

Matrix models as non-local hidden variables theories

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ABSTRACT

It is shown that the matrix models which give non-perturbative definitions of string and M theory may be interpreted as non-local hidden variables theories in which the quantum observables are the eigenvalues of the matrices while their entries are the non-local hidden variables. This is shown by studying the bosonic matrix model at finite temperature, with T taken to scale as $1/N$. For large N the eigenvalues of the matrices undergo Brownian motion due to the interaction of the diagonal elements with the off diagonal elements, giving rise to a diffusion constant that remains finite as $N \rightarrow \infty$. The resulting probability density and current for the eigenvalues are then found to evolve in agreement with the Schroedinger equation, to leading order in $1/N$, with \hbar proportional to the thermal diffusion constant for the eigenvalues. The quantum fluctuations and uncertainties in the eigenvalues are then consequences of ordinary statistical fluctuations in the values of the off-diagonal matrix elements. Furthermore, this formulation of the quantum theory is background independent, as the definition of the thermal ensemble makes no use of a particular classical solution. The derivation relies on Nelson's stochastic formulation of quantum theory, which is expressed in terms of a variational principle.

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1 Introduction

In this paper we describe a new proposal concerning the relationship between general relativity and the quantum theory. This is that matrix formulations of string, or \mathcal{M} theory[1, 2, 3, 4], which are known to reproduce general relativity, at least perturbatively, are also hidden variables theories. More precisely, given the dynamics postulated by these models, we show that the finite temperature classical statistical mechanics of the matrix elements reproduces the quantum theory of the matrix eigenvalues. This happens when the temperature is scaled appropriately, as $T \approx 1/N$ as N is taken to infinity. Thus, these theories contain not only a non-perturbative definition of string theory, they contain a reformulation of quantum theory in terms of the ordinary statistical mechanics of a set of non-local variables which are the matrix elements.

By a hidden variables theory we mean generally a theory which purports to give a more detailed description of individual events and processes for which the quantum mechanics yields only probabilistic predictions. It is thus a theory whose ordinary statistical mechanics reproduces, at least to a certain order of approximation, the conventional quantum mechanics. The name hidden variables refers to the existence of degrees of freedom beyond the observables of the quantum theory, whose statistical fluctuations are the source of the quantum fluctuations and uncertainties of quantum theory.

We know from the observed violations of the Bell inequalities that any viable hidden variable theory should be non-local. This leads to a natural suggestion, which is that the non-local hidden variables describe relationships between the local degrees of freedom[5]. A simple hypothesis is that there is such a relational hidden variable for each pair of degrees of freedom of a quantum theory. This then suggests formulating a non-local hidden variables theory in which the fundamental, hidden degrees of freedom are elements of matrices while the observables are the eigenvalues of the matrices.

Such a hidden variables theory was developed some years ago and shown to reproduce the predictions of non-relativistic quantum mechanics for an N -body system[5]. When the elements of an $N \times N$ matrix are put into a thermal bath, it was found that under certain conditions, the probability density and current for the eigenvalues behaved to leading order in $1/\sqrt{N}$, in a way that reproduced the N -body Schroedinger equation. Other non-local hidden variables have been formulated and studied, for example, the Bohm model[6].

Thus, it is not very difficult to make non-local hidden variables theories. What is more challenging is to invent such a theory that solves other problems besides the issues in the foundations of quantum mechanics. Since a hidden variables theory must be non-local, it seems very likely that a true one would have something to do with the structure of space and time, and hence with quantum gravity¹.

It is then remarkable that matrix models which make use of essentially the same idea as the hidden variables theory described in[5]-that the real physical degrees of freedom are matrix elements, while the eigenvalues correspond to the observables-were developed as rep-

¹A different proposal to unify relativity with quantum theory in the context of a hidden variables theory has been proposed by 't Hooft[9].

representations of string and \mathcal{M} theory. These matrix formulations of string theory have solved certain problems, but during their study a certain puzzle has emerged concerning their quantization.

This is a consequence of the fact that one defines string theories from compactifications of the matrix models. To define a quantum string theory one first picks a classical solution of the matrix model, which gives information about the background fields including the space time geometry. One then proceeds to define a quantum theory with respect to that classical background.

Of course, one can define formal quantizations, for example by means of path integral, which appear to be background independent. However as soon as one attempts to define the quantum theory precisely, for example, in terms of a *BRST* quantization, one finds that the precise definitions of the Hilbert space and *BRST* charge Q do depend on the background. Or, if one wants to define the quantum theory precisely in a path integral context, one faces the problem that the Lorentzian path integral does not appear to be well defined. However, the usual solution to this problem, which is to make a Euclidean continuation, is itself background dependent, because it depends on a choice of time in a particular classical spacetime manifold. No background independent definition of continuation between Euclidean and Lorentzian theories is known.

This problem, of the background dependence of the quantizations, is a major problem for string and \mathcal{M} theory. As the theory classically has many solutions, which define different spacetime backgrounds, it seems that there ought to be a background independent way to define the quantum theory. If the fundamental theory is a quantum theory it should be that the different quantizations defined around the different backgrounds should be approximations to a single exact quantum theory.

This is part of the motivation for the present proposal. Perhaps the background independent theory is not a conventional quantum theory, but some deeper theory, which can be approximated by conventional quantum theories when the state defines a fixed spacetime background. Such a theory might be approached in different ways, but here we investigate the hypothesis that it might be a hidden variables theory. As we will show, the hint from earlier work is correct, and a matrix model of the kind studied in string and \mathcal{M} theory can serve as a non-local hidden variables theory which can then reproduce a quantum theory for the eigenvalues of the matrices to leading order in $1/N$.

That is, rather than *quantizing* the *classical* matrix model in some conventional fashion, we will simply assume that the off diagonal elements of the matrix model are in a classical thermal state. We will find that the quantum theory for the eigenvalues can be reproduced so long as the temperature is scaled in a certain way with N . This formulation of the quantum theory is by definition background dependent, because the definition of the thermal ensemble makes no reference to any particular classical solution.

To see how this happens it is useful to consider the diagonal elements of the matrices, which become increasingly free at low temperatures, as analogous to the pollen grains in classical Brownian motion. The off diagonal elements are then analogous to the molecules whose constant collisions with the grains cause them to move with a Brownian motion.

Indeed, the diagonal elements are subject to random forces from their interactions with the off-diagonal elements. The off diagonal elements are small at low temperature, but as we increase N their effects on the diagonal elements are greater due to their greater number. The result is that the interaction of a diagonal elements with a large number of off-diagonal elements introduces a Brownian motion, which is transferred to a Brownian motion of the eigenvalues at low temperature. That is, the randomness of the local variables-the diagonal elements and the eigenvalues is due to their interactions with a much larger number of non-local variables.

To find interesting behavior we have to scale T in an appropriate way with N as we take the former to zero and the latter to infinity. In fact, we find that the model behaves critically when we scale the temperature so that $T \approx 1/N$. In this case the off diagonal matrix elements are of order $1/\sqrt{N}$. However their collective effects on the diagonal elements, and hence the eigenvalues, remain as $N \rightarrow \infty$. One such effect is that the diffusion constant which measures the Brownian motion of the eigenvalues remains finite as $N \rightarrow \infty$.

