Review

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Matrix Methods for Estimating the Coherence Functions from Estimates of the Cross-Spectral Density Matrix

It is shown that the usual method for estimating the coherence functions (ordinary, partial, and multiple) for a general multiple-input/multiple-output problem can be expressed as a modified form of Cholesky decomposition of the cross-spectral density matrix of the input and output records. The results can be equivalently obtained using singular value decomposition (SVD) of the cross-spectral density matrix. Using SVD suggests a new form of fractional coherence. The formulation as a SVD problem also suggests a way to order the inputs when a natural physical order of the inputs is absent. © *1996 John Wiley & Sons, Inc.*

INTRODUCTION

Understanding partial and multiple coherence for the analysis of multiple-input/multiple-output systems is at best difficult. It is particularly difficult when the inputs themselves are partially coherent. The traditional method for computing partial coherence (Bendat and Piersol, 1986) can be expressed in the form of a Cholesky factorization of the cross-spectral density matrix of the input and output records. The removal of the influence of each input in the traditional method is dependent on the order of the inputs. Each input is assumed to be caused by a linear combination of the previous inputs and an independent contribution. Sometimes a natural physical reason exists for ordering the inputs. But more commonly, no physical reason exists for assuming a cause and effect relationship between the inputs. Without additional information it is unclear which input is the dominate source of the coherent response.

The Cholesky factorization depends on the input cross-spectral density being positive definite. This is not always true and the method will sometimes fail. A method to avoid this problem is presented. A model of the system explaining the coherence using singular value decomposition (SVD) is possible. SVD makes no assumptions of the order of the inputs, but will model the inputs as a vector of independent sources monotonically decreasing in power. SVD does not require a positive definite matrix and thus removes the flaw (failure of Cholesky factorization for a singular matrix) in the traditional methods based on Cholesky factorization. While the model may not have physical significance (sometimes the traditional methods do not have physical significance either), the method provides a convenient way to explain the coherence of the inputs and their influence on the outputs. The method introduces a new concept of fractional coherence to supplement the concept of partial coherence in the tradi-

tional methods. It is shown that the traditional partial coherence still can be found using SVD.

DEFINITIONS

The vector of records is defined as a column vector, for example $\mathbf{x} = \{x(t)\}$. The Fourier transforms of the vector of records is defined as $\mathbf{X} = \{X(f)\}$. For convenience the dependence on t and f is not usually stated. Ideally the equations involve a limit as the record length, $T \to \infty$, but the limit notation is omitted for clarity. In practice with finite records the limiting operation is never done, and only estimates of the quantities are available. The cross-spectral density matrix of the vector of records will be defined as

$$\mathbf{G}_{xx} = \frac{2}{T} E[\mathbf{XX'}]. \tag{1}$$

The diagonal elements will be the autospectral densities of each record and the off the diagonal elements will be the cross-spectral densities between pairs of records. Note that this is the complex conjugate of the cross-spectral density as defined by Bendat and Piersol (1986). The cross-spectral density between two vectors $\{x\}$ and $\{y\}$ is defined as

$$\mathbf{G}_{xy} = \frac{2}{T} E[\mathbf{XY'}]. \tag{2}$$

An element in a cross-spectral density matrix will be defined as

$$G_{ij}^{xy} = \frac{2}{T} E[X_i \ Y_j^*], \tag{3}$$

and $\mathbf{G}_{\cdot j}^{xy}$ is the jth column of $\mathbf{G}_{xy}$. Sometimes the superscript will be omitted if it is clear from the context. Note that

$$\mathbf{G}_{xy}' = \mathbf{G}_{yx}, \tag{4}$$

but this is not generally true for the frequency response function matrix, $\mathbf{H}_{xy}$, to be defined later. $\mathbf{G}_{xx}$ and $\mathbf{G}_{yy}$ are square, but $\mathbf{G}_{xy}$ is square only if the number of inputs and the number of outputs is the same. The notation $x_{m.n!}$ will be used to denote a conditioned record. $x_{m.1}$ is the record m with the effects of the first record removed, and $x_{m.n!}$ is the record m with the effects of records 1

through n removed. A similar notation will be used for a conditioned cross-spectral density matrix. For example, $\mathbf{G}_{yy.x}$ is the conditioned cross-spectral density of the records y with the effect of the records x removed. $\mathbf{G}_{yy:x}$ is the fraction of the cross-spectral density matrix of the records y that are linearly related to the records x. $\mathbf{G}_{yy:x}$ is the part of the cross-spectral density matrix of y that is coherent with the records x. This can be read as the cross-spectral density matrix of y given x. The total cross-spectral density matrix of the records y is the sum of these two values.

$$\mathbf{G}_{yy} = \mathbf{G}_{yy:x} + \mathbf{G}_{yy.x}. \tag{5}$$

Similarly $\mathbf{G}_{yy:x_{i.(-1)!}}$ is the conditioned cross-spectral density matrix of the records, y, given the record x_i with the effects of the records 1 through $i - 1$ removed.

