MATRIX STRUCTURAL ANALYSIS

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SUMMARY

A survey is made of the basic matrix equations discretizing the strain-displacement, equilibrium and constitutive equations of linear elasticity theory. The finite element approach requires in addition a discretized form of the transitional equations between subdomains and a progressive treatment from the element level to the assembled structural level.

Special attention has been given to a rational incorporation of body loads and internal degrees of freedom.

The direct stiffness method, force method and combined methods are related to discretized formulations of variational principles. The saddle point character of the many-field principles is put in evidence.

1.1. INTRODUCTION

As a result of any type of discretization of a continuous structure, there are essentially 3 groups of matrix equations describing its behaviour :

- the kinematic group $S^T q = e$ (1)
  describing the relations between (generalized) displacements and strains,
- the static group $S s = p$ (2)
  describing the relations between (generalized) stresses and applied loads,
- the constitutive equations relating stresses and strains. For a linear elastic body they assume the form

$$s = J e \qquad \text{or} \qquad e = F s \tag{3}$$

where J is a positive definite matrix of generalized elastic moduli and $F = J^{-1}$ is a positive definite matrix of compliances.

Equations (1) constitute a discretized form of the linearized strain measures of a continuum

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \\ 0 & D_3 & D_2 \\ D_3 & 0 & D_1 \\ D_2 & D_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

presented here in cartesian coordinates, where $D_i = \partial/\partial x_i$, and whose matrix form will be written as

$$\varepsilon = D\, u \tag{4}$$

Equations (2) are the corresponding discretizations of the continuum equilibrium equations

$$D^T \sigma + \phi = 0 \tag{5}$$

relating the elements

$$(\tau_{11} \quad \tau_{12} \quad \tau_{33} \quad \tau_{23} \quad \tau_{31} \quad \tau_{12}) = \sigma^T$$

of the stress tensor and the vector ϕ of body loads, and

$$N^T \sigma = t \tag{6}$$

which are surface equilibrium equations, where t is the vector of surface tractions. Matrix N derives from the matrix differential operator D by replacing D_i by n_i, the direction cosines of the outward normal to the surface under consideration. Equations (3) are the discretized form of generalized relations between stresses and strains

$$\sigma = H \varepsilon \tag{7}$$

where the symmetrical positive definite Hooke matrix H contains the 21 elasticity moduli of the meterial that may vary from point to point in the case of a non homogeneous body.

Solving these equations under prescribed boundary conditions constitutes the aim of matrix structural analysis. When the discretization is performed by finite element methods, the matrix equations (1), (2) and (3) are first obtained for each individual element. The ensuing problem of connecting the elements together, in order to establish the same equations at the global structural level, becomes also one of the important steps in matrix structural analysis. Considering the continuum to be divided into a finite number of subdomains later to be discretized and become the finite elements, the connection problem involves discretization of the transition conditions between face + and face - of each interface :

$$u^+ = u^- \tag{8}$$

$$t^+ + t^- = 0 \tag{9}$$

expressing respectively continuity of displacements and reciprocity of surface tractions. If necessary, those transitions conditions can be generalized to introduce dislocations and application of external surface loads at an interface.

It will be observed that the presence of the same differential operator D in equations (4) and (5) of the continuum provides an indication of the existence of dual properties between kinematics and statics. This duality appears in the discretized forms (1) and (2) through the presence of the same "kinematical matrix" S.

1.2. KINEMATICS AT ELEMENT LEVEL

It will be assumed that the rigid body displacements are included in the degrees of freedom retained where discretizing a finite element. Even under very small strain, a region of thin-walled structure can undergo large displacements, whose major part is then of rigid body type. In the geometrically non-linear theory of elasticity the inclusion of rigid body displacements is thus imperative. The situation is different in linear elasticity, where local rotations are of the same small order of magnitude as strains. However both intuition and experience show that the inclusion of rigid body degrees of freedom, although not absolutely necessary for convergence with respect to reduction in mesh size, is a practical requirement for good accuracy with economical sizes of elements To the approximation of geometrically linear strain, the rigid body displacements are infinitesimal and members of a linear finite dimensional space. In other words any linear combination of rigid body displacements is itself a rigid body displacement. This leads to a matrix representation of the rigid body modes as

$$q = R\, r \qquad (10)$$

where q is the vector of generalized boundary displacements of the element and the columns of R constitute a fundamental set (complete and linearly independent) of particular rigid body modes.

We say then that R is a "base matrix" of the subspace of rigid body modes.

The vector r is that of coefficients of an arbitrary linear combination of those columns. Let us give two examples.

Example 1. The constant strain membrane triangle (Fig. 1)

The displacement field is completely determined by the local displacement vectors at the nodes. The local displacement vectors elsewhere are, by definition of the discretization, obtained by linear interpolation. The set of generalized displacements is indicated, each component is some orthogonal (covariant) projection of a nodal displacement vector on one of the sides of the triangle. A simple base matrix R is obtained by taking as columns generalized displace-

ments generated by rotations about the nodes 1, 2 and 3 respectively.

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{11}$$

Example 2. The constant stress membrane triangle (Fig. 2).

As the name suggests, this element is derived from a discretization of the stress field instead of the displacement field. Under constant stress the surface tractions are uniform along the edges and can be characterized by the corresponding resultants U_{12} , V_{12} , U_{23} , V_{23} , U_{31} , V_{31} , listed in that order in the generalized boundary load vector g. The conjugate boundary displacements are defined by virtual work. For instance the generalized displacement u_{12} , conjugate to the load U_{12} must be such that

$$u_{12}\, U_{12} = \int_1^2 t_u\, u\, ds$$

However, since the surface traction component t_u is uniform, we have

$$t_u = U_{12}/c_{12}$$

where c_{12} is the length of side 12. Therefore

$$u_{12} = \frac{1}{c_{12}} \int_1^2 u\, ds \tag{12}$$

and similar definitions for the other generalized displacements.

From this example can be understood that a discretization needs not necessarily be based on local values of fields but involves, in general, <u>linear functionals</u> of the field. In the present example the displacement field is known only by its average values taken along the edges.

To obtain rigid body modes, it is necessary to compute expressions like (12) for a rigid body displacement field. If, for instance, the element be rotated infinitesimally about node 1, the u component of displacement along

12 varies linearly between zero and $-(y_2 - y_1)\, d\omega$.
In that case (12) gives

$$u_{12} = -\frac{1}{2}\, d\omega\, (y_2 - y_1) \quad .$$

By similar calculations we obtain, suppressing the factor $\frac{1}{2}\, d\omega$, the base matrix

$$R = \begin{pmatrix} -y_2 + y_1 & y_2 - y_1 & -y_1 - y_2 + 2y_3 \\ x_2 - x_1 & -x_2 + x_1 & x_1 + x_2 - 2x_3 \\ -y_2 - y_3 + 2y_1 & -y_3 + y_2 & y_3 - y_2 \\ x_2 + x_3 - 2x_1 & x_3 - x_2 & -x_3 + x_2 \\ -y_3 + y_1 & -y_3 - y_1 + 2y_2 & -y_1 + y_3 \\ x_3 - x_1 & x_3 + x_1 - 2x_2 & x_1 - x_3 \end{pmatrix} \tag{13}$$

The first column is the rotation mode about node 1, the second is about node 2, the third about node 3.
An interesting observation is that the displacement functionals coincide for rigid body motions with the local values of displacments at the mid-side points.
This is natural from virtual work considerations, since the resultant boundary loads may be considered to be applied at these "virtual nodes".

The elimination of the n(r) elements of r between the n(q) equations (10) results in a set of

$$n(q) - n(r) = n(e) \tag{14}$$

independent equations relating the elements of q

$$S^T q = 0 \tag{15}$$

S^T is a n(e) x n(q) matrix of linearly independent rows, and (15) is the necessary and sufficient condition for q to belong to the subspace of rigid body modes.
By construction (10) with arbitrary r, is the general solution of (15), hence

$$S^T R = 0 \qquad (16)$$

This shows S, that has linearly independent columns, to be a base matrix for the orthogonal complementary subspace to that generated by R.
Consider now the more general inhomogeneous equation

$$S^T q = e \qquad (1)$$

related to (15). Since e vanishes when q belongs to the subspace of rigid body modes it is appropriate to consider e as a measure of strain of the element; its elements are called the generalized strains.

Consider now non singular transformations of the quantities r and e

$$r = (r/\hat{r})\ \hat{r} \qquad \hat{r} = (\hat{r}/r)\ r \qquad (\hat{r}/r) = (r/\hat{r})^{-1}$$

$$e = (e/\hat{e})\ \hat{e} \qquad \hat{e} = (\hat{e}/e)\ e \qquad (\hat{e}/e) = (e/\hat{e})^{-1}$$

We have $q = \hat{R}\ \hat{r}$ and $\hat{S}^T q = \hat{e}$,

$\hat{R} = R(r/\hat{r})$ $\hat{S}^T = (\hat{e}/e)\ S^T$,

so that $\hat{S}^T \hat{R} = (\hat{e}/e)\ S^T R(r/\hat{r}) = 0$

This shows that R and S are only defined within equivalence classes, each member of an equivalence class being derived from another by postmultiplication by a non singular matr

Homogeneous constraints placed on q

$$R^{\#} q = 0 \qquad (17)$$

can be devised to prevent the occurence of rigid body modes.
This implies that $R^{\#} R\ r = 0$ must only possess the trivial solution $r = 0$.
In other words the n(r) columns of $R^{\#}R$ must be linearly independent. Its number of rows, equal to that of $R^{\#}$, must be superior or equal to n(r).
Hence the minimal number of constraints capable of suppressing rigid body modes is n(r).
Any matrix $R^{\#}$ of size n(r) x n(r) with the property

$$R^{\#} R\ r = 0 \quad \rightarrow \quad r = 0$$

is said to define a set of external isostaticity constraints.
Since premultiplication of (17) by a non singular matrix does obviously not modify the set of constraints, $R^{\#T}$ is defined within an equivalence class. It is obviously possible to select $R^{\#T}$, within its equivalence class so that $R^{\#}R$, which is non singular, becomes the identity matrix :

$$R^{\#}R = (r/r) = R^{T}R^{\#T} \qquad (18).$$

Consider now the general solution of (17) when $R^{\#}$ defines a set of external isostaticity constraints. It is of the form

$$q = S^{\#T}\, e \qquad (19)$$

$S^{\#T}$ is a base matrix of the subspace defined by (17). Each of its columns is a straining mode of the element, since rigid body modes were prevented to take place.
The elements of the arbitrary vector e, that combine the columns linearly, can rightfully be called generalized strains; in fact, as the notation prefigures, the generalized strains defined by (19) can be rendered identical to those defined by (1); the condition therefore will be equation (23). As (19) provides the general solution to (17) (e arbitrary)

$$R^{\#}\, S^{\#T} = 0 \qquad (20)$$

holds true and $R^{\#T}$ and $S^{\#T}$ are base matrices of orthogonal complementary subspaces.
Once again it appears that $S^{\#T}$ is only representative of an equivalance class and that (20) is true wathever be the separate choices of $R^{\#T}$ and $S^{\#T}$ within their respective equivalence classes.

