n}^{T}+\left(0, \cdots, 0, \hat{x}_{n}\right) L_{n}^{T}
$$

Consequently from (9)

$$
\begin{equation*}
\hat{Y}_{n}^{T} A \hat{X}_{n}=P_{n} J_{n}^{T}=J_{n} P_{n} . \tag{10}
\end{equation*}
$$

The three term recurrence relation now provides the algorithm

$$
\begin{aligned}
& \hat{x}_{-1}=\hat{y}_{-1}=0, \hat{x}_{0}=x_{0}, \hat{y}_{0}=y_{0} \\
& \hat{x}_{n+1}=\left(A-\alpha_{n+1} I\right) \hat{x}_{n}-\beta_{n}^{2} \hat{x}_{n-1} \\
& \hat{y}_{n+1}=\left(A^{T}-\alpha_{n+1} I\right) \hat{y}_{n}-\beta_{n}^{2} \hat{y}_{n-1} \\
& \quad n=0,1, \cdots, \nu-1
\end{aligned}
$$

in which, on comparing coefficients in (9) and (10),

$$
\beta_{n}^{2}=\frac{\hat{y}_{n}^{T} \hat{x}_{n}}{\hat{y}_{n-1}^{T} \hat{x}_{n-1}}, \quad \alpha_{n+1}=\frac{\hat{y}_{n}^{T} A \hat{x}_{n}}{\hat{y}_{n}^{T} \hat{x}_{n}}
$$

Moreover

$$
c_{n}^{(0)}=\prod_{k=0}^{n-1} \hat{y}_{k}^{T} \hat{x}_{k}
$$

This is the essence of the Lanczos algorithm [21] for tridiagonalization. If the algorithm can be completed $(\nu=N)$, which is possible if $A$ is symmetric with distinct eigenvalues and $x_{0}, y_{0}$ are chosen appropriately, then $A$ is similar to $J_{N}{ }^{T}$ :

$$
\hat{X}_{N}^{-1} A \hat{X}_{N}=J_{N}^{T} .
$$

In particular if $A$ is symmetric and $y_{0}=x_{0}$ then $\tilde{X}_{N} \equiv \hat{X}_{N} P_{N}^{-1 / 2}$ is orthogonal and $\tilde{X}_{N}{ }^{T} A \tilde{X}_{N}=\tilde{J}_{N}$ is also symmetric. It is always theoretically possible to complete a modified version of the algorithm [16].
3. Concluding remarks. The material of this section will be developed in detail elsewhere, but is mentioned here for the sake of completeness.

The Lanczos polynomials $\left\{p_{n}\right\}_{0}^{\infty}$ and $\left\{q_{n}\right\}_{0}^{\infty}$ may be generalized to maintain their connection with the Pade numerators and denominators, which are defined even though nontrivial blocks may occur in the Pade table; see [11]. The $\left\{q_{n}\right\}$ remain monic of degree $n$ but the 'orthogonality' is lost. The matrix $P=L C L^{T}$ is now block diagonal and the diagonal blocks are lower triangular ( $\Delta$ ) Hankel matrices which are nonsingular, except possibly for a last one which is the
infinite null matrix ( $f(z)$ rational). This 'left triangular congruence' arose in connection with the determination of the signature of a general Hankel matrix and certain theorems from [4] were used in the proof. The matrix $J$ becomes block tridiagonal and the diagonal blocks are companion matrices. The off diagonal blocks are null apart from their lower left element which is unity for the superdiagonal and nonnull for the subdiagonal. These results are intimately related to $P$-fractions [22, 23, 24] which, however, are developed in the ascending power notation. A generalization of the algorithm of theorem 2 follows from the theory of continued fractions.