GENERAL MULTIPLE-INPUT/MULTIPLE-OUTPUT SYSTEM

The linear relationship between a vector of n inputs, $\mathbf{x}$, and a vector of m outputs, $\mathbf{y}$, can be visualized as a system coupled through a matrix of frequency response functions, $\mathbf{H}_{yx}$, as shown in Fig. 1, where

$$\mathbf{Y} = \mathbf{H}_{yx}\mathbf{X}. \tag{6}$$

The cross-spectral density matrix between the input and output is given by

$$\mathbf{G}_{yx} = \mathbf{H}_{yx}\mathbf{G}_{xx}. \tag{7}$$
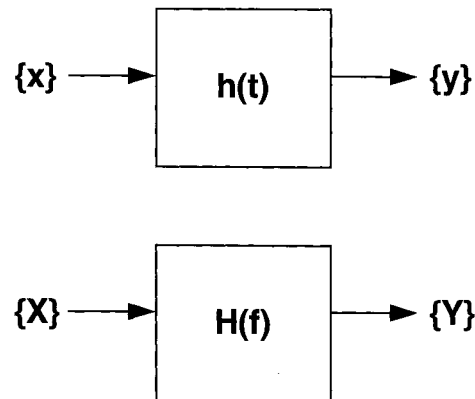

FIGURE 1 Multiple-input/multiple-output system.

If $\mathbf{G}_{xx}$ is not singular this gives

$$\mathbf{H}_{yx} = \mathbf{G}_{yx}(\mathbf{G}_{xx})^{-1}. \tag{8}$$

The output spectral density matrix coherent with the inputs, x, is given by

$$\mathbf{G}_{yy:x} = \mathbf{H}_{yx}\mathbf{G}_{xx}\mathbf{H}'_{yx}. \tag{9}$$

Combining Eqs. (8) and (9) gives

$$\mathbf{G}_{yy:x} = \mathbf{G}_{yx}(\mathbf{G}_{xx})^{-1}\mathbf{G}_{xy} = \mathbf{G}_{yx}(\mathbf{G}_{xx})^{-1}\mathbf{G}'_{yx}. \tag{10}$$

The part not coherent with the input is given by

$$\mathbf{G}_{yy.x} = \mathbf{G}_{yy} - \mathbf{G}_{yy:x}, \tag{11}$$

or the total cross-spectral density is the sum of the coherent and noncoherent parts.

$$\mathbf{G}_{yy} = \mathbf{G}_{yy.x} + \mathbf{G}_{yy:x} \tag{12}$$

It is often convenient to examine the effects of removing the effects of some of the inputs while retaining others. To do this it is convenient to express the vector of input and output records as a single vector of records

$$\mathbf{z} = \left\{ \begin{array}{c} \mathbf{x} \\ \mathbf{y} \end{array} \right\}. \tag{13}$$

The cross-spectral density matrix of z can be partitioned into

$$\mathbf{G}_{zz} = \begin{bmatrix} \mathbf{G}_{xx} & \mathbf{G}_{xy} \\ \mathbf{G}_{yx} & \mathbf{G}_{yy} \end{bmatrix}. \tag{14}$$

The inputs whose effects are to be removed are placed in the primary $\mathbf{p}$ vector and the remaining inputs, $\mathbf{\bar{x}}$, along with the outputs, $\mathbf{y}$, are placed in residual vector $\mathbf{r}$.

$$\mathbf{z} = \left\{ \begin{array}{c} \mathbf{p} \\ \mathbf{r} \end{array} \right\}. \tag{15}$$

As before the cross-spectral density matrix of z can be partitioned into

$$\mathbf{G}_{zz} = \left\{ \begin{array}{cc} \mathbf{G}_{pp} & \mathbf{G}_{pr} \\ \mathbf{G}_{rp} & \mathbf{G}_{rr} \end{array} \right\}. \tag{16}$$

The conditioned spectral densities $\mathbf{G}_{rr.p}$ and $\mathbf{G}_{rr:p}$ can be found from the above equations [Eqs. (10)–(12)] by just substituting p and r for x and y, respectively. This result will be used later when discussing the coherence functions.

If the inputs are independent the effects of an input on an output are easily computed because $\mathbf{G}_{xx}$ is diagonal. If the inputs are not independent the relationships are not as easily studied. One technique to examine the system is to model the inputs, $\mathbf{X}$, as a system of independent inputs, $\mathbf{W}$, coupled through a matrix of frequency response functions that will produce the vector of inputs, $\mathbf{X}$. The model may or may not have a physical significance. Equation (9) suggests that any factorization of the form

$$\mathbf{G}_{xx} = \mathbf{H}_{xw}\mathbf{G}_{ww}\mathbf{H}'_{xw}, \tag{17}$$

where $\mathbf{G}_{ww}$ is a diagonal matrix, will accomplish this goal. If $\mathbf{G}_{ww}$ is a diagonal matrix, the signal in the vector $\mathbf{w}$ are independent because the cross spectrum between every pair is zero.

