MATRIX REPRESENTATIONS OF INVERSE SEMIGROUPS

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In his paper [1], W. D. Munn determines the irreducible matrix representations of an arbitrary inverse semigroup. Munn also gives a necessary and sufficient condition upon a 0-simple inverse semigroup for it to have a non-trivial matrix representation and for such semigroups gives a complete account of their representations. Munn's results rest upon the earlier work of Clifford [2] in which the representations of Brandt semigroups were determined. An alternative account of such representations was given by Munn in [3]. This earlier work is presented in Sections 5.2 and 5.4 of [4].

In this paper we obtain a complete determination of the matrix representations of inverse semigroups. We restrict ourselves to inverse semigroups with a zero, and there is clearly no loss in generality in so doing. We show that any representation (without a null component) decomposes into what we term primitive components. Each primitive component in turn decomposes into representations which are determined by representations of certain associated Brandt semigroups. The set of Brandt semigroups involved is determined by what we call the representation ideal series of the representation.

Conversely, representation ideal series are abstractly characterized and it is shown that starting with a representation ideal series and the Brandt semigroups it determines then representations of these Brandt semigroups determine, in a unique fashion, a representation of the original semigroup.

The methods used are a development of those used by Munn in [1]. The results of Munn may be easily inferred from our results.

The terminology and notation followed will be that of [4]. Certain differences from the terminology already used by Munn in earlier work are adopted for systematic reasons in conformity with [10]. Concepts and terminology not in [4] will be defined.

The main theorem of this paper was announced in [13].

1. Primitive regular semigroups

It will be convenient to adopt as a shorthand phrases such as 'Let $S = S^0$ be a semigroup' to convey that S is a semigroup, that $|S| > 1$ and

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that S has a zero element 0. This is in conformity with the meaning usually attached to S^0 (cf. [4]).

A semigroup $S = S^0$ will be said to be *primitive* if each of its non-zero idempotents is primitive.

A semigroup $S = S^0$ is said to be the 0-*direct union* of its subsemigroups $S_i (i \in I)$ if $0 \in S_i$ for all $i \in I$, if S is the union of the S_i and if $S_i S_j = 0$ for $i \neq j$ and $i, j \in I$. It follows that each S_i is a (two-sided) ideal of S and that products in S are known once they are known for the S_i. The semigroups S_i are called *summands* of the 0-direct union $S = \cup \{S_i : i \in I\}$.

The following theorem, fundamental for our results, was conjectured by H. Schneider [1] in a letter to the author and the proof below is, but for small changes, a copy of the proof given in the authors' reply [2]. The results have been independently proved by P. S. Venkatesan [12].

THEOREM 1. *A semigroup $S = S^0$ is a primitive regular semigroup if and only if it is the* 0-*direct union of a set of completely* 0-*simple subsemigroups. The completely* 0-*simple summands of a primitive regular semigroup are uniquely determined: the summand containing the non-zero idempotent e is SeS.*

PROOF. Since S is regular, every principal left ideal is generated by an idempotent ([4], lemma 1.13).

Suppose that $L = Sf$ $(f^2 = f)$ is a principal left ideal which is not 0-minimal. Then L contains properly another non-zero principal left ideal L', say, and $L' = Se$ for some idempotent e. Since $e \in Sf$, $ef = e$. Now $fe \neq f$; for otherwise $Sf = Sfe \subseteqq Se$. Let $g = fe$. Then

$$g^2 = f(ef)e = fe^2 = fe = g.$$

Furthermore, $ge = fe^2 = fe = g$. Hence $Sg = Sge \subseteqq Se$. But

$$eg = e(fe) = (ef)e = e^2 = e.$$

Hence $Se = Seg \subseteqq Sg$. Thus $Se = Sg$.

Now $gf = (fe)f = f(ef) = fe = g$, and $fg = f(fe) = fe = g$. Hence, since f is primitive, $g = f$ or $g = 0$. The possibility $g = 0$ is excluded because $Sg = Se \neq 0$. Hence $g = f$. Thus $L' = Se = Sg = Sf = L$. This is a contradiction. Hence, in fact, L must be 0-minimal.

Now consider $M = LS$. M is a two-sided ideal of S. $M \neq 0$, for $f \in M$. Furthermore, M is a completely 0-simple semigroup. For, firstly, M contains a primitive idempotent, namely f. Secondly M is 0-simple. For let $z \in M \setminus 0$. Then $z \in Lx$ for some x in S, and Lx is a 0-minimal ideal of S ([4], Lemma 2.32). Thus $Sz = Lx$ and so $MzM = LSzLS = LLxLS$. Now $L^2 = L$

[1] July 28, 1959.
[2] September 15, 1959.

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because L^2 is a left ideal of S containing $f^2 = f \neq 0$ and because L is 0-minimal. Similarly, $LxLx = Lx$, and consequently $LxL \neq 0$, which gives $LxL = L$. Hence $LLxL = L$, whence $MzM = LS = M$. This completes the proof that M is completely 0-simple.

Every element of S belongs to some principal left ideal and so to some 0-minimal left ideal. Hence every element of S belongs to some ideal of S which is completely 0-simple. Thus S is a union of completely 0-simple semigroups. Let M_1 and M_2 be two distinct completely 0-simple ideals of S. Then $M_1 M_2$ is an ideal of S contained in, and therefore an ideal of, each of M_1 and M_2. Since M_1 and M_2 are 0-simple we must have $M_1 M_2 = 0$. Thus S is the 0-direct union of completely 0-simple subsemigroups. That the completely 0-simple summands are uniquely determined by S is clear, for the above discussion shows that the summand containing the idempotent $e \neq 0$, is SeS.

Conversely, since each non-zero idempotent of a completely 0-simple semigroups is primitive, it easily follows that all the non-zero idempotents of any 0-direct union of completely 0-simple semigroups are also primitive. Since any 0-direct union of regular semigroups is itself regular, this completes the proof of the theorem.

For inverse semigroups we have the following corollary. This corollary is an evident inference from Theorem 1 in [5]. (In [5] the term 'primitive', as applied to inverse semigroups, was used in a wider sense than here.)

COROLLARY 2. *A semigroup $S = S^0$ is a primitive inverse semigroup if and only if it is the 0-direct union of a set of Brandt semigroups. The Brandt semigroup summands of a primitive inverse semigroup are uniquely determined.*

2. Homomorphisms onto primitive regular semigroups

In [6] W. D. Munn introduced the following conditions on a semigroup $S = S^0$.

C1. If a, b, and c are elements of S such that $abc = 0$, then either $ab = 0$ or $bc = 0$.

C2. If M and N are non-zero ideals of S then so also is $M \cap N$.

I shall say that a semigroup satisfying condition C1 is *categorical at zero*. When condition C2 is satisfied, I shall say (in analogy with a terminology from the theory of commutative rings) that the zero (or the zero ideal) of S is *indecomposable*. In the contrary case, the zero of S will be said to be decomposable.

A congruence ρ on a semigroup $S = S^0$ will be called *0-restricted* if $\{0\}$ is a ρ-class. A homomorphism ϕ of $S = S^0$ will be called *0-restricted* if the congruence $\phi \circ \phi^{-1}$ naturally induced on S by ϕ is 0-restricted. This terminology differs from that of Munn [6] where 0-restricted congruences are termed proper congruences. We make the change (see [10]) because proper is also used to describe representations (see [4, p. 177], [1] and below in § 3). A homomorphic image under a 0-restricted homomorphism will be called *0-restricted homomorphic image.*

Munn showed that an inverse semigroup $S = S^0$ has a Brandt semigroup as a 0-restricted homomorphic image if and only if it has an indecomposable zero and it is categorical at zero. We shall extend this result and show that the condition of being categorical at zero is a necessary and sufficient condition upon an inverse semigroup $S = S^0$ for it to have a primitive inverse semigroup as a 0-restricted homomorphic image.

LEMMA 3. *Let ρ be a 0-restricted congruence on a semigroup $S = S^0$. Suppose that S/ρ is a primitive regular semigroup. Then S is categorical at zero.*

PROOF. Let $abc = 0$ in S. Let ϕ denote the natural homomorphism of S upon S/ρ. Then $(abc)\phi = a\phi \cdot b\phi \cdot c\phi = 0$ in S/ρ. If $a\phi$, $b\phi$ and $c\phi$ do not all belong to the same completely 0-simple summand of S/ρ, then either $a\phi \cdot b\phi = (ab)\phi = 0$, or $b\phi \cdot c\phi = (bc)\phi = 0$. Since ϕ is 0-restricted therefore either $ab = 0$ or $bc = 0$. If $a\phi$, $b\phi$ and $c\phi$ belong to the same completely 0-simple summand then the proof of Theorem 1.1 in [6] applies to show that either $ab = 0$ or $bc = 0$. Thus S is categorical at zero.

When a regular semigroup $S = S^0$ possesses a 0-restricted homomorphic image which is a primitive regular semigroup then there exists a maximal such homomorphic image through which every 0-restricted homomorphism with primitive image can be factored. This is established in the following lemma.

LEMMA 4. *Let $S = S^0$ be a regular semigroup and ρ a 0-restricted congruence on S such that S/ρ is primitive. Then there exists a 0-restricted congruence π on S such that S/π is primitive and such that if σ is any 0-restricted congruence with S/σ primitive, then $\pi \subseteqq \sigma$.*

PROOF. Let $\{\rho_i : i \in I\}$ denote the set of all 0-restricted congruences ρ_i on S with the property that S/ρ_i is primitive. By assumption this set is non-empty. Let

$$\pi = \cap \{\rho_i : i \in I\}.$$

Then π is 0-restricted on S, for $(a, 0) \in \pi$ implies that $(a, 0) \in \rho_i$ and hence, since ρ_i is 0-restricted, that $a = 0$. Also, for each i, $\pi \subseteqq \rho_i$. Thus to complete the proof of the lemma it only remains to show that every non-zero idempotent of S/π is primitive.

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Let E, F be any two non-zero idempotents of S/π such that $EF = FE = F$. We have to show that $E = F$. Let $e \in E$ and $f \in F$. Since E is idempotent, $(e, e^2) \in \pi$; and, similarly, $(f, f^2) \in \pi$. Since $EF = FE = F$, also $(ef, f) \in \pi$ and $(fe, f) \in \pi$. Hence, since $\pi \subseteqq \rho_i$, for each i, we have that (e, e^2), (f, f^2), (ef, f) and (fe, f) belong to each ρ_i. If we denote by E_i and F_i the ρ_i-classes containing e and f, respectively, then this means that $E_i^2 = E_i$, $F_i^2 = F_i$, $E_iF_i = F_i$ and $F_iE_i = F_i$, for each i in I. Since, by assumption, E and F are non-zero elements of S/π and each ρ_i is 0-restricted, therefore each E_i and F_i is non-zero. Hence, using the fact that S/ρ_i is primitive, it follows that $E_i = F_i$ for each i in I. Thus $(e, f) \in \rho_i$ for each i in I, whence, from the definition of π, $(e, f) \in \pi$. This shows that $E = F$, and completes the proof of the lemma.

It will have been observed that, in part, the above argument holds if S is not regular. However, when S is not regular and possesses primitive regular 0-restricted homomorphic images it does not necessarily possess a maximum (in the sense of the lemma) primitive regular 0-restricted homomorphic image. For example let S be an infinite cyclic semigroup to which a zero has been adjoined. Then the intersection of all 0-restricted congruences ρ such that S/ρ is primitive regular is the identity congruence on S.

For inverse semigroups we have the following extension of Munn's Theorem 2.7 [6].

THEOREM 5. *Let $S = S^0$ be an inverse semigroup.*

(i) *S possesses a primitive inverse semigroup as a 0-restricted homomorphic image if and only if S is categorical at zero.*

(ii) *Let S be categorical at zero. Define the relation π on S thus:*

(1) $$\pi = \{(x, y) \in S \times S : ax = ay \neq 0 \text{ for some } a \in S\} \cup \{(0, 0)\}.$$

Then π is a 0-restricted congruence on S and S/π is a primitive inverse semigroup. Furthermore, π is the finest congruence (i.e. is contained in any other such) on S with these properties.