Of course, the idea that quantum statistics might just be ordinary statistics in an unusual context is an old one. In particular, Nelson[7] has proposed a *stochastic formulation of quantum theory* according to which the quantum description of a particle is derived by modifying the classical description solely by the addition of a universal Brownian motion, which satisfies certain special properties. Chief amongst them is that the Brownian motion is non-dissipative, in that energy and momentum are still conserved. Nelson shows in [7] that when a classical particle is subject to such a non-dissipative Brownian motion, its probability density and current evolve in a way which is equivalent to that given by the Schroedinger equation.

Nelson's formulation plays a key role in the present work, in that we show that the stochastic formulation of quantum theory is recovered to leading order in $1/N$, for the eigenvalues of the matrices in our model².

In the present paper we study a bosonic matrix model, and show that it is indeed a non-local hidden variables theory. The extensions of this work to the supersymmetric matrix models associated with string and \mathcal{M} theory are in progress with Stephon Alexander[10].

In the next section we describe the matrix model we will study. A variational principle related to Nelson's formulation of quantum theory is presented in section 3, where the basics of the theory of Brownian motion are reviewed for those unfamiliar with it. In section 4 we describe the basic physical picture which suggests the connection between the classical statistical mechanics of the matrix model and quantum theory. In section 5 we estimate the dependence of the relevant diffusion constants on N , T and other parameters. Finally, the derivation of the Schroedinger equation for the eigenvalues is given in section 6.

²Note that one can quantize the free bosonic string directly using Nelson's stochastic quantum theory[11].

2 The model

We study a bosonic matrix model which is the bosonic part of the models used in string and \mathcal{M} theory[1, 2, 3, 4]. The degrees of freedom are d $N \times N$ real symmetric matrices X_{ai}^j , with $a = 1, \dots, d$ and $i, j = 1, \dots, N$. The action is,

$$S = \mu \int dt Tr \left[\dot{X}_a^2 + \omega^2 [X_a, X_b][X^a, X^b] \right] \quad (1)$$

We choose the matrices X^a to be dimensionless. ω is a frequency and μ has dimensions of mass \cdot length². We do not assume $\hbar = 1$, in fact, as we aim to derive quantum mechanics from a more fundamental theory, \hbar is not yet meaningful. We will introduce \hbar as a function of the parameters of the theory when we derive the Schroedinger equation as an approximate evolution law. We may note that the parameters of the theory define an energy $\epsilon = \mu\omega^2$. The basic idea is that the off diagonal matrix elements of X^a will be the non-local hidden variables. The physical observables will be defined to be the eigenvalues λ_i^a of the matrices. We will put the system at a small, but finite temperature, the result of which will be that the matrix elements undergo Brownian motion as they oscillate in the potential. It follows from linear algebra that the eigenvalues also undergo Brownian motion. We will see that the parameters of the theory can be scaled with N in such a way that Nelson's stochastic formulation of quantum mechanics is realized for the eigenvalues. When $T = 0$ the matrices must commute with each other so as to achieve the vanishing of the potential energy. This means that it is possible to simultaneously diagonalize them. When T is finite, but small compared to ϵ , the off diagonal elements will on average be small. As a result, it is useful to split the matrices into diagonal and off-diagonal pieces,

$$X_{ai}^j = D_{ai}^j + Q_{ai}^j \quad (2)$$

where $D^a = \text{diagonal}(d_1^a, \dots, d_N^a)$ is diagonal and Q_{ai}^j has no diagonal elements. Since the Q_{ai}^j are dimensionless we will expect them to scale like a power of $T/\mu\omega^2$. We then write the action 1 as

$$S = \int dt \left[\mathcal{L}^d + \mathcal{L}^Q + \mathcal{L}^{int} \right] \quad (3)$$

The theory of the d 's alone is free,

$$\mathcal{L}^d = \mu \sum_{ai} (\dot{d}_i^a)^2, \quad (4)$$

while the theory of the Q 's alone has the same quartic interaction

$$\mathcal{L}^Q = \mu \left[\sum_{aij} (\dot{Q}_{ai}^j)^2 + \omega^2 [Q_a, Q_b][Q^a, Q^b] \right] \quad (5)$$

The interaction terms between the diagonal and off-diagonal elements are

$$\mathcal{L}^{int} = 2\mu\omega^2 \sum_{abij} \left[-(d_i^a - d_j^a)^2 (Q_{bi}^j)^2 - (d_i^a - d_j^a)(d_i^b - d_j^b) Q_{ai}^j Q_{bj}^i + 2(d_i^a - d_j^a) Q_i^{bj} [Q_a, Q_b]_j^i \right] \quad (6)$$

We note that the model has a translation symmetry given by

$$d_i^a \rightarrow d_i^a + v^a. \quad (7)$$

The result is that the center of mass momentum of the system is conserved

3 The statistical variational principle

We now introduce the basic ideas behind the stochastic formulation of quantum mechanics of Edward Nelson. We introduce Nelson's formulation by means of a variational principle, which we call the statistical variational principle. This is closely related to variational principles previously introduced by Guerra[8] and by Nelson[7]. We then will formulate it in hamiltonian language, which will provide insight into how the linearity of the state space of quantum theory emerges from the theory of Brownian motion.

3.1 S ensembles

Consider a dynamical system living on an n -dimensional configuration space coordinatized by x^a . The dynamics can be described in terms of a Hamilton-Jacobi function, $S(x, t)$ which solves the Hamilton-Jacobi equation

$$\dot{S} + \frac{1}{2m}(\partial_a S)^2 + U = 0 \quad (8)$$

A particular solution S defines a family of classical trajectories whose momenta at any point x^a are defined by

$$p_a(x) = \partial_a S \quad (9)$$

There are many solutions to the Hamilton-Jacobi equation, each of which defines a congruence of classical trajectories. A statistical description of the system may be given in terms of a probability density $\rho(x, t)$ and a probability current $v^a(x, t)$, which together satisfy

$$\dot{\rho} + \partial_a(\rho v^a) = 0 \quad (10)$$

Since the probability is conserved we may always assume $\int d^n x \rho(x, t) = 1$.

Now we will do something unusual. Let us restrict attention to an ensemble of classical trajectories whose evolution is determined by a particular solution S of the Hamilton-Jacobi equation. We may call this an S -ensemble. These have a probability density $\rho_S(x, t)$. Since the momentum is determined by S , so must be the probability current. We then have,

$$m v_a = \partial_a S \quad (11)$$

so that the probability conservation equation is now

$$\dot{\rho} + \frac{1}{m} \partial_a(\rho \partial^a S) = 0 \quad (12)$$

The restriction of attention to a statistical ensemble with fixed solution S is unusual, but it does not take us out of the domain of classical physics. The total probability density may be recovered formally as

$$\rho_{total} = \sum_S \rho_S \quad (13)$$

where the sum is over all solutions to the Hamilton-Jacobi function. Of course, the probability current does not add. Now we may notice that such ensembles are given by the solutions of a simple variational principle

$$I[\rho, S] = \int dt \int d^n x \rho(x, t) \left[\dot{S} + \frac{1}{2m} (\partial_a S)^2 + U(x) \right] \quad (14)$$

The equations that arise from varying ρ and S are, respectively, the Hamilton-Jacobi equation (8) and the probability conservation equation (12). Thus, the variational principle describes an ensemble of trajectories, each of which evolves according to the Hamilton-Jacobi equation, so that the current velocity is proportional to $\partial_a S$