This method can be used for both the analysis and the generation of partially coherent random signals (Smallwood and Paez, 1993). The cross-spectral density matrix relating the modeled independent sources $\mathbf{w}$ and the outputs $\mathbf{y}$ can be developed as follows.

$$\mathbf{X} = \mathbf{H}_{xw}\mathbf{W}, \tag{18}$$

$$\mathbf{W}' = \mathbf{X}'(\mathbf{H}'_{xw})^{-1}, \tag{19}$$

which results in

$$\mathbf{YW}' = \mathbf{YX}'(\mathbf{H}'_{xw})^{-1}. \tag{20}$$

Taking the expectation

$$\mathbf{G}_{yw} = \mathbf{G}_{yx}(\mathbf{H}'_{xw})^{-1}, \tag{21}$$

the equation is unchanged by writing as

$$\mathbf{G}_{yw} = [\mathbf{G}_{yx}(\mathbf{H}'_{xw})^{-1}\mathbf{G}_{ww}^{-1}]\mathbf{G}_{ww}. \tag{22}$$

The inverse of $\mathbf{G}_{ww}$ will exist if none of the diagonal elements of $\mathbf{G}_{ww}$ are zero. Comparing Eqs. (7) and (22) gives

$$\mathbf{H}_{yw} = \mathbf{G}_{yx}(\mathbf{H}'_{xw})^{-1}\mathbf{G}_{ww}^{-1}. \tag{23}$$

The cross-spectral density of y is then given by

$$\mathbf{G}_{yy:x} = \mathbf{G}_{yy:w} = \mathbf{H}_{yw}\mathbf{G}_{ww}\mathbf{H}'_{yw}. \qquad (24)$$

Of course p and r can be substituted for x and y in the above development.

TRADITIONAL METHODS USING CHOLESKY FACTORIZATION

Cholesky Factorization

If a matrix $\mathbf{G}$ is positive definite, the matrix can be factored using Cholesky factorization into a product of a lower triangular matrix, $\mathbf{L}$, and its conjugate transpose such that

$$\mathbf{G} = \mathbf{LL}'. \qquad (25)$$

Without loss of generality this can be written as

$$\mathbf{G} = \mathbf{LIL}', \qquad (26)$$

where $\mathbf{I}$ is the identity matrix. Equation (26) has the form of Eq. (9). Using this model the inputs are modeled by a vector of independent white sources with unity amplitude. The scaling of the noise sources is maintained in the lower triangular matrix $\mathbf{L}$. If the matrix $\mathbf{G}$ is not positive definite one or more of the diagonal elements of $\mathbf{L}$ will be zero. And because the computation of a column of $\mathbf{L}$ requires division by the diagonal element, the factorization will fail. To avoid this problem, if a diagonal element of $\mathbf{L}$ is zero the corresponding column of $\mathbf{L}$ is set to zero, i.e., $L_{ji} = 0$ if $L_{ii} = 0$, $j > i$.

Cholesky Factorization with Ones on the Diagonal

If each column of $\mathbf{L}$ is divided by its corresponding diagonal element to give $\overline{\mathbf{L}}$, $\overline{\mathbf{L}}$ is a lower triangular matrix with ones on the diagonal and

$$\mathbf{G} = \overline{\mathbf{L}}\mathbf{C}\overline{\mathbf{L}}'. \qquad (27)$$

$\mathbf{C}$ is a diagonal matrix where the elements are defined as

$$C_{ii} = L_{ii}^2. \qquad (28)$$

This will be called a modified Cholesky factorization with ones on the diagonal. This procedure is

also called Cholesky factorization without square roots (Lawson and Hanson, 1974; Smallwood and Paez, 1993). If the matrix $\mathbf{G}$ is not positive definite one or more of the diagonal elements of $\mathbf{C}$ will be zero and the corresponding diagonal element in $\mathbf{L}$ will be zero. And because the computation of a column of $\overline{\mathbf{L}}$ and $\mathbf{L}$ requires division by the diagonal element, the factorization will fail. To avoid this problem, if a diagonal element of $\mathbf{C}$ is zero the corresponding off the diagonal elements of $\overline{\mathbf{L}}$ are set to zero, i.e., $L_{ji} = 0$ if $C_{ii} = 0$, $j > i$.