PROOF. (i) The necessity of the condition follows directly from Lemma 3, for a primitive inverse semigroup is a primitive regular semigroup. Sufficiency will follow once we have proved (ii).

(ii) In his proof of Theorem 2.7 [6] (which differs from our theorem only in that S is assumed also to have an indecomposable zero), Munn shows that π is a 0-restricted congruence on S. (He comments that only the fact that S is categorical at zero is used in showing this.) We have to show that every non-zero idempotent of S/π is primitive. Let E, F be non-zero idempotents of S/π. Recall that there then exist idempotents e, say, in E and f, say, in F (V. V. Vagner [7]). If $ef \neq 0$, then $e(ef) = ef \neq 0$

shows that $(ef, f) \in \pi$; and similarly it follows that $(ef, e) \in \pi$. Hence $(e, f) \in \pi$. Hence $EF = 0$ or $E = F$. In other words every non-zero idempotent of S/π is primitive.

Finally, that π is the unique finest congruence on S with this property follows by a straightforward extension of the appropriate part of Munn's proof of his Theorem 2.7 in [6].

REMARK. The lack of symmetry in the definition of π is only apparent, for we could equally well define

$$\pi = \{(x, y) \in S \times S : xa = ya \neq 0 \text{ for some } a \in S\} \cup \{(0, 0)\},$$

and then carry out the proof of Theorem 5 with the appropriate modifications.

3. Primitive inverse semigroups of matrices

An application of A. H. Clifford's representation theory [2] for Brandt semigroups gives us canonical forms, to within equivalence, for Brandt semigroups of matrices. Applying Clifford's results to the summands of primitive inverse semigroups of matrices enables us to derive canonical forms for these semigroups. We derive these in this section.

Let Φ be a field and n a positive integer. Then $(\Phi)_n$ will denote the set of all $n \times n$ matrices over Φ and will be regarded as a (multiplicative) semigroup or as an algebra over Φ as the context demands.

By a *representation of degree n over the field Φ* of a semigroup $S = S^0$ we shall mean a homomorphism Γ of S into the semigroup $(\Phi)_n$ which maps the zero of S upon the zero matrix. Γ will be said to be *proper* if it is non-null and does not decompose into two representations, one of which is null [1].

We recall that Brandt semigroups can be characterized as semigroups isomorphic to Rees matrix semigroups $B = \mathscr{M}^0 (G; I, I; \Delta)$ over a group with zero G^0, where Δ is the $I \times I$ identity matrix over G^0 and where G is the structure group of B. The *rank* of a Brandt semigroup is defined to be the cardinal of its set of non-zero idempotents [1]. When the Brandt semigroup is given in the above form, thus $|I|$ is its rank. (See Clifford [2] or [4], Theorem 3.9.)

When I is finite, $|I| = k$, say, we shall write (following Munn) $\mathscr{M}^0 (G; k, k; \Delta_k)$ instead of $\mathscr{M}^0 (G; I, I; \Delta)$, Δ_k denoting the $k \times k$ identity matrix over G^0.

THEOREM 6 (Clifford). (i) *A Brandt semigroup admits a non-null representation if and only if its rank is finite.*

(ii) *Let $B = \mathscr{M}^0 (G; k, k; \Delta_k)$ be a Brandt semigroup of finite rank k. Let Φ be a field. Let $\Gamma^\dagger$ be a proper representation of G^0 of degree l over Φ and*

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let Γ^* be a mapping of B into $(\Phi)_{kl}$ defined by the rule that $\Gamma^*((a; i, j))$ is that $k\times k$ matrix of $l\times l$ blocks which has $\Gamma^\dagger(a)$ in its (i, j)-th block and zeros elsewhere. Then Γ^* is a proper representation of B.

(iii) *Every proper representation Γ^* of $B = \mathscr{M}^0(G; k, k; \Delta_k)$ is, to within equivalence, constructed as in* (ii) *from a proper representation $\Gamma^\dagger$ of G^0.*

(iv) *The correspondence $\Gamma^* \leftrightarrow \Gamma^\dagger$ established in* (ii) *and* (iii) *preserves reduction and decomposition.*

We need an extension of the concept of rank from Brandt semigroups to primitive inverse semigroups. Let S be a primitive inverse semigroup and let E be its set of non-zero idempotents. Then the cardinal $|E|$ of E is defined to be the *rank* of S. When S is a Brandt semigroup its rank in this sense coincides with its rank as already defined.

Let S be a primitive inverse semigroup. Then, by Corollary 2, S is the 0-direct union of a uniquely determined set of Brandt semigroups, B_j, say, for $j \in J$. The rank of S is then the sum of the ranks of the B_j, $j \in J$.

The *semigroup algebra* $\Phi[S]$ of the semigroup S over the field Φ is the vector space with basis S and with multiplication induced by that of S ([4], § 5.2). If $S = S^0$, then $Z = \Phi[\{0\}]$, where 0 is the zero of S, is an ideal of $\Phi[S]$ and the *contracted semigroup algebra* is $\Phi_0[S] = \Phi[S]/Z$. The zero of S may be identified with that of $\Phi_0[S]$ and the non-zero elements of S with a basis of $\Phi_0[S]$.

LEMMA 7. *Let S be the 0-direct union of the semigroups S_j, $j \in J$. Then $\Phi_0[S]$ is the direct sum of its two-sided ideals $\Phi_0[S_j]$, $j \in J$.*

PROOF. Let $x \in \Phi_0[S_j]$ and $y \in \Phi_0[S_k]$, where $k \neq j$. Then

$$x = \sum_{t=1}^{p} \lambda_t s_{tj},$$

$$y = \sum_{u=1}^{q} \mu_u s_{uk}$$

for λ_t, $\mu_u \in \Phi$, $s_{tj} \in S_j$, $s_{uk} \in S_k$. Since $S_j S_k = S_k S_j = 0$, by assumption on S, it follows that $xy = yx = 0$. Furthermore, every element of S in an element of some S_j. Hence the $\Phi_0[S_j]$, $j \in J$, span $\Phi_0[S]$. The assertion of the lemma therefore follows.

For primitive inverse subsemigroups of $(\Phi)_n$ we have the following preliminary result.

LEMMA 8. *Let $S = S^0$ be a primitive inverse subsemigroup of $(\Phi)_n$, where Φ is a field. Let Γ denote the identical mapping of S, and suppose that Γ is a proper representation of S in $(\Phi)_n$.*

Then S has only a finite number of non-zero idempotents. Let $e_1, e_2, \cdots, e_t$

be these idempotents. Let r_i be the matrix rank of e_i, $i = 1, 2, \cdots, t$. Then

$$\sum_{i=1}^{t} e_i = I_n,$$

where I_n is the $n \times n$ identity matrix, and there exists a representation Γ^* of S equivalent to Γ, such that

$$\Gamma^*(e_i) = \begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 & \cdot \\ 0 & \cdot & & & \cdot & \\ \cdot & \cdot & & & \cdot & \\ 0 & \cdot & \cdot & \cdot & I_{r_i} & \cdot \\ \cdot & \cdot & & & \cdot & \end{pmatrix},$$

where the matrix on the right is partitioned in rows and columns according to the partition $n = r_1 + r_2 + \cdots + r_t$, and where I_{r_i} denotes the $r_i \times r_i$ identity matrix, situated in the (i, i)-th partition position.

Proof. Let W be an n-dimensional vector space over Φ and, choosing a basis for W, regard the elements of $(\Phi)_n$ as linear transformations of W.

Let $e_1, e_2, \cdots, e_u$ be distinct non-zero idempotents of S. Then, setting $E_i = We_i$, the image of W under e_i, E_i is a linear subspace of W on which e_i induces the identity transformation. Let w belong to E_u and also to the subspace of W generated by $E_1, E_2, \cdots, E_{u-1}$. Thus

$$w = w_1 + w_2 + \cdots + w_{u-1},$$

where $w_i \in E_i$. Since e_i is the identical transformation on E_i,

$$w = w_1 e_1 + w_2 e_2 + \cdots + w_{u-1} e_{u-1}.$$

Hence

$$we_u = w_1 e_1 e_u + w_2 e_2 e_u + \cdots + w_{u-1} e_{u-1} e_u = 0,$$

since the product of two distinct idempotents of S is zero. Thus $w = we_u = 0$. By induction, it therefore follows that the subspaces $E_1, E_2, \cdots, E_u$ of W generate their direct sum. Since W is of finite dimension, the number of non-zero idempotents of S must be finite. Let them be $e_1, e_2, \cdots, e_t$.

Consider $\sum_{i=1}^{t} e_i = e$, say, an element of the algebra $\Phi_0[S]$. Let a be any non-zero element of S. Since S is a 0-direct union of Brandt semigroups, there is a unique non-zero idempotent of S, e_a, say, such that $e_a a = a$; for all other idempotents g of S, $ga = 0$. Hence $ea = a$, in $\Phi_0[S]$. It follows that e is the identity element of $\Phi_0[S]$.

Since Γ was assumed to be proper it follows that $e = I_n$ and this in turn implies that

$$W = E_1 \oplus E_2 \oplus \cdots \oplus E_t.$$

Hence, choosing an appropriate basis for W, we obtain Γ^*, equivalent to Γ, such that $\Gamma^*(e_i)$, $i = 1, 2, \cdots, t$, have the form given in the lemma.

We can now infer a canonical form for primitive inverse semigroups of matrices.

THEOREM 9. *Let $S = S^0$ be a primitive inverse subsemigroup of $(\Phi)_n$, where Φ is a field. Let Γ denote the identical representation of S. Suppose that Γ, as a mapping of S into $(\Phi)_n$, is a proper representation of S. Then*

(i) *S is of finite rank t, say;*

(ii) *S is the 0-direct union of a uniquely determined finite set $B_1, B_2, \cdots, B_u$, say of Brandt semigroups, so that if t_j is the rank of B_j, $t = t_1+t_2+\cdots+t_u$;*

(iii) *the semigroup algebra $\Phi_0[S]$ decomposes into the direct sum of (two-sided) ideals $\Phi_0[B_j]$:*

$$\Phi_0[S] = \Phi_0[B_1] \oplus \Phi_0[B_2] \oplus \cdots \oplus \Phi_0[B_u];$$

(iv) *Γ is equivalent to a representation*

$$\Gamma^* = \Gamma_1^* + \cdots + \Gamma_u^*,$$

where Γ_j^* is a representation of B_j, $j = 1, 2, \cdots, u$ and where each Γ_j^* is of the form given in Theorem 6 (ii);

(v) *if n_j is the degree of the representation Γ_j^* and if d_j is the rank of each non-zero matrix $\Gamma(x_j)$ for $x_j \in B_j$, $j = 1, 2, \cdots, u$, then*

$$n = n_1+n_2+\cdots+n_u$$

and

$$d_j t_j = n_j,\ j = 1, 2, \cdots, u.$$

PROOF. (i), (ii), and (iii) follow directly from Lemma 8, Lemma 7 and Corollary 2. Once (iv) is established (v) is clear.

Attach subscripts to the non-zero idempotents of S, so that $e_1, e_2, \cdots, e_{t_1}$ are the non-zero idempotents of B_1, $e_{t_1+1}, \cdots, e_{t_1+t_2}$ are the non-zero idempotents of B_2, and so on. The non-zero idempotents of S are then $e_1, e_2, \cdots, e_t$ and by applying Lemma 8 we obtain a representation Γ^*, say, of S, equivalent to Γ, such that each $\Gamma^*(e_i)$ is a diagonal matrix of the form given in Lemma 8, where r_i denotes, as in Lemma 8, the rank of e_i.