The assembled matrix $(R\ S^{\#T})$ has, according to (14), n(r) + n(e) = n(q) columns, and is consequently square. Furthermore the homogeneous equation

$$R\ r + S^{\#T}\, e = 0$$

is easily seen to possess only the trivial solution. If we premultiply by $R^{\#}$ there follows already r = 0 on account of (20) and the non singularity of $R^{\#}R$.

Then $e = 0$ follows because of the linear independence of the columns of $S^{\#T}$.
Consequently an arbitrary generalized boundary displacement vector q has a unique expansion in terms of a complete base matrix $(R\ S^{\#T})$

$$q = R\ r + S^{\#T}\ e \tag{21}$$

corresponding to a separation in rigid body modes and kinematically admissible straining modes (i.e. not violating a certain choice of isostaticity fixations). Premultiplying by S^T, there comes on account of (16)

$$S^T\ q = S^T\ S^{\#T}\ e \tag{22}$$

The matrix $S^T S^{\#T}$ is square, its number of rows, equal to that of S^T was defined to be n(e), the number of columns of $S^{\#T}$ is complementary to that of $R^{\#T}$, or $n(q)-n(r)=n(e)$ again. It is also non singular; for $S^T(S^{\#T}e) = 0$, implies that $S^{\#T}e$ belong to the complementary subspace generated by R, i.e.

$$S^{\#T}\ e = -\ R\ r \qquad \text{for some } r$$

However such a relation was seen to imply both $r = 0$ and $e = 0$.
This proves that (22) constitutes at most a non singular transformation of generalized strains between their definitions according to (1) and their definitions according to (19). Both definitions become identical by selecting $S^{\#T}$ within its equivalence class to obtain

$$S^T\ S^{\#T} = (e/e) = S^{\#}\ S \tag{23}$$

To determine r in the expansion (21) we premultiply by $R^{\#}$ and use (18) and (20) to obtain

$$r = R^{\#}\ q \tag{24}$$

Comparison of this with (17), suggests for r the interpretation of vector of generalized displacements of the supports.

1.3. VIRTUAL WORK AT ELEMENT LEVEL

The q vector defines the boundary displacements of the element.
It sometimes also defines unambiguously the internal displacements.
There are many cases however where internal degrees of freedom are introduced with fixed boundaries (q = 0) to improve the conditioning of the stiffness matrix or better represent the effects of an eventual body loading of the element.
It is extremely important to note that, for a given discretization, there is no unique way of defining the generalized displacements related to the additional internal degrees of freedom. This is because there is no unique way of defining the internal displacements which are forced by the motion of the boundaries.
Between any two possible definitions b or c of vectors of generalized internal displacements belonging to the same discretization, there is a relation

$$b = c + U\,q \tag{25}$$

where U is some suitable matrix. For fixed boundaries both definitions give identical displacements but if we set c = 0, a motion of the boundaries will have different internal effects than setting b = 0.
A step towards uniqueness is provided by the requirement that generalized internal displacements may be regarded as internal strains as well. In other words b would have to vanish like e, whenever the element undergoes a rigid body displacement.
Suppose that, for the original choice c, we find

$$c = C\,r \qquad \text{to be linked with} \qquad q = R\,r$$

in the rigid body motions. We would then require U to satisfy

$$0 = (C + U\,R)\,r \qquad \text{for arbitrary } r$$

and be presented the problem of solving

$$C + U\,R = 0 \qquad \text{or} \qquad R^T U^T = -C^T$$

From (18) and the transpose of (16) the general solution of this is

$$U = -C\,R^{\#} + A\,S^T$$

where A is an arbitrary matrix of suitable dimensions, and we find

$$b = c - C R^{\#} q + A S^T q = c - C r + A e$$

we then say that b has a strain-type definition.

Introduce now the generalized loads through the virtual work equation.
We distinguish between the loads applied at the boundary of the element (except along the structural boundary and in the presence of interface loading, they are internal connexion loads at the structural level) and the body loads.
Thus the boundary load vector g produces virtual work on q alone, while the body load vector has a virtual work associated with both internal and boundary displacements and falls in two parts : f which is conjugate to q and h which is conjugate to b.
The virtual work at element level has the structure

$$\mathcal{V}_e = q^T (g + f) + b^T h \qquad (26).$$

1.4. STATICS AT ELEMENT LEVEL

Equilibrium of the element requires that the virtual work of loads vanishes for any displacement of rigid body type; thus, provided b has a strain-type definition, when b = 0 and q = R r.
It follows then from (26) that

$$R^T (g + f) = 0 \qquad (27)$$

represent the equilibrium conditions constraining the loads. From (16) it is found that the general solution of (27) can be written

$$g + f = S s \qquad (28)$$

and s is a vector of generalized stresses. Premultiplying this result by q^T and considering (1) we obtain

$$q^T (g + f) = (S^T q)^T s = e^T s \qquad (29)$$

showing the stresses so defined to be conjugate to the strains e.
Finally the external form of virtual work (26) can be equated to an internal form

$$\mathcal{V}_e = e^T s + b^T h = \mathcal{V}_i \tag{30}$$

and this is a discretized version of the virtual work theorem of continuum mechanics.
It will be noted that the strain-type definition of b, enables its conjugate h to be considered both as a generalized load and a generalized stress. Another consequence is that, as representing a load system, h is self-equilibrating since its virtual work vanishes under rigid body displacements.
In contrast to this the body load system represented by f is generally not self-equilibrated. Being conjugate to q it is possible to balance it statically by reaction loads from the isostaticity constraints. Thus in a virtual work approach to static equilibrium

$$(R\ r)^T f + r^T j = 0$$

must hold for arbitrary r. There follows for the reaction loads

$$j = - R^T f \tag{31}$$

The general solution of this, considering j to be given and f unknown, is

$$f = - R^{\#T} j + S\ s_f \tag{32}$$

the first term being a particular solution as can be verified from (18).
The second term is, according to (16) in transpose, a general solution of the homogeneous equation.
The expansion (32) decomposes f into a part absorbed by reactions j against the isostaticity constraints as given by (31) and a self-equilibrated part, whose generalized stresses s_f are given by (see (20) and (23))

$$s_f = S^{\#} f \tag{33}$$

A similar expansion is valid for the generalized boundary loads, since (28) and (32) are equivalent to

$$g = R^{\#T} j + S\ s_g \tag{34}$$

with $s_f + s_g = s$ (35)

In (34) we find for given g the unique values

$$j = R^T g = - R^T f \qquad s_g = S^{\#} g \qquad (36).$$

The unicity of the expansion is otherwise obvious because $(R^{\#T} S)$ is non singular and a natural base matrix for g-space or f-space. Its transpose is the inverse of our base matrix for q-space

$$\begin{pmatrix} R^{\#} \\ S^T \end{pmatrix} \quad (R \quad S^{\#T}) = \begin{pmatrix} (r/r) & 0 \\ 0 & (e/e) \end{pmatrix} \qquad (37)$$

as is readily verified from (18), (20), (16) and (23).

1.4. KINEMATICS AND STATICS AT THE STRUCTURAL LEVEL

We now affect the previously defined matrices by a bracketed subscript (i) to indicate the element to which they belong. Similar quantities without bracket will be defined at the assembled structural level.

The operation of connecting the elements together has a kinematical aspect which consists simply in identifying each generalized boundary displacements $q_{(i)}$ of each element, with one of the elements listed in a nodal displacement vector q.
The possibility of doeing so implies of course a compatibility of definitions of generalized displacements at each node. Remember also that the nodal displacements need not be local values but simply linear functionals. The kinematical connexions are then expressible as

$$q_{(i)} = L_{(i)} q \qquad (38)$$

for each element. The Boolean matrices $L_{(i)}$ are convenient tools for presenting formulas and manipulating them. In actual software practice they are replaced by more efficient algorithms.