Relationship Between Cholesky Factorization with Ones on the Diagonal and Conditioned Spectral Densities

As explained above, one way to factor the input spectral density matrix, $\mathbf{G}_{xx}$, is with a modified Cholesky factorization with ones on the diagonal. In this case let

$$\mathbf{G}_{xx} = \overline{\mathbf{L}}\mathbf{C}\overline{\mathbf{L}}' = \mathbf{H}_{xc}\mathbf{G}_{cc}\mathbf{H}'_{xc}. \qquad (29)$$

It can be shown that there is a one to one correspondence between the elements of $\mathbf{H}_{xc}$ and the terms L_{ij} from Bendat and Piersol (1986, Chap. 7; L_{ji}, from Bendat and Piersol $= H_{ij}^{xc}$, this article), and a one to one correspondence between $\mathbf{G}_{ii}^{cc}$ and the terms $G_{ii.(i-1)!}$ from Bendat and Piersol (1986). Thus it is seen that the solution for L_{ij} and $G_{ii.(i-1)!}$ from Bendat and Piersol (1986) are equivalent to a modified Cholesky factorization with ones on the diagonal where

$$\{c\} = \begin{Bmatrix} x_1 \\ x_{2.1} \\ x_{3.2!} \\ \vdots \end{Bmatrix}. \qquad (30)$$

Bendat and Piersol (1986) do not explain the procedure to use if an element in $\mathbf{G}_{ii}^{cc}$ is zero. This procedure is explained above. Equation (23) gives

$$\mathbf{H}_{yc} = \mathbf{G}_{yx}(\mathbf{H}'_{xc})^{-1}\mathbf{G}_{cc}^{-1}. \qquad (31)$$

Because the inputs c are independent, the conditioned response from each conditioned input $\{c\}$ can be found from

$$G_{yy:x_{i.(i-1)!}} = G_{yy:c_i} = H_{:j}^{yc}G_{ii}^{cc}(H_{:j}^{yc})' \\ j = 1:m, i = 1:n, \qquad (32)$$

where j is the index of the output and i is the index of the input. The coherent power at the jth output due to a single conditioned input c_i is given by

$$G_{jj}^{yy:c_i} = |H_{ji}^{yc}|^2 G_{ii}^{cc} \quad j = 1:m, i = 1:n. \quad (33)$$

The output power of the jth output with $i!$ conditioned inputs removed is given by

$$G_{jj.i!}^{yy} = G_{jj}^{yy} - \sum_{k=1}^{i} G_{jj}^{yy:c_k} \quad j = 1:m, i = 1:n. \quad (34)$$

The cross-spectral density matrix of the outputs, given the vector of conditioned responses, is given by

$$\mathbf{G}_{yy:c} = \mathbf{H}_{yc}\mathbf{G}_{cc}\mathbf{H}_{yc}', \quad (35)$$

and the cross-spectral density matrix of the outputs with all the inputs removed is given by

$$\mathbf{G}_{yy.c} = \mathbf{G}_{yy} - \mathbf{G}_{yy.c}. \quad (36)$$

Thus Eqs. (32)–(36) are an alternate solution to the methods in Bendat and Piersol (1986) for the conditioned spectral densities and have the advantage of being written in matrix notation, which is easy to implement using a modern matrix algebra package. The ordinary coherence is given by

$$(\gamma_{ji}^{yx})^2 = \frac{|G_{ji}^{yx}|^2}{G_{ii}^{xx}G_{jj}^{yy}} \quad j = 1:m, i = 1:n. \quad (37)$$

The terms on the right side of Eq. (37) are all terms from the original cross-spectral density matrix [Eq. (2)]. The partial coherence is given by

$$(\gamma_{j1.0!}^{yx})^2 = (\gamma_{ji}^{yx})^2 \qquad j = 1:m; \quad (38a)$$

$$(\gamma_{ji.(i-1)!}^{yx})^2 = \frac{|G_{ji}^{yc}|^2}{G_{ii}^{cc}G_{jj.(i-1)!}^{yy}} \quad j = 1:m, i = 1:n. \quad (38b)$$

The term in the numerator on the right side of Eq. (38b) can be computed from Eq. (21) with the subscript c substituted for the subscript w. $\mathbf{H}_{xc}$ is computed from Eq. (29). The first term in the denominator is an element from the factorization in Eq. (29). The second term in the denominator is from Eq. (34).

The multiple coherence is given by

$$(\gamma_{j:n!})^2 = 1 - \prod_{i=1}^{n} (1 - (\gamma_{ji.(i-1)!}^{yx})^2)$$

$$j = 1:m, i = 1:n. \quad (39)$$

PROCEDURE USING SVD

SVD

The SVD of a matrix $\mathbf{G}$ is defined as: $\mathbf{S}$ is a diagonal matrix of the same dimension as $\mathbf{G}$, with nonnegative diagonal elements in decreasing order, and unitary matrices $\mathbf{U}$ and $\mathbf{V}$ so that

$$\mathbf{G} = \mathbf{USV}'. \quad (40)$$

An advantage is that because $\mathbf{U}$ and $\mathbf{V}$ are orthogonal,

$$\mathbf{U}^{-1} = \mathbf{U}' \quad \text{and} \quad \mathbf{V}^{-1} = \mathbf{V}'. \quad (41)$$