Consider the elements of $\Gamma^*(B_j)$. $\Gamma^*(B_j)$ is a Brandt semigroup of matrices equivalent to B_j and its idempotents are diagonal matrices of the form described. If $x \in \Gamma^*(B_j)$ and $x \neq 0$, then there are unique non-zero idempotents, e and f, say, in $\Gamma^*(B_j)$, such that $ex = x = xf$. Let g be any idempotent of $\Gamma^*(S)$. If $g \neq e$, then $gx = 0$; if $g \neq f$, then $xg = 0$. Conversely, if e and f are any non-zero idempotents of $\Gamma^*(B_j)$, then there exists a non-zero element $x \in \Gamma^*(B_j)$ such that $ex = x = xf$. It follows that each

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of the idempotent matrices $\Gamma^*(e_i)$, $i = t_{j-1}+1, \cdots, t_{j-1}+t_j$, of $\Gamma^*(B_j)$ have the same rank, d_j, say, and that each $\Gamma^*(x)$, for $x \in B_j$, consists of a $d_j \times d_j$ sub-matrix bounded by zeros.

More precisely, straightforward arguments show that if we denote by $\Gamma_j^*(x)$ the submatrix of $\Gamma^*(x)$ of order $n_j \times n_j$, where $n_j = t_j d_j$, whose principal diagonal is in the position occupied by the non-zero elements of

$$\sum \{\Gamma^*(e_i) : i = t_{j-1}+1, \cdots, t_{j-1}+t_j\},$$

then Γ_j^* is a proper representation of B_j of the form described in Theorem 6 (ii). Moreover it is then clear that

$$\Gamma^* = \Gamma_1^* \oplus \Gamma_2^* \oplus \cdots \oplus \Gamma_u^*.$$

4. Maximal primitive ideals

An ideal A of a regular semigroup $S = S^0$ will be called a *primitive ideal* if every non-zero idempotent of A is primitive in S. A primitive ideal is necessarily regular, for any ideal of a regular semigroup is regular. For the ideal A to be primitive it suffices that each of its non-zero idempotents is primitive in A. For if $e^2 = e$, $f^2 = f$, $ef = fe = f$ and $e \in A$, then clearly, since A is an ideal, $f \in A$. (Note that the term *primitive ideal* is here used differently from in the author's paper [5].)

LEMMA 10. *Let $S = S^0$ be a regular semigroup. Suppose that S contains a non-zero (i.e. not equal to $\{0\}$) primitive ideal. Then S contains a unique maximal primitive ideal.*

PROOF. The union of all primitive ideals of S, non-zero by assumption, is clearly an ideal every non-zero idempotent of which is primitive in S.

The following lemma may be compared with the analogous result of R. J. Koch [11] for (arbitrary) semigroups without zero.

LEMMA 11. *Let $S = S^0$ be a regular semigroup and let e be a non-zero idempotent of S. Then e is primitive if and only if SeS is completely 0-simple.*

PROOF. If SeS is completely 0-simple then all of its idempotents are primitive and hence, in particular, e is primitive.

Conversely, assume that e is primitive. To show that SeS is completely 0-simple, it suffices to show that it is 0-simple. That this is so was proved in the fourth paragraph of the proof of Theorem 1.

From the preceding two lemmas combined with Theorem 1, we have

THEOREM 12. *Let $S = S^0$ be a regular semigroup. Then S contains a non-zero primitive ideal if and only if it contains a primitive idempotent. If E, assumed non-empty, is the set of all primitive idempotents in S, then*

SES is the (unique) maximum primitive ideal of S. SES is a primitive semigroup and its completely 0-simple summands are the semigroups SeS, $e \in E$.

5. 1-complete ideals

If B is a Brandt semigroup and $a \in B$ then, either $a = 0$, when $ae = 0$ for all idempotents e in B or $a \neq 0$, when $ae \neq 0$ for precisely one idempotent e in B. The sole idempotent e for which $ae \neq 0$ is the right unit $e = a^{-1}a$ of a. Similarly the left unit aa^{-1} is the sole idempotent e in B such that $ea \neq 0$. These properties of the non-zero elements of a Brandt semigroup are shared by the non-zero elements of any 0-direct union of Brandt semigroups, i.e. of any primitive inverse semigroup.

These observations lead us to frame a definition. We introduce it first in a special case.

Let P be a primitive ideal of an inverse semigroup $S = S^0$. An element x of S will be said to be *n-linked to P* if $xe \neq 0$ for precisely n non-zero idempotents e in P. Thus, from the above remarks, the zero of S is 0-linked to P and each non-zero element of P is 1-linked to P.

The next lemma shows that we could equally have defined n-linked in terms of multiplications by idempotents on the left.

LEMMA 13. *Let P be a primitive ideal of an inverse semigroup $S = S^0$. Let $x \in S$. Then x is n-linked to P if and only if $ex \neq 0$ for precisely n idempotents e in P.*

PROOF. For the purposes of this proof let us call n-linked 'right n-linked' and refer to the left-right dual of n-linked as 'left n-linked'. We have to show that right n-linked means the same thing as left n-linked.

Suppose then that $x \in S$ and that x is right n-linked to P. Suppose that $xe \neq 0$ and $xf \neq 0$, where e and f are idempotents in P. Let g be the left unit of xe. Then $g(xe) = xe \neq 0$. Thus $gx \neq 0$ and, since $g \in P$, $gx \in P$. Hence e is the sole idempotent in P such that $(gx)e \neq 0$. Similarly, if h is the left unit of xf, f is the sole idempotent in P such that $(hx)f \neq 0$. Hence, if $e \neq f$, then $gx \neq hx$ and so $g \neq h$.

It follows that there are at least n distinct idempotents k of P such that $kx \neq 0$. Conversely, a similar argument shows that if we assume x to be left n-linked to P then there are at least n distinct idempotents k of P such that $xk \neq 0$. Consequently, x is left n-linked to P if and only if it is right n-linked.

We now extend the concept of n-linkage to the following more general situation. Let $S = S^0$ be an inverse semigroup and let V and P be ideals of S such that $V \subset P$ and P/V has a 0-restricted homomorphic primitive image, i.e., in view of Theorem 5, such that P/V is categorical at zero. Let π (cf.

Theorem 5) be the finest 0-restricted congruence on P/V such that $(P/V)/\pi$ is primitive. Denote by x^* the natural image in $(P/V)/\pi$ of an element x of P (obtained by following the natural mapping of P onto P/V by the natural mapping of P/V onto $(P/V)/\pi$). Let $x \in S$. Then x is said to be *n-linked to P modulo V* if it is possible to find n, and not more than n, idempotents $e_1, e_2, \cdots, e_n$, say, of P such that $e_1^*, e_2^*, \cdots, e_n^*$ are all distinct and such that $(xe_i)^* \neq 0$ (in $(P/V)/\pi$) for $i = 1, 2, \cdots, n$. We easily see that although a choice is in general possible for $e_1, e_2, \cdots, e_n$, the idempotents $e_1^*, e_2^*, \cdots, e_n^*$ are uniquely determined by x. An argument similar to that used to prove Lemma 13 shows that this definition coincides in meaning with its left-right dual. When $V = 0$ and P is primitive, 'n-linked modulo V' means the same as 'n-linked'.

We continue to attach the above meanings to and make the above assumptions about P, V, π, and $*$ throughout this section. The following lemma deals with a detail of technique required to prove the above assertions and which we shall need again.

Lemma 14. *Let f and g be idempotents of P and let $x \in S$. Let $f^* = g^*$. Then $(fx)^* = (gx)^*$ and $(xf)^* = (xg)^*$.*

Proof. We prove the first equation; the other follows similarly. Let h be an idempotent of P such that h^* is the right unit of $(fx)^*$. Then

$$\begin{aligned}(fx)^* &= (fx)^*h^* = (fxh)^* = f^*(xh)^* \\ &= g^*(xh)^* = (gxh)^* = (gx)^*h^*,\end{aligned}$$

since the elements starred belong to P. If $(fx)^*$ is non-zero, thus $(gx)^*h^* \neq 0$ and hence h^* is the right unit of $(gx)^*$. Thus $(gx)^*h^* = (gx)^*$, whence $(fx)^* = (gx)^*$. Similarly the equation holds if $(gx)^* \neq 0$; which completes the proof.

Denote by $L_n(P)$ the set of all elements of S that are m-linked to P modulo V for $m \leqq n$.

Lemma 15. *For each integer $n \geqq 0$, $L_n(P)$ is an ideal of S.*

Proof. Let s be an element of S which is n-linked to P modulo V. It will suffice to prove that, for any $a, b \in S^1$, asb is m-linked for some $m \leqq n$.

Suppose that $e \in P$ and that $(asbe)^* \neq 0$. Then $be \in P$ and $(be)^* \neq 0$. Let f^*, where $f \in P$, be the left unit of $(be)^*$. Let g^*, where $g \in P$, be the left unit of $(asbe)^*$. Then

$$\begin{aligned}(asbe)^* &= g^*(asbe)^* = (gasbe)^* = (gas)^*(be)^* \\ &= (gas)^*f^*(be)^* = (gasf)^*(be)^* = (ga)^*(sf)^*(be)^*,\end{aligned}$$

since the starred elements all belong to P and $*$ is a homomorphism. Thus

$(sf)^* \neq 0$ and so f^* is one of the uniquely determined set of n idempotents of $(P/V)/\pi$ with this property. By Lemma 14, e^* is uniquely determined by f^* as the right unit of $(fb)^*$. It follows that, for a given element asb, there can be at most n such elements e^*. This completes the proof of the lemma.

The ideal P will be said to be *n-complete* in S modulo V if $L_n(P) \subseteqq P$.

If P is not 1-complete then we can extend P and V to obtain a 1-complete ideal and leave $(P/V)/\pi$ unchanged to within isomorphism. This is achieved in the following theorem.

THEOREM 16. *Set $L_1(P) = P_c$ and $L_0(P) = V_c$. Then $P_c \supseteqq P$, $V_c \supseteqq V$, $P_c \supset V_c$, P_c/V_c is categorical at zero and, denoting by π_c the finest 0-restricted congruence on P_c/V_c with primitive image, $(P_c/V_c)/\pi_c \cong (P/V)/\pi$ under the natural mapping $x\pi \to x\pi_c$. Moreover, P_c is 1-complete in S modulo V_c.*

PROOF. It has already been observed that $P_c \supseteqq P$ and it is clear that $V_c \supseteqq V$. Since $P \supset V$ and $P \setminus V \subseteqq P_c \setminus V_c$, therefore $P_c \supset V_c$. To see that P_c/V_c is categorical at zero, consider a, b, c in P_c/V_c and suppose that $abc = 0$. To show that P_c/V_c is categorical at zero we must show that either $ab = 0$ or $bc = 0$.

If any of a, b, c is zero, then clearly one of these equations holds. In the contrary case $a, b, c \in P_c \setminus V_c$ and, evaluating the product in P_c, $abc \in V_c$. We are to show that either $ab \in V_c$ or $bc \in V_c$.

Suppose that $ab \notin V_c$. Then $ab \in P_c$ and so there exist idempotents e^* and f^* in $(P/V)/\pi$, where e and f are idempotents in P, such that $(eab)^* \neq 0$ and $(abf)^* \neq 0$. The idempotents e^* and f^* are unique with these properties, and by Lemma 14, $(eab)^*$ is determined by e^* and $(abf)^*$ is determined by f^*. Let g^* be the right unit of $(eab)^*$, where $g = g^2 \in P$. Then

$$(eab)^* = (eab)^* g^* = (eabg)^* = e^*(abg)^*,$$

since * is a homomorphism. Thus $(abg)^* \neq 0$; whence, from the uniqueness of f^*, $f^* = g^*$ and, by Lemma 14, $(abg)^* = (abf)^*$. Similarly it follows that e^* is the left unit of $(abf)^*$. It follows that

$$(eab)^* = (eab)^* f^* = (eabf)^* = (ea)^*(bf)^* \neq 0;$$

and so $(ea)^* \neq 0$ and $(bf)^* \neq 0$. Let h^* be the right unit of $(ea)^*$ and let k^* be the left unit of $(bf)^*$. Then

$$(eab)^* = (ea)^*(bf)^* = (ea)^* h^* k^* (bf)^* \neq 0;$$

and so $h^* = k^*$.