The statical aspect of the connexions can be derived from virtual work equations :

$$\mathcal{V} = \sum_i \{ q_{(i)}^T g_{(i)} + q_{(i)}^T f_{(i)} + b_{(i)}^T h_{(i)} \} \qquad (39)$$

expresses the virtual work of all forces acting externally on all the, yet unconnected, elements. If the transitions conditions (8) and (9) in the continuum were exactly satisfied, there would be no contributions from the interfaces to the virtual work. More generally, even if some external loads were applied at the interfaces, the virtual work of internal connexion loads would vanish. This property is assumed to hold after the discretization and, as a result, (39) as modified by (38) can be equated to the virtual work performed by the external loads (external at the structural level) only :

$$q^T \left(\sum_i L_{(i)}^t g_{(i)} + \sum_i L_{(i)}^T f_{(i)} \right) + \sum_i b_{(i)}^T h_{(i)}$$

$$= q^T g + q^T \sum_i L_{(i)}^T f_{(i)} + \sum_i b_{(i)}^T h_{(i)} \qquad (40)$$

Indeed the body loads on each element are still external loads for the global structure and g condenses the external loads applied at the structural boundary and possibly at the interfaces. Simplification of (40) and consideration of the fact that it must hold for any nodal vector q, yields the relation

$$g = \sum_i L_{(i)}^T g_{(i)} \qquad (41)$$

At the structural boundary (41) is a discretized form of the surface equilibrium equations (6). At an interior node where there is no contribution to g from external loads, it is a discretized form of the reciprocity relations (9) between internal connecting loads.

The kinematical connection rules (38) allow to deduce the generalized boundary strains of each element from the nodal displacement vector :

$$e_{(i)} = S_{(i)}^T q_{(i)} = (S_{(i)}^T L_{(i)}) q \qquad (42)$$

By definition, the generalized strain vector e of the structure is built from the sequence, in some conventional order, of the strains $e_{(i)}$

$$e^T = (e_{(1)}^T \quad e_{(2)}^T \cdots e_{(N)}^T) \tag{43}$$

There follows that the kinematical relation (1) holds true at the structural level if a structural kinematical matrix S be assembled as follows

$$S = (L_{(1)}^T S_{(1)} \quad L_{(2)}^T S_{(2)} \cdots L_{(N)}^T S_{(N)}) \tag{44}$$

The dual role of this structural kinematical matrix is obtained by combining equations (28) for each element

$$g_{(i)} + f_{(i)} = S_{(i)} s_{(i)}$$

into the single equation

$$\sum_i L_{(i)}^T g_{(i)} + \sum_i L_{(i)}^T f_{(i)} = \sum_i L_{(i)}^T S_{(i)} s_{(i)} \tag{45}$$

The first term is identified by (41), the second suggests the definition

$$f = \sum_i L_{(i)}^T f_{(i)} \tag{46}$$

of a body load vector f conjugate to q, the right hand side term involves the structural kinematical matrix (44) and the structural stress vector s

$$s^T = (s_{(1)}^T \quad s_{(2)}^T \quad \cdots \quad s_{(N)}^T) \tag{47}$$

conjugate to e. Hence (45) turns into equation (2) and at the structural level is completely analogous to (28) :

$$p = g + f = S\, s \tag{48}.$$

The virtual work theorem for the structure becomes

$$q^T p = q^T g + q^T f = q^T S\, s = e^T s \tag{49}$$

and is identical to (29). Both the general equilibrium equations of the structure (48) and the kinematical equations

$$e = S^T q \qquad (50)$$

expressing compatibility of the structural strains are necessary for the virtual work theorem to hold.

1.5. TOPOLOGICAL ANALYSIS OF THE STRUCTURAL CONNECTIONS

Clearly (48) and (50) are dual systems of equations, involving the same kinematical matrix and describing the topology of interelement connection from the dual aspects of kinematics and statics.
Their properties are important because of their wide range of applicability; since no constitutive equations relating stresses and strains are implied in this investigation, its results will apply to any structure where a geometrically linear measure of strain is acceptable.

The kinematical problem : solve

$$S^T q = e$$

when the strains are given and the displacements are the unknowns, is the discretized equivalent to the problem of constructing a displacement field in a continuum, knowing the strain tensor field. It is known that local and possibly global compatibility conditions must be satisfied by the strains for a solution to exist and this solution may not be unique.

The statical problem : solve

$$S\ s = p$$

when the loads are given and the stresses are the unknowns is known to be generally undetermined. The existence and unicity conditions of those two problems are interrelated because of their adjoint character and should consequently be treated together.
We note that concerning the dimensions of the problem, we have

$$n(q) = n(p) = \nu \qquad n(e) = n(s) = \sigma$$

In equation (50) each column of S^T is, by definition, a compatible strain vector associated to some unit displacement vector. The columns are however not linearly independent and the homogeneous problem

$$S^T q = 0 \quad \rightarrow \quad q = R\, r \quad \rightarrow \quad S^T R = 0 \qquad (51)$$

has a set of non trivial solutions, represented as in the case of a single element by linear combinations of the columns of a matrix R.
These columns, which constitute a maximal set of linearly independent solutions, are $n(r) = \rho$ in number so that R is a $\nu \times \rho$ matrix.
Each solution corresponds either to a rigid body displacement of the structure or a "mechanism", they are both referred to as kinematically admissible modes of the structure and the elements of r as kinematical coordinates.

Similarly in equation (48) each column of S is a self-equilibrating load vector associated to some unit stress vector. The columns are generally not linearly independent and the homogeneous problem

$$S\, s = 0 \quad \rightarrow \quad s = X\, x \quad \rightarrow \quad S\, X = 0 \qquad (52)$$

has a certain number of linearly independent solutions. The general solution is a linear combination, whose coefficients are collected in x, of a fundamental set represented by the columns of X. Each solution is appropriately called a self-stressing, since it indicates a state of internal stress existing without externally applied loads.
The element of x are "redundancy coordinates". We denote $n(x) = \xi$, so that X is a $\sigma \times \xi$ matrix.
Linear, non singular transformations of the kinematical coordinates or the redundancy coordinates merely change the choice of R or X within their respective equivalence classes. Wathever be the choices, the orthogonality conditions (51) and (52) remain satisfied.
The condition for problem (50) to have a solution is the orthogonality of the given strain vector e to all the solutions of the homogeneous adjoint problem $S\, s = 0$.
Hence

$$e^T X\, x = 0 \qquad \text{for all } x \quad \rightarrow \quad X^T e = 0 \qquad (53)$$

That the condition is necessary is immediatly seen by premultiplying (50) by X^T and using (52) in transpose. That it is sufficient will be demonstrated by actually constructing a solution.

Similarly the existence condition for a solution of problem (48) is the orthogonality of the given load vector to all the solution of the homogeneous adjoint problem $S^T q = 0$. Hence

$$p^T R r = 0 \quad \text{for all } r \quad \rightarrow \quad R^T p = 0 \qquad (54)$$

Necessity follows from premultiplication of (48) by R^T and use of (51) in transpose. Again a solution will then be constructed.
The existence condition (53) is the discretized form of the compatibility conditions to be satisfied by strains.
The existence condition (54) expresses that the virtual work performed by the external loads on any kinematically admissible mode must vanish.
Stated otherwise, the load vector p must be self-equilibrated. The transpose of (51) states that each column of S is a self-equilibrated load vector.

As for isolated elements, a minimal set of constraints can be selected that prevents the occurence of kinematically admissible modes in the structure.
We express the constraints in the same notation

$$R^* q = 0 \qquad R^* \text{ a } \rho \text{ x } \nu \text{ with linearly independent rows (55).}$$

Again R^* can be chosen within an equivalence class to satisfy

$$R^* R = (\rho/\rho) = R^T R^{*T} \qquad (56)$$

The general solution to (55) is

$$q = W w \qquad W \text{ a } \nu \text{ x } (\nu-\rho) \text{ with linearly independent columns (57),}$$

with the property

$$R^* W = 0 \qquad \text{or} \qquad W^T R^{*T} = 0 \qquad (58)$$

expressing that R^{*T} and W are base matrices spanning complementary orthogonal subspaces.

Theorem 1 . (R W) is a base matrix for q-space.
It has the required number of columns, ρ for R and $\nu - \rho$ for W.
The linear independence of the columns follows from the fact that the homogeneous

equations

$$R\, r + W\, w = 0$$

possess only the trivial solution. Indeed, if we premultiply by $R^{\#}$ and consider (56) and (58) there follows already $r = 0$. The equations are then reduced to $W\, w = 0$, that imply $w = 0$ because of the recognized linear independence of the columns of W.

Corollary 1.1. Any nodal displacement vector has a unique expansion of the type

$$q = R\, r + W\, w \tag{59}$$

The first part contains the kinematically admissible modes of the structure, the second the displacement modes compatible with the applied constraints.

Corollary 1.2. The σ x $(\nu - \rho)$ matrix $S^T W$ has linearly independent columns. Indeed by reference to (51), equation $S^T(W\, w) = 0$ has the general solution

$$W\, w = R\, r$$

but, from the theorem, this holds only for $w = 0$ and $r = 0$. Hence $S^T\, W\, w = 0$, having only the trivial solution, has linearly independent columns.
Then, from $S^T q = e = S^T\, W\, w$, we conclude that the columns of $S^T W$ constitute a maximal set of linearly independent compatible strain vectors.

Working by analogy, we introduce the concept of internal isostaticity constraints or "generalized releases". They are homogeneous relations imposed on the stress vector

$$X^{\#}\, s = 0 \qquad X^{\#} \text{ a } \xi \text{ x } \sigma \text{ matrix with linearly independent rows} \tag{60}$$

and capable of preventing the occurence of self-stressings. This implies that $X^{\#}X$ be non singular. Again, as $X^{\#T}$ can be selcted within an equivalence class, we can obtain that

$$X^{\#}\, X = (x/x) = X^T\, X^{\#T} \tag{61}$$

The general solution of equation (60) is

$$s = Y\, y \qquad Y \text{ a } \sigma \text{ x } (\sigma - \xi) \text{ matrix with linearly independent columns} \tag{62},$$

and the property

$$X^{\#}\, Y = 0 \qquad \text{or} \qquad Y^T\, X^{\#T} = 0 \tag{63}$$

Theorem 2 . (X Y) is a base matrix for s-space.
It has the required number of columns, ξ for X and $(\sigma - \xi)$ for Y. ,
Premultiplying

$$X\,x + Y\,y = 0$$

by $X^{\#}$ and using (61) and (63), there follows $n = 0$. Then $Y\,y = 0$ implies $y = 0$ because of the linear independence of the columns of Y. The columns of X and Y are thus linearly independent.

