# Matrix theory of elastic wave scattering

P. C. Waterman

*8 Baron Park Lane, Burlington, Massachusetts 01803*
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Upon invoking Huygen's principle, matrix equations are obtained describing the scattering of waves by an obstacle of arbitrary shape immersed in an elastic medium. New relations are found connecting surface tractions with the divergence and curl of the displacement, and conservation laws are discussed. When mode conversion effects are arbitrarily suppressed by resetting appropriate matrix elements to zero, the equations reduce to a simultaneous description of acoustic and electromagnetic scattering by the obstacle at hand. Unification with acoustic/electromagnetics should provide useful guidelines in elasticity. Approximate numerical equality is shown to exist between certain of the scattering coefficients for hard and soft spheres. For penetrable spheres, explicit analytical results are found for the first time.

Subject Classification: [43]20.15, [43]20.30.

## INTRODUCTION

The scattering of waves in an elastic solid finds important application in a variety of fields ranging from nondestructive testing to seismic exploration, properties of composite materials, and questions of dynamic stress concentration.

Such problems were first considered, from the point of view of mathematical boundary value problems, by Clebsch, in 1863![1] An excellent history of the subject from Clebsch to the present day is given by Pao and Mow in their book,[2] and a comprehensive discussion of applications in solid-state physics may be found in the text of Truell, Elbaum, and Chick.[3]

The classical papers employing separation of variables in modern notation are those of White, for cylindrical obstacles,[4] and Ying and Truell,[5] and Einspruch *et al.*[6] who considered plane-wave incidence on spheres. A modification of White's work has recently been made by Lewis and Kraft.[7] Numerical applications of the sphere work are discussed by Johnson and Truell,[8] as well as McBride and Kraft.[9]

The purpose of this paper is to present a matrix theory of scattering by elastic obstacles of general shape. The theory is based on Huygens' principle, in the form given by Morse and Feshbach,[10,11] supplemented by new relations connecting surface tractions with the divergence and curl of the field at the surface. Insofar as possible, we try to preserve the notation employed in earlier developments of acoustic[12] and electromagnetic[13] scattering.

It is worth noting that other versions of Huygens' principle exist which might equally well serve as a starting point. One of these has been given by Pao and Varatharajulu, in which one works directly with displacements and surface tractions, but at the expense of in most cases a more complex kernel.[14] An integral representation can also be given in terms of scalar and vector potentials, as shown by Banaugh[15]; here the fields are expressed in simplest form, but enforcement of boundary conditions becomes somewhat more intractable. The situation is perhaps analogous to what happens in electromagnetic theory, where one has a choice of working with the Franz representation,[16] the Stratton–Chu formulas,[17] or scalar and vector potentials, each in principle equivalent but each having their individual nuances.

In other related work, a time-dependent version of Huygens' principle has been given by Knopoff,[18] a subject further examined by Pao and Varatharajulu.[14] A Neumann-series development appropriate for low frequencies has been presented by Hsiao and co-workers, using a regularized version of the Betti formulas.[19]

We go on to describe constraints of symmetry and unitarity on the transition matrix, based on time-reversal invariance and energy conservation. The basic boundary conditions are then taken up individually; the rigid body, the cavity, the fluid-filled cavity, and the elastic obstacle. In each instance, the equations may be specialized to a spherical object. At that point a surprise is in store; we find that Huygens' principle yields fundamentally simpler results than separation of variables! Reasons for this are discussed, along with implications on existing numerical computations.

When a purely transverse (solenoidal) or longitudinal (irrotational) wave is incident on an obstacle, in general both transverse and longitudinal waves are generated. This phenomenon is known as *mode conversion*, and is expressed in our theory by the presence of certain nonvanishing matrix elements. If mode conversion be artificially suppressed, by resetting the matrix elements in question to zero, then the present equations reduce to an independent superposition of the matrix equations for acoustic[12] and electromagnetic[13] scattering. We thus have a unified theory of acoustic, electromagnetic and elastic wave scattering by an obstacle of specified geometry. Such unification should prove invaluable, by providing the entire body of theoretical and experimental results from acoustics and electromagnetics to use as comparison standards in the elastic case.

## I. HUYGENS' PRINCIPLE

We seek the scattering from an object bounded by the closed surface $\sigma$, as shown in Fig. 1, upon illumination with a given incident wave having particle displacement $\vec{u}^i$. The object is situated in a homogeneous, iso-

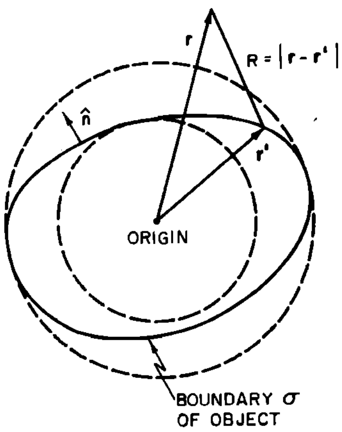


FIG. 1. Geometry of an obstacle embedded in an elastic medium.

tropic elastic medium of density $\rho$, and Lamé (stiffness) constants $\lambda$, $\mu$, within which wave motions are governed by the dynamical equation

$$\mu\nabla\times\nabla\times\vec{u}-(\lambda+2\mu)\nabla\nabla\cdot\vec{u}=-\rho\partial^2\vec{u}/\partial t^2 \tag{1}$$

in the absence of body forces. For the monochromatic waves to be considered here, with time dependence $e^{-i\omega t}$ suppressed, both incident wave $\vec{u}^i$ and scattered wave $\vec{u}^s$ obey the reduced wave equation

$$(1/\kappa^2)\nabla\times\nabla\times\vec{u}-(1/k^2)\nabla\nabla\cdot\vec{u}-\vec{u}=0\ , \tag{1$'$}$$

with transverse and longitudinal propagation constants

$$\kappa^2\equiv(\omega/c_t)^2=\omega^2\rho/\mu\ , \tag{2a}$$

$$k^2\equiv(\omega/c_l)^2=\omega^2\rho/(\lambda+2\mu)\ , \tag{2b}$$

$c_t$ and $c_l$ being the respective phase velocities.