If also $bc \notin V_c$ then it similarly follows that there is a unique idempotent m^* in $(P/V)/\pi$, with $m = m^2 \in P$, such that $(bcm)^* \neq 0$ and such that m^* is the right unit of $(bcm)^*$ and also of $(cm)^*$. Since f^* is the right unit of $(bf)^*$, it follows that f^* is the left unit of $(cm)^*$. We now have

$$(ea)^*(bf)^*(cm)^* \neq 0$$

in $(P/V)/\pi$. Hence

$$\begin{aligned}(eabc)^*m^* &= (eabcm)^* = (eab)^*(cm)^*\\ &= (ea)^*(bf)^*(cm)^* \neq 0,\end{aligned}$$

and so $(eabc)^* \neq 0$. Consequently, $abc \notin V_c$. This contradicts our original assumption. Hence either $ab \in V_c$ or $bc \in V_c$, as required.

We now show that $x\pi \to x\pi_c$ is an isomorphism of $(P/V)/\pi$ onto $(P_c/V_c)/\pi_c$. The mapping is clearly a homomorphism into. To show that it is one-to-one consider x_1, x_2 in $P\setminus V$ such that $x_1\pi_c = x_2\pi_c$, i.e. such that there exists an element a, say, in P_c/V_c, such that $x_1a = x_2a \neq 0$. Since $x_1a \notin V_c$, there exists an idempotent f in P, such that $(x_1af)^* \neq 0$. Thus $x_1(af) = x_2(af) \neq 0$, whence, since $af \in P$, $x_1\pi = x_2\pi$.

It remains to show that the mapping is onto. Let y be any element of P_c/V_c. If $y = V_c$, then $y\pi_c = 0\pi_c = 0\pi$, with the usual varying interpretation of 0. If $y \in P_c\setminus V_c$, then there exists an idempotent f in P such that $(yf)^* \neq 0$. Then $yf \in P\setminus V$ and $(yf)f = yf$ implies that $y\pi_c = (yf)\pi_c$. Since $yf \in P\setminus V$ this shows that the mapping $x\pi \to x\pi_c$ is onto.

To see, finally, that P_c is 1-complete in S, let $x \in L_1(P_c)$. If x is 0-linked to P_c, then x is 0-linked to P and so $x \in V_c \subseteqq P_c$. If x is 1-linked to P_c then there exists an idempotent f in $P_c\setminus V_c$ such that $(xf)\pi_c \neq 0$ and the idempotent $f\pi_c$ of $(P_c/V_c)/\pi_c$ so determined is unique. Because of the isomorphism established already between $(P/V)/\pi$ and $(P_c/V_c)/\pi_c$, there is an idempotent g, say, in P such that $g\pi_c = f\pi_c$. Then

$$\begin{aligned}(xg)\pi_c &= (xg^2)\pi_c = (xg)\pi_c \cdot g\pi_c\\ &= (xg)\pi_c \cdot f\pi_c\\ &= (xgf)\pi_c = (xfg)\pi_c\\ &= (xf)\pi_c \cdot g\pi_c = (xf)\pi_c \cdot f\pi_c\\ &= (xf^2)\pi_c = (xf)\pi_c.\end{aligned}$$

Hence, using the isomorphism again, since $xg \in P$, $(xg)\pi = (xg)^* \neq 0$.

Suppose, conversely, that h is an idempotent of P such that $(xh)^* \neq 0$. Then, because of the isomorphism between $(P/V)/\pi$ and $(P_c/V_c)/\pi_c$, $(xh)\pi_c \neq 0$, whence it follows that $h\pi_c = f\pi_c$. Consequently, $h\pi = g\pi$, i.e. $h^* = g^*$; and this proves that x is 1-linked to P, i.e. that $x \in P_c$. Thus P_c is 1-complete in S modulo V_c.

6. The primitive component of a representation

Let $S = S^0$ be an inverse semigroup and let Γ be a proper representation of S of degree n over the field Φ (see § 3). Since a proper representation is non-null, $\Gamma(S)$ contains matrices other than the zero matrix $\Gamma(0) = 0$.

Let $r = r(\Gamma)$ be the minimal rank of the non-zero elements of $\Gamma(S)$ and let $R = \{x \in \Gamma(S)\colon \text{rank of } x \leqq r(\Gamma)\}$.

LEMMA 17. *R is a non-zero primitive ideal of $\Gamma(S)$.*

PROOF. R is clearly an ideal of $\Gamma(S)$. We need to prove that each non-zero idempotent of R is primitive.

Let e, f be non-zero idempotents with e in R and f in $\Gamma(S)$ and suppose that $ef = fe = f$. Thus also $f \in R$ and rank e = rank $f = r$. Let us regard, as in the proof of Lemma 8, the elements of $\Gamma(S)$ as linear transformations of the vector space W. Set $We = E$ and $Wf = F$. Then

$$F = Wf = Wfe \subseteqq We = E,$$

whence since the dimension of F equals that of E $(= r)$, $F = E$.

Let $w \in W$. Then $we \in E = F$. Since f is the identity transformation on F, therefore $(we)f = we$, i.e. $w(ef) = we$. But $ef = f$, by assumption. Hence $wf = we$. This shows that $e = f$, whence e is primitive; which completes the proof of the lemma.

The above lemma enables us to apply Lemma 10 to infer that $\Gamma(S)$ contains a unique maximal primitive ideal, consisting of the ideal generated by all the primitive idempotents of $\Gamma(S)$ (Theorem 12). Throughout this section we shall denote *this maximal primitive ideal by $P^* = P^*(\Gamma)$*. Simple examples show that P^* may contain R properly. Further we define

$$P = P(\Gamma) = \{x \in S : \Gamma(x) \in P^*\},$$

and, following Munn [1], we set

$$V = V(\Gamma) = \{x \in S : \Gamma(x) = 0\},$$

V being the *vanishing ideal* of the representation Γ. Since P^* is an ideal of $\Gamma(S)^*$, P is an ideal of S and clearly contains the ideal V.

We shall show that Γ decomposes into representations Γ_P and Δ_P determined by P and that, denoting by Q the vanishing ideal of Δ_P, Q contains P, P contains V properly, P/V is categorical at zero, P is 1-complete in Q and $(P/V)/\pi$ is of finite rank, where π (see below) has its earlier meaning. Γ_P will be shown to be determined by the restriction of Γ to P, and will be termed the primitive component of Γ. We establish these results in a series of lemmas.

That P contains V properly follows from the fact that P^* is non-zero. Moreover, $\Gamma(P) = P^*$ and this homomorphism Γ of P onto P^* induces a 0-restricted homomorphism of P/V onto P^*. Consequently, since P^* is primitive, Theorem 5 gives that P/V is categorical at zero. Hence, with the notation of the previous section, there is a 0-restricted congruence π, say, on P/V, finest among the congruences on P/V which give a primitive quotient. Equation (1) of Theorem 5, taking S to be P/V, defines π.

LEMMA 18. *Let $x, y \in P \setminus V$ and $(x, y) \in \pi$. Then $\Gamma(x) = \Gamma(y)$.*

PROOF. By the definition of π there exists a in P such that $ax = ay \notin V$. Thus $\Gamma(ax) = \Gamma(ay) \neq 0$, i.e. $\Gamma(a)\Gamma(x) = \Gamma(a)\Gamma(y) \neq 0$ in P^*. But P^* is primitive. Hence $\Gamma(x) = \Gamma(y)$.

Denote by $x \to x^* (x \in P)$ the result of following the natural mapping of P onto P/V by that of P/V onto $(P/V)/\pi$. Because of Lemma 18, we may define a representation Γ^* of $(P/V)/\pi$ by $\Gamma^*(x^*) = \Gamma(x)$, $(x \in P)$. We state as a lemma, for later use, the fact that this equation also serves to define $\Gamma(x)$ for x in P.

LEMMA 19. *Γ restricted to P is determined by Γ^*: $\Gamma(x) = \Gamma^*(x^*)$, $(x \in P)$.*

PROOF. This is merely a rephrasing of the preceding lemma. For we merely have to show that $x^* = y^*$ implies that $\Gamma(x) = \Gamma(y)$; and this follows directly from Lemma 18.

LEMMA 20. *$(P/V)/\pi$ is of finite rank (equal to that of P^*). Γ^* maps the set of Brandt semigroup summands of $(P/V)/\pi$ in a one-to-one fashion onto the set of Brandt semigroup summands of P^*.*

PROOF. Γ^* is a homomorphism of $(P/V)/\pi$ onto P^*. By Theorem 9, P^* is of finite rank. We shall show that $(P/V)/\pi$ is of rank equal to that of P^*.

Observe first that $\Gamma^*(x^*) = 0$ if and only if $x^* = 0$ in P/V, i.e. if and only if $x \in V$, because of the definition of Γ^* and because V is the vanishing ideal of Γ. Hence Γ^* induces a non-null representation on each of the Brandt semigroup summands of $(P/V)/\pi$ (cf. Corollary 2). By a result in [8] the rank of a non-trivial homomorphic image of a Brandt semigroup B is the same as the rank of B. Hence each Brandt semigroup of $(P/V)/\pi$ is of the same finite rank as its image under Γ^*.

Let B_1 and B_2 be two distinct Brandt semigroup summands of $(P/V)/\pi$. Let e_1 and e_2 be non-zero idempotents in B_1 and B_2, respectively. Then $e_1 e_2 = 0$. Hence $\Gamma^*(e_1) \cdot \Gamma^*(e_2) = \Gamma^*(e_1 e_2) = 0$. Consequently, since $\Gamma^*(e_1)$ and $\Gamma^*(e_2)$ are each non-zero idempotents, $\Gamma^*(e_1) \neq \Gamma^*(e_2)$. It follows that $\Gamma^*(B_1) \cap \Gamma^*(B_2) = 0$; whence distinct Brandt semigroup summands of $(P/V)/\pi$ are mapped by Γ^* onto distinct Brandt semigroup summands of P^*. This suffices to complete the proof of the lemma.

Choose idempotents e_i, $i = 1, 2, \cdots, t$ in P, so that $e_i^* \neq e_j^*$ if $i \neq j$ and so that $e_1^*, e_2^*, \cdots, e_t^*$ are the nonzero idempotents of $(P/V)/\pi$, where we suppose that the rank of $(P/V)/\pi$, finite by Lemma 20, is t. Then, setting $e = \sum_{i=1}^{t} e_i$, so that e is an element of $\Phi_0[P]$, and regarding Γ as extended to $\Phi_0[S]$ in the usual way,

$$\Gamma(e) = \sum_{i=1}^{t} \Gamma(e_i) = \sum_{i=1}^{t} \Gamma^*(e_i^*).$$

$\Gamma(e)$ is then an identity for $\Gamma(P)$:

$$\Gamma(e)\Gamma(x) = \Gamma(x) = \Gamma(x)\Gamma(e)$$

for $x \in P$; although of course, in general, $\Gamma(e) \notin \Gamma(P)$. Since $\Gamma(e)$ is idempotent, there is a representation $\Gamma^\dagger$, say, of S, equivalent to Γ such that

$$\Gamma^\dagger(e) = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix},$$

where I_m denotes the $m \times m$ identity matrix and where the matrix on the right is partitioned according to $n = m + (n-m)$. Since our aim is to determine Γ to within equivalence, we *may assume that* $\Gamma^\dagger = \Gamma$.

From the fact that $\Gamma(e)$ is an identity for $\Gamma(P)$ the next lemma now follows.