We note that the action and equations of motion are invariant under time reversal with

$$t \rightarrow -t, \quad S \rightarrow -S, \quad \rho \rightarrow \rho \quad (15)$$

3.2 A brief review of the theory of Brownian motion

Nelson's stochastic version of quantum theory may be formulated in the language we have just introduced. To do this we assume that in addition to the classical motion, the particles in our ensemble are subject to a Brownian motion. This Brownian motion is, however, unusual, in that it does not alter the condition that each trajectory in the ensemble is governed by the same solution of the Hamilton-Jacobi equation. We will see that this requirement can be met, by altering in a small way the Hamilton-Jacobi equation itself, to take into account the fact that trajectories undergoing Brownian motion are not differentiable. Nelson calls the resulting Brownian motion dissipationless Brownian motion, as energy and momentum are still conserved. To describe this dissipationless Brownian motion we may use the language of stochastic differential equations³ In this language the small change in time of a trajectory is given by

$$Dx^a = b^a(x(t), t)dt + \Delta w^a \quad dt > 0 \quad (16)$$

for small positive changes in time and

$$D^*x^a = -b^{*a}(x(t), t)dt + \Delta w^{*a} \quad dt < 0 \quad (17)$$

for small negative changes in time. b^a and b^{*a} are called the forward and backwards drift velocities. They describe the average motion of the particles in the ensemble. They are defined by

$$b^a(x, t) = \lim_{\Delta t \rightarrow 0} \left\langle \frac{x^a(t + \Delta t) - x^a(t)}{\Delta t} \right\rangle_{x(t)=x} \quad (18)$$

³For reviews see [7].

and

$$b^{*a}(x, t) = \lim_{\Delta t \rightarrow 0} \left\langle \frac{x^a(t) - x^a(t - \Delta t)}{\Delta t} \right\rangle_{x(t)=x} \quad (19)$$

The different elements of the ensemble are distinguished by their Brownian motion, which is given by a Markov process defined by

$$\langle \Delta w^a \Delta w^b \rangle = - \langle \Delta w^{*a} \Delta w^{*b} \rangle = \nu dt q^{ab} \quad (20)$$

and

$$\langle \Delta w^a \Delta w^{*b} \rangle = 0 \quad (21)$$

Here q^{ab} is a metric on the configuration space and ν is the diffusion constant. The averages $\langle \dots \rangle$ are defined with respect to the ensemble. Thus, for any function $F(x)$ on the phase space

$$\langle F \rangle = \int d^n x \rho(x, t) F(x) \quad (22)$$

From these basic definitions one can derive easily the forward and backwards Fokker-Planck equations

$$\dot{\rho} = -\partial_a(\rho b^a) + \nu \nabla^2 \rho \quad (23)$$

$$\dot{\rho} = -\partial_a(\rho b^{*a}) - \nu \nabla^2 \rho \quad (24)$$

From these the current conservation equation follows with

$$v^a = \frac{1}{2}(b^a + b^{*a}) \quad (25)$$

The difference between the forward and backward drift velocities is called the osmotic velocity. From the Fokker-Planck equation it satisfies

$$u^a = \frac{1}{2}(b^a - b^{*a}) = \nu \partial^a \ln \rho \quad (26)$$

We thus see that the diffusion constant measures the extent to which the paths are non-differentiable, so that the forward and backwards drift velocities are not equal. This is of course possible because they are defined in eqs. (18,19) in such a way that the limit $\Delta t \rightarrow 0$ is taken *after* averaging over the ensemble.

3.3 Quantum Brownian motion and Nelson's stochastic formulation of quantum theory

With this quick survey of Brownian motion over, we return to the case of interest, which is an ensemble of trajectories that share the same Hamilton-Jacobi function, S . We want to preserve the property (11) that the current velocity is proportional to the gradient of the Hamilton-Jacobi function, but we want to find a way to include Brownian motion within the ensemble defined by a particular Hamilton-Jacobi function. One way to approach this is to

modify the statistical variational principle to include the effects of Brownian motion. It is not hard to see that this is possible, and that the right extension of the variational principle is

$$I^\nu[\rho, S] = \int dt \int d^n x \rho(x, t) \left[\dot{S} + \frac{1}{2m} (\partial_a S)^2 + \frac{m\nu^2}{2} (\partial_a \ln \rho)^2 + U(x) \right] \quad (27)$$

To see why let us use the fact that $\partial_a S$ is proportional to the momentum. Thus we have for smooth motion, on solutions to the variational principle,

$$\begin{aligned} \int dt \int d^n x \rho(x, t) \frac{1}{2m} (\partial_a S)^2 &= \int dt \int d^n x \rho(x, t) \frac{1}{2m} (p^a)^2 \\ &= \int dt \int d^n x \rho(x, t) \frac{m}{2} (\dot{x}^a)^2 \\ &= \int dt \int d^n x \rho(x, t) \frac{m}{2} \lim_{\Delta t \rightarrow 0} \left(\frac{x^a(t + \Delta t) - x^a(t)}{\Delta t} \right)^2 \end{aligned} \quad (28)$$

However, for a Brownian motion process, the limit in the last line is not defined. So this is not a consistent variational principle when $\nu \neq 0$. To define a variational principle that is well defined for the case of Brownian motion where the paths are non-differentiable we need to take the limit that defines the derivative outside of the ensemble average. Thus, we may define instead,

$$\int dt \int d^n x \rho(x, t) \frac{1}{2m} (p^a)^2 \equiv \int dt \lim_{\Delta t \rightarrow 0} \int d^n x \rho(x, t) \frac{m}{2} \left(\frac{x^a(t + \Delta t) - x^a(t)}{\Delta t} \right)^2 \quad (29)$$

This form of the integrand appears, however, to lack invariance under time reversals, eq. (15). This is because when the paths are non-differentiable b^a and b^{*a} may not be equal. However, we may notice that under the time integral we can write

$$\begin{aligned} \int dt \lim_{\Delta t \rightarrow 0} \int d^n x \rho(x, t) \frac{m}{2} \left(\frac{x^a(t + \Delta t) - x^a(t)}{\Delta t} \right)^2 &= \int dt \lim_{\Delta t \rightarrow 0} \int d^n x \frac{m}{4} [\rho(x, t) \left(\frac{x^a(t + \Delta t) - x^a(t)}{\Delta t} \right)^2 \\ &\quad + \rho(x, t - \Delta t) \left(\frac{x^a(t) - x^a(t - \Delta t)}{\Delta t} \right)^2] \end{aligned} \quad (30)$$

Now, we notice that

$$\rho(x, t - \Delta t) = \rho(x, t) - \Delta t \dot{\rho}(x, t) \quad (31)$$

As $\dot{\rho}(x, t)$ is given by the Fokker Planck equation the second term leads to terms that are well behaved and vanish as $\Delta t \rightarrow 0$. Thus, we can take the average and then the limit, using (18,19) to find,

$$\begin{aligned} \int dt \lim_{\Delta t \rightarrow 0} \int d^n x \rho(x, t) \frac{m}{2} \left(\frac{x^a(t + \Delta t) - x^a(t)}{\Delta t} \right)^2 &= \int dt \int d^n x \rho(x, t) \frac{m}{4} [(b^a)^2 + (b^{*a})^2 + C] \\ &= \int dt \int d^n x \rho(x, t) \frac{m}{2} [(v^a)^2 + (u^a)^2 + C] \end{aligned} \quad (32)$$