Corollary 2.1. Any generalized stress vector has a unique expansion of the type

$$s = X\,x + Y\,y$$

the first part contains the self-stressings; the second those stressing modes that comply with the chosen set of releases.

Corollary 2.2. The $\nu \times (\sigma - \xi)$ matrix S Y has linearly independent columns.
Indeed, by reference to (52) the equation $S(Y\,y) = 0$ has the general solution

$$Y\,y = X\,x \quad .$$

But, by theorem 2, this can only hold for $y = 0$ and $x = 0$. Hence $S\,Y\,y = 0$ having only the trivial solution has linearly independent columns.
From $S\,s = p = S\,Y\,y$ we conclude that the columns of S Y constitute a maximal set of linearly independent self-equilibrated loading modes.

Theorem 3 . The matrix $(Y^T\ S^T W)$ is non singular.
Consider the homogeneous problem

$$Y^T\,(S^T\,W\,w) = 0$$

Since Y and $X^{\#T}$ span complementary orthogonal subspaces, its general solution is

$$S^T\,W\,w = X^{\#T}\,v \qquad (65)$$

with arbitrary v. Premultiplying this by X^T we get, in view of (52) in transpose and (61) that $v = 0$. Then corollary 1.2. implies that also $w = 0$.
Our homogeneous problem has only the trivial solution and the columns of $Y^T\ S^T\ W$ are linearly independent. This requires

$$\sigma - \xi \geq \nu - \rho \qquad (66)$$

Conversely, consider the homogeneous problem

$$y^T (Y^T S^T W) = 0 \quad \text{equivalent to} \quad W^T (S Y y) = 0 \ .$$

Since W and $R^{\#T}$ span complementary orthogonal subspaces, the general solution is

$$S Y y = R^{\#T} z \qquad \text{where } z \text{ is arbitrary.} \tag{67}$$

Premultiplication by R^T yields $z = 0$, in view of (51) in transpose and (56) after which Corollary 2.2. enforces $y = 0$. This shows that $W^T S Y$ has linearly independent columns and the matrix of the theorem linearly independent rows. Consequently

$$\nu - \rho \geq \sigma - \xi \tag{68}$$

Comparison of (66) and (68) establishes that

$$\nu - \rho = \sigma - \xi \tag{69}$$

Thus $Y^T S^T W$ is square and having linearly independent rows and columns is non singular. The same is of course true of its transpose $W^T S Y$.

Corollary 3.1. $(S^T W \quad X^{\#T})$ is a base matrix for e-space.
It has the required number of columns, $\nu - \rho = \sigma - \xi$ for $S^T W$ and ξ for $X^{\#T}$.
The linear independence of all the columns is a direct consequence of the proof that problem (65) has only the trivial solution.
This base matrix provides for any strain vector a unique expansion

$$e = S^T W w + X^{\#T} v \tag{70}$$

The first part collects the linearly independent compatible straining modes (for which a nodal displacement vector q exists), the second part is made of generalized dislocation modes.

Corollary 3.2. $(S Y \quad R^{\#T})$ is a base matrix for p-space.
$S Y$ has $\sigma - \xi = \nu - \rho$ columns and $R^{\#T}$ has ρ , making the required total of ν .
Their linear independence is a consequence of the proof that problem (67) has only the trivial solution.
The unique expansion

$$p = S Y y + R^{\#T} z \tag{71}$$

of an arbitrary loading vector p, divides it in a self-equilibrated part and a part reacted upon by the set of external isostaticity constraints.

Note 1 . While $R^{\#T}$ and $X^{\#T}$ can be selected in their respective equivalence classes to implement (56) and (61), it is also possible to select either W or Y in their equivalence classes to reduce the matrix of theorem 3 to an identity matrix :

$$Y^T S^T W = (w/w) = (y/y) = W^T S Y \tag{72}$$

Note 2 . Computing the virtual work through the expansions (59) and (71), we find

$$q^T p = r^T z + w^T (W^T S Y) y \tag{73}$$

Similarly the virtual work computed from (70) and (64) is

$$e^T s = v^T x + w^T (W^T S Y) y \tag{74}$$

As observed before in connection with the virtual work theorem (49), the equality between (73) and (74) is ensured when v = 0, which means that the strain vector is compatible, and z = 0, which implies that the external loading is self-equilibrated. Those expressions also exhibit the conjugate nature of the couple (r,z) and the couple (v,x).
Under the conditions of Note 1, w and y would also be conjugate.

We now pass to the effective construction of solutions of problems (48) and (50) when the existence conditions are satisfied.

Premultiplication of equations (50) by the non singular matrix $(X\ Y)^T$ subdivises their solution in two parts. The first

$$X^T e = X^T S^T q = (S X)^T q = 0 \qquad \text{because of (52)}$$

is the known statement (53) of the existence condition for a solution. The second

$$Y^T e = Y^T S^T q$$

can be solved by calling on theorem 3. The solution is in fact unique if we add the requirement that it should satisfy the isostaticity constraints (55).
The displacement vector is then of type (57) and the problem becomes

$$Y^T e = (Y^T S^T W) w \tag{75}$$

with the unique solution, based on theorem 3

$$w = (Y^T S^T W)^{-1} Y^T e \quad .$$

The final result is that, provided the compatibility conditions (52) for strains are satisfied, problem (50) has the particular solution

$$q = S^{T\#} e \qquad S^{T\#} = W (Y^T S^T W)^{-1} Y^T \tag{76}$$

satisfying the isostaticity constraints (55).
If we disregard such constraints, the general solution is obviously

$$q = S^{T\#} e + R r \qquad \text{arbitrary} \tag{77}$$

the sum of a particular solution and the general solution for e = 0, which consists of kinematically admissible displacements. This result is in fact equivalent to the expansion (59) where w has been expressed in terms of the strain vector e by means of (75).
A completely analogous procedure is applicable to the solution of equation (48). After premultiplication by the non singular matrix $(R\ W)^T$ they are split into

$$R^T p = R^T S s = (S^T R)^T s = 0 \qquad \text{by virtue (51)},$$

which is a restatement of the existence condition (54), and

$$W^T p = W^T S s$$

If we here add the requirements that s should satisfy a set of releases (60), then s is of type (62) and we have a unique solution

$$y = (W^T S Y)^{-1} W^T p$$

or $$s = S^{\#} p \qquad S^{\#} = Y (W^T S Y)^{-1} W^T \tag{78}$$

The general solution without releases is

$$s = S^{\#}\, p + X\, x \qquad x \text{ arbitrary} \tag{79}$$

By comparing (78) to (76) we see that

$$S^{\#T} = S^{T\#} \tag{80}.$$

1.6. PSEUDO INVERSES AND PROJECTORS

Because

$$q = S^{T\#}\, e \tag{76}$$

is, in some sense, a solution of the singular problem

$$S^{T}\, q = e \tag{50}$$

$S^{T\#}$ is called a (oblique) pseudo-inverse of S^{T}. While in (76) we obtain a displacement vector for any strain vector (even no compatible), the strain vector obtained through substitution of this displacement vector into (50) is necessarily compatible. Hence the elimination of q between the two equations generates the product $S^{T}\, S^{T\#}$ that is not an identity matrix, as would be the case with genuine inverses. It is an operator that transforms an arbitrary strain vector into a compatible one. This operator is a projection into the subspace $X^{T}\, e = 0$; the projection rays (or vectors annihilated by the projector) are the dislocation modes $X^{\#T}\, v$ of expansion (70). This is to be verified by looking at the structure (76) of $S^{T\#}$ and using (63). Similarly, by eliminating e between the two equations, we obtain the product $S^{T\#}S^{T}$ as an operator linking two displacement vectors. Again $S^{T\#}$ is not a genuine left-inverse to S^{T} because the operator transforms an arbitrary displacement into one that satisfies the isostaticity constraints $R^{\#}\, q = 0$. It is in fact a projector of an arbitrary displacement into the subspace $R^{\#}\, q = 0$; the projection rays R r are the kinematically admissible modes.

One way to construct an identity matrix is to complete (76)

$$q = S^{T\#}\, e + R\, r$$

in order to give back to q its arbitrariness. Replace in it

$$e = S^T q \qquad \text{and} \qquad r = R^{\#} q \qquad \text{(justified by (56) and (58))}$$

to obtain an identity in q, whereby

$$S^{T\#} S^T + R R^{\#} = (q/q) = S S^{\#} + R^{\#T} R^T \tag{81}$$

Proceeding in the same manner with the equations

$$s = S^{\#} p \qquad \text{and} \qquad p = S s$$

we construct the operators

$S^{\#} S$ projecting an arbitrary stress vector into the subspace of stresses complying with the releases $X^{\#} s = 0$ and annihilating the self-stressings;

$S S^{\#}$ projecting an arbitrary load into a subspace of self-equilibrating ones by annihilating the reactions from isostaticity constraints.