LEMMA 21. *For* $x \in P$,

$$\Gamma(x) = \begin{pmatrix} x_P & 0 \\ 0 & 0 \end{pmatrix},$$

where x_P *is an* $m \times m$ *matrix and the matrix on the right is partitioned according to* $n = m + (n-m)$. (*Note the above assumption about the replacement of* Γ, *if necessary, by an equivalent representation.*) *The mapping* $x \to x_P$ *is a representation of degree* m *of* P *over* Φ.

We can now define Γ_P:

$$\Gamma_P(x) = (xe)_P,\ x \in S.$$

Here the representation $x \to x_P$, $x \in P$, of Lemma 21, has been extended in the usual way to a representation of $\Phi_0[P]$; and this gives $(xe)_P$ a meaning.

LEMMA 22. Γ_P *is a proper representation of* S *of degree* m *over* Φ. *The equation*

$$\Gamma_P(x) = (ex)_P,\ x \in S$$

may also be used to define Γ_P.

PROOF. Let $x, y \in S$. Then

$$\begin{aligned} \Gamma_P(x) \cdot \Gamma_P(y) &= (xe)_P(ye)_P \\ &= I_m(xe)_P(ye)_P \\ &= e_P(xe)_P(ye)_P \\ &= (exe)_P(ye)_P \\ &= (ex)_P e_P(ye)_P \\ &= (ex)_P(ye)_P \\ &= (exye)_P \\ &= e_P(xye)_P \\ &= (xye)_P \\ &= \Gamma_P(xy), \end{aligned}$$

using frequently the fact that $a \to a_P$ is a homomorphism of $\Phi_0[P]$.

Thus Γ_P is a representation of S, clearly of degree m. Γ_P is proper because $\Gamma_P(e) = I_m$.

It remains to show that $(ex)_P = (xe)_P$ for $x \in S$. This follows because

$$\begin{aligned}(ex)_P &= (ex)_P I_m = (ex)_P e_P \\ &= (exe)_P = e_P(xe)_P \\ &= I_m(xe)_P = (xe)_P.\end{aligned}$$

This completes the proof of the lemma.

We now reinterpret Lemma 19 in the form that we require it: *Since Γ_P is determined by the restriction of Γ to P, Lemma* 19 *implies that Γ_P is uniquely determined by Γ^*, the induced representation of $(P/V)/\pi$.*

LEMMA 23. *Γ_P is a component of Γ: for x in S, if $n > m$, we have*

$$\Gamma(x) = \begin{pmatrix} \Gamma_P(x) & 0 \\ 0 & \Delta_P(x) \end{pmatrix},$$

where Δ_P is a proper representation of S of degree $n-m$ over Φ.

PROOF. If $n = m$ then $\Gamma_P = \Gamma$ is trivially a component of Γ. Suppose that $n > m$ and, for x in S, write

$$\Gamma(x) = \begin{pmatrix} \Gamma_{11}(x) & \Gamma_{12}(x) \\ \Gamma_{21}(x) & \Gamma_{22}(x) \end{pmatrix},$$

where the rows and columns of the matrix on the right are partitioned according to the partition $n = m+(n-m)$ of n. Then

$$\begin{aligned}\Gamma(x)\Gamma(e) &= \Gamma(x) \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \Gamma_{11}(x) & 0 \\ \Gamma_{21}(x) & 0 \end{pmatrix}.\end{aligned}$$

But $\Gamma(x)\Gamma(e) = \Gamma(xe)$ and $xe \in \Phi_0[P]$. Hence, by Lemma 21,

$$\Gamma(x)\Gamma(e) = \begin{pmatrix} (xe)_P & 0 \\ 0 & 0 \end{pmatrix},$$

where $(xe)_P$ is an $m \times m$ matrix. Comparing these two equations shows that $\Gamma_{21}(x) = 0$. Similarly, by Lemma 21,

$$\Gamma(e)\Gamma(x) = \Gamma(ex) = \begin{pmatrix} (ex)_P & 0 \\ 0 & 0 \end{pmatrix}$$

from which we infer $\Gamma_{12}(x) = 0$.

Setting $\Gamma_{22}(x) = \Delta_P(x)$, the result of the lemma follows, Δ_P being proper because Γ is proper.

We call Γ_P the *primitive component* of Γ.

We shall now determine the vanishing ideal of Δ_P. Define

$$Q = Q(\Gamma) = (\Gamma^{-1}(\Gamma(\Phi_0[P]))) \cap S.$$

LEMMA 24. *$Q = V(\Delta_P)$, the vanishing ideal of Δ_P.*

PROOF. Let $y \in Q$ so that $\Gamma(y) \in \Gamma(\Phi_0[P])$. Thus

$$\Gamma(y) = \sum \alpha_i \Gamma(x_i)$$

for some finite sum, where $\alpha_i \in \Phi$ and $x_i \in P$. From the definition of Δ_P and from Lemma 21, $\Delta_P(x) = 0$ if $x \in P$. Hence $\Delta_P(y) = 0$.

Conversely, suppose that $\Delta_P(y) = 0$ for y in S. Then

$$\begin{aligned} \Gamma(y) &= \begin{pmatrix} \Gamma_P(y) & 0 \\ 0 & \Delta_P(y) \end{pmatrix} \\ &= \begin{pmatrix} \Gamma_P(y) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \Gamma_P(y) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \\ &= \Gamma(y)\Gamma(e) \\ &= \Gamma(ye). \end{aligned}$$

But $ye \in \Phi_0[P]$. Hence $y \in Q$.

This completes the proof of the lemma.

LEMMA 25. *P is contained in the ideal Q and P is 1-complete in Q modulo V.*

PROOF. Suppose that $y \in Q$ and that $y \in L_1(P)$. Thus there exists at most one idempotent f^*, say, of P^*, such that $(yf)^* \neq 0$, where, when there is such an idempotent, we may take f to be one of the idempotents $e_1, e_2, \cdots, e_t$ of P chosen as before, so that $e = \sum e_i$ and $e_1^*, e_2^*, \cdots, e_t^*$ are the non-zero idempotents of $(P/V)/\pi$. By Lemma 24, $\Delta_P(y) = 0$ and hence

$$\begin{aligned} \Gamma(y) = \begin{pmatrix} \Gamma_P(y) & 0 \\ 0 & 0 \end{pmatrix} &= \Gamma(y)\Gamma(e) \\ &= \Gamma(ye) \\ &= \sum_{i=1}^{t} \Gamma(ye_i) \\ &= \sum_{i=1}^{t} \Gamma^*(ye_i)^*, \end{aligned}$$

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from the definition of Γ^*. But at most one of the $(ye_i)^*$ is non-zero. Hence, either $\Gamma(y) = 0$ or $\Gamma(y) = \Gamma(yf)$, where $f \in P$. In the former event, $y \in V$. In the latter event $\Gamma(y) \in P^*$, whence $y \in P$, since $P = \Gamma^{-1}(P^*)$. Hence in both cases $y \in P$; which completes the proof of the lemma.

Finally we wish to show that $V(\Gamma_P) = V$. The proof turns on the following well-known result on matrices.

LEMMA 26. *Let h and k be idempotent matrices in $(\Phi)_n$. Suppose that $h \leqq k$, i.e. that $hk = kh = h$. Then, if $h \neq k$, the rank of h is less than the rank of k.*

LEMMA 27. $V(\Gamma_P) = V$.

PROOF. Clearly $V \subseteqq V(\Gamma_P)$. Let $x \in V(\Gamma_P)$ so that

$$\Gamma(x) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_P(x) \end{pmatrix}.$$

From $x^{-1}xx^{-1} = x^{-1}$ and $\Gamma_P(x) = 0$ we infer $\Gamma_P(x^{-1}) = 0$. Hence

$$\Gamma(x^{-1}) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_P(x^{-1}) \end{pmatrix}$$

and

$$\Gamma(xx^{-1}) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_P(xx^{-1}) \end{pmatrix}.$$

$\Gamma(xx^{-1})$ is an idempotent of $\Gamma(S)$. If it is non-zero, from Lemma 26, there is a non-zero primitive idempotent of $\Gamma(S)$ under it. Let h be such a primitive idempotent. By the definition of P^*, $h \in P^*$ and h is one of the matrices

$$\begin{pmatrix} \Gamma_P(e_i) & 0 \\ 0 & 0 \end{pmatrix},$$

$i = 1, 2, \cdots, t$. Hence $h\Gamma(xx^{-1}) = 0$. This contradicts the assumption that $h\Gamma(xx^{-1}) = h \neq 0$. Consequently $\Gamma(xx^{-1}) = 0$ and so $\Gamma(x) = 0$, i.e. $x \in V$.

This completes the proof of the lemma.

COROLLARY 28. *P is 0-complete in S modulo V.*

PROOF. Let x be 0-linked to P modulo V. Then $(xe_i)^* = 0$, i.e. $xe_i \in V$ for $i = 1, 2, \cdots, t$. Hence $\Gamma(xe_i) = 0$ and so $\Gamma(xe) = 0$, where

$$e = e_1 + e_2 + \cdots + e_t.$$

Thus, by definition, $\Gamma_P(x) = 0$. Hence, by the lemma, $x \in V$ and so $x \in P$.

We collect our results together in the following theorem.

THEOREM 29. *Let $S = S^0$ be an inverse semigroup and let Γ be a proper* *representation of S of degree n over the field* Φ. *Then* $\Gamma(S)$ *contains a unique maximum primitive ideal* P^*, *say. Set* $P = \Gamma^{-1}(P^*) \cap S$. *Then P is an ideal of S properly containing the vanishing ideal* $V = V(\Gamma)$ *of* Γ. *Moreover* P/V *is categorical at zero and* $(P/V)/\pi$ *the maximum* 0*-restricted primitive homomorphic image of* P/V *has finite rank equal to that of* P^*. *Setting* $Q = (\Gamma^{-1}(\Gamma(\Phi_0[P]))) \cap S$, *Q is an ideal of S containing P, P is* 1*-complete in Q modulo V, and P is* 0*-complete in S modulo V.*

Let $e_1, e_2, \cdots, e_t$ *be idempotents of P, where t is the rank of* $(P/V)/\pi$, *chosen so that* $\{e_1^*, e_2^*, \cdots, e_t^*\}$ *is the set of distinct non-zero idempotents of* $(P/V)/\pi$, *where* $x \to x^* (x \in P)$ *is the natural mapping of P onto* $(P/V)/\pi$. *Set*

$$e = \sum_{i=1}^{t} e_i.$$

Then Γ *is equivalent to a representation of S such that*

$$\Gamma(e) = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix},$$

where I_m *is the* $m \times m$ *identity matrix over* Φ *and* $m \leqq n$. *If* $m = n$, $\Gamma(x) = \Gamma(xe)$ *for all x in S. In this event we define* $\Gamma_P = \Gamma$. *Otherwise, if* $m < n$, *for all x in S,* $\Gamma(xe)$ *has the form*

$$\Gamma(xe) = \begin{pmatrix} \Gamma_P(x) & 0 \\ 0 & 0 \end{pmatrix},$$

where Γ_P *is an* $m \times m$ *matrix; and taking this equation as defining* $\Gamma_P(x)$,

$$\Gamma(x) = \begin{pmatrix} \Gamma_P(x) & 0 \\ 0 & \Delta_P(x) \end{pmatrix}.$$

Γ_P *is then a proper representation of S of degree m over* Φ, *the primitive component of* Γ. *V is the vanishing ideal of* Γ_P *and the representation* Γ^*, *say, induced on* $(P/V)/\pi$ *by* Γ_P *is* 0*-restricted and proper. Moreover* Γ_P *is uniquely determined by* Γ^* *by the equation*

$$\Gamma_P(x) = \Gamma^*(xe)^*.$$

The component Δ_P *of* Γ *is a proper representation of S of degree* $n-m$ *over* Φ, *and Q is its vanishing ideal.*

7. Representation ideal series

Let Γ be a proper representation of degree n over the field Φ of the inverse semigroup $S = S^0$. Let Γ_P be the primitive component of Γ. Then, either $\Gamma = \Gamma_P$ or, to within equivalence,

$$\Gamma(x) = \begin{pmatrix} \Gamma_P(x) & 0 \\ 0 & \Delta_P(x) \end{pmatrix},$$

as in Theorem 29. It will be convenient now to call Γ_P the *first primitive component of* Γ. We may now apply Theorem 29 to the representation Δ_P of S, decomposing Δ_P into its primitive component, which we shall call the *second primitive component of* Γ and, if Δ_P is not equal to its primitive component, into a further component. Theorem 29 may be applied to this latter component; and so on, the process terminating, after a finite number r, say, of steps, when we reach a component equal to its own primitive component, this primitive component then being called the *r-th primitive component of* Γ.