Here C is an infinite constant, which is equal to

$$C = \nu d \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \quad (33)$$

Thus, we have found that we can extract an infinite constant from the action, leaving us with a finite piece that is well defined and time reversal invariant. So we have,

$$\int dt \lim_{\Delta t \rightarrow 0} \int d^n x \rho(x, t) \frac{m}{2} \left(\frac{x^a(t + \Delta t) - x^a(t)}{\Delta t} \right)^2 = \int dt \int d^n x \rho(x, t) \left[\frac{1}{2m} (\partial_a S)^2 + \frac{m\nu^2}{2} (\partial_a \ln \rho)^2 \right] + \frac{mC}{2} \quad (34)$$

Constants, even infinite constants, play no role in classical action principles. Hence, putting this last result back in the definition of the action principle we see that the effect of exchanging the order of the integral and the limit is the new action principle (27). Thus we see that while we can have a Brownian motion within the trajectories defined by a given Hamilton-Jacobi functional, the definition of Brownian motion requires that we modify the Hamilton-Jacobi equation, so that we have a consistent variational principle, even in the presence of non-differentiable paths.

From the argument just given we see that the variational principle (27) is equivalent to defining the time derivatives so that they are taken outside the ensemble averages. This yields the standard variational principle when the trajectories are smooth, because then it doesn't matter in which order we take the limits involved in the definition of the time derivatives and the ensemble average. But for non-differentiable paths the order matters and we see that we must take the limit defining the time derivative after that defining the ensemble average. As we have seen, up to an infinite constant which may be ignored, this is equivalent to adding the new term $\frac{m\nu^2}{2} (\partial_a \ln \rho)^2$ to the variational principle.

The new Hamilton-Jacobi equation follows from (27) by varying by ρ . We find that

$$\dot{S} + \frac{1}{2m} (\partial_a S)^2 - 2m\nu^2 \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} + U = 0 \quad (35)$$

The Hamilton-Jacobi functional is thus modified by a new potential term which is a function of the probability density. We recall that this unusual feature follows because it is the gradient of the probability density that measures the importance of the non-differentiability of the paths. The probability conservation, however is not modified. Now, the big surprise is that (35) and (12) are nothing else than the real and imaginary parts of the Schrodinger equation, with

$$\Psi(x, t) = \sqrt{\rho(x, t)} e^{iS/\hbar} \quad (36)$$

and with

$$\hbar = 2\nu m. \quad (37)$$

Thus, the variational principle (27), in the presence of Brownian motion is equivalent to

$$i\hbar \frac{d\Psi(x, t)}{dt} = \left[-\frac{\hbar^2}{2m} \nabla^2 + U(x) \right] \Psi(x, t) \quad (38)$$

One way to understand this is the following. From the point of view presented here, a quantum state is nothing more nor less than an ordinary statistical ensemble of Brownian motion trajectories, which share a single hamilton-jacobi function, S , where that S is itself a solution to a modified Hamilton-Jacobi equation, modified to take into account the change in the definition of the momentum necessary when the motion is Brownian. This is the basic message of Nelson's stochastic quantum theory.

3.4 Hamiltonian formulation of the statistical variational principle

To find the hamiltonian formulation of the statistical variational principle note that (27) can be written

$$I^\nu[\rho, S] = \int dt \int d^n x [S\dot{\rho} - \mathcal{H}[\rho, S, x]] \quad (39)$$

where the Hamiltonian density is

$$\mathcal{H}[\rho, S, x] = \rho \left[\frac{1}{2m} (\partial_a S)^2 + \frac{m\nu^2}{2} (\partial_a \ln \rho)^2 + U(x) \right]. \quad (40)$$

Thus we see that the probability density ρ can be considered as a conjugate coordinate with S its conjugate momentum, so that we have an infinite dimensional phase space with

$$\{\rho(x), S(x')\} = \delta^n(x', x) \quad (41)$$

The Hamiltonian,

$$H = \int d^n x \mathcal{H} \quad (42)$$

is then conserved in time. It is easy to check that Hamilton's equations of motion are the Hamilton-Jacobi equation, (35) and the probability conservation equation (12). We note that this is true for any value of the diffusion constant ν so that this is true in both classical and quantum theory. To get more insight into how the linearity of quantum theory has emerged from the theory of Brownian motion, we can write out the conserved hamiltonian,

$$H = \int d^n x \rho \left[\frac{1}{2m} (\partial_a S)^2 + \frac{m\nu^2}{2} (\partial_a \ln \rho)^2 + U(x) \right] \quad (43)$$

This seems a very non-linear expression, but it is nothing but the expectation value of a linear operator. To see this we rewrite it slightly as

$$H = \int d^n x \sqrt{\rho} e^{-iS/\hbar} \left[\frac{1}{2m} (\partial_a S)^2 - 2m\nu^2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} + U(x) \right] \sqrt{\rho} e^{iS/\hbar} \quad (44)$$

Using (36) and (37) this is easily seen to be equal to

$$H = \int d^n x \bar{\Psi} \hat{\mathcal{H}} \Psi \quad (45)$$

with

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m}\nabla^2 + U(x). \quad (46)$$

Thus, the very non-linear seeming equation $\dot{H} = 0$ is seen to be actually equivalent to the linear Schroedinger equation. We further see that the conserved Hamiltonian which arises from the statistical variational principle is exactly equal to the expectation value of the Hamiltonian operator in the quantum theory. Thus, the conservation of the Hamiltonian in the statistical variational principle is equivalent to the conservation of the expectation value of the Hamiltonian operator in the quantum theory.

4 The physical picture

We will now return to the matrix model we introduced in section 2. We will show that the ordinary statistical physics of this model has a critical behavior when the off diagonal sector is heated to finite temperature and the large N limit is taken with the temperature scaled so that $T \approx 1/N$. We will further see that a feature of the critical behavior in this limit is to reproduce quantum mechanics. That is, to leading order in $1/N$ the evolution of the probability density and current for the eigenvalues of the matrixes is equivalent to that given by the free Schroedinger equation. To show this we will apply what we have learned in the last section to the matrix model. We will formulate an S ensemble for the matrix model in terms of the statistical variational principle. At the fundamental level the dynamics is formulated in terms of the variational principle without Brownian motion, i.e. the simple variational principle (14). Thus, the whole matrix model is in an ordinary statistical ensemble. In the next section we will study the behavior the matrix model at low temperature and large N . We see that when we pick $T \approx 1/N$ the off diagonal elements scale as $1/\sqrt{N}$; this makes sense as they must go to zero at $T = 0$. We also see that to leading order the off diagonal elements can be seen to move harmonically in an average field given by the average values of all the other off-diagonal elements. The diagonal elements are not required to vanish as $T \rightarrow 0$, so they remain of order unity. However, the diagonal elements move in a random potential given by the oscillations of all the off diagonal elements. The result is that the diagonal elements pick up a random Brownian motion, on top of their free motion. This Brownian motion is then also experienced by the eigenvalues. We will see that when the model is scaled critically, the diffusion constants for the diagonal elements and eigenvalues go to constant limits as $N \rightarrow \infty$ and $T \rightarrow 0$. We then want to study the effect of the Brownian motion of the eigenvalues. To do this we derive an effective statistical action for the probability distribution of the eigenvalues by averaging the statistical variational principle for the whole model over the values of the matrix elements. This is the task of section 6. We see that the Brownian motion term in (27) emerges naturally as a term in the effective statistical action for the eigenvalues, as a result of the induced Brownian motion just described. Furthermore, in that limit the conserved energy of the variational principle of the whole model reduces to the conservation of the expectation value of the free

hamiltonian operator for the eigenvalues. Thus, in the large N limit Nelson's stochastic formulation of quantum theory emerges naturally as a description of the statistical behavior of the eigenvalues.