A pseudoinverse can be defined as

$$\mathbf{G}^{-1} = \mathbf{VS}^{-1}\mathbf{U}'. \quad (42)$$

The pseudoinverse has the properties that

$$\mathbf{G}^{-1}\mathbf{G} = \mathbf{I} \quad \text{and} \quad \mathbf{GG}^{-1} = \mathbf{I}. \quad (43)$$

The pseudoinverse reduces to the normal inverse for a square matrix whose determinate is nonzero. $\mathbf{S}^{-1}$ is trivial to compute as a diagonal matrix whose elements are the inverse of the diagonal elements of $\mathbf{S}$. If a diagonal element of $\mathbf{S}$ is zero the corresponding element of $\mathbf{S}^{-1}$ is defined as zero.

If $\mathbf{G}$ is Hermitian, as is a cross-spectral density matrix,

$$\mathbf{U} = \mathbf{V} \quad (44)$$

and

$$\mathbf{G} = \mathbf{USU}'. \quad (45)$$

The pseudoinverse reduces to

$$\mathbf{G}^{-1} = \mathbf{US}^{-1}\mathbf{U}'. \quad (46)$$

An important property of SVD is that the ratio of the smallest diagonal element in $\mathbf{S}$ to S_{11} is a measure of the condition of the matrix. Roughly,

if the ratio of a particular S_{ii} to S_{11} is less than the relative error in the elements of G, the matrix can be assumed to be computationally singular of order $n - i$, where n is the number of rows and columns in G. Values of S_{ii} less than this ratio can be set to zero, and the corresponding rows of U and V can be deleted. SVD is a robust technique and seldom fails.

Relationship Between SVD and Conditioned Spectral Densities

SVD can also be used to decompose the cross-spectral density matrix G_{xx}.

$$G_{xx} = H_{xs}G_{ss}H'_{xs}. \tag{47}$$

This is very convenient as Eq. (23) reduces to

$$H_{ys} = G_{yx}H_{xs}G_{ss}^{-1}, \tag{48}$$

because for SVD

$$H'_{xs} = H_{xs}^{-1}. \tag{49}$$

Because the inputs are independent, the conditioned response from each conditioned input s can be found from

$$G_{yy:s_i} = H_{:j}^{ys}G_{ii}^{ss}(H_{:j}^{ys})' \quad j = 1:m, i = 1:n. \tag{50}$$

The coherent power at the jth output due to a single conditioned input s_i, is given by

$$G_{jj}^{yy:s_i} = |H_{ji}^{ys}|^2 G_{ii}^{ss} \quad j = 1:m, i = 1:n. \tag{51}$$

The output power of the jth output with $i!$ conditioned inputs removed is given by

$$G_{jj.i!}^{yy} = G_{jj}^{yy} - \sum_{k=1}^{i} G_{jj}^{yy:s_k} \quad j = 1:m, i = 1:n. \tag{52}$$

The matrix of conditioned responses is given by

$$G_{yy:s} = H_{ys}G_{ss}H'_{ys} \tag{53}$$

and

$$G_{yy.s} = G_{yy} - G_{yy:s}. \tag{54}$$

The results can be improved by first conditioning the cross-spectral density terms by removing any

singular values less than the relative errors in the cross-spectral density matrix (Golub and Van Loan, 1989). The results can also be improved by scaling the cross-spectral density matrix to remove the artificial conditioning effects caused by using different physical units and other scaling factors (Fletcher et al., 1995). The scaling is not unique and serves only to make the dynamic range of the data as small as possible. If an estimate of the noise floor for each channel is available, one way to scale the data is to divide each channel by the estimate of its noise floor. This will tend to make the noise level in each channel the same. If it is believed that each channel has a range of good data from the noise floor to Q times the noise floor, then singular values less than the maximum singular value divided by Q will be rejected.

There are several advantages to using SVD instead of Cholesky factorization. The method seldom fails and does not suffer from the conditioning problems of Cholesky factorization. A unified method for dealing with noise in the spectral estimates is available. The method does not depend on any particular order of the inputs. Using the conditioned spectral density matrices the ordinary, fractional, and multiple coherence functions can be computed as follows. The ordinary coherence is given by

$$(\gamma_{ji}^{yx})^2 = \frac{|G_{ji}^{yx}|^2}{G_{ii}^{xx}G_{jj}^{yy}}. \tag{55}$$

The fractional coherence, analogous to the partial coherence using Cholesky factorization, is given by

$$(\bar{\gamma}_{ji.0!}^{yx})^2 = (\gamma_{j1}^{yx})^2, \tag{56a}$$

$$(\bar{\gamma}_{ji.(i-1)!}^{yx})^2 = \frac{|G_{ji}^{ys}|}{G_{ii}^{ss}G_{jj.(i-1)!}^{yy}}. \tag{56b}$$