Corresponding to this decomposition of Γ there is an ideal series of S,

$$(2) \qquad 0 \subseteqq V_1 \subset P_1 \subseteqq V_2 \subset P_2 \subseteqq \cdots \subseteqq V_r \subset P_r \subseteqq V_{r+1} = S,$$

of length $2r+1$, obtained as follows. Firstly we write V_1 and P_1 for V and P, respectively, of our earlier notation. The first primitive component of Γ is then Γ_{P_1}. For Q of our earlier notation we write V_2 so that V_2 is now the vanishing ideal of Δ_{P_1}. Γ_{P_2} then vanishes on V_2 and is the primitive component of Δ_{P_1}, i.e. the second primitive component of Γ; and so on. Applying Theorem 29 to each representation Δ_{P_i}, writing $\Gamma = \Delta_{P_0}$, it follows that the above ideal series satisfies the following conditions, each of which holds for $i = 1, 2, \cdots, r$.

R1. P_i/V_i is categorical at zero.

R2. Denoting by π_i the finest 0-restricted congruence on P_i/V_i from the set of such congruences determining a primitive quotient, $(P_i/V_i)/\pi_i$ is of finite rank.

R3. P_i is 0-complete in S modulo V_i.

R4. P_i is 1-complete in V_{i+1} modulo V_i.

We shall call an ideal series of the form (2) of length $2r+1$ and satisfying conditions R1—R4, for $i = 1, 2, \cdots, r$, a *representation ideal series for* S.

The representation ideal series (2) which was determined by Γ will be called the *representation ideal series of* Γ.

The i-th primitive component Γ_{P_i} of Γ is, by Theorem 29, determined by the representation Γ_i^*, say, induced in $(P_i/V_i)/\pi_i$ by Γ. Hence, since Γ decomposes into the Γ_{P_i}, Γ is determined, to within equivalence, by the representation ideal series (2) and by the Γ_i^*, $i = 1, 2, \cdots, r$. We now consider the converse and show that a representation ideal series (2) together with 0-restricted representations of its associated $(P_i/V_i)/\pi_i$ determine in a unique fashion a representation Γ with (2) as its representation ideal series.

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THEOREM 30. *Let $S = S^0$ be an inverse semigroup and let* (2) *be a representation ideal series for S. Let Γ_i^* be a 0-restricted proper representation of $(P_i/V_i)/\pi_i$, of degree m_i over the field Φ, for $i = 1, 2, \cdots, r$. Denote by $x \to x^*$ the natural homomorphism P_i onto $(P_i/V_i)/\pi_i$. (Using the same notation for each of these r homomorphisms will lead to no ambiguity, for in each case only one interpretation will be possible.) Let t_i be the rank of $(P_i/V_i)/\pi_i$ and let*

$$e_i^j, j = 1, 2, \cdots, t_i$$

be idempotents of $P_i \setminus V_i$, for $i = 1, 2, \cdots, r$, such that $(e_i^j)^*, j = 1, 2, \cdots, t_i$, are the t_i distinct non-zero idempotents of $(P_i/V_i)/\pi_i$. For each i, set

$$e_i = \sum \{e_i^j : j = 1, 2, \cdots, t_i\},$$

an element of $\Phi_0[P_i]$.

For $x \in S$ define $\Gamma_i(x)$ thus:

$$\Gamma_i(x) = \Gamma_i^*(xe_i)^*,$$

*where Γ_i^* and the homomorphism * are regarded as extended in the natural way to $\Phi_0[(P_i/V_i)/\pi_i]$ and $\Phi_0[P_i]$, respectively. Now define Γ, thus:*

$$\Gamma(x) = \begin{pmatrix} \Gamma_1(x) & 0 & \cdot & \cdot & \cdot \\ 0 & \Gamma_2(x) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \Gamma_r(x) \end{pmatrix},$$

for $x \in S$.

Then Γ is a proper representation of S of degree $n = m_1 + m_2 + \cdots + m_r$, (2) *is the representation ideal series of Γ and Γ_i is the i-th primitive component of Γ for $i = 1, 2, \cdots, r$.*

PROOF. It will follow that $\Gamma(x)\Gamma(y) = \Gamma(xy)$ once we show that $\Gamma_i(x)\Gamma_i(y) = \Gamma_i(xy)$ for $i = 1, 2, \cdots, r$.

By Lemma 8, $\Gamma_i^*(e_i)^*$ is the identity matrix I_{m_i}. Hence from the definition of Γ_i we have (cf. the proof of Lemma 22)

$$\begin{aligned} \Gamma_i(x)\Gamma_i(y) &= \Gamma_i^*(xe_i)^*\Gamma_i^*(ye_i)^* \\ &= I_{m_i}\Gamma_i^*(xe_i)^*\Gamma_i^*(ye_i)^* \\ &= \Gamma_i^*(e_i)^*\Gamma_i^*(xe_i)^*\Gamma_i^*(ye_i)^* \\ &= \Gamma_i^*(e_i^*(xe_i)^*)\Gamma_i^*(ye_i)^* \\ &= \Gamma_i^*((e_ix)^*e_i^*)\Gamma_i^*(ye_i)^* \\ &= \Gamma_i^*(e_ix)^*\Gamma_i^*(e_i)^*\Gamma_i^*(ye_i)^* \\ &= \Gamma_i^*(e_ix)^*\Gamma_i^*(ye_i)^* \\ &= \Gamma_i^*((e_ix)^*(ye_i)^*) \end{aligned}$$

$$\begin{aligned} &= \Gamma_i^*(e_i xy e_i)^* \\ &= \Gamma_i^*(e_i)^* \Gamma_i^*(xye_i)^* \\ &= \Gamma_i^*(xye_i)^* \\ &= \Gamma_i(xy). \end{aligned}$$

Consequently Γ is a representation of S, clearly of degree n. Since, for each i, $\Gamma_i(e_i) = I_{m_i}$, Γ is proper.

Let V be the vanishing ideal of Γ. Then, if $x \in V$, in particular, $\Gamma_1(x) = 0$, i.e. $\Gamma_1^*(xe_1)^* = 0$. Hence $\Gamma_1^*(xe_1)^* \Gamma_1^*(e_j^1)^* = 0$, i.e.

$$\Gamma_1^*((xe_1)^*(e_1^j)^*) = \Gamma_1^*(xe_1e_1^j)^* = \Gamma_1^*(x_1e_1^j)^* = 0.$$

Since Γ_1^* is 0-restricted, therefore $(x_1e_1^j)^* = 0$ for $j = 1, 2, \cdots, t_1$. Hence x is 0-linked to P_1 modulo V_1. From condition R3, which the series (2) satisfies, therefore $x \in V_1$. Thus $V \subseteqq V_1$.

Conversely, let $x \in V_1$. Then $x \in V_i$ for $i = 1, 2, \cdots, r$. Hence $(xe_i)^* = 0$ for $i = 1, 2, \cdots, r$. Thus $\Gamma_i(x) = 0$ for $i = 1, 2, \cdots, r$, and so $\Gamma(x) = 0$. Hence $V_1 \subseteqq V$. Combined with the earlier inequality this gives $V = V_1$.

Let P^* be the unique maximal primitive ideal of $\Gamma(S)$ and let $P = \Gamma^{-1}(P^*) \cap S$. We shall show that $P = P_1$.

Let us show firstly that $\Gamma(e_1^j)$, for $j = 1, 2, \cdots, t_1$, are primitive non-zero idempotents of $\Gamma(S)$. It is clear that $\Gamma(e_1^j) \neq 0$; for $(e_1^j)^* \neq 0$ and Γ_1^* is 0-restricted. Suppose that $\Gamma(f)$ is a non-zero idempotent of $\Gamma(S)$ under $\Gamma(e_1^j) : \Gamma(f)\Gamma(e_1^j) = \Gamma(f) \neq 0$. If $i > 1$, then $e_1^j e_i \in \Phi_0[P_1] \subseteqq \Phi_0[V_i]$ and so $\Gamma_i^*(e_1^j e_i)^* = 0$. Hence the equation $\Gamma(f)\Gamma(e_1^j) = \Gamma(f)$ implies that $\Gamma_i^*(fe_i)^* = 0$ for $i > 1$. We shall show that $\Gamma_1^*(fe_1)^* = \Gamma_1^*(e_1^j)^*$.

Since $\Gamma_i^*(e_1^j e_i)^* = 0$, if $i > 1$, and the $\Gamma_1^*(e_1^j)^*$ are t_1 distinct primitive non-zero idempotents of $\Gamma_1^*((P_1/V_1)/\pi_1)$, therefore $\Gamma(e_1^j)\Gamma(e_1^k) = 0$, if $j \neq k$. Hence the equation $\Gamma(f)\Gamma(e_1^j) = \Gamma(f) \neq 0$ holds for precisely one j. Hence $\Gamma_1^*(fe_1^j e_1)^* = \Gamma_1^*(fe_1^j)^* \neq 0$, i.e., since Γ_1^* is 0-restricted, $(fe_1^j)^* \neq 0$ for precisely one j. Thus f is 1-linked to P_1 modulo V_1. If $r = 1$, then $f \in V_2 = S$. If $r > 1$, then from $\Gamma_2^*(fe_2)^* = 0$ we infer that $(fe_2)^* = 0$ and so f is 0-linked to P_2 modulo V_2. By condition R3, therefore $f \in P_2$ and so $f \in V_2$. We now have f 1-linked to P_1 modulo V_1 and $f \in V_2$; hence by condition R4, $f \in P_1$.

It now follows that $\Gamma_1^*(fe_1)^*$ is one of the non-zero idempotents of $\Gamma_1^*((P_1/V_1)/\pi_1)$. Hence $\Gamma_1^*(fe_1)^* = \Gamma_1^*(e_1^j)$. Consequently, $\Gamma(f) = \Gamma(e_1^j)$; and this completes the proof that each $\Gamma(e_1^j)$ is primitive in $\Gamma(S)$.

We now return to proving $P = P_1$. Let $x \in P_1$. Then, setting $f = xx^{-1}$, $\Gamma(f) = 0$ or $\Gamma(f) = \Gamma(e_1^j)$, for some j, and so, as we have shown, $\Gamma(f)$ is primitive in $\Gamma(S)$, i.e. $\Gamma(f) \in P^*$. Hence $f \in P$; whence $x = fx \in P$. Thus $P_1 \subseteqq P$.

Conversely, let $x \in P$. Then, if $f = xx^{-1}$, $\Gamma(f) = 0$ or $\Gamma(f)$ is primitive, by the definition of P^*. If $\Gamma(f) = 0$, then $f \in V = V_1 \subseteqq P_1$. If $\Gamma(f) \neq 0$, then either $\Gamma(f) = \Gamma(e_1^j)$ for some j, or $\Gamma(f)$ is different from all the $\Gamma(e_1^j)$. In the former event, $\Gamma_2^*(fe_2)^* = 0$ whence we infer, as before, that $f \in V_2$. Then, again as in an earlier argument, $\Gamma_1^*(fe_1)^* = \Gamma_1^*(e_1^j)^*$ ensures that f is 1-linked to P_1 modulo V_1; whence we infer, from condition R4, that $f \in P_1$.