5 Estimates for the diffusion constants at low temperature

In this section we thus investigate the consequences of putting the matrix model in a thermal bath at a temperature T . We have two tasks. The first is to understand how various quantities of interest scale with T and N , at low temperatures. The second is to derive estimates for the diffusion constants for the matrix elements and eigenvalues that are good for low temperature and large N . By low temperature, T , we will mean that the ratio $T/\mu\omega^2$ is small. It will be convenient to scale this ratio with N , so we define

$$\frac{T}{8(d-1)\mu\omega^2} = \frac{t}{N^p} \quad (47)$$

with t dimensionless and p a power. The factor of $8(d-1)$ is inserted for later convenience. We begin by recalling how the potential is written in terms of diagonal and off diagonal elements (eq. (6)).

$$\begin{aligned} U(d, Q) = \mathcal{L}_{int} = \mu\omega^2 Tr & \left[-2(d_a^i - d_a^j)(d^{ai} - d^{aj})Q_{bij}Q^{bij} + 2(d_a^i - d_a^j)(d^{bi} - d^{bj})Q_{bij}Q^{aji} \right. \\ & \left. + 4(d_a^i - d_a^j)Q_{bij}[Q^a, Q^b]^{ji} + [Q^a, Q^b]^2 \right] \end{aligned} \quad (48)$$

The classical equations of motion are then,

$$\ddot{d}_a^i = \omega^2 \left[-8(d_a^i - d_a^j)Q_{bij}Q^{bij} + 8(d^{bi} - d^{bj})Q_{bij}Q^{aji} + 8Q_{bij}[Q^a, Q^b]^{ji} + 4[Q_b, [Q^a, Q^b]_{ij}] \right] \quad (49)$$

$$\ddot{Q}_{ij}^a = \omega^2 \left[-4(\delta_b^a(d_a^i - d_a^j))^2 - (d_a^i - d_a^j)(d^{bi} - d^{bj})Q_{ij}^b + 4(d_a^i - d_a^j)Q_{bij}^2 - 8(d_b^i - d_b^j)[Q_a, Q_b]_{ij} \right] \quad (50)$$

Now we will consider how each matrix element moves in an effective potential created by the averaged motions of the other elements. To see this we make a mean field approximation, good at large N . We assume that the statistical averages satisfy relations consistent with the symmetry of the theory. This gives us

$$\langle Q_a^{ij}Q_b^{kl} \rangle = q^2 \delta_{ab}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad (51)$$

$$\langle (d_a^i - d_a^j)(d_b^k - d_b^l) \rangle = r^2 \delta_{ab}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \quad (52)$$

$$\langle Q_{aij} \rangle = \langle d_a^i \rangle = \langle dQ \rangle = \langle Q^3 \rangle = \langle d^3 \rangle = 0 \quad (53)$$

We assume also that averages of four matrix elements factor into pairs of averages of two in all ways. Our goal will be to solve for the value of q when the off diagonal elements are in

a thermal bath. We now write out the effective potential for the matrix elements moving in the averaged fields of the other elements, to quadratic order. This gives us,

$$\langle U \rangle = \frac{\mu\Omega_Q^2}{2} Q_{aij} Q^{aij} + \frac{\mu\Omega_d^2}{2} (d_a^i - d_a^j)^2 \quad (54)$$

with

$$\Omega_Q^2 = 4(d-1)\omega^2[(N-1)q^2 + 2r^2] \quad (55)$$

$$\Omega_d^2 = 4(d-1)\omega^2 q^2 \quad (56)$$

Thus, each off diagonal element moves in a harmonic potential created by the averaged values of the other elements. The diagonal elements are, to leading order a system of points, each connected to all the others by springs with the same spring constant. We will make no assumption about the statistical distribution of the diagonal elements. We will see that this is consistent so long as p is chosen so that the diagonal spring constant Ω_d vanishes as $N \rightarrow \infty$. We will assume that the Q_{aij} 's are in thermal equilibrium with each other. This implies that at temperature T ,

$$\frac{\mu}{2} q^2 \Omega_Q^2 = \frac{T}{4} \quad (57)$$

which tells us that

$$\frac{T}{8(d-1)\mu\omega^2} = (N-1)q^4 + 2r^2 q^2 \quad (58)$$

Now let us assume that we can neglect the term in $r^2 q^2$ for estimating the order of magnitudes of quantities and the dependence on N . This will be consistent so long as $r^2 \leq Nq^2$. We then arrive at the estimate that for large N ,

$$q = \frac{1}{N^{1/4}} \left(\frac{T}{8(d-1)\mu\omega^2} \right)^{1/4} = t^{1/4} \frac{1}{N^{\frac{p+1}{4}}}. \quad (59)$$

We see that for $p = 1$, it is true that Nq^2 is of order unity, so this is consistent also with r being of order unity as $N \rightarrow \infty$ and $T \rightarrow 0$. Note that we now have

$$\Omega_d^2 = 4\omega^2(d-1)[2r^2 + \sqrt{t}N^{\frac{1-p}{2}}] \quad (60)$$

We see that for Ω_Q not to diverge as $N \rightarrow \infty$ we must have $p \geq 1$. We then see that $p = 1$ corresponds to a critical point at which Ω_Q is fixed as $N \rightarrow \infty$. We also see from (59) that this corresponds to $q \approx 1/\sqrt{N}$. We see also that,

$$\frac{\Omega_d^2}{\Omega_Q^2} = \frac{q^2}{Nq^2 + 2r^2} = \frac{1}{N + 2\frac{r^2}{q^2}} \quad (61)$$

Thus we see that if we choose $p = 1$ so Ω_Q is fixed in the limit $N \rightarrow \infty$ then in the same limit Ω_d vanishes, so the diagonal elements remain free in the mean field approximation. From now on, unless otherwise specified, we will take $p = 1$. Now we would like to estimate

the effects on the matrix elements of the fluctuations around the thermal averages for small temperature and large N . Let us consider for example the contribution of the last term in eq. (49) to the motion of d_a^i . We have, neglecting the other terms,

$$\ddot{d}_a^i = 4\omega^2 \sum_{bj} [Q_b, [Q^a, Q^b]]_{ij} \quad (62)$$

Counting the commutators, this is a sum of dN^3 terms, for each a and i . These add with random signs. Each impulse of a fixed sign lasts an average time of Ω_Q^{-1} , as this is the time over which the values of the Q_{aij} 's oscillate. The result is that d_a^i has a random Brownian motion on top of the free motion given by the $N \rightarrow \infty$ limit. To estimate the diffusion constant that results note that if, in a time Δt the average displacement resulting from the random forces is Δd , the diffusion constant is

$$\nu_d = \frac{(\Delta d)^2}{\Delta t} \quad (63)$$

If a_{total} is the total acceleration given by the sum of the random forces over the time Δt we have $\Delta d = \frac{1}{2}a_{total}(\Delta t)^2$. Thus, we have,

$$\nu_d = \frac{1}{4}a_{total}^2(\Delta t)^3 \quad (64)$$

For a_{total} we may take

$$a_{total} = \omega^2 q^3 \sqrt{N^3 d} \quad (65)$$

because we are adding $N^3 d$ terms with random signs and average magnitude $\omega^2 q^3$. Taking $\Delta t = \Omega_Q^{-1}$ we have for large N ,

$$\nu_d = \frac{\omega d}{4(d-1)^{3/2}} \frac{q^3 N^{3/2}}{[1 + \frac{2r^2}{Nq^2}]^{3/2}} = \frac{\omega dt^{3/4}}{4(d-1)^{3/2} N^{\frac{3}{4}(p-1)}} \frac{1}{[1 + \frac{2r^2}{Nq^2}]^{3/2}} \quad (66)$$