Huygens' principle is now obtained by applying the divergence theorem separately to the quantities

$$(\nabla\times\vec{u})\times\mathcal{G}+\vec{u}\times(\nabla\times\mathcal{G})\qquad(\mathcal{G}\text{ the Green's dyadic})$$

and

$$(\nabla\cdot\vec{u})\mathcal{G}-\vec{u}(\nabla\cdot\mathcal{G})\ ,$$

then taking an appropriate linear combination of the two resulting equations. The first expression, incidentally, was employed by Franz for electromagnetic problems,[16] the second is appropriate for acoustic problems. We identify $\vec{u}=\vec{u}^i+\vec{u}^s$ with the total field; $\mathcal{G}$ is to be the free-space Green's dyadic, defined by

$$(1/\kappa^2)\nabla\times\nabla\times\mathcal{G}-(1/k^2)\nabla\nabla\cdot\mathcal{G}-\mathcal{G}=\mathcal{I}\delta(\vec{r}-\vec{r}')\ , \tag{3}$$

with $\mathcal{I}$ the identity dyadic. Further requiring that the scattered wave be outgoing at infinity we can now write, following Morse and Feshbach,[10,11]

$$\begin{aligned}\vec{u}^i(\vec{r})+\kappa\int d\sigma'\{&\mathcal{G}\cdot[\hat{n}'\times(\nabla'\times\vec{u})_+]+(\hat{n}'\times\vec{u}_+)\cdot(\nabla'\times\mathcal{G})\\&-(\kappa/k)^2[(\hat{n}'\cdot\mathcal{G})(\nabla'\cdot\vec{u})_+-(\nabla'\cdot\mathcal{G})(\hat{n}'\cdot\vec{u}_+)]\}\\&=\begin{cases}\vec{u}(\vec{r}), & \vec{r}\text{ outside }\sigma\ ,\\ 0\ , & \vec{r}\text{ inside }\sigma\ .\end{cases}\end{aligned} \tag{4}$$

In this equation the unit normal $\hat{n}'$ points outward, away from the volume enclosed by $\sigma$, and $\mathcal{G}(\kappa,k,|\vec{r}-\vec{r}'|)$ of course depends on distance from integration point to field point in the usual manner. Plus (or minus) subscripts appearing on $\vec{u}$ and its derivatives indicate that corresponding quantities are to be evaluated on the surface in the limit approached from the outside (or inside).

In the exterior region $\vec{r}$ outside $\sigma$, Eq. (4) gives a prescription for evaluating the scattered wave, given by the integral, by quadrature of the (presently unknown) surface field, and its divergence and curl. In the interior, on the other hand, one sees that the field expressed by the surface integral must precisely extinguish the incident wave. We will make use of this assertion, but first we must introduce a set of basis functions.

As noted by Morse and Feshbach,[10] the outgoing spherical partial wave solutions of Eq. (1) require four indices for their specification; following a notation used earlier for the transverse functions,[13] we write $\vec{\psi}_{\tau\sigma mn}(\vec{r})$, where $\tau=1,2,3$ distinguishes the two transverse waves and the longitudinal wave, respectively, $\sigma=e$, 0 (even, odd) specifies azimuthal parity, $m=0$, $1,\ \ldots,\ n$ specifies rank, and $n=0,\ 1,\ 2,\ \ldots$ order of the spherical harmonics involved. Whenever the various indices needn't appear specifically, economy of notation is achieved by writing simply $\vec{\psi}_n(\vec{r})$, with the understanding that $n$ now runs through all cases included in $(\tau\sigma mn)$. Sometimes it will be convenient to exhibit $\tau$ explicitly; in that event, we write $\vec{\psi}_{\tau n}(\vec{r})$.

With all this in mind, the basis functions are given by

$$\begin{aligned}\vec{\psi}_{1\sigma mn}(\vec{r})&=(1/\kappa)\nabla\times\vec{\psi}_{2\sigma mn}=\gamma_{mn}^{1/2}[n(n+1)]^{-1/2}\nabla\\&\quad\times[\vec{r}Y_{\sigma mn}(\vartheta,\varphi)h_n(\kappa r)]\\&=\gamma_{mn}^{1/2}\vec{A}_{1\sigma mn}(\vartheta,\varphi)h_n(\kappa r)\\&\xrightarrow[\kappa r\gg1]{}\gamma_{mn}^{1/2}(-i)^{n+1}(1/\kappa r)\,e^{i\kappa r}\vec{A}_{1\sigma mn}(\vartheta,\varphi)\ ,\\ \vec{\psi}_{2\sigma mn}(\vec{r})&=(1/\kappa)\nabla\times\vec{\psi}_{1\sigma mn}=\gamma_{mn}^{1/2}[n(n+1)]^{-1/2}(1/\kappa)\nabla\\&\quad\times\nabla\times[\vec{r}Y_{\sigma mn}(\vartheta,\varphi)h_n(\kappa r)]\\&=\gamma_{mn}^{1/2}\{\vec{A}_{2\sigma mn}(\vartheta,\varphi)(1/\kappa r)[\kappa r h_n(\kappa r)]'\\&\quad+[n(n+1)]^{1/2}\vec{A}_{3\sigma mn}(\vartheta,\varphi)(1/\kappa r)h_n(\kappa r)\}\\&\xrightarrow[\kappa r\gg1]{}\gamma_{mn}^{1/2}(-i)^n(1/\kappa r)\,e^{i\kappa r}\vec{A}_{2\sigma mn}(\vartheta,\varphi)\ ,\\ \vec{\psi}_{3\sigma mn}(\vec{r})&=\gamma_{mn}^{1/2}(k/\kappa)^{3/2}(1/k)\nabla[Y_{\sigma mn}(\vartheta,\varphi)h_n(kr)]\\&=\gamma_{mn}^{1/2}(k/\kappa)^{3/2}\{\vec{A}_{3\sigma mn}(\vartheta,\varphi)h_n'(kr)\\&\quad+[n(n+1)]^{1/2}\vec{A}_{2\sigma mn}(\vartheta,\varphi)(1/kr)h_n(kr)\}\\&\xrightarrow[\kappa r\gg1]{}\gamma_{mn}^{1/2}(k/\kappa)^{3/2}(-i)^n(1/kr)\,e^{ikr}\vec{A}_{3\sigma mn}(\vartheta,\varphi)\ ,\end{aligned} \tag{5a}$$