The multiple coherence is given by

$$\gamma_{j:q!}^2 = 1 - \prod_{i=1}^{q}(1 - (\bar{\gamma}_{ji.(i-1)!}^{yx})^2). \tag{57}$$

The *fractional coherence* is not the same as *partial coherence*. The fractional coherence suggests that the set of inputs can be modeled by a set of independent sources in order of *decreasing magnitude*. The inputs are then a linear combination of the independent sources. The first frac-

tional coherence expresses the fraction of output in a least squares sense that can be modeled by a single independent source. The second fractional coherence expresses the fraction of the output that can be modeled by the two independent sources, etc.

To compute partial coherence using SVD requires a special procedure. The partial coherence of any input, with respect to any set of inputs removed, can be solved by just arranging the inputs in the z vector. The inputs whose effects are to be removed are placed in the p vector and the remaining inputs x̄, along with the outputs y, are placed in the residual vector r. The complete set of partial coherences formed by removing the inputs one at a time can be found by solving a succession of problems, each problem removing one more input. The vector z is partitioned into primary (the removed input records) and residual components (the remaining input and the output records).

$$z = \begin{Bmatrix} p \\ r \end{Bmatrix}. \qquad (58)$$

The primary inputs are factored using SVD.

$$G_{pp} = H_{ps}G_{ss}H'_{ps}. \qquad (59)$$

The conditioned spectral density is then found from Eqs. (48) and (53).

$$H_{rs} = G_{rp}H_{ps}G_{ss}^{-1}, \qquad (60)$$

$$G_{rr:s} = H_{rs}G_{ss}H'_{rs}, \qquad (61)$$

and

$$G_{rr.s} = G_{rr} - G_{rr:s}. \qquad (62)$$

The conditioned matrix is partitioned into

$$G_{rr.s} = \begin{bmatrix} G_{\bar{x}\bar{x}.s} & G_{\bar{x}y.s} \\ G_{y\bar{x}.s} & G_{yy.s} \end{bmatrix}. \qquad (63)$$

The partial coherence of any of the remaining inputs x̄ with the inputs in the p vector removed is the ordinary coherence from the conditioned matrix.

$$(\gamma^{yx}_{jk.(i-1)!})^2 = \frac{|G^{y\bar{x}.s}_{jk}|^2}{G^{\bar{x}\bar{x}.s}_{kk}G^{yy.s}_{jj}} \quad j = 1{:}m, k = i{:}n, \qquad (64)$$

where k is the index of the desired remaining input in x̄. More specifically the first row of the reduced conditioned set $G_{rr.s}$ is the partial coherence for first input in the r vector with the effects of the p vector removed.

$$(\gamma^{yx}_{ji.(i-1)!})^2 = \frac{|G^{y\bar{x}.s}_{ji}|^2}{G^{\bar{x}\bar{x}.s}_{11}G^{yy.s}_{jj}} \quad j = 1{:}m, i = 1{:}n, \qquad (65)$$

where $(i - 1)$ is the length of the vector p, or the number of inputs removed.

This method can also be used with Cholesky factorization by substituting a Cholesky factorization for the SVD. Using this method the partial coherences of all the outputs with respect to all the remaining inputs can be evaluated with the effects of the primary inputs removed.

SUGGESTED METHOD FOR ORDERING INPUTS WHEN USING CHOLESKY FACTORIZATION

One of the problems using Cholesky factorization for the computation of the coherence functions is the selection of the input order. The discussion using SVD that orders the inputs in a monotonic decreasing order suggests a method. First the ordinary coherence is computed. The input with the largest ordinary coherence is selected to be first. If more than one output is present the selection can be made on the basis of a weighting criterion, for example, the largest ordinary coherence in a least squares sense with respect to all the outputs. Another approach would be to accomplish the ordering for each output separately. The effects of this input are removed to form the conditioned spectral density matrix of the remaining inputs and the outputs with the selected input removed. This will remove the most power possible for the effects of a single input from the conditioned power of the outputs. The partial coherence of the remaining inputs with the first input removed is generated from ordinary coherence of this conditioned spectral density matrix. This order was also suggested by Bendat and Piersol (1986, p. 217). The input with the largest partial coherence is selected next. Placing this input second is hinted at by Bendant and Piersol but not stated explicitly. Again this will remove the maximum power from the conditioned output for any of the remaining inputs. The process is then repeated removing the effects of the second input, etc., until all the inputs are removed. With the order

of the inputs now established, the traditional Cholesky decomposition can be used to find the conditioned spectral density and the corresponding coherence functions. Because the spectral density functions are a function of frequency, the additional small burden of keeping track of the input order as a function of frequency must be maintained. But input order can reveal significant information about the influence of the various inputs on the output as a function of frequency. This will result in the conditioned output power being reduced in a fastest descent as the effects of successive inputs are removed.