So we see that if we pick $p = 1$, and so long as $2r^2$ is of order one, the diffusion constant for the diagonal elements goes to a limit which is N -independent and hence T independent as $N \rightarrow \infty$ and $T \rightarrow 0$. Under these same conditions we can show that as $N \rightarrow \infty$ these terms make the dominant contribution to the random forces on the diagonal elements coming from eq. (49). A similar analysis starting from eq. (50) allows us to estimate the diffusion constant for the Q 's coming from the random forces to be,

$$\nu_Q = \omega \frac{r^2 t}{N^{7p/4+5/4}} \quad (67)$$

The result is that for $p \geq 1$, ν_Q vanishes in the limit $N \rightarrow \infty$. Now a classic result of random matrix theory is that if the matrix elements of a matrix undergo Brownian motion, so do its eigenvalues. In a case like ours it is clear, as the eigenvalues will be close to the diagonal values, as the off diagonal values vanish as $N \rightarrow \infty$. But we have to be careful

about the contributions from higher order terms in perturbation theory. By making use of the standard formula,

$$\lambda_i^a = d_I^a + \sum_j \frac{Q_{ij}^a Q_{ji}^a}{d_i^a - d_j^a} + \dots \quad (68)$$

we can show that the diffusion constant for the eigenvalues is given by

$$\nu_\lambda = \nu_d + \frac{N\nu_Q q^2}{r^2} + \dots \approx \omega \left[\frac{dt^{3/2}}{4(d-1)^{3/2} N^{\frac{3}{4}(p-1)}} + \frac{t^{3/2}}{N^{9p/4+3/4}} \right] \quad (69)$$

This tends to ν_d as $N \rightarrow \infty$. Thus, we see that the dominant contribution to the Brownian motion of the eigenvalues comes from the Brownian motion of the diagonal elements. These in turn are fluctuating because they are perturbed by their interactions with the off diagonal elements, which are moving in a harmonic potential, created by their averaged values at finite temperature. The result is that a randomness is introduced into the motions of the eigenvalues, coming from the interactions of the diagonal elements with a very large number of random variables, which are the off diagonal elements. This then illustrates that idea that a local degree of freedom can have its motion randomized by interaction with a large number of non-local degrees of freedom. In the next section we will see that this may result in behavior that for large N is indistinguishable from that predicted by the Schroedinger equation.

6 Derivation of the Schroedinger equation

We are now ready to derive the Schroedinger equation for the eigenvalues of the matrices. As we described above this is a three step process,

- **STEP 1** Formulate the statistical variational principle for the matrix model.
- **STEP 2** Make assumptions about the statistical ensemble. In particular, we assume that the model is in an S -ensemble, heat it to finite temperature T and then study the large N limit with $T \approx 1/N$.
- **STEP 3** Derive an effective statistical variational principle for the eigenvalues by averaging over the variational principle of the matrix elements and show that when $N \rightarrow \infty$ this is equivalent to Schroedinger quantum theory for the eigenvalues.

6.1 STEP 1: The statistical variational principle for the matrix model

We begin by defining an S -ensemble for the matrix elements. That is, we begin with the variational principle

$$I[\rho, S] = \int dt \int (dd)(dQ) \rho(d, Q, t) \left[\dot{S}(d, Q) + \frac{1}{2\mu} \left(\frac{\delta S(d, Q, t)}{\delta d_i^a} \right)^2 + \frac{1}{2\mu} \left(\frac{\delta S(d, Q, t)}{\delta Q_{ij}^a} \right)^2 + U(d, Q) \right] \quad (70)$$

where $U(d, Q)$ is the interaction term \mathcal{L}^{int} given by (6).

6.2 STEP 2: Physical assumptions

We then will state the physical assumptions we make concerning ρ and S . These are assumed only to hold to leading order in $1/N$

- The Q system is in a distribution that is to leading order in $1/N$ statistically independent of the distribution of the eigenvalues. This means that to leading order the probability density factorizes

$$\rho(d, Q) = \rho_d(d) \rho_Q(Q) + O(1/N) \quad (71)$$

- The Q subsystem is in thermal equilibrium at a temperature T . So we have

$$\rho_Q(Q) = \frac{1}{Z} e^{-H(Q)/T} \quad (72)$$

where $H(Q)$ is the hamiltonian corresponding to the Q system alone

$$H(Q) = \mu \left[\sum_{aij} (\dot{Q}_{ai}^j)^2 - \omega^2 [Q_a, Q_b][Q^a, Q^b] \right] \quad (73)$$

and

$$Z = \int dQ e^{-H(Q)/T} \quad (74)$$

As a result of these assumptions our variational principle reads,

$$I[\rho_d, S, T] = \int dt \int (dd)(dQ) \rho_d(d) \rho_Q(Q) \left[\dot{S}(d, Q) + \frac{1}{2\mu} \left(\frac{\delta S}{\delta d_i^a} \right)^2 + \frac{1}{2\mu} \left(\frac{\delta S}{\delta Q_{ij}^a} \right)^2 + U(d, Q) \right] \quad (75)$$

6.3 STEP 3: Derive an effective variational principle for the eigenvalues

Now we want to derive an effective variational principle to describe the evolution of the probability distribution for the eigenvalues. We will do this by averaging the variational principle (75) over the values of the matrix elements, and then extracting the leading behavior for large N and small T . We begin by inserting the factor unity in the form

$$1 = \int \prod_{ai} d\lambda_i^a \delta \left(\lambda_i^a - d_i^a - \sum_j \frac{Q_{ij}^a Q_{ji}^a}{d_i^a - d_j^a} + \dots \right) \quad (76)$$

Thus, we have,

$$I[\rho_d, S, T] = \frac{1}{Z} \int dt \int dddQ \int d\lambda \delta \left(\lambda_i^a - d_i^a - \sum_j \frac{Q_{ij}^a Q_{ji}^a}{\lambda_i^a - \lambda_j^a} + \dots \right) \rho_d e^{-H(Q)/T} \left[\dot{S} + \frac{1}{2\mu} \left(\frac{\delta S}{\delta d_i^a} \right)^2 + \frac{1}{2\mu} \left(\frac{\delta S}{\delta Q_{ij}^a} \right)^2 + U(d, Q) \right] \quad (77)$$