in terms of the spherical Hankel functions of the first kind $h_n(\kappa r)$, a prime designating the derivative with respect to the entire argument. The scalar and vector spherical harmonics are given in terms of the associated Legendre functions $P_n^m$ as

$$\begin{aligned}Y_{(e/0)mn}(\vartheta,\varphi)&=\binom{\cos}{\sin}m\varphi P_n^m(\cos\theta)\ ,\\ \vec{A}_{1\sigma mn}(\vartheta,\varphi)&=-\hat{r}\times\vec{A}_{2\sigma mn}=[n(n+1)]^{-1/2}\nabla\times(\vec{r}Y_{\sigma mn})\ ,\\ \vec{A}_{2\sigma mn}(\vartheta,\varphi)&=\hat{r}\times\vec{A}_{1\sigma mn}=[n(n+1)]^{-1/2}r\nabla Y_{\sigma mn}\ ,\\ \vec{A}_{3\sigma mn}(\vartheta,\varphi)&=\hat{r}Y_{\sigma mn}\ ,\end{aligned} \tag{5b}$$

with orthogonality properties

$$\vec{A}_{\tau\sigma mn}\cdot\vec{A}_{\tau'\sigma mn}=0 \text{ for } \tau'\neq\tau\ ,$$

$$\gamma_{mn}\int d\Omega Y_{\sigma mn}Y_{\sigma' m' n'}=\gamma_{mn}\int d\Omega\vec{A}_{\tau\sigma mn}\cdot\vec{A}_{\tau\sigma' m' n'} \tag{5c}$$

$$=\pi\delta_{\sigma\sigma'}\delta_{mm'}\delta_{nn'}\ ,$$

integration being carried over the unit sphere. The purely transverse or longitudinal nature of the $\vec{A}_n$, and hence in the farfield the $\vec{\psi}_n$, is evident. The normalizing constants are defined as

$$\gamma_{mn}=\epsilon_m(2n+1)(n-m)!/4(n+m)!\ , \tag{5d}$$

where the Neumann factor $\epsilon_m=1$ for $m=0$, $\epsilon_m=2$ otherwise. Except for normalization the $\vec{\psi}$ functions are precisely the $\vec{M}$, $\vec{N}$, and $\vec{L}$ functions employed by Morse and Feshbach; and $\vec{A}$ functions correspond exactly to $\vec{C}$, $\vec{B}$, and $\vec{P}$.[10] Our functions are normalized so that, to within a *common* factor, each $\vec{\psi}_n$ carries unit energy flux out of any closed surface containing the origin.

The $\{\vec{\psi}_n\}$ are a complete set suitable to represent the scattered wave everywhere outside of the spherical surface circumscribing the object (Fig. 1). We also require the wave functions $\{\mathrm{Re}\vec{\psi}_n\}$ regular at the origin, obtained by taking the real part of the $\vec{\psi}_n$ (yielding Bessel rather than Hankel function radial dependence). The incident and scattered waves can now be written

$$\begin{aligned}\vec{u}^i(\vec{r})&=\sum a_n\,\mathrm{Re}\vec{\psi}_n(\vec{r}), \quad r<r_{\min\text{ on }\sigma}\\ \vec{u}^s(\vec{r})&=\sum f_n\,\vec{\psi}_n(\vec{r}), \quad r>r_{\max\text{ on }\sigma}\end{aligned} \tag{6}$$

for field points inside the inscribed sphere, or outside the circumscribed sphere of Fig. 1, respectively, where the incident wave has been assumed to have no sources in the interior of the object, although they may be present anywhere outside the boundary $\sigma$. Assuming linear boundary conditions, our main goal will be to determine the *transition matrix* $T$ which computes the scattered wave directly from the incident wave by the prescription

$$f_n=\sum T_{nn'}a_{n'},\quad n=1,2,\ \dots$$

or, in obvious matrix notation,

$$f=Ta\ . \tag{7}$$

The situation is complicated by the fact that we are not able to work directly on the boundary with the expansions of Eq. (6). Fortunately, however, Huygens' principle enables us to overcome this difficulty. The Green's dyadic is first expanded as[10]

$$\mathcal{G}(\kappa,k,|\vec{r}-\vec{r}'|)=(i/\pi)\sum\vec{\psi}_n(\vec{r}_>)\,\mathrm{Re}\vec{\psi}_n(\vec{r}_<)\ . \tag{8}$$

This expansion is uniformly convergent for $r\neq r'$, with $r_>$, $r_<$ respectively the greater and lesser of $r$, $r'$. Now substituting Eq. (8) in Eq. (4), the scattered wave in the exterior region is found precisely in the form given by Eq. (6), with expansion coefficients

$$f_n=(i\kappa/\pi)\int d\sigma\{(\mathrm{Re}\vec{\psi}_n)\cdot[\hat{n}\times(\nabla\times\vec{u})_+]+(\nabla\times\mathrm{Re}\vec{\psi}_n)\cdot(\hat{n}\times\vec{u}_+)$$
$$-(\kappa/k)^2[(\hat{n}\cdot\mathrm{Re}\vec{\psi}_n)(\nabla\cdot\vec{u})_+-(\nabla\cdot\mathrm{Re}\vec{\psi}_n)(\hat{n}\cdot\vec{u}_+)]\}\ .$$
$$n=1,2,\ \dots\ . \tag{9a}$$