Inclusion disks for the approximants of positive definite ' $J$-fractions' $F(z)$ are classical $[12,32,36,1]$. When $\mu$ has a restricted set $S$ of points of increase these classical regions are not best possible. In [1] the best regions are described when $S$ is the complement of a finite interval, in [15] the Stieltjes case $S=[0,+\infty)$ is treated and in $[9,10]$ the extended Hausdorff case $S=[a, b]$. The best regions are now intersections of two circular disks, or lunes. It will now be indicated how to construct the best inclusion lunes $L_{n}(z)$ for $f(z)=\int_{a}^{b} d \mu(t) /(z-t) \quad(-\infty<a<b<+\infty)$ when the moments $\left\{c_{k}\right\}_{0}^{n}$ are known.

Thus let $K_{n}(z)$ be the classical disk for $f(z)$ when $\left\{c_{k}\right\}_{0}^{2 n-2}$ are known. The functions

$$
\begin{aligned}
f_{a}(z) & =\int_{a}^{b} \frac{(t-a) d \mu(t)}{z-t}=(z-a) f(z)-c_{0} \\
f_{b}(z) & =\int_{a}^{b} \frac{(b-t) d \mu(t)}{z-t}=(b-z) f(z)+c_{0} \\
f_{a b}(z) & =\int_{a}^{b} \frac{(t-a)(b-t) d \mu(t)}{z-t} \\
& =(z-a)(b-z) f(z)+(z-a-b) c_{0}+c_{1}
\end{aligned}
$$

each have (convergent) $J$-fraction expansions. Let $\hat{K}_{n}{ }^{a}(z), \hat{K}_{n}{ }^{b}(z)$ and $\hat{K}_{n}{ }^{a b}(z)$ be the corresponding classical disks. Transforming back to $f(z)$ gives

$$
\begin{aligned}
K_{n}^{a}(z) & \equiv\left[\hat{K}_{n}^{a}(z)+c_{0}\right] /(z-a), \\
K_{n}^{b}(z) & \equiv\left[\hat{K}_{n}^{b}(z)-c_{0}\right] /(b-z), \\
K_{n}^{a b}(z) & \equiv\left[\hat{K}_{n}^{a b}(z)-(z-a-b) c_{0}-c_{1}\right] /(z-a)(b-z) .
\end{aligned}
$$

Observe that $K_{n}{ }^{a}(z), K_{n}{ }^{b}(z)$ require the moments $\left\{c_{k}\right\}^{2 n-1}$ and $K_{n}{ }^{a b}(z)$ requires $\left\{c_{k}\right\}_{0}^{2 n}$. The lunes are thus given by

$$
\begin{aligned}
L_{2 n-1}(z) & =K_{n}^{a}(z) \cap K_{n}^{b}(z), \\
L_{2 n}(z) & =K_{n+1}(z) \cap K_{n}^{a b}(z)
\end{aligned}
$$

An application of Christoffel's formula, theorem 3.3, then shows that the vertices of the lunes $L_{n}(z)$ may be expressed in terms of $\left\{p_{k}(z)\right\}$, $\left\{q_{k}(z)\right\},\left\{q_{k}(a)\right\}$ and $\left\{q_{k}(b)\right\}$. In particular only one application of the continued fraction algorithm of theorem 2 is necessary.

The advantage of this technique over those using continued fractions of special form is its extension to the case when $S$ is the union of a number of disjoint intervals. For example if $S=\left[a_{1}, b_{1}\right] \cup\left[a_{2}, b_{2}\right]$ then each inclusion region is the intersection of four circular disks. The polynomials corresponding to $1, t-a, b-t,(t-a)(b-t)$ above are now $1, t-a_{1}, b_{2}-t,\left(t-a_{1}\right)\left(b_{2}-t\right),\left(t-b_{1}\right)\left(t-a_{2}\right)$, $\left(t-a_{1}\right)\left(t-b_{1}\right)\left(t-a_{2}\right),\left(t-b_{1}\right)\left(t-a_{2}\right)\left(b_{2}-t\right)$ and $\left(t-a_{1}\right)\left(t-b_{1}\right)$ $\left(t-a_{2}\right)\left(b_{2}-t\right)$.

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