Now we would like to integrate over the d 's, which will express the action in terms of only the λ 's and Q 's. However, before doing this we need to take into account that while to the order we are working, the d 's and the λ 's will be undergoing Brownian motion because the diagonal elements are moving in a random potential given by the values of the Q 's. There are also additional contributions to the diffusion constant coming from the terms in Q that contribute to the eigenvalues at higher order. So we will have to be careful about the definitions of the velocities. In particular, we will have to recall that in the theory of stochastic processes the limits which define time derivatives are taken *after* the averages over probability distributions, not before. So as before we must write

$$\begin{aligned} \int dt \int dddQ \rho_d(d) \rho_Q(Q) \frac{1}{2\mu} \left(\frac{\delta S}{\delta d_i^a} \right)^2 &= \int dt dddQ \rho_d(d) \rho_Q(Q) \mu (V(d)_i^a)^2 \\ &= \int dt \lim_{\Delta t \rightarrow 0} \int dddQ \rho_d(d) \rho_Q(Q) \mu \left(\frac{d_i^a(t + \Delta t) - d_i^a(t)}{\Delta t} \right)^2 \\ &= \int dt \lim_{\Delta t \rightarrow 0} \int dddQ \rho_d(d) \rho_Q(Q) \frac{\mu}{2} \left\{ \left(\frac{d_i^a(t + \Delta t) - d_i^a(t)}{\Delta t} \right)^2 + \left(\frac{d_i^a(t) - d_i^a(t - \Delta t)}{\Delta t} \right)^2 \right\} \end{aligned} \quad (78)$$

Note that the last equation follows trivially, for smooth motion, but it will have non trivial consequences once we have averaged over the Q 's because the result for large N is to induce Brownian motion for the off diagonal elements and eigenvalues. Now we perform the integral over the d 's. It is useful to write

$$d_i^a = \lambda_i^a + \Delta \lambda_i^a \quad (79)$$

where

$$\Delta \lambda_i^a(Q, \lambda) = - \sum_j \frac{Q_{ij}^a Q_{ji}^a}{\lambda_i^a - \lambda_j^a} + \dots \quad (80)$$

has to be treated as a stochastic variable, taking into account its dependence on the Q 's which are themselves fluctuating due to the assumption that they are in equilibrium in a potential. We then have, to leading order in $1/N$,

$$I[\rho_d, S_d, T] = \int dt \int d\lambda \rho_d(\lambda, t) \int dQ \rho_Q(Q) \left[\dot{S}(\lambda + \Delta\lambda, Q) + \frac{1}{2\mu} \left(\frac{\delta S(\lambda + \Delta\lambda, Q)}{\delta Q_{ij}^a} \right)^2 + U(\lambda, Q) \right] + K.E. \quad (81)$$

where to leading order, the kinetic energy terms for the d 's have become,

$$K.E. = \int dt \lim_{\Delta t \rightarrow 0} \int d\lambda \rho_d(\lambda, t) \int dQ \rho_Q(Q) \frac{\mu}{2} \left\{ \left(\frac{(\lambda_i^a(t + \Delta t) - \lambda_i^a(t))^2}{\Delta t^2} \right) + \left(\frac{(\lambda_i^a(t) - \lambda_i^a(t - \Delta t))^2}{\Delta t^2} \right) \right\} \quad (82)$$

We now are ready to integrate over the Q 's. The key point is that the dependence of the λ 's on the Q 's through a sum of a large number of independent terms, $\sum_j \frac{Q_{ij}^a Q_{ji}^a}{d_i^a - d_j^a}$, as well as the coupling of the λ 's with the Q 's coming from the terms in $U(\lambda, Q)$ turns the λ 's into stochastic variables, described by a stochastic differential equation of the form

$$D\lambda_i^a = b_i^a(\lambda, t)dt + \Delta\lambda_i^a \quad \Delta t > 0 \quad (83)$$

$$D\lambda_i^a = b_i^{*a}(\lambda, t)dt + \Delta^*\lambda_i^a \quad \Delta t < 0 \quad (84)$$

with

$$\langle \Delta\lambda_i^a \Delta\lambda_j^b \rangle = \delta^{ab} \delta_{ij} \nu_\lambda dt dt > 0 \quad (85)$$

$$\langle \Delta^*\lambda_i^a \Delta^*\lambda_j^b \rangle = -\delta^{ab} \delta_{ij} \nu_\lambda dt \quad dt < 0 \quad (86)$$

Here the brackets mean

$$\langle F(\lambda, Q) \rangle = \int dQ \rho_Q(Q) F(\lambda, Q) \quad (87)$$

We note that we can use the value of ν_λ , (69) computed in the last section as all the assumptions we made there have been carried over here, so long as we work to leading order in $1/N$ with T scaled as $T \approx 1/N$. We also have, from the Focker-Planck equations that the current velocity is

$$v_i^a(\lambda) = \frac{1}{2}(b_i^a + b_i^{*a}) \quad (88)$$

while the osmotic velocity is

$$u_i^a(\lambda) = \frac{1}{2}(b_i^a - b_i^{*a}) = \nu_\lambda \frac{\delta \ln \rho_\lambda}{\delta \lambda_i^a} \quad (89)$$

From these we can derive,

$$\lim_{\Delta t \rightarrow 0} \int dQ \rho_d(\lambda, t) \rho_Q(Q) \frac{1}{2} \left(\frac{(\lambda_i^a(t + \Delta t) - \lambda_i^a(t))^2}{\Delta t^2} \right) = \rho_d(\lambda, t) [b_i^a(\lambda, t)^2 + NC] \quad (90)$$

and

$$\lim_{\Delta t \rightarrow 0} \int dQ \rho_d(\lambda, t) \rho_Q(Q) \frac{1}{2} \left(\frac{(\lambda_i^a(t) - \lambda_i^a(t - \Delta t))^2}{\Delta t^2} \right) = \rho_d(\lambda, t) [b_i^{*a}(\lambda, t)^2 + NC] \quad (91)$$

where C is the infinite constant we defined in eq. (33).

To go further we need to define the effective Hamilton-Jacobi function for the eigenvalues. We define

$$S_\lambda(\lambda) = \int (dQ) \rho_Q(Q) S(\lambda, Q) \quad (92)$$

We now show that, to leading order in $1/N$,

$$\mu v_i^a = \frac{\delta S_\lambda(\lambda)}{\delta \lambda_i^a}. \quad (93)$$

Consider the probability conservation law that follows from the statistical variational principle that defines the dynamics of our matrix model, eq. (70).

$$\dot{\rho}(d, Q) = -\frac{1}{\mu} \left[\frac{\delta}{\delta d_{ai}} (\rho(d, Q) \frac{\delta S(d, Q)}{\delta d_{ai}}) + \frac{\delta}{\delta Q_{aij}} (\rho(d, Q) \frac{\delta S(d, Q)}{\delta Q_{aij}}) \right] \quad (94)$$

But using (71) and (72) we have that

$$\dot{\rho}(d, Q) = \dot{\rho}_d(d) \rho_Q(Q). \quad (95)$$

We also have, by the same assumptions, since a thermal distribution is stationary and has no current velocity,

$$v^{aij}(Q) = \frac{1}{\mu} \frac{\delta S(d, Q)}{\delta Q_{aij}} = O(1/N) \quad (96)$$

Thus, we have, integrating over the Q 's,

$$\dot{\rho}_d(d) = -\frac{1}{\mu} \frac{\delta}{\delta d_{ai}} (\rho_d(d) \frac{\delta}{\delta d_{ai}} \int dQ \rho_Q(Q) S(d, Q)) + O(1/N). \quad (97)$$