We find the identity

$$S^{\#} S + X X^{\#} = (s/s) = S^T S^{T\#} + X^{\#T} X^T \tag{82}.$$

If the base matrices for s and e are taken to be identical, which is always possible although sometimes artificial from a physical point of view, and the same is done for the base matrices of q and p; the following simplifications occur

$$X^{\#T} \equiv X \qquad Y \equiv S^T W \qquad R^{\#T} \equiv R \qquad W \equiv S Y \tag{83}$$

Each base matrix is split into orthogonal complementary parts

$$X^T Y = 0 \qquad\qquad R^T W = 0 \tag{84}$$

The pseudo inverse $S^{\#}$ becomes the so-called Moore-Penrose generalized inverse of S

$$S^{\dagger} = S^T W (W^T W)^{-1} W^T = Y (Y^T Y)^{-1} Y^T S^T \tag{85}.$$

1.7. ALGORITHMS FOR A DIRECT DETERMINATION OF PSEUDO-INVERSES

For structures made of assembled beams the determination of redundancies was traditionnally done by releases, that is by cutting members or at least some of their load transmission capabilities, untill the internal forces in the released structure could be determined by statics. This procedure which largely relies for its success on physical intuition becomes slow, tedious and prone to error for the large arrays of more complicated finite elements. The automatic determination of all the self-stressing states by the computer itself is a considerable step towards salvation of one of the most time-honored method of dealing with hyperstatic systems. It was pioneered by P. DENKE and J. ROBINSON. As will be shown the algorithm they introduced, which is the Jordan version of the Gauss elimination algorithm, as applied to the structural kinematical matrix, provides on constructive answer to the problem of determining all the base matrices involved in the topological analysis of section 1.5.

The Gauss-Jordan algorithm consists in the step by step determination of a $\nu \times \nu$ non singular premultiplier P that transforms the kinematical matrix S into a matrix

$$P\,S = \begin{pmatrix} I_{\sigma-\xi} & Q \\ 0 & 0 \end{pmatrix} \qquad (86)$$

having ρ last rows of zeros (ρ is in fact determined by the algorithm itself). $I_{\sigma-\xi}$ is the identity matrix in $\sigma - \xi = \nu - \rho$ dimensions and Q becomes some $(\sigma-\xi) \times \xi$ matrix.

Each factor of the premultiplier depends on the choice of a **pivot.**

If in the first row of S a pivot $s_{1\alpha}$ is selected, the first factor of the premultiplier is constructed with the remaining elements of the αth column of S.

$$P_1 = \left(\begin{array}{c|c} 1/s_{1\alpha} & 0 \\ \hline - s_{2\alpha}/s_{1\alpha} & \\ \vdots & I_{\nu-1} \\ - s_{\nu\alpha}/s_{1\alpha} & \end{array}\right)$$

The αth column of $P_1 S$ thereby turns into a unit vector. Denoting by $s^{(1)}_{\eta\xi}$ the elements of $P_1 S$, a pivot $S^{(1)}_{2\beta}$ is selected on the second row and with the other elements of the βth column the non singular premultiplier is constructed.

$$P_2 = \left(\begin{array}{c|c|c} 1 & -s^{(1)}_{1\beta}/s^{(1)}_{2\beta} & 0 \\ \hline 0 & 1/s^{(1)}_{2\beta} & 0 \\ \hline 0 & -s^{(1)}_{3\beta}/s^{(1)}_{2\beta} & \\ \cdot & \cdot & \\ \cdot & \cdot & I_{\nu-2} \\ \cdot & \cdot & \\ 0 & -s^{(1)}_{\nu\beta}/s^{(1)}_{2\beta} & \end{array}\right),$$

while the αth column of $P_2 P_1 S$ remains the same unit vector, the βth column is now turned into the next unit vector. Whenever in pursuing this algorithm with the successive rows of the sucessive transforms of S a row of zeros in encountered, it is sent to the last row by a renumbering of the elements of p. The procedure terminates when the last rows to be treated are all composed of zeros.

Finally the sequence (α, β, γ ...) of columns is here transformed into the sequence (1, 2, 3 ...) by renumbering the elements of s.

The final structure of the transformed S is then given by (86) and the structure of the premultiplier is

$$P_{\nu-\rho} \cdots P_2 \cdot P_1 = P = \begin{pmatrix} N & 0 \\ T & L \end{pmatrix} \qquad (87)$$

N is a non singular $(\nu - \rho) \times (\nu - \rho)$, L is non singular $\rho \times \rho$ and lower triangular, T is a $\rho \times (\nu - \rho)$.

Splitting s and p to suit the partitioning of P and P S, problem (2), premultiplied by the non singular P, takes the new form

$$s_{(1)} + Q\, s_{(2)} = N\, p_{(1)} \qquad (88)$$

$$0 = T\, p_{(1)} + L\, p_{(2)} \qquad (89)$$

The corresponding homogeneous problem (p = 0), or self-stressing problem has clearly from (88) a solution where $s_{(2)}$ can be taken as the set of redundancy coordinates and $s_{(1)}$ is determined from its knowledge. This determines a possible matrix X in (52)

$$s_{(1)} = - Q\, x \qquad s_{(2)} = x \qquad X = \begin{pmatrix} - Q \\ I_\xi \end{pmatrix} \tag{90}.$$

Furthermore, as we can take $s_{(2)} = 0$ as set of releases, a possible $X^{\#}$ matrix is

$$X^{\#} = (0 \qquad I_\xi) \tag{91}$$

and, since the general solution of (60) is simply $s_{(2)} = 0$, $s_{(1)}$ arbitrary, we can identify

$$s_{(1)} = y \qquad Y = \begin{pmatrix} I_{\sigma-\xi} \\ 0 \end{pmatrix} \tag{92}$$

Equation (89) that does no more contain s, is obviously the existence condition for a solution of the equilibrium equations. Thus a possible R^T matrix in (54) is

$$R^T = (T \qquad L) \tag{93}$$

Finally comparing (88) to (79) we obtain a possible pseudo-inverse

$$S^{\#} = \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix} \tag{94}$$

Confirmation of all this is obtained in the modified form taken by the dual problem (1) when we expand q as in (59) by

$$q = P^T \begin{pmatrix} w \\ r \end{pmatrix} = \begin{pmatrix} N^T & T^T \\ 0 & L^T \end{pmatrix} \begin{pmatrix} w \\ r \end{pmatrix} = \left(\begin{array}{c|c} N^T & R \\ 0 & \end{array}\right) \begin{pmatrix} w \\ r \end{pmatrix} = \begin{pmatrix} N^T \\ 0 \end{pmatrix} w + R\, r$$

where R has been taken as suggested by (93). Thus we can take

$$W = \begin{pmatrix} N^T \\ 0 \end{pmatrix} \tag{95}$$

Substitution of this into problem (1) results in

$$e = S^T P^T \begin{pmatrix} w \\ r \end{pmatrix} = (P\ S)^T \begin{pmatrix} w \\ r \end{pmatrix} = \begin{pmatrix} I_{\sigma-\xi} & 0 \\ Q^T & 0 \end{pmatrix} \begin{pmatrix} w \\ r \end{pmatrix}$$

or $e = (I_{\sigma-\xi} \quad Q)^T w$ for the compatible strains.

A set of external isostaticity constraints is furnished by

$$R^{\#} = (0 \quad (L^T)^{-1}) \tag{96}$$

that verifies (56) and (58). The non singular matrix of theorem 3, turns out to be an identity matrix and formula (78) is then seen to verify the result (94).

1.8. COMPACT SELF-STRESSINGS

Different choices of pivots in the Gauss-Jordan algorithm lead to different sets of redundancy coordinates and matrices X describing the self-stressings of the structure While S is sparsely populated, its transformation by the Gauss-Jordan process will usually result in a fully populated Q matrix and self-stressings with a tendancy to diffuse in the whole structure. From a physical point of view this is unsatisfactory since it is now well known that compact self-stressings, involving only few members (finite elements) of the structure are the more numerous. In fact they represents the local compatibility conditions for strains in the continuum and must therefore be concentrated in a few number of adjacent members. Only the global compatibility conditior in multiply connected domains correspond to extensive redundancy "circuits".
The compact self-stressings can of course be obtained by dismantling the structure into its constitutive (finite)elements and reassembling a small number of them untill a redundancy is observed.
This procedure becomes however tedious for large structures and it would be desirable to program the computer to automate the search for compact self-stressings. This would result in a sparsely populated X matrix with the attendant advantages of best-conditioning of matrix force programs.
In some cases where a discretization of the stress field of the continuum is possible

in terms of stress-functions, the process is trivially achieved by the nodal values of the stress functions, which are in fact local redundancy coordinates. When the stress-function approach is not viable the search of compact self-stressings via appropriate selection of pivots in the Gauss-Jordan algorithm is still a subject of research.

1.9. EXTERNAL HYPERSTATICITY

In many cases, especially in Civil Engineering structures, the external constraints placed on q do more than prevent the kinematically admissible displacements; they can introduce additional self-stressings. Such cases of redundancy of supports are most easily handled by considering the set of nodal displacements subjected to supporting constraints to belong to an additional virtual finite element with its own elemental kinematical matrix and strain vector. Adding this strain vector to the sequence in e , we increase n(e) but not n(q) and the number of independent self-stressings is corresponding ly increased; they now include the self-stressings due to redundancy of the supports. It will be observed that only non zero strains in the additional element can induce strains in the structure, a rigid body displacement of the additional element produces a corresponding kinematical mode in the structure.