On the other hand, for field points inside the inscribed sphere of Fig. 1, the entire left-hand side of Eq. (4) takes the form of an expansion in the regular functions $(\mathrm{Re}\vec{\psi}_n)$. Because of orthogonality, each coefficient must vanish separately, giving

$$a_n=-(i\kappa/\pi)\int d\sigma\{\vec{\psi}_n\cdot[\hat{n}\times(\nabla\times\vec{u})_+]+(\nabla\times\vec{\psi}_n)\cdot(\hat{n}\times\vec{u}_+)$$
$$-(\kappa/k)^2[(\hat{n}\cdot\vec{\psi}_n)(\nabla\cdot\vec{u})_+-(\nabla\cdot\vec{\psi}_n)(\hat{n}\cdot\vec{u}_+)]\}\ ,$$
$$n=1,2,\ \dots\ . \tag{9b}$$

These latter equations are necessary and sufficient conditions for satisfying Eq. (4) within the inscribed sphere. Because of the continuation property of solutions of elliptic partial differential equations, it follows that Eq. (4) will then in actuality be satisfied throughout the interior of the object.

A problem will arise when we attempt to apply boundary conditions. The physical description of behavior at the boundary will involve the elastic displacement $\vec{u}$ along with the surface traction $\vec{t}$, whereas Eqs. 9 are expressed in terms of $\vec{u}$ and its divergence and curl. To resolve this difficulty we make a brief sojourn into differential geometry. Introduce a set of orthogonal curvilinear coördinates $(v_1,\ v_2,\ v_3)$ known as Dupin coördinates, as follows[20,21]: $v_3$ = constant defines the surface $\sigma$ of our object, so that $\hat{v}_3=\hat{n}$ = unit normal. $v_1$ and $v_2$, which span the surface, are chosen along the lines of curvature (that is, those arcs along the surface for which consecutive normals intersect). Differential length $dl$ in space on and just outside the surface is given by

$$(dl)^2=h_1^2(dv_1)^2+h_2^2(dv_2)^2+h_3^2(dv_3)^2\ , \tag{10}$$

in terms of the metric coefficients

$$h_1^2=(\partial\vec{r}/\partial v_1)\cdot(\partial\vec{r}/\partial v_1)\ ,$$
$$h_2^2=(\partial\vec{r}/\partial v_2)\cdot(\partial\vec{r}/\partial v_2)\ ,$$
$$h_3=1\ ,$$

where $\vec{r}(v_1,\ v_2)$ is the position vector to points on the surface. The *surface gradient*, *divergence*, and *curl* are now defined as[20,21]

$$\nabla_s f=(1/h_1)\hat{v}_1\partial f/\partial v_1+(1/h_2)\hat{v}_2(\partial f/\partial v_2)\ ,$$
$$\nabla_s\cdot\vec{F}=(1/h_1h_2)[\partial(h_2F_1)/\partial v_1+\partial(h_1F_2)/\partial v_2]+JF_3\ , \tag{11}$$
$$\nabla_s\times\vec{F}=(\nabla_sF_3)\times\hat{n}-(F_2/R_2)\hat{v}_1+(F_1/R_1)\hat{v}_2+\hat{n}\nabla_s\cdot(\vec{F}\times\hat{n})\ ,$$

where $J\equiv(R_1)^{-1}+(R_2)^{-1}$ is the *mean curvature*, and the principal radii $R_i$ are taken as positive if the surface is convex when viewed from the outside.

The surface traction is defined as $\vec{t}=\hat{n}\cdot\overleftrightarrow{\sigma}$. Invoking Hooke's law relating the stress dyadic $\overleftrightarrow{\sigma}$ to the displacements gives[22]

$$\vec{t}=\lambda\hat{n}(\nabla\cdot\vec{u})+\mu\hat{n}\cdot(\nabla\vec{u}+\vec{u}\nabla)\ . \tag{12}$$

Writing out $\vec{t}$, as well as $\nabla\cdot\vec{u}$ and $\hat{n}\times(\nabla\times\vec{u})$, in Dupin coördinates, a linear combination of the three quantities can be chosen so as to eliminate normal derivatives of the displacement [which are really the troublesome terms in Eqs. (10) and (11)]. Comparing with Eq. (11), we end up with the equation

$$\vec{t}+\mu\hat{n}\times(\nabla\times\vec{u})-(\lambda+2\mu)\hat{n}(\nabla\cdot\vec{u})=2\mu[\hat{n}\times(\nabla_s\times\vec{u})-\hat{n}(\nabla_s\cdot\vec{u})] , \tag{13}$$

the tangential and normal components of which give

$$\hat{n}\times(\nabla\times\vec{u})=\mu^{-1}[2\mu\hat{n}\times(\nabla_s\times\vec{u})-\vec{t}_{\tan}] , \tag{13a}$$

$$\nabla\cdot\vec{u}=(\lambda+2\mu)^{-1}[2\mu(\nabla_s\cdot\vec{u})+\hat{n}\cdot\vec{t}] . \tag{13b}$$

These equations allow us to re-express the unknown surface fields appearing in Eq. (9b) entirely in terms of $\vec{u}_+$ and its derivatives *along* the surface, and $\vec{t}_+$. Note that the surface curl and divergence are invariants, so that we may in fact choose whatever coördinates we desire to describe the surface; expressions for these quantities in general coördinates are given by Weatherburn.[20]

Letting $\tau$ take on each of its three allowed values, Eq. (9b) constitutes three sets of equations for six scalar unknowns, the components of $\vec{u}_+$ and $\vec{t}_+$ [or, alternately, $\vec{u}_+$, $\hat{n}\times(\nabla\times\vec{u})_+$ and $(\nabla\cdot\vec{u})_+$]. Three additional equations must be provided by the boundary conditions. Once all surface fields have been determined, the scattered wave coefficients can be computed from Eq. (9a).