EXAMPLE. Assume a problem with 3 inputs and 1 output where the cross-spectral density matrix and modified Cholesky decomposition are given by

$$G_{zz} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \end{bmatrix}, \bar{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notice that a traditional Cholesky decomposition would fail for this example. The SVD for this example is given by

$$U = \begin{bmatrix} 0.3389 & -0.5774 & 0.2277 & -0.7071 \\ 0.3389 & -0.5774 & 0.2277 & 0.7071 \\ 0.5577 & 0 & -0.83 & 0 \\ 0.6777 & 0.5774 & 0.4554 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 5.6458 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.3542 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If noise of magnitude 0.01 is added to each input and the output, the spectral density matrix becomes

$$G_{zz} = \begin{bmatrix} 1.01 & 1 & 1 & 1 \\ 1 & 1.01 & 1 & 1 \\ 1 & 1 & 2.01 & 2 \\ 1 & 1 & 2 & 3.01 \end{bmatrix}.$$

The diagonal of the matrix C and the singular values become diag(C) = [1.01 0.0199 1.015 1.0199] and diag(S) = [5.6558 1.01 0.3642 0.01]. Notice that the small C_{22} indicates a problem, but the magnitude does not directly reflect the noise level. The small value of S_{44} also indicates a problem, and the value is directly related to the noise level. The cross-spectral density matrix can be conditioned by setting S_{44} to zero. If the inputs are reordered by the criteria stated above, the input order becomes [3 1 2]. The values for the ordinary coherence, partial coherence, and multiple coherence for the various formulations of the example are given in Table 1.

The row in Table 1 labeled the original problem can be considered to be the correct answer for this example. When noise is added the values of the coherence are corrupted. The ordinary coherence is reduced and the partial coherence of input 2 is no longer zero. When the spectral density matrix is conditioned by removing the smallest singular value, the coherence values are improved. The reordering of the inputs reveals an

Table 1. Coherence Values for Various Statements of Example Problem

Problem	Ordinary Coherence			Partial Coherence			Multiple Coherence
	1	2	3	1	2	3	
Original	0.333	0.333	0.667	0.333	0	0.5	.667
With noise added	0.329	0.329	0.661	0.329	0.002	0.494	.661
Conditioned	0.331	0.331	0.661	0.331	0	0.494	.661
Input order changed to 312 No noise	0.333	0.333	0.667	0	0	0.667	.667
Noise added and conditioned	0.331	0.331	0.661	0	0	0.661	.661

Inputs are 1, 2, and 3.

Ordinary coherence with respect to: -.-.input 1 & input 2 ___input 3
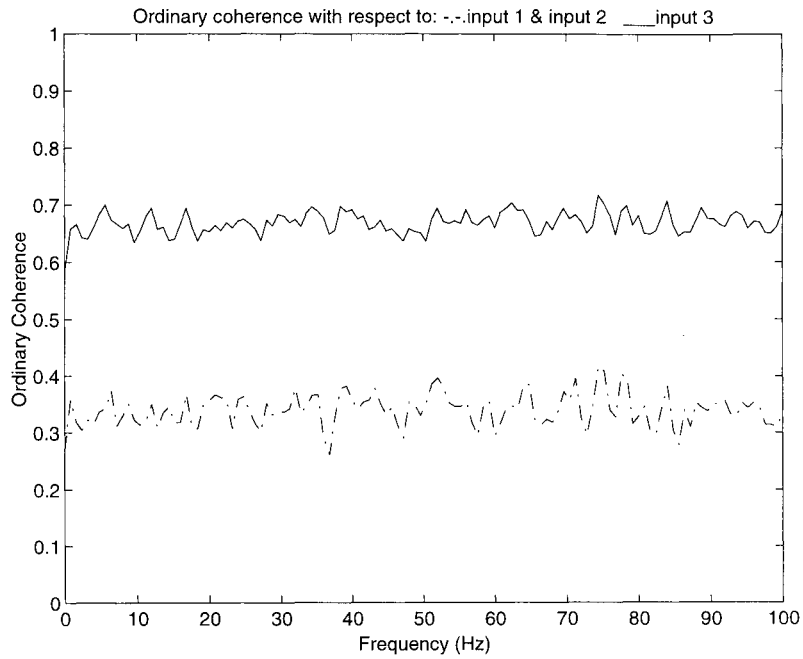

FIGURE 2 Ordinary coherence for the example.