1.10. STIFFNESS AND FLEXIBILITY AT ELEMENT LEVEL

In approximations based on a discretization of the displacement field, the strain energy is obtained directly as a quadratic form in the boundary and internal displacement coordinates q and b :

$$W = \frac{1}{2} \begin{pmatrix} q^T & b^T \end{pmatrix} \begin{pmatrix} K_{qq} & K_{qb} \\ K_{bq} & K_{bb} \end{pmatrix} \begin{pmatrix} q \\ b \end{pmatrix} \geqslant 0 \qquad (97)$$

The symmetrical matrix of the form is known as the stiffness matrix and is non negative. In a rigid body displacement

$$q = R\, r \qquad \text{and} \qquad b = C\, r$$

the strains vanish and the strain energy also; thus

$$K_{qq}\, R + K_{qb}\, C = 0 \qquad\qquad K_{bq}\, R + K_{bb}\, C = 0 \qquad (98)$$

Let dq and db denote differential increases in the generalized displacements and apply the energy conservation theorem, equating the strain energy increase to the virtual work performed by the loads

$$dW = dq^T (K_{qq}\, q + K_{qb}\, b) + db^T (K_{bq}\, q + K_{bb}\, b) = dq^T (f + g) + db^T\, h \quad .$$

Since this is valid for arbitrary dq and db, we obtain the Castigliono type formulas

$$\begin{aligned} K_{qq}\, q + K_{qb}\, b &= g + f \\ K_{bq}\, q + K_{bb}\, b &= h \end{aligned} \qquad (99)$$

Let us investigate the effects of modifying the definition of internal displacement coordinates according to the formula

$$b = c + U\, q \qquad (25)$$

The strain energy is modified into

$$\frac{1}{2} \begin{pmatrix} q^T & c^T \end{pmatrix} \begin{pmatrix} \hat{K}_{qq} & K_{qc} \\ K_{cq} & K_{bb} \end{pmatrix} \begin{pmatrix} q \\ c \end{pmatrix}$$

$$\hat{K}_{qq} = K_{qq} + U^T K_{bq} + K_{qb}\, U + U^T K_{bb}\, U$$

$$K_{qc} = K_{qb} + U^T K_{bb} = K_{cq}^T$$

The virtual work becomes

$$dq^T (g + f + U^T h) + dc^T\, h$$

and our stiffness formulas

$$\hat{K}_{qq}\, q + K_{qc}\, c = g + f + U^T h$$

$$K_{cq}\, q + K_{bb}\, c = h$$

The following consequences of this transformation must be stressed :

- the boundary displacement direct stiffness matrix K_{qq} is modified to $\hat{K}_{qq}$,
- the generalized body load conjugate to q is modified from f to $f + U^T h$,
- the internal stiffness matrix K_{bb} and the body load conjugate to the internal displacement remain invariant,
- the stiffness coupling $K_{qb} = K_{bq}^T$ is modified.

If the new definition of internal displacements is of strain-type, we have, by definition, c = 0 together with q = R r in rigid body displacements and the corresponding properties

$$\hat{K}_{qq}\, R = 0 \qquad\qquad K_{cq}\, R = 0 \quad .$$

Holding the boundary fixed (q = 0) and displacing the inside necessarily generates internal strains and a positive strain energy. For this reason the internal stiffness matrix K_{bb} is necessarily positive definite, which opens the possibility of solving the stiffness equations for the internal degrees of freedom. Elimination of the internal displacements is the more indicated that it can be done economically at the element level and does not impeach the interconnexion problem that concerns the boundary displacements only. From (99)

$$b = K_{bb}^{-1} (h - K_{bq}\, q) \qquad (100)$$

Substitution of this in both the strain energy and the virtual work is equivalent to an internal coordinate transformation (25) with

$$U = - K_{bb}^{-1} K_{bq}$$

and the additional advantage that

$$K_{qc} = K_{qb} - K_{qb}\, K_{bb}^{-1} K_{bb} = 0$$

the stiffness coupling is eliminated. Moreover the transformation is such that c is zero under rigid body displacements, or, stated otherwise, the new, internal coordinates are of strain-type definition. This follows directly from the absence of stiffness coupling; the strain energy is reduced to

$$W = \frac{1}{2} q^T K q + \frac{1}{2} c^T K_{bb} c \tag{101}$$

with $$K = K_{qq} - K_{qb} K_{bb}^{-1} K_{bq} \tag{102}$$

and vanishing of the strain energy, which implies rigid body motion, implies $c = 0$ because K_{bb} is positive definite, while

$$K R = 0 \tag{103}$$

The same result is of course obtained from formulas (98) as modified by the transformation.

We summarize the situation after elimination of the internal degrees of freedom, as it will be applied in the sequel.
Starting from the quadratic form (97) as obtained with an arbitrary definition of the internal degrees of freedom, we eliminate b by formula (100) that also enables us to retrieve the internal displacements once q has been calculated. We thus obtain the boundary-stiffness relation

$$K q = g + f \tag{104}$$

where K is given by (102) and, for economy of notations, f denotes the body load effectively conjugate to q

$$f = f_b - K_{qb} K_{bb}^{-1} h \tag{105}$$

where f_b stands for the body load conjugate to q under the original choice b for the internal coordinates. The stiffness matrix K is non negative. Its singular character is demonstrated by formula (103).
Transforming the energy due to boundary displacements by the expansion (21) we obtain in view of (103) the positive definite form

$$\frac{1}{2} q^T K q = \frac{1}{2} e^T J e$$

where $$J = S^{\#} K s^{\#T} \qquad (106)$$

is a positive definite matrix called the core stiffness matrix. It clearly depends on the choice of e, that is on the choice of the kinematical matrix S within its equivalence class.

As the following transformations show

$$s = S^{\#} (g + f) = S^{\#} K q = S^{\#} K (S^{\#T} e + R r) = J e \qquad (107)$$

J is a matrix of generalized elastic moduli connecting the stresses and the strains of the element.

The inverse relation

$$e = F s \qquad (108)$$

defines the positive core flexibility matrix, or matrix of compliances.

From (107)

$$f + g = S s = S J e = S J S^T q$$

we obtain the inverse of (106)

$$K = S J S^T \qquad (109).$$

In connexion with this result, the question arises whether it is also possible to define an extended flexibility matrix relating loads and displacements, providing in fact an inversion of the stiffness relations (104). Starting from (21) we derive

$$q = S^{\#T} e + R r = S^{\#T} F s + R r = S^{\#T} F S^{\#} (g + f) + R r$$

or $$q = G (g + f) + R r \qquad (110)$$

$$G = S^{\#T} F S^{\#} \qquad \text{extended flexibility matrix} \qquad (111)$$

As was to be expected from the singular character of equations (104), this result holds because g + f is self-equilibrated and the existence condition (27) is satisfied, but the solution is not unique. The part of q governed by G is the one satisfying a certain choice of isostaticity constraints implicit in $S^{\#}$. From (18) and (20) we obtain

$$R^{\#} q = r$$

the constraint violations, or generalized support displacements, of the general solution. The converse of (111) is easily found to be

$$F = S^T G S \qquad (112).$$

1.11. THE DIRECT STIFFNESS OR DISPLACEMENT METHOD

Having determined the stiffness matrix of each element, the stiffness relations

$$g_{(i)} + f_{(i)} = K_{(i)} q_{(i)} \qquad i = 1, 2, \ldots N$$

are multiplied to the left by $L_{(i)}^T$ and added to produce in view of (38), (41) and (46)

$$p = g + f = K q \qquad (113)$$

where $$K = \sum_i L_{(i)}^T K_{(i)} L_{(i)} \qquad (114)$$

K is the master stiffness matrix of the unsupported structure, as the right hand side demonstrates it can be obtained quite simply by a proper addressing procedure of the elemental stiffnesses.

We recall that while f, the body load conjugate to q, must also be built up from the individual $f_{(i)}$ according to (46), the load g is obtained directly by computing the virtual work q^T g of the loads externally applied at the structural boundaries or, as the case may be, at the interfaces.

Because the structure is as yet unsupported, the master stiffness matrix is singular. If a base matrix of kinematically admissible displacements R is known, the master stifness obeys the equation K R = 0. It is however tedious for a large structure to

set up a base matrix R from inspection.

In aerospace applications, where the structure is normally not supported, a Gauss-Jordan algorithm applied to K can again produce a base matrix R together with a pseudo-inverse of the master stiffness relations (113).

With the same notations as in section 1.7., let the premultiplier P receive the structure

$$P = \begin{pmatrix} N & 0 \\ T & L \end{pmatrix} \qquad \text{and} \qquad P\,K = \begin{pmatrix} I & Q \\ 0 & 0 \end{pmatrix}$$

However, in this case the symmetry of K induces two important properties.

Noting that $P\,K\,P^T$ is also symmetrical and computing

$$(P\,K)\,P^T = \begin{pmatrix} I & Q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} N^T & T^T \\ 0 & L^T \end{pmatrix} = \begin{pmatrix} N^T & T^T + Q\,L^T \\ 0 & 0 \end{pmatrix}$$

we obtain at once

$$N^T = N \tag{115}$$

$$T^T + Q\,L^T = 0 \qquad \text{or} \qquad T + L\,Q^T = 0 \tag{116}$$

The premultiplication of (113) by the non singular P matrix reduces the stiffness relations to

$$q_{(1)} + Q\,q_{(2)} = N\,p_{(1)} \tag{117}$$

$$0 = T\,p_{(1)} + L\,p_{(2)} \tag{118}$$

The first can be rewritten as

$$q = \begin{pmatrix} q_{(1)} \\ q_{(2)} \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_{(1)} \\ p_{(2)} \end{pmatrix} + \begin{pmatrix} -Q \\ I \end{pmatrix} r$$

a pseudo-inversion of (113)

$$q = K^{\#} p + R r \tag{119}$$

where $$K^{\#} = \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix} = K^{\#T} \qquad R = \begin{pmatrix} -Q \\ I \end{pmatrix} \tag{120}$$

and $q_{(2)}$ is identified with the kinematical coordinates r of the structure.
The second (118) is obviously the existence condition for the solvability of (113) with respect to q, or equilibrium conditions to be satisfied by the loads to avoid excitation of the kinematically admissible displacements. As such it must be equivalent to

$$R^T p = -Q^T p_{(1)} + p_{(2)} = 0$$

This is not contradictory with (118), which can in fact be placed in the form $L R^T p = 0$ by virtue of (116). Since L is non singular both forms are thus equivalent.
In the general case the known external loads are applied on a set of coordinates distinguished by the subscript (1), in other words a set $p_{(1)}$ of external loads is given. For the complementary set, the values of displacements is known : $q_{(2)}$ is given (zero for fixed supports).
We then modify the Gauss-Jordan algorithm applied to K by forbidding the choice of pivots on the columns belonging to the second set. When the algorithm stops, the premultiplier has still the same structure, but now

$$P K = \begin{pmatrix} I & Q \\ 0 & V \end{pmatrix}$$

If V has no zero rows, the structure is at least isostatically supported.
If it has zero rows it is only partially supported.
The stiffness equations are immediatly resolved as follows

$$q_{(1)} = N p_{(1)} - Q q_{(2)} \tag{121}$$

gives the unknown displacements in terms of the known loads and support displacements,

$$p_{(2)} = L^{-1} (V q_{(2)} - T p_{(1)}) \qquad (122)$$

yields the reaction loads in the supports.