A useful identity emerges if we consider the trivial case for which the obstacle is indistinguishable from its surroundings, so that in fact no scattering occurs. In this case, the total field is simply equal to the incident wave everywhere inside and outside the obstacle. From Eq. (6),

$$\vec{u}=\sum a_n \operatorname{Re}\vec{\psi}_n ,$$

and substitution in Eq. (9b) leads to

$$a=-iC'a , \tag{14a}$$

the prime designating transpose, where the matrix elements of $C$ are defined as

$$C_{nn'}=(\kappa/\pi)\int d\sigma\{[\hat{n}\times(\nabla\times\operatorname{Re}\vec{\psi}_n)]\cdot\vec{\psi}_{n'}+(\hat{n}\times\operatorname{Re}\vec{\psi}_n)\cdot(\nabla\times\vec{\psi}_{n'}) - (\kappa/k)^2[(\nabla\cdot\operatorname{Re}\vec{\psi}_n)(\hat{n}\cdot\vec{\psi}_{n'})-(\hat{n}\cdot\operatorname{Re}\vec{\psi}_n)(\nabla\cdot\vec{\psi}_{n'})]\} . \tag{14b}$$

Note, incidentally, that many terms in the integrand are identically zero because of the relations $\nabla\cdot\vec{\psi}_{1n}=\nabla\cdot\vec{\psi}_{2n}=\nabla\times\vec{\psi}_{3n}=0$, etc. If Eq. (14a) is to be satisfied, it must be true that

$$-iC'=\text{Identity matrix}. \tag{14c}$$

Using the definitions of the wave functions, and the electromagnetic and acoustic divergence theorems mentioned earlier, one can verify that this identity is satisfied. Furthermore, Eq. (9a) for the scattered wave becomes

$$f=i\operatorname{Re}(C')a=0 , \tag{14d}$$

which vanishes in view of Eq. (14c), so that, as expected, no scattering occurs.

## II. CONSERVATION LAWS

The requirements of energy conservation and time-reversal invariance impose constraints determining about three-quarters of the degrees of freedom of the transition matrix.[13] To derive these constraints, notice that the total field may be written

$$\vec{u}=a'\operatorname{Re}\vec{\psi}+f'\vec{\psi} , \tag{15a}$$

where,

$$f=Ta \tag{15b}$$

and the prime denotes transpose, so that $a'$ is a *row* vector having entries $a_1, a_2, \ldots$. The quantity $\operatorname{Re}\vec{\psi}$ is a column vector, each entry of which is a vector in the ordinary sense, i.e., $\operatorname{Re}\vec{\psi}_1$, $\operatorname{Re}\vec{\psi}_2$, .... This may be changed over to a mathematically equivalent basis of outgoing and ingoing waves, $\{\vec{\psi}_n\}$ and $\{\vec{\psi}_n^*\}$, respectively. Noting that $2\operatorname{Re}\vec{\psi}_n=\vec{\psi}_n+\vec{\psi}_n^*$ we have

$$\vec{u}=\tfrac{1}{2}(a'\vec{\psi}^*+b'\vec{\psi}) , \tag{16a}$$

in which the *scattering matrix* $S$ can be defined to compute the outgoing waves from the ingoing, i.e.,

$$b=Sa . \tag{16b}$$

Comparison of these equations shows that

$$S\equiv 1+2T . \tag{16c}$$

Now compute from Eq. (16a) the net energy flux out of any spherical surface enclosing the obstacle, which of course must vanish if there is no dissipation. Because of orthonormality, we readily find that

$$0=b'^*b-a'^*a=a'^*(S'^*S-1)a .$$

But the incident-wave coefficients are quite arbitrary; it follows that

$$S'^*S=1 \ (\text{or } T'^*T=-\operatorname{Re}T) , \tag{17a}$$

i.e., $S$ is unitary.

In addition, the field $\vec{u}$ must remain a solution upon time reversal. This corresponds to taking the complex conjugate of $\vec{u}$, giving

$$\vec{u}^*=\tfrac{1}{2}(a'^*\vec{\psi}+b'^*\vec{\psi}^*) .$$

In terms of $S$ one has $a^*=Sb^*$, or

$$a=S^*b=S^*Sa .$$

Again because of arbitrariness of the $\{a_n\}$ it is necessary that $S^*S=1$, and comparing with Eq. (17a) we find

$$S'=S \ (\text{or } T'=T) , \tag{17b}$$

i.e., $S$ (or $T$) is symmetric.

For plane-wave incidence, one consequence of this symmetry is that the scattering coefficients relating to mode conversion, transverse-to-longitudinal and longitudinal-to-transverse, must be equal. In two dimensions, this equality was demonstrated by White for circular cylinders under certain boundary conditions,[4] and more recently by Lewis and Kraft for all cases.[7] In neither instance, however, was the equality recognized to be due to conservation laws.

Anticipating later results, the matrix equation determining $S$ has the general form

$$QS = -Q^* \text{ (or } QT = -\mathrm{Re}\, Q) , \tag{18}$$

where $Q$ is a known matrix, the exact nature of whose elements depends on the boundary conditions at hand. This equation may be solved, subject to constraints of symmetry and unitarity, by the following formal procedure.[13] We first convert $Q$ to a unitary matrix $\hat{Q}$. This is done by truncating to $N$ rows and columns, then employing Schmidt orthogonalization on row vectors, starting at the bottom. In matrix form one has simply

$$\hat{Q} = MQ , \tag{19}$$

with $M$ upper-triangular. The bottom row of $M$ (single nonzero entry) normalizes the last row vector of $Q$, the next-to-bottom row (two nonzero entries) chooses an appropriate linear combination of the last two rows of $Q$, and so on. We agree during this process to choose diagonal elements of $M$ to be real, which can be done.