To leading order we can replace everywhere the dependence on d_{ai} with dependence on λ_{ai} , since the terms by which they differ are also higher order in $1/N$. Thus we have

$$\dot{\rho}_d(\lambda) = -\frac{1}{\mu} \frac{\delta}{\delta \lambda_{ai}} (\rho_d(\lambda) \frac{\delta S_\lambda(\lambda)}{\delta \lambda_{ai}}) + O(1/N) \quad (98)$$

But by (88) we must have

$$\dot{\rho}_d(\lambda) = -\frac{\delta \rho_d(\lambda) v^{ai}(\lambda)}{\delta \lambda_{ai}} \quad (99)$$

This establishes eq. (93). With this result we have the key relation that,

$$\frac{\mu}{2} (b^2 + b^{*2}) = \frac{\mu}{2} (v^2 + u^2) = \left\{ \frac{1}{2\mu} \left(\frac{\delta S_\lambda(\lambda)}{\delta \lambda_i^a} \right)^2 + \frac{\mu v_\lambda^2}{2} \left(\frac{\delta \ln \rho_\lambda(\lambda)}{\delta \lambda_i^a} \right)^2 \right\}. \quad (100)$$

We also define

$$E_Q = \int dQ \rho_Q(Q) \left[\frac{1}{2\mu} \left(\frac{\delta S_Q(Q)}{\delta Q_{ij}^a} \right)^2 + \frac{\mu\omega^2}{2} \text{Tr}[Q, Q]^2 \right] \quad (101)$$

and

$$\frac{\mu}{2} \Omega_d^2 \sum_{aij} (\lambda_i^a - \lambda_j^a)^2 = \int dQ \rho_Q(Q) U^{int}(\lambda, Q) \quad (102)$$

We can estimate that $E_Q = TN(N-1)/4 \approx N\mu\omega^2$ so this is a divergent constant in the limit. The result is

$$I[\rho_d, S, T] = \int dt \int d\lambda [\rho_d(\lambda, t) \dot{S}_\lambda - H^{eff}(S_\lambda, \rho_d, T)] \quad (103)$$

where, the effective hamiltonian for the eigenvalues is

$$H^{eff}(S_\lambda, \rho_\lambda, T) = \rho_d(\lambda) \left[E'_Q + \left\{ \left(\frac{\frac{1}{2\mu} \delta S_\lambda(\lambda)}{\delta \lambda_i^a} \right)^2 + \frac{\mu\nu_\lambda^2}{2} \left(\frac{\delta \ln \rho_\lambda(\lambda)}{\delta \lambda_i^a} \right)^2 \right\} + \frac{\mu\Omega_d^2}{2} \sum_{aij} (\lambda_i^a - \lambda_j^a)^2 \right] \quad (104)$$

where $E'_Q = E_Q + N\mu C$ contains both infinite constants. The resulting equations of motion are

$$E'_Q + \dot{S}_\lambda + \frac{1}{2\mu} \left(\frac{\delta S_d(\lambda)}{\delta \lambda_i^a} \right)^2 + \frac{\mu\Omega_d^2}{2} \sum_{aij} (\lambda_i^a - \lambda_j^a)^2 + U^{quantum} = 0 \quad (105)$$

and the current conservation equation

$$\dot{\rho}_\lambda = -\frac{1}{\mu} \partial^{ai} \rho_\lambda (\partial_{ai} S_\lambda) \quad (106)$$

The so-called ‘‘quantum potential’’ is given by

$$U^{quantum} = \mu\nu_\lambda^2 \left\{ \left(\frac{\delta \ln \rho_\lambda(\lambda)}{\delta \lambda_i^a} \right)^2 + \frac{1}{\rho_\lambda} \partial_{ai} (\rho_\lambda \partial^{ai} \ln \rho_\lambda) \right\} = -\mu\nu_\lambda^2 \frac{1}{\sqrt{\rho_\lambda(\lambda)}} \nabla^2 \sqrt{\rho_\lambda(\lambda)} \quad (107)$$

These we recognize as the real and imaginary parts of the Schroedinger equation, when we write

$$\Psi(\lambda, t) = \sqrt{\rho_\lambda} e^{S_\lambda/\hbar} \quad (108)$$

with

$$\hbar = \mu\nu_\lambda = \mu\omega \frac{dt^{3/2}}{4(d-1)^{3/2}} \quad (109)$$

So, finally, we have in the limit $N \rightarrow \infty$,

$$i\hbar \frac{d\Psi(\lambda, t)}{dt} = \left[-\frac{\hbar^2}{2\mu} \frac{\delta^2}{\delta(\lambda_i^a)^2} + \frac{\mu\Omega_d^2}{2} \sum_{aij} (\lambda_i^a - \lambda_j^a)^2 + E'_Q \right] \Psi(\lambda, t) \quad (110)$$

Finally, by repeating the argument of section 3.4 we can show that the conserved energy of the original theory splits into two pieces,

$$H = H^\Psi + E'_Q \quad (111)$$

where

$$H^\Psi = \int d\lambda \bar{\Psi} \left(-\frac{\hbar^2}{2\mu} \frac{\delta^2}{\delta(\lambda_i^a)^2} + \frac{\mu\Omega_d^2}{2} \sum_{aij} (\lambda_I^a - \lambda_j^a)^2 \right) \psi. \quad (112)$$

Since E'_Q is an infinite constant the result is that H^Ψ , which is the quantum mechanical energy, is conserved as $N \rightarrow \infty$. We can then renormalize the wavefunctional so that

$$\Psi_r(\lambda) = e^{iE'_Q t/\hbar} \Psi(\lambda) \quad (113)$$

Finally, we note that as $\Omega_d^2 \approx 1/N$ the eigenvalues become free in the limit $N \rightarrow \infty$. Thus, when $N \rightarrow \infty$ the probabilities evolve according to the free Schroedinger equation,

$$i\hbar \frac{d\Psi_r(\lambda, t)}{dt} = \left[-\frac{\hbar^2}{2\mu} \frac{\delta^2}{\delta(\lambda_i^a)^2} \right] \Psi_r(\lambda, t) \quad (114)$$

7 Conclusions

What we have shown here may be summarized by saying that matrix theory may not only give rise to string theory, and hence gravity, it may also give rise to quantum theory, in the sense that the quantum evolution of the eigenvalues may appear, at large N , to be a consequence of the classical statistical physics of the matrices. This means that we may be able to solve the daunting conceptual problems of quantum theory by means of a simple physical hypothesis: that the theory of gravity and hence spacetime arises from a non-local background independent theory in which geometry initially plays no role and the physical degrees of freedom represent relational rather than intrinsic properties. There remain many open questions. A short list is:

- Is it possible to extend these results to the actual supersymmetric models that arise in string theory⁴?
- Is it possible that the existence of dualities that connect certain quantum field theory observables to the classical limit of string theory are related to the fact that the classical matrix theory can in a certain limit reproduce a quantum theory?
- How does Lorentz invariance and relativistic causality arise in the matrix models which describe relativistic string and membrane theories, and how is this compatible with the non-local dynamics of the off diagonal elements?

⁴This is in progress with Stephon Alexander[10].

- Can this be extended to truly background independent matrix models, such as the cubic matrix models[12] and the matrix models which have been developed for spin foams[13]?
- Are there any practical experimental predictions that follow from these theories?

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