interesting observation that when the third input is removed first, the partial coherence of both inputs 1 and 2 are zero. Four 50,000 point sequences of data were generated using Smallwood and Paez's method (1993) as a realization of a random process having a flat cross-spectral density matrix given by the original G_{zz} in the example. The cross-spectral density matrix was then estimated using Welch's method (1967) (block size = 256, overlap = 75%, window = Hanning, sample rate = 204.8). The coherence, the partial coherence, and the singular values for these se-
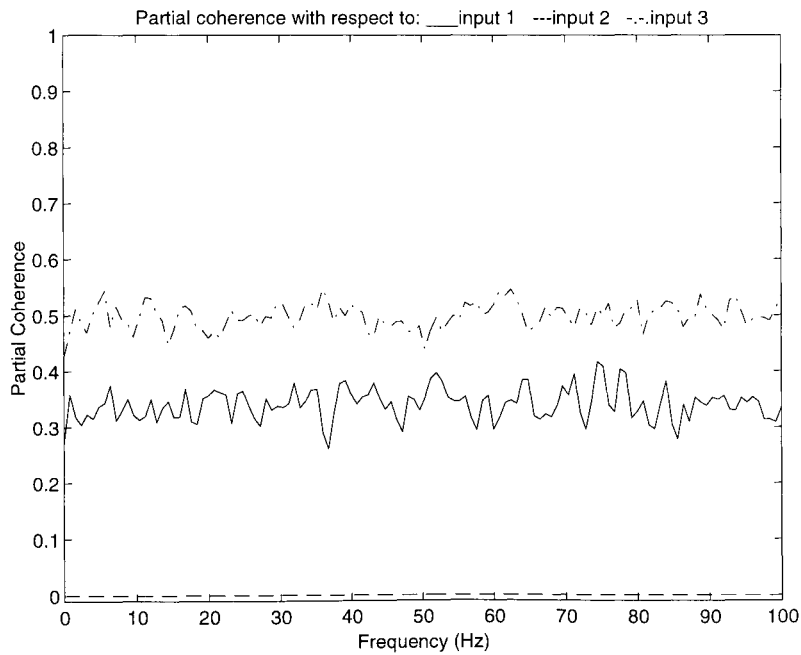
Partial coherence with respect to: ___input 1 ---input 2 -.-.input 3

FIGURE 3 Partial coherence for the example.

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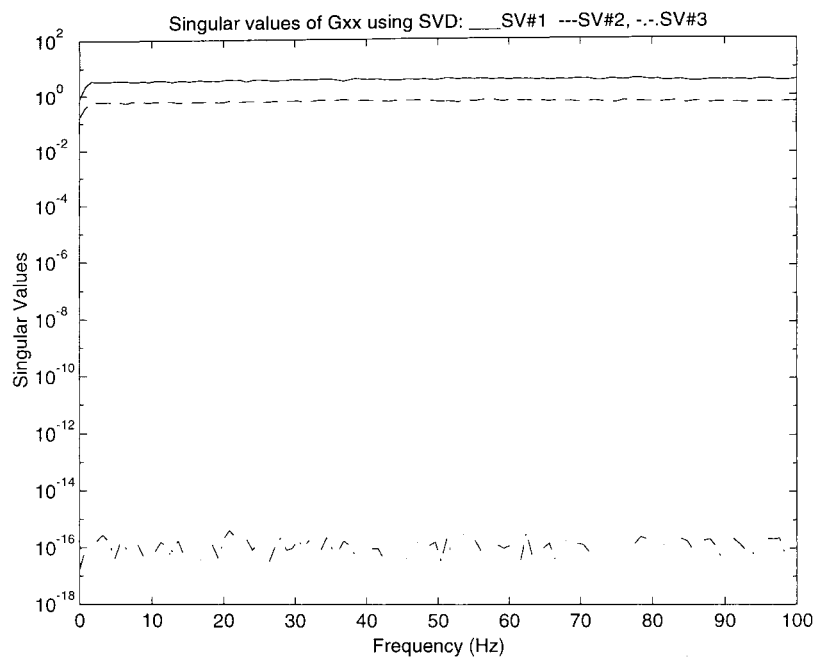Singular values of Gxx using SVD: ___SV#1 ---SV#2, -.-.SV#3

FIGURE 4 Singular values of the input cross-spectral density matrix for the example.

quences were then estimated and are plotted as Figs. 2–4. This data illustrates that the statistical errors associated with the estimation of spectral densities from a realization of random process are present in the coherence estimates; however, they are not present in the singular values, and hence do not affect the condition of the spectral density matrix.

CONCLUSIONS

It is shown that the traditional methods for computing the coherence functions can be efficiently formulated as a modified Cholesky decomposition of the cross-spectral density matrix. An alternate solution using SVD is developed. The use of SVD suggests a new concept of fractional coherence. The development of the method using SVD also suggests a way to order the inputs when using the traditional Cholesky decomposition method. The input with the largest ordinary coherence is placed first. The influence of this input is removed. The input with the largest partial coherence with the first input removed is placed second. The influences of the first two inputs are removed and the process is repeated until all the inputs have been ordered. The use of SVD to condition the estimated cross-spectral density matrix can also improve the results.

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