Upon premultiplying Eq. (18) by $M$ one gets

$$\hat{Q}S = -MQ^* = -MM^{*-1}M^*Q^* = -MM^{*-1}\hat{Q}^* ,$$

or

$$S = -\hat{Q}'^*(MM^{*-1})\hat{Q}^* . \tag{20}$$

From the fact that $M$ is upper triangular with real diagonal elements, it readily follows that the product $MM^{*-1}$ is unit upper-triangular, i.e., all diagonal elements equal one. Furthermore, in the limit of infinite matrix size symmetry of $S$ implies, from Eq. (20), that $MM^{*-1}$ must also be symmetric. At this point, $MM^{*-1}$ can only be the identity matrix (that is, $M$ is real). Using this limit in Eq. (20) gives a new sequence of truncated solutions

$$S = -\hat{Q}'^*\hat{Q}^* \text{ (or } T = -\hat{Q}'^* \mathrm{Re}\, \hat{Q}) , \tag{21}$$

which are, at each truncation, exactly symmetric and unitary.

Detailed numerical comparison of solutions in the form of Eq. (21), versus standard matrix inversion techniques applied to Eq. (18), for the somewhat simpler electromagnetic case, reveals that Eq. (21) is far superior from the point of view of numerical convergence.[13] The computer program for carrying this process out has been documented elsewhere.[23]

The invariance properties of $T$ can also be used to simplify the computation of scattering cross section. First we define farfield scattering amplitudes, using the asymptotic form of the wave functions in Eq. (6) to get

$$\vec{u}^s(\vec{r}) \xrightarrow[\kappa r \gg 1]{} (e^{i\kappa r}/\kappa r)\vec{F}_t(\hat{\kappa}) + (e^{ikr}/kr)\, F_l(\hat{k}) , \tag{22}$$

where the transverse and longitudinal amplitudes are given, respectively, by

$$F_t(\hat{\kappa}) \equiv \sum (-i)^n \gamma_{mn}^{1/2} [-i\vec{A}_{1n}(\hat{\kappa}) f_{1n} + \vec{A}_{2n}(\hat{\kappa}) f_{2n}] , \tag{22a}$$

$$F_l(\hat{k}) \equiv (k/\kappa)^{3/2} \sum (-i)^n \gamma_{mn}^{1/2} \vec{A}_{3n}(\hat{k}) f_{3n} . \tag{22b}$$

The energy flux associated with $\vec{u}^s$ may now be integrated over $4\pi$ steradians to get the scattering cross section[13]

$$\sigma_s = (16\pi/\kappa^2) \sum |f_n|^2 , \tag{23}$$

with coefficients of both transverse and longitudinal scattered waves included in the summation. Now, using our shorthand notation along with the invariance properties of Eqs. (17a) and (17b), the summation may be transformed as follows:

$$\sum |f_n|^2 = f'^* f = a'^* T'^* Ta = -a'^* \mathrm{Re}\,(T)a = -\mathrm{Re}\,(a'^* Ta),$$

which gives us the alternate formula

$$\sigma_s = -(16\pi/\kappa^2)\, \mathrm{Re}\Big(\sum a_n^* f_n\Big) . \tag{23'}$$

All of this may be specialized to plane wave incidence. From the closed-form expression for the Green's dyadic[10,11] we find, letting the source point $r_>$ go to infinity in the direction $-\hat{\kappa}_i = -\hat{k}_i$,

$$\mathcal{G} \xrightarrow[\kappa r_> \gg 1]{} (1/4\pi\kappa r_>) \exp[i(\kappa r_> + \hat{\kappa}_i \cdot \vec{r}_<)](\vec{1} - \hat{\kappa}_i\hat{\kappa}_i) + (k/\kappa)^3 (1/4\pi k r_>) \exp[i(kr_> + \vec{k}_i \cdot \vec{r}_<)]\hat{k}_i\, \hat{k}_i .$$

Comparing this with the asymptotic form of Eq. (8), the transverse and longitudinal incident plane wave expansions are found to be

$$\vec{u}^i(\vec{r}) = \hat{u}_0 \exp(i\vec{\kappa}_i \cdot \vec{r}) \quad (\hat{u}_0 \cdot \hat{\kappa}_i = 0)$$
$$= -4\sum (i)^{n+1} \gamma_{mn}^{1/2} [i\vec{A}_{1n}(\hat{\kappa}_i)\, \mathrm{Re}\vec{\psi}_{1n}(\vec{r}) + \vec{A}_{2n}(\hat{\kappa}_i)\, \mathrm{Re}\vec{\psi}_{2n}(\vec{r})] , \tag{24a}$$

$$\vec{u}^i(\vec{r}) = \hat{u}_0 \exp(i\vec{k}_i \cdot \vec{r})(\hat{u}_0 = \hat{k}_i)$$
$$= -4(\kappa/k)^{3/2} \sum (i)^{n+1} \gamma_{mn}^{1/2} \vec{A}_{3n}(\hat{k}_i)\, \mathrm{Re}\vec{\psi}_{3n}(\vec{r}) , \tag{24b}$$

respectively. Upon inserting these values of the incident wave expansion coefficients in Eq. (23'), we find

$$\sigma_s = -(64\pi/\kappa^2)\, \mathrm{Im}[\hat{u}_0 \cdot \vec{F}_t(\hat{\kappa}_i)] \tag{25a}$$

and

$$\sigma_s = -(64\pi/\kappa^2)\, \mathrm{Im}[\hat{u}_0 \cdot \vec{F}_l(\hat{k}_i)] , \tag{25b}$$

for transverse and longitudinal plane wave incidence, respectively. Equations (25a) and (25b) were obtained originally by Barrett and Collins by a quite different method.[24] These equations, or by the same token Eq. (23'), have a computational advantage over the more commonly employed Eq. (23) in that the coefficients associated with mode conversion no longer appear.