If $\Gamma(f)$ is not equal to any of the $\Gamma(e_1^j)$, then since $\Gamma(f)$ is primitive, $\Gamma(f)\Gamma(e_1^j) = 0$ for all j. Thus $\Gamma(fe_1) = 0$, whence $\Gamma_1^*(fe_1)^* = 0$. This implies that f is 0-linked to P_1 modulo V_1. From condition R3, therefore $f \in P_1$ and so $f \in V_1$. Hence, $\Gamma(f) = 0$, contrary to assumption.

Hence, in all cases $f \in P_1$; whence $x = fx \in P_1$. Thus $P \subseteqq P_1$; whence $P = P_1$.

What we have shown so far suffices to show that Γ_1 is the first primitive component of Γ. We have $\Gamma = \Gamma_1 \oplus \Delta_1$, say, where $\Delta_1 = \Delta_{P_1}$, in our earlier notation. Now apply the argument we have just applied to Γ and to S instead to Δ_1 and to S. The conditions upon the ideal series (2) then ensure that V_2 is the vanishing ideal of Δ_1, that Γ_2 is the first primitive component of Δ_1 and that $P_2 = \Delta_1^{-1}(P_2^*) \cap S$, where P_2^* denotes the unique maximal primitive ideal of $\Delta_1(S)$; and so on.

Hence, for each i, Γ_i is the i-th primitive component of Γ and the series (2) is the representation ideal series of Γ.

This completes the proof of the theorem.

When the series (2) is a representation ideal series let us call the semigroups $(P_i/V_i)/\pi_i$, $i = 1, 2, \cdots, r$, the *primitive factors* of the series. From the remarks preceding Theorem 30 and from Theorem 30, it follows that any proper representation of S determines a representation ideal series and 0-restricted proper representations of its primitive factors, and that, in turn, these representations of the primitive factors determine the primitive components of the original representation and so determine this representation to within equivalence. And, conversely, starting with a given representation ideal series, and assigning 0-restricted proper representations to each of its primitive factors, we thereby determine in a unique fashion a proper representation of the whole semigroup with the given representation ideal series as its representation ideal series and determining in turn the given assigned representations of its primitive factors.

The one-to-one relationship we have thus established between proper representations of S and 0-restricted proper representations of the primitive factors of a representation ideal series is a relationship which involves a particular means of constructing, to within equivalence, each from the other. The wider question of characterizing equivalent proper representations in terms of their representation ideal series and representations of their primitive factors we treat in a later section. As a preliminary to this analysis we examine in the next section, more closely, the representations of the semigroups P_i/V_i.

What we have proved shows that an inverse semigroup $S = S^0$ admits a proper representation if and only if it possesses a representation ideal series. We complete this section by some comments on the construction of such series.

Isolating the portion

$$0 \subseteqq V_r \subset P_r \subseteqq V_{r+1} = S$$

of the representation ideal series (2), and observing that this is itself a representation ideal series, it follows that S possesses representation ideal series if and only if it contains ideals V and P such that

$$0 \subseteqq V \subset P \subseteqq S \tag{3}$$

is a representation ideal series. Theorem 16 then applies to show that such a representation ideal series can be constructed provided we can find an ideal series (3) satisfying merely the conditions that P/V is categorical at zero and $(P/V)/\pi$ is of finite rank.

Suppose that $(P/V)/\pi$ is of finite rank and that $\{B_j^* : j = 1, 2, \cdots, u\}$, say, is its set of Brandt semigroup summands. Let B_j be the set of all elements of P mapped onto B_j^* under the natural mapping of P onto $(P/V)/\pi$. Since π is 0-restricted, therefore $B_j \cap B_k \subseteqq V$ if $j \neq k$. Consequently P/V is the 0-direct union of the B_j/V, $j = 1, 2, \cdots, u$. Denote by π_j the restriction of π to B_j/V. Then $(B_j/V)/\pi_j \cong B_j^*$. Hence

$$0 \subseteqq V \subset B_j \subseteqq S$$

satisfies the two conditions: B_j/V is categorical at zero and $(B_j/V)/\pi_j$ is a Brandt semigroup of finite rank.

Munn showed in [6] that a semigroup $T = T^0$, say, has a 0-restricted homomorphic image which is a Brandt semigroup if and only if it is categorical at zero and its zero is indecomposable. Moreover Munn also showed that if T satisfies these conditions then any non-zero ideal of T satisfies these conditions and has the same (to within isomorphism) maximal 0-restricted homomorphic Brandt semigroup image (*loc. cit.*).

Taking P to be one of the B_j's or an ideal of one of the B_j's properly containing V, we therefore see that S has a representation ideal series if and only if it possesses an ideal series

$$0 \subseteqq V \subset P \subseteqq S$$

such that (i) P/V is categorical at zero and has an indecomposable zero and (ii) $(P/V)/\pi$ is (a Brandt semigroup) of finite rank.

From our earlier results we infer

THEOREM 31. *Let $S = S^0$ be an inverse semigroup. Then S possesses a* *non-trivial matrix representation over a field if and only if there exists an ideal series*

$$0 \subseteqq V \subset P \subseteqq S$$

such that P/V is categorical at zero and $(P/V)/\pi$ is of finite rank. In this condition it may also be assumed that P/V has an indecomposable zero, in which event $(P/V)/\pi$ is a Brandt semigroup of finite rank.

Munn's necessary and sufficient conditions for a 0-simple inverse semigroup to have a non-trivial matrix representation ([1], Theorem 2.6 (is) are a special case of the above result. For Brandt semigroups, as shown earlier by Clifford [2], the conditions mean that the Brandt semigroup has to have finite rank. We complete the section with a further example of a 0-simple inverse semigroup with only trivial matrix representations.

The example is the inverse semigroup S generated, as a subsemigroup of the symmetric inverse semigroup $\mathscr{I}_N$, by the two one-to-one mappings, α and β of the set N of natural numbers into itself defined thus:

$$\alpha\colon n \to 2n \ (n \in N);$$
$$\beta\colon n \to 2n+1 \ (n \in N).$$

The zero of S is the zero of $\mathscr{I}_N$, the empty mapping 0. We have $\alpha\alpha^{-1} = \iota_N$, the identical mapping of N. Hence $\iota_N \in S$. Thus $\alpha\iota_N\beta^{-1}$ is a product of elements in S. Moreover $\alpha\iota_N = \alpha$ and $\iota_N\beta^{-1} = \beta^{-1}$; but $\alpha\iota_N\beta^{-1} = \alpha\beta^{-1} = 0$. Hence S is not categorical at zero.

Once we show that S has no proper non-zero ideals, i.e. that S is 0-simple, then it will follow from the preceding theorem that S has only null matrix representations. To see that S is 0-simple, consider the elements of S. We easily see that $\alpha^i\beta^{-j} = 0$ for any positive integers i, j; and similarly (or consequently) $\beta^j\alpha^{-i} = 0$ for i, j positive. Giving the meaning ι_N to each of α^0 and β^0, the powers of each of α and β form infinite cyclic groups with ι_N as identity element. From these remarks we see that the non-zero elements of S can be reduced to two kinds (i) products of non-negative powers of α and β (ii) products of negative powers of α and β. Let

$$x = \alpha^{i_1}\beta^{j_1}\alpha^{i_2}\beta^{j_2}\cdots\alpha^{i_k}\beta^{i_k}$$

be an element of the first kind. Then its left unit is ι_N. Hence $SxS = S$.

Each element x of the second kind is an inverse of an element of the first kind. Hence $SxS = S$ for such elements also. Hence S is 0-simple-as asserted.

8. Representations of semigroups which are categorical at zero

We have seen in the previous section that the general representation of an inverse semigroup is determined by the 0-restricted representations of the semigroups P_i/V_i associated with a representation ideal series (2). These semigroups P_i/V_i are categorical at zero and such that the corresponding primitive factors $(P_i/V_i)/\pi_i$ are of finite rank. In this section we examine more closely these representations.

If S is an inverse semigroup which is categorical at zero, then we define $\beta = \beta(S)$ on S thus

(4) $\beta = \{(x, y) \in S \times S$: *there exist* a, b, z *in* S *such that* ax, az, zb *and* yb *are all non-zero*$\}$.

THEOREM 32. *Let* S *be an inverse semigroup which is categorical at zero. Define* β *by* (4).

Then the restriction of β *to* $S \setminus 0$ *is an equivalence relation on* $S \setminus 0$.

Let $\{B_j : j \in I\}$ *be such that* (i) $0 \in B_j$ *for each* j *and* (ii) $\{B_j \setminus 0 : j \in I\}$ *is the set of* β-*classes in* $S \setminus 0$.

Then each B_j *is a subsemigroup of* S *which is both categorical at zero and with an indecomposable zero. Moreover* S *is the* 0-*direct union of the* $\{B_j : j \in I\}$.

Furthermore, if T *is an ideal of* S *in which* 0 *is indecomposable, then* $T \subseteqq B_j$ *for some* $j \in I$.

Alternatively, the B_j *may be defined thus. Let* π *be defined on* S *by equation* (1) *of Theorem* 5. *Then* S/π *is the* 0-*direct union of a set of Brandt semigroups* $\{B_j^* : j \in I\}$, *say. For each* $j \in I$, *define* $B_j = B_j^*(\pi^\natural)^{-1}$.

PROOF. Define the sets B_j as in the final paragraph of the theorem. The B_j are then subsemigroups of S. Further, since π is 0-restricted, S is the 0-direct union of the $\{B_j : j \in I\}$. Hence, since $B_j \cap B_k = 0$ if $j \neq k$, $\{B_j \setminus 0 : j \in I\}$ forms a partition of $S \setminus 0$. Denote by σ the equivalence relation determining this partition. We shall show that σ is the restriction of β to $S \setminus 0$.

Let $x, y \in B_j \setminus 0$. Then $x\pi, y\pi \in B_j^* \setminus 0$. Since B_j^* is 0-bisimple (i.e. $B_j^* \setminus 0$ forms a $\mathscr{D}$-class) there exists z in $B_j \setminus 0$ such that $(x\pi, z\pi) \in \mathscr{R}$ and $(z\pi, y\pi) \in \mathscr{L}$, where $\mathscr{L}$ and $\mathscr{R}$ denote Green's equivalence relations (see [4], Chapter 2) on B_j^*. Let a, b be elements of $B_j \setminus 0$ such that $a\pi$ is an idempotent $\mathscr{R}$-equivalent to $x\pi$ and $b\pi$ is an idempotent $\mathscr{L}$-equivalent to $y\pi$. Then

$$(ax)\pi = (a\pi)(x\pi) = x\pi, \quad (az)\pi = (a\pi)(z\pi) = z\pi,$$
$$(zb)\pi = (z\pi)(b\pi) = z\pi, \text{ and } (yb)\pi = (y\pi)(b\pi) = y\pi.$$

Since π is 0-restricted and $x\pi$, $y\pi$ and $z\pi$ are all non-zero, therefore ax, az, zb and yb are all non-zero. Hence $(x, y) \in \beta$. Consequently, $\sigma \subseteqq \beta$.

Conversely, let x, y be non-zero and suppose that $(x, y) \in \beta$. Thus ax, az, zb, yb are all non-zero for some a, z, b in S. Hence $(a\pi)(x\pi)$, $(a\pi)(z\pi)$, $(z\pi)(b\pi)$, and $(y\pi)(b\pi)$ are non-zero in S/π. Thus $(a\pi)^{-1}(a\pi)$ is the left unit

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of $x\pi$ and of $z\pi$; whence $x\pi$ and $z\pi$ belong to the same Brandt summand of S/π. Similarly, $z\pi$ and $y\pi$ belong to the same Brandt summand. Hence $x\pi$ and $y\pi$ belong to the same summand, B_j^*, say. Thus $x, y \in B_j \setminus 0$ and so $(x, y) \in \sigma$. We have thus shown that the restriction of β to $S \setminus 0$ coincides with σ.