1.12. THE FORCE METHOD

While the direct stiffness method completely by passes the construction of the structural kinematical matrix, the force method is actually based on its topological analysis. This is because, as its other name indicates, the direct stiffness method selects the generalized displacements as the basic unknowns. The strains and stresses are then derived from the generalized displacements, not necessarily from matrix equations, but often by returning to the field equations in each subdomain. The force method aims directly at the stresses and begins therefore by the determination of the redundancies. We start from equation (79)

$$s = S^{\#} p + X x$$

where a pseudo-inverse $S^{\#}$ and a base matrix X have been obtained by a Gauss-Jordan algorithm, as explained in section 1.7.
The strains are given through the core flexibility relations

$$e_{(i)} = F_{(i)} s_{(i)}$$

established for each element. Considering the definitions of e and s, respectively (43) and (47), for the assembled structure, the set of those relations can be presented in the form e = F s, where F is built up from diagonal $F_{(i)}$ blocks

$$F = \left(\begin{array}{c|c|c|c} F_{(1)} & 0 & 0 & 0 \\ \hline 0 & F_{(2)} & 0 & 0 \\ \hline 0 & 0 & - - & 0 \\ \hline 0 & 0 & 0 & F_{(N)} \end{array}\right) .$$

Thus $e = F S^{\#} p + F X x$

The compatibility conditions for the strains were found to be expressed through equation (53) and we must have

$$X^T F S^* p + (X^T F X) x = 0 \quad . \tag{123}$$

Since F is positive definite and X has linearly independent columns, this set of linear equations for the unknown redundancy coordinates is non singular and its inversion furnishes

$$x = - (X^T F X)^{-1} X^T F S^* p$$

and the generalized stresses are given in terms of the applied loads

$$s = S^* p - X (X^T F X)^{-1} X^T F S^* p \tag{124}$$

Given the compatible strains which follow from $e = F s$, the displacements can be found from

$$q = S^{*T} e + R r \tag{125}.$$

This procedure solves the problem of the unsupported or "free" structure, as well as that of isostatic or hypostatic supports, for which r can be determined at least partially without invalidating (124).

Imagine now that we incorporate an additional element whose boundary displacements $q_{(o)}$ do not introduce new generalized displacements but are taken from the existing set q.

The addition in a given truss of a new bar connecting two existing nodes is the simplest example. In more complicated examples the new element can possess, in isolation, its own non empty set of self-stressings.

In any case

$$q_{(o)} = L_{(o)} q$$

and the new element has its own set of generalized strains

$$e_{(o)} = S^T_{(o)} q_{(o)} = S^T_{(o)} L_{(o)} q \tag{126}$$

so that the structural kinematical matrix receives additonal columns, that will be assumed to be listed in front of the others

$$S = (L_{(o)}^{T} \; S_{(o)} \qquad S_{(f)}) \quad ,$$

$S_{(f)}$ denoting the kinematical matrix of the "free" structure.

Theorem 4

The self-stressing matrix X of augmented structure can be partitioned as follows

$$\begin{pmatrix} s_{(o)} \\ s_{(f)} \end{pmatrix} = \begin{pmatrix} X_{oo} & X_{ot} \\ 0 & X_{ft} \end{pmatrix} \begin{pmatrix} x_{(o)} \\ x_{(t)} \end{pmatrix} \qquad (127)$$

where $x_{(o)}$ is a complete set of redundancy coordinates for the additional element in isolation. X_{oo} and X_{ft} have linearly independent columns.

Proof

It is obvious that the self-stressings $s_{(o)} = X_{oo}\, x_{(o)}$ of the isolated new element induce no loads on the free structure when it is being connected, so that $s_{(f)}$ does not depend on $x_{(o)}$. By definition the columns of X_{oo} are linearly independent.
Assume now that the columns of X_{ft} are not linearly independent. A non zero $x_{(t)}$ would then exist such that $s_{(f)} = X_{ft}\, x_{(t)} = 0$. For this $x_{(t)}$ we must have $s_{(o)} = X_{ot}\, x_{(o)} \neq 0$, otherwise the columns of X related to $x_{(t)}$ would be linearly dependent, which contradicts the definition of X.
Hence this $x_{(t)}$ would generate a self-stressing in the isolated new element alone, which contradicts the assumption that $x_{(o)}$ is a complete set.

Note

If $x_{(t)}$ is properly choosen, X_{ot} itself can have a set of zero columns, the corresponding columns of X_{ft} representing self-stressings of the free structure alone.
The remaining columns of X_{ot} and X_{ft} represent the new self-stressings that may appear as a result of the interconnexion.
The partitioning of X, considered in theorem 4, is one obtained from a Gauss-Jordan transformation in which the pivots are first selected amongst the $L_{(o)}^{T}\, S_{(o)}$ columns of S. It yields for $s_{(f)}$ the expansion

$$s_{(f)} = S^{\#}_{(f)}\, p + X_{ft}\, x_{(t)} \tag{128}$$

From the core flexibility relation of the free structure follows

$$e_{(f)} = F_{(f)}\, s_{(f)} = F_{(f)}\, S^{\#}_{(f)}\, p + F_{(f)}\, X_{ft}\, x_{(t)}$$

Apply now the compatibility conditions on the strains

$$X^T \begin{pmatrix} e_{(o)} \\ e_{(f)} \end{pmatrix} = \begin{pmatrix} X^T_{oo} & 0 \\ X^T_{ot} & X^T_{ft} \end{pmatrix} \begin{pmatrix} e_{(o)} \\ e_{(f)} \end{pmatrix} = 0 \quad ,$$

yielding in particular

$$X^T_{ot}\, e_{(o)} + X^T_{ft}\, e_{(f)} = 0 \quad ,$$

or, in view of (127),

$$X^T_{ot}\, e_{(o)} + X^T_{ft}\, F_{(f)}\, S^{\#}_{(f)}\, p + (X^T_{ft}\, F_{(f)}\, X_{ft})\, x_{(t)} = 0 \tag{129}.$$

Now $F_{(f)}$ being positive definite and X_{ft} having linearly independent columns, equation (129) can be solved for the unknown $x_{(t)}$.
When the result is substituted into (127), there follows

$$s_{(f)} = \{ S_{(f)} - X_{ft}\, (X^T_{ft}\, F_{(f)}\, X_{ft})^{-1}\, X^T_{ft}\, F_{(f)}\, S^{\#}_{(f)} \}\, p$$

$$- X_{ft}\, (X^T_{ft}\, F_{(f)}\, X_{ft})^{-1}\, X^T_{ot}\, e_{(o)} \tag{130}$$

This establishes the stresses in the original structure as a result of the application of external loads and the application of given strains in the additional element. Obviously we need only consider loads p applied to the set q complementary to $q_{(o)}$. As no flexibility of the additional element is involved, this formula is applicable to the problem of the hyperstatically supported structure. The additional element is then the virtual element suggested in section 1.9. incorporating the set of displacements $q_{(o)}$ which are either fixed ($q_{(o)} = 0$ and thus $e_{(o)} = 0$) or whose values are prescribed.

An interesting feature of the force method, which is in fact related to reciprocity properties and consequently to virtual work, is the Pasternak reduction principle. Evaluating the internal form of the virtual work theorem, it can be verified from (123) and (124) in the simplified form

$$s = s^{\#} + X\,x \qquad (s^{\#} = S^{\#}\,p)$$

$$e = F\,s = e^{\#} + F\,X\,x \qquad (e^{\#} = F\,s^{\#})$$

$$x = -\,(x^{T}\,F\,X)^{-1}\,X^{T}\,e^{\#}$$

that $$e^{T}\,s = e^{\#T}\,s = e^{T}\,s^{\#} \qquad (131)$$

In other words, the virtual work can be evaluated by reducing either the strains, or the stresses, to their particular statically determined part.

1.13. THE COMBINED METHOD

In the combined method, stresses and displacements are solved simultaneously. Elimination of the strains between equations (1) and (3)

$$F\,s - S^{T}\,q = 0 \qquad (132)$$

and equation (2), generate a combined linear system

$$\begin{pmatrix} F & S^{T} \\ S & 0 \end{pmatrix} \begin{pmatrix} s \\ -\,q \end{pmatrix} = \begin{pmatrix} 0 \\ p \end{pmatrix} \qquad (133)$$

whose matrix is still symmetrical.