## III. THE RIGID BODY

Having established the basic machinery for the computation, we now consider various boundary conditions. The simplest case arises when the obstacle is rigid and fixed (the limit of very stiff, very dense material) so that

$$\vec{u}_+ = 0, \text{ on } \sigma . \tag{26}$$

As noted by Pao and Mow,[2] the physical interpretation of this case is subject to some question, as it does not have Rayleigh behavior at low frequencies; it is nevertheless an instructive mathematical example.

Two terms now vanish in the integrands of Eqs. (9a) and (9b), leaving us with

$$f_n = (i\kappa/\pi)\int d\sigma\{\mathrm{Re}\,\vec{\psi}_n\cdot[\hat{n}\times(\nabla\times\vec{u})_+] - (\kappa/k)^2(\hat{n}\cdot\mathrm{Re}\,\vec{\psi}_n)(\nabla\cdot\vec{u})_+\}\,, \tag{27a}$$

$$a_n = -(i\kappa/\pi)\int d\sigma\{\vec{\psi}_n\cdot[\hat{n}\times(\nabla\times\vec{u})_+] - (\kappa/k)^2(\hat{n}\cdot\vec{\psi}_n)(\nabla\cdot\vec{u})_+\}\,. \tag{27b}$$

Knowing the incident-wave coefficients $\{a_n\}$, we must use Eqs. (27b) to determine the surface fields $\hat{n}\times(\nabla\times\vec{u})_+$ and $(\nabla\cdot\vec{u})_+$; the desired scattering coefficients $\{f_n\}$ are then given explicitly by Eqs. (27a).

Although we will not pursue it further, it is worth noting that Eqs. (27a) and (27b) take on remarkably simple form in terms of the surface tractions. If $\vec{u}_+ = 0$, it follows that the *surface* divergence and curl of $\vec{u}_+$ will also vanish, and using Eqs. (13a) and (13b) we can write

$$\begin{aligned}\vec{\psi}_n\cdot[\hat{n}\times(\nabla\times\vec{u})_+] - (\kappa/k)^2(\hat{n}\cdot\vec{\psi}_n)(\nabla\cdot\vec{u})_+ &= -\mu^{-1}[\vec{\psi}_n\cdot\vec{t}_{\tan} + (\hat{n}\cdot\vec{\psi}_n)(\hat{n}\cdot\vec{t})] \\ &= -\mu^{-1}\vec{\psi}_n\cdot\vec{t}\;,\end{aligned}$$

from which we find

$$f_n = -(i\kappa/\pi\mu)\int d\sigma\,\mathrm{Re}(\vec{\psi}_n)\cdot\vec{t}\;, \tag{27a$'$}$$

$$a_n = (i\kappa/\pi\mu)\int d\sigma\,\vec{\psi}_n\cdot\vec{t}\;. \tag{27b$'$}$$

These equations offer an alternate method for solving the problem directly in terms of surface tractions, and could have been derived directly from the Pao–Varatharajulu representation.[14]

Returning to the first form of the equations, we proceed to expand the unknown surface fields essentially in terms of the regular wave functions $\{\mathrm{Re}\,\vec{\psi}_n\}$. This is precisely the technique used with good success in both acoustics[12] and electromagnetics.[13] Completeness of the regular wave functions in this sense has been demonstrated for the acoustic case[12,25]; from this it follows that such expansions will converge in mean square sense. Thus assume that

$$\begin{aligned}\hat{n}\times(\nabla\times\vec{u})_+ &= \sum_{\tau=1,2}\alpha_n\,\hat{n}\times(\nabla\times\mathrm{Re}\,\vec{\psi}_n) \\ (\nabla\cdot\vec{u})_+ &= \sum_{\tau=3}\alpha_n(\nabla\cdot\mathrm{Re}\,\vec{\psi}_n)\quad \vec{r}\text{ on }\sigma\,.\end{aligned} \tag{28}$$

Note that both sets of transverse wave functions are needed to represent the tangential components of $(\nabla\times\vec{u})_+$ (two degrees of freedom); the single set of longitudinal waves suffices for $(\nabla\cdot\vec{u})_+$, which of course is a scalar.

Introducing the matrix $Q$ with elements

$$Q_{nn'} = (\kappa/\pi)\int d\sigma\{[\hat{n}\times(\nabla\cdot\mathrm{Re}\vec{\psi}_n)]\cdot\vec{\psi}_{n'} - (\kappa/k)^2(\nabla\cdot\mathrm{Re}\vec{\psi}_n)(\hat{n}\cdot\vec{\psi}_{n'})\}\,, \tag{29}$$

the substitution of Eq. (28) into Eqs. (27a) and (27b) yields

$$f = i\,\mathrm{Re}(Q')\,\alpha\,, \tag{30a}$$

$$a = -iQ'\alpha\,. \tag{30b}$$

The latter equation may be solved in truncation to get the surface fields in terms of the known incident wave $a$, the scattered wave $f$ then being given by Eq. (30a). On the other hand, the surface fields may be eliminated, and the physically more interesting scattered wave determined directly; i.e., $f = -\mathrm{Re}(Q')(Q')^{-1}a$. By comparison with Eq. (7), the transition matrix is now seen to be determined by (using the fact that $T$ is symmetric)

$$QT = -\mathrm{Re}\;Q\;. \tag{31}$$

The recommended procedure for solving this equation was discussed earlier.