That each B_j is categorical at zero is immediate because S is categorical at zero. That B_j has an indecomposable zero follows from Munn's Theorem 1.1 in [6], since B_j^* is its homomorphic image.

Now let T be an ideal of S which has an indecomposable zero. Let $T_j = T \cap B_j$, $j \in I$. Then T_j is an ideal of T, for each j, and $T_j \cap T_k = 0$, if $j \neq k$. Since the zero of T is indecomposable it follows that $T_k = 0$ except for at most one $k \in J$. Hence $T = T_j \subseteqq B_j$, for some $j \in J$.

The proof of the theorem is complete.

We now examine the semigroups B_j more closely. They are semigroups categorical at zero and with an indecomposable zero or, as we shall say, more shortly, *categorical at an indecomposable zero*. Munn showed [6] that an inverse semigroup $S = S^0$ is categorical at an indecomposable zero if and only if it has a Brandt semigroup as a 0-restricted homomorphic image. Munn also showed that the maximal such homomorphic image is the same, to within isomorphism, for S and for any of its non-zero ideals ([6] Theorem 3.6).

We recall that if x is an element of a semigroup $S = S^0$, then $J(x)$ denotes the principal ideal generated by x, $I(x)$ denotes the subset of $J(x)$ consisting of those elements of $J(x)$ which generate a principal ideal of S properly contained in $J(x)$, and J_x denotes $J(x) \setminus I(x)$. $I(x)$ is non-empty when $x \neq 0$, and is an ideal of S. The semigroups $J(x)/I(x)$ are the principal factors ($\neq 0$) of S. When S is an inverse semigroup, each of the principal factors $J(x)/I(x)$ is 0-simple. J_x is the set of generators of $J(x)$ and is the set of non-zero elements of $J(x)/I(x)$. (See [4], § 2.6.)

When S is categorical at an indecomposable zero and has a 0-minimal ideal K, say, then K is (isomorphic to) a principal factor of S and, from the above remarks, S and K have the same maximal Brandt homomorphic image. This is the case, in particular, when S is finite. As we shall show by an example shortly there need not be a principal factor of S with the same Brandt homomorphic image.

Following Munn, if ρ is a 0-restricted congruence on $S = S^0$ such that S/ρ is a Brandt semigroup, we shall call ρ a *Brandt congruence*.

LEMMA 33. *Let S be categorical at an indecomposable zero. Let $x, y \in S \setminus 0$ and suppose that $J(x) \subseteqq J(y)$. Let ρ be a Brandt congruence on S. Then, for each element a in J_y there is an element b in J_x such that $(a, b) \in \rho$.*

PROOF. Let $a \in J_y$. Since $J(x) \subseteqq J(y) = J(a)$ there exist $u, v \in S$ such that $x = uav$. Set $b = u^{-1}xv^{-1}$. Then

$$ubv = uu^{-1}xvv^{-1} = uu^{-1}uavv^{-1}v = uav = x.$$

Thus $b \in J_x$. Furthermore, $b = u^{-1}uavv^{-1}$, which implies that

$$b\rho = (u\rho)^{-1}(u\rho)(a\rho)(v\rho)(v\rho)^{-1}.$$

Since ρ is 0-restricted, $b\rho \neq 0$. Hence $(u\rho)^{-1}(u\rho)$ is the left unit of $a\rho$ in the Brandt semigroup S/ρ and, similarly, $(v\rho)(v\rho)^{-1}$ is the right unit. Hence $a\rho = b\rho$.

COROLLARY 34. *Let S be categorical at an indecomposable zero. Let ρ be a Brandt congruence on S and suppose that S/ρ is finite.*

Then there exists a $\mathscr{J}$-class J, say, such that $J\rho^\natural = (S/\rho)\setminus 0$. Furthermore, if J' is a non-zero $\mathscr{J}$-class and $J' \leqq J$, then $J'\rho^\natural = (S/\rho)\setminus 0$.

PROOF. Choose $y_1, y_2, \cdots, y_m$, say, in S such that $(y_1\rho), (y_2\rho), \cdots, (y_m\rho)$ are the non-zero elements of S/ρ. Then

$$J(y_1) \cap J(y_2) \cap \cdots \cap J(y_m) \neq 0,$$

because otherwise 0 is decomposable. Choose a non-zero element x belonging to all the $J(y_i)$. Then, by the lemma, since $J(x) \subseteqq J(y_i)$, there is an element x_i, say, in J_x, such that $x_i\rho = y_i\rho$. This holds for $i = 1, 2, \cdots, m$. Hence, taking $J = J_x$, $J\rho^\natural = (S/\rho)\setminus 0$. If $0 \neq J' \leqq J$, then the lemma immediately gives $J'\rho^\natural = J\rho^\natural$.

In a later paper the structure of inverse semigroups which are categorical at an indecomposable zero will be explored further. Meanwhile we give the example mentioned earlier.

Let G be an abelian group of type p^∞, with generators, a_i, $i = 1, 2, \cdots$, which satisfy the defining relations $a_{i+1}^p = a_i$, and $a_1^p = 1$. We now define an inverse semigroup S which has G^0 as its maximal Brandt semigroup image. S is commutative, has a zero 0, and generators b_i, e_i, $i = 1, 2, \cdots$, satisfying the defining equations

$$\begin{aligned} b_i^{p^i} &= e_i, && i = 1, 2, \cdots, \\ b_i^m b_j^n &= b_j^{mp^{j-i}+n}, && \text{if } i \leqq j. \end{aligned}$$

It may then be checked that each $\mathscr{J}$-class of S is a finite cyclic group, the principal factors being obtained by adjoining zeros. In fact the $\mathscr{J}$-classes are the subgroups of S generated by the b_i, $i = 1, 2, \cdots$. Each principal factor is thus a finite Brandt semigroup. Define ϕ as the homomorphism of S which maps b_i onto a_i, $i = 1, 2, \cdots$. Then $S\phi = G^0$, and it may be shown that G^0 is the maximal Brandt semigroup 0-restricted image of S. In particular, G^0 is not the homomorphic image of any principal factor of S.

We return now to the representations of semigroups categorical at zero that are involved as primitive components of the general representation. They are 0-restricted representations which have a primitive image. For these the following theorem gives the extension of the result of Theorem 9 needed. We shall say that the representation Γ of a semigroup S is *primitive* if $\Gamma(S)$ is primitive.

THEOREM 35. *Let S be an inverse semigroup categorical at zero and such that its maximal* 0*-restricted primitive image is of finite rank. Let $B_j, j = 1, 2, \cdots, u$, be the* 0*-direct summands of S with indecomposable zeros of Theorem* 32*, i.e., such that $\{B_j \setminus 0 : j = 1, 2, \cdots, u\}$ is the set of β-classes, where β is defined by* (4).

Let Γ be a 0*-restricted primitive representation of S. Then Γ decomposes into* 0*-restricted primitive representations Γ_j:*

$$\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_u,$$

where Γ_j is a representation of B_j.

PROOF. From Theorem 32, the semigroups B_j may be identified as the semigroups $B_j^*(\pi^\natural)^{-1}$, where π is defined by equation (1) of Theorem 5. By assumption, $\Gamma(S)$ is primitive and 0-restricted and hence, by Theorem 5, $\pi \subseteqq \Gamma \circ \Gamma^{-1}$. Set $\Gamma(S) = P^*$ and let $P_1^*, P_2^*, \cdots, P_v^*$, say, be the Brandt semigroup summands of P^*.

Let $x, y \in B_j \setminus 0$. Then $\Gamma(x) \neq 0$ and $\Gamma(y) \neq 0$, since Γ is 0-restricted. Let $\Gamma(x) \in P_k^*$, $\Gamma(y) \in P_l^*$; then $x \in \Gamma^{-1}(P_k^*)$, $= T_k$, say, and $y \in \Gamma^{-1}(P_l^*)$, $= T_l$, say. By Munn's Theorem (loc. cit. [6]), T_k and T_l have indecomposable zeros. Hence, by Theorem 32, T_k and T_l are each contained in one of the semigroups $B_1, B_2, \cdots, B_u$. Since $x, y \in B_j$, therefore T_k and T_l are contained in B_j. Were $k \neq l$, then it would follow that $T_k \cap T_l = 0$ and this would conflict with the fact that B_j has an indecomposable zero. Hence $k = l$; and we have shown that each $\Gamma(B_j)$ is contained in one of the P_i^*.

Since Γ is 0-restricted, distinct B_j are contained in distinct P_i^*. There is thus a one-to-one correspondence between the B_j and the P_i^*. Hence $u = v$ and we may assume that the P_i^* are enumerated so that $\Gamma(B_j) \subseteqq P_j^*$, $j = 1, 2, \cdots, u$. Indeed, since

$$\cup \Gamma(B_j) = \Gamma(S) = \cup P_j^*$$

and $\Gamma(B_j) \cap \Gamma(B_k) = 0$, if $j \neq k$, we have $\Gamma(B_j) = P_j^*$, for each j.

If we denote the restriction of Γ to B_j by Γ_j, it now follows immediately from Theorem 9, that Γ decomposes into the representations Γ_j. It is clear that the Γ_j are 0-restricted and primitive.

9. Equivalent representations

Let Γ_1 and Γ_2 be equivalent representations of the inverse semigroup S and let

$$0 \subseteqq V_1 \subset P_1 \subseteqq V_2 \subset \cdots \subset P_r \subseteqq V_{r+1} = S \tag{2}$$

be the representation ideal series of Γ_1.

Since $\Gamma_1(x) = 0$ if and only if $\Gamma_2(x) = 0$, V_1 is also the vanishing ideal of Γ_2. Let P^* be the maximal primitive ideal of $\Gamma_1(S)$. Then $P_1 = \Gamma_1^{-1}(P^*)$. Let Q^* be the maximal primitive ideal of $\Gamma_2(S)$. Then we shall show that also $P_1 = \Gamma_2^{-1}(Q^*)$.

By assumption there is a non-singular matrix A, say, such that $A^{-1}\Gamma_1(x)A = \Gamma_2(x)$ for all $x \in S$. It is clear therefore that $\Gamma_2(x)$ is an idempotent if and only if $\Gamma_1(x)$ is an idempotent and that the idempotent $\Gamma_2(x_1)$ is under the idempotent $\Gamma_2(x_2)$ if and only if $\Gamma_1(x_1)$ is under $\Gamma_1(x_2)$. Hence if E denotes the set of primitive idempotents of $\Gamma_1(S)$ and F that of $\Gamma_2(S)$, $F = A^{-1}EA$ $(= \{A^{-1}\Gamma_1(x)A : \Gamma_1(x) \in E\})$.

Let $\Gamma_1(x) \in P^*$. Then $\Gamma_1(x)\Gamma_1(y) = \Gamma_1(x)$, where $\Gamma_1(y) \in E$ and $\Gamma_1(y)$ is the right unit of $\Gamma_1(x)$. Hence $\Gamma_2(x)\Gamma_2(y) = \Gamma_2(x)$, where $\Gamma_2(y) \in F$; whence $\Gamma_2(x) \in Q^*$. The reverse argument holds. Hence $\Gamma_1(x) \in P^*$ if and only if $\Gamma_2(x) \in Q^*$, i.e. $P_1 = \Gamma_2^{-1}(Q^*)$.

Thus we have shown that the first two terms of the representation ideal series for Γ_2 coincide with the first two terms, V_1 and P_1, of the representation ideal series for Γ_1. Consideration of the second primitive component of Γ_1 shows that the next two terms of the representation ideal series of Γ_1 and Γ_2 coincide; and so on. Thus we have proved,

THEOREM 36. *Two equivalent representations of an inverse semigroup $S = S^0$ have the same representation ideal series.*

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