1.14. DISCRETE VARIATIONAL PRINCIPLES

The formulations of the direct stiffness, combined, or force method can be presented as the Euler-Lagrange equations of discrete variational principles.
The variational principle of displacements, or principle of minimum of the total potential

$$\frac{1}{2} q^T K q - q^T p \qquad \text{minimum} \qquad (134)$$

features the strain energy expressed in terms of the generalized displacements and the potential energy $- q^T p$ of the loads p assumed to be known and invariant (dead loads). If the displacements actually occuring are incremented by "variations" δq, the conditions under which the total energy remains stationary to first order in the variations is

$$\delta q^T (K q - p) = 0$$

and, since the increments are arbitrary, we obtain equations (113) of the direct stiffness method. The increment of the total potential to second order (the second variation)

$$\delta q^T K \delta q$$

is non negative and the total potential is thus a minimum.
More general boundary conditions consist in specifying loads on a subset $p_{(1)}$ of p and displacements on the complementary subset $q_{(2)}$ of q. The principle takes the form

$$\frac{1}{2} q_{(1)}^T K_{11} q_{(1)} + q_{(1)}^T K_{12} q_{(2)} + \frac{1}{2} q_{(2)}^T K_{22} q_{(2)} - q_{(1)}^T p_{(1)} \qquad \text{minimum}$$

The third term is in fact a specified constant that could be dropped from the principle, were is not of interest to adopt the Lagrangian multiplier technique and restore to $q_{(2)}$ its freedom by augmenting the functional with the term $- p_{(2)}^T (q_{(2)} - u_{(2)})$.
Here $p_{(2)}$ is a set of Lagrangian multipliers, $u_{(2)}$ the set of specified values of $q_{(2)}$.
The Euler equations pertaining to variations on $q_{(1)}$ and $q_{(2)}$ are then respectively

$$K_{11} q_{(1)} + K_{12} q_{(2)} - p_{(1)} = 0$$

$$K_{21}\, q_{(1)} + K_{22}\, q_{(2)} - p_{(2)} = 0 \qquad (K_{21} = K_{12}^T)$$

The Lagrangian multipliers are thus identified with the loads $p_{(2)}$ reacting against the imposition of displacements.

If the conditions $q_{(2)} = 0$ prevent any admissible kinematical freedom of the structure, K_{11} is non singular and

$$q_{(1)} = K_{11}^{-1} (p_{(1)} - K_{12}\, q_{(2)}) \tag{135}$$

an equation that solves the problem of determining the remaining displacements in terms of the given data. Also

$$p_{(2)} = K_{21}\, K_{11}^{-1} (p_{(1)} - K_{12}\, q_{(2)}) + K_{22}\, q_{(2)} \tag{136}$$

an equation determining the reaction loads. This situation, that often prevails, is to be compared with the solution given previously by equations (121) and (122).

Consider now the Friedrichs transformation of principle (134) by which the strain energy is now expressed in terms of strains through use of (109)

$$\frac{1}{2}\, q^T K\, q = \frac{1}{2}\, q^T S\, J\, S^T q = \frac{1}{2}\, (S^T q)^T J\, (S^T q) = \frac{1}{2}\, e^T J\, e$$

and the kinematical equations (1) are incorporated through a set s of Lagrangian multipliers

$$\frac{1}{2}\, e^T J\, e + s^T (S^T q - e) - q^T p \qquad \min_{q} \left[\max_{s} \left(\min_{e} \right) \right] \tag{137}$$

variations on e produce as Euler equations

$$J\, e - s = 0 \tag{138}$$

which, in view of (3), identifies the multipliers with the stresses associated with the strains in the constitutive equations. Considering the positive definiteness of J, it is clear that for fixed q and s, (137) is a minimum with respect to that choice of e.

Variations on s produce as Euler equations the kinematical constraints (1); variations on q the equilibrium equations (2).
Consequently this 3-field variational principle contains all three basic equations (1), (2) and (3) of our problem as Euler equations.
It remains to justify its minimum-maximum character with respect to the choices on q and s.
Accepting the constitutive equations (3) or (138) as satisfied a priori and substituting the minimizing choice of e into (137), we obtain

$$-\frac{1}{2}\, s^T F\, s + s^T S^T q - q^T p \qquad \min_{q} \left(\max_{s} \right) \qquad (139)$$

Indeed, it is clear that the quadratic form in the stresses is negative definite and produces a maximum of the functional for the choice corresponding to the variational derivative (Euler equation) with respect to s :

$$-F\, s + S^T q = 0 \qquad (140)$$

This is of course equivalent to the compatibility equations (1) as modified by the constitutive equations.
The variational derivative of (139) with respect to q yields again the equilibrium equations (2). The 2-field principle (139) is <u>canonical</u> in the sense given to it by Friedrichs. In the context of linear elasticity it is better known as the Reissner principle. Should we substitute for s its maximizing value $s = J\, S^T q$, taken from (140), the principle reverts to the one given by (134), where the choice of q is minimizing.

If on the contrary we write (139) in the equivalent form

$$-\frac{1}{2}\, s^T F\, s + q^T (S\, s - p) \qquad \min_{q} \left(\max_{s} \right)$$

and assume the equilibrium equations (2) to be satisfied a priori, we obtain the single-field principle

$$-\frac{1}{2}\, s^T F\, s \qquad \max_{s} \qquad (141)$$

By changing the sign this turns into the minimum complementary energy principle.
A direct application of this principle requires that we solve the equilibrium equations. When this has been done by a suitable definition of redundancy coordinates, we can substitute equation (79) and obtain

$$\frac{1}{2}\,(p^T S^{\#T} + x^T X^T)\, F\, (S^{\#} p + X x) \qquad \min_x \qquad (142),$$

whose Euler equations are identical to (123).
Consequently as our initial single-field principle (134) is related to the direct stiffness method, the dual single-field principle (141) is related to the force method.
The variational equations of the Reissner two-field principle, equations (140) and (2) are seen to correspond to the final system (133) of the combined method.
A second two-field principle can be derived from the general principle (137) by observing that the a priori satisfaction of the equilibrium equations (2) reduces it to

$$\frac{1}{2}\, e^T J\, e - s^T e \qquad \min_e$$

Its direct applicability requires again that the equilibrium equations be solved, it then appears in the new form

$$\frac{1}{2}\, e^T J\, e - e^T (S^{\#} p + X x) \qquad \max_x \,[\min_e] \qquad (143)$$

Indeed, its variational derivative with respect to e is

$$J\, e - S^{\#} p - X x = 0 \qquad (144)$$

and, when this minimizing value of e is substituted, it turns into the functional (142) of x with a change of sign, whereby x becomes maximizing. The direct variational derivative of (143) with respect to x produces the compatibility equations

$$X^T e = 0$$

When they are solved simultaneously with (144)

$$\begin{pmatrix} J & X \\ X^T & 0 \end{pmatrix} \begin{pmatrix} e \\ -x \end{pmatrix} = \begin{pmatrix} S^{\#} p \\ 0 \end{pmatrix} \qquad (145)$$

we have a new type of combined method.

All the discrete variational principles so reviewed have corresponding formulations for the continuum. When the displacement, strain tensor or stress-tensor fields they contain are replaced by discrete representations, some of the constraints represented by equilibrium, compatibility or constitutive equations are relaxed. The principle of minimum total potential relaxes only the equilibrium equations. Its dual, the principle of minimum complementary energy relaxes only the compatibility conditions. Reissner's principle relaxes both, but retains exact constitutive equations. The second two-field principle relaxes compatibility and constitutive equations but satisfies rigorously equilibrium.

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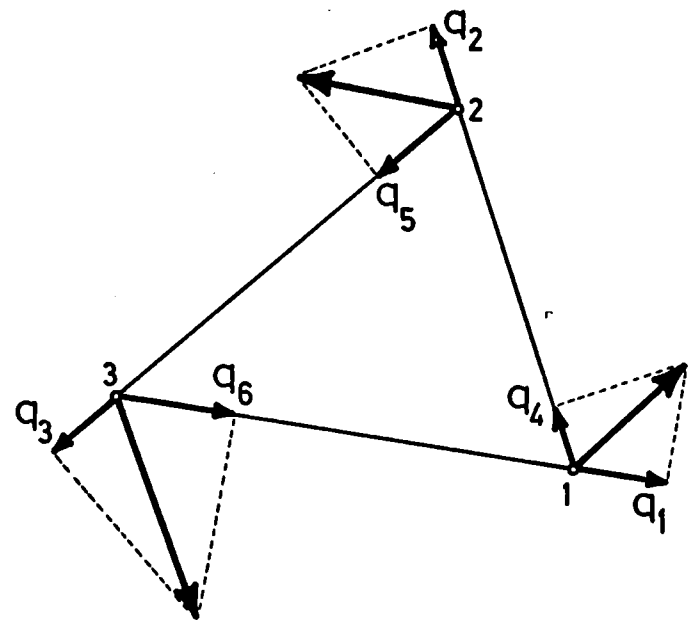

FIGURE 1 CONSTANT STRAIN TRIANGLE
COVARIANT NATURAL DISPLACEMENTS

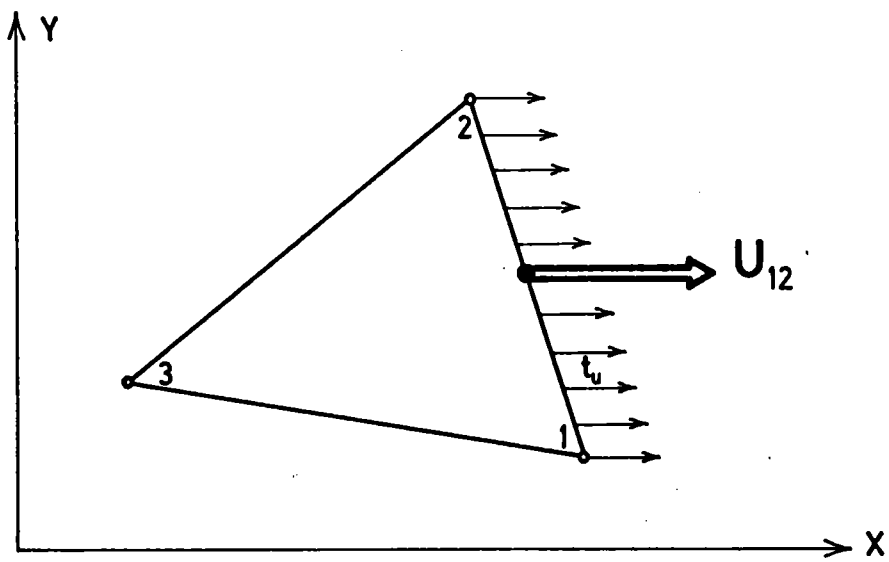

FIGURE 2 CONSTANT STRESS TRIANGLE
GENERALIZED BOUNDARY LOADS
IN GLOBAL COORDINATES