It is instructive to look more closely at the behavior of $Q$ versus the individual transverse ($\tau = 1, 2$) or longitudinal ($\tau = 3$) modes involved. From Eq. (29) we have

$$Q_{\tau n\tau' n'} = \begin{cases}(\kappa/\pi)\int d\sigma[\hat{n}\times(\nabla\times\mathrm{Re}\,\vec{\psi}_{\tau n})]\cdot\vec{\psi}_{\tau' n'}\,, & \tau = 1, 2\,, \\ -(\kappa/\pi)(\kappa/k)^2\int d\sigma(\nabla\cdot\mathrm{Re}\,\vec{\psi}_{\tau n})(\hat{n}\cdot\vec{\psi}_{\tau' n'})\,, & \tau = 3\,.\end{cases} \tag{32}$$

Suppose $Q$ to be partitioned into a supermatrix, each "element" of which is a $3\times3$ matrix generated as $\tau, \tau'$ run through allowed values. Now mode conversion from transverse to longitudinal, or vice versa, occurs because of the presence of nonzero elements $(\tau\tau') = 13$, 23, 31, and 32. If we arbitrarily reset these four entries to zero for the moment, then the element of $Q$ takes the form

$$Q_{\tau\tau'} = \begin{pmatrix} X & X & 0 \\ X & X & 0 \\ 0 & 0 & X\end{pmatrix}\,. \tag{33}$$

Comparing with earlier work, we find that the $2\times2$ array in Eq. (33) is precisely the $Q$ matrix element for electromagnetic scattering by a perfectly conducting body[13]; similarly, the 33 element in Eq. (33) is exactly that for acoustic scattering by a hard body.[12] For the electromagnetic case, $\rho$ and $\mu$ are identified with the dielectric constant, and the reciprocal of the magnetic permeability, respectively. Identifications for the acoustic case are obvious. This of course isn't too surprising, because the elastic wave Eq. (1) reduces to the Maxwell equation, or the scalar Helmholtz equation, given that $\vec{u}$ is either solenoidal or irrotational. Notice also that both boundary conditions, vanishing of tangential electric field, or normal acoustic displacement, are effectively present in the elastic boundary condition, Eq. (26).

Physically, this says that the elastic scattering problem would be an independent superposition of the electromagnetic and acoustic cases were it not for the presence of mode conversion. From the computational point of view, we see that it is possible to write a unified computer program, including a switch to reset to zero elements indicated in Eq. (33), which would with little further effort encompass acoustic, electromagnetic and elastic wave scattering from a given body. Such a unified approach offers great leverage for the elastic case, as noted in the Introduction.

If we specialize to the sphere of radius $r = a$, then elements of $Q$ may be evaluated using the orthogonality relations for the spherical harmonics. Most of the elements vanish; the index notation can be simplified by writing (Kronecker delta)

$$Q_{\tau\sigma mn\tau'\sigma'm'n'} = \delta_{\sigma\sigma'}\,\delta_{mm'}\,\delta_{nn'}\,Q_{\tau\tau'n}. \tag{34}$$

Whenever a Bessel function appears times some function of its argument, we agree to enclose the product in parentheses and omit the argument of the Bessel function, e.g., $(kaj_n) \equiv kaj_n(ka)$, $(h_n/\kappa a) \equiv h_n(\kappa a)/\kappa a$. We also write $(\kappa aj_n)' \equiv d(\kappa aj_n)/d(\kappa a)$, etc. The elements of $Q$ now become

$$\begin{aligned}
Q_{11n} &= (\kappa aj_n)'(\kappa ah_n),\\
Q_{22n} &= (\kappa aj_n)(\kappa ah_n)',\\
Q_{23n} &= [n(n+1)\,k/\kappa]^{1/2}(\kappa aj_n)\,h_n(ka),\\
Q_{32n} &= [n(n+1)\,\kappa/k]^{1/2}(kaj_n)\,h_n(\kappa a),\\
Q_{33n} &= (kaj_n)(kah_n'),\\
Q_{\tau\tau'n} &= 0 \text{ otherwise.}
\end{aligned} \tag{35a}$$

The supermatrix element mentioned earlier thus takes the form [compare Eq. (33)]

$$Q_{\tau\tau'} = \begin{pmatrix} X & 0 & 0\\ 0 & X & X\\ 0 & X & X \end{pmatrix}, \tag{33'}$$

as is apparently true in general for spherical obstacles. The $\tau=1$ mode (in electromagnetic scattering the "magnetic" mode) is reflected with no coupling to the other modes, whereas the electric ($\tau=2$) and acoustic ($\tau=3$) modes are always coupled. This behavior can be traced directly to the defining Eqs. (5a)–(5d) for the wave functions: The $\vec{\psi}_{1n}$ involve only the vector spherical harmonics of the first type $\vec{A}_{1n}$, whereas the functions $\vec{\psi}_{2n}$ and $\vec{\psi}_{3n}$ each involve both $\vec{A}_{2n}$ and $\vec{A}_{3n}$.

In computing the transition matrix, any real factors in Eq. (35a) that are common to both sides of Eqs. (31) and (31b) may be dropped, simplifying $Q$ to

$$\begin{aligned}
Q^0_{11n} &= h_n(\kappa a),\\
Q^0_{22n} &= (\kappa ah_n)',\\
Q^0_{23n} &= [n(n+1)\,k/\kappa]^{1/2}\,h_n(ka),\\
Q^0_{32n} &= [n(n+1)\,\kappa/k]^{1/2}\,h_n(\kappa a),\\
Q^0_{33n} &= (kah_n'),\\
Q^0_{\tau\tau'n} &= 0 \text{ otherwise.}
\end{aligned} \tag{35b}$$

For each fixed value of $n$, Eq. (31) in general consists of one equation in one unknown, plus a $2\times 2$ matrix equation. Solving these, the scattering coefficients are found to be

$$f_{\tau n} = \sum_{\tau'} T_{\tau\tau'n}\,a_{\tau'n}, \tag{36a}$$

where nonzero elements of the transition matrix are given by

$$\begin{aligned}
T_{11n} &= -\mathrm{Re}(Q_{11n})/Q_{11n},\\
\Delta T_{22n} &= Q_{23n}\,\mathrm{Re}\,Q_{32n} - Q_{33n}\,\mathrm{Re}\,Q_{22n},\\
\Delta T_{23n} &= Q_{23n}\,\mathrm{Re}\,Q_{33n} - Q_{33n}\,\mathrm{Re}\,Q_{23n},\\
\Delta T_{32n} &= Q_{32n}\,\mathrm{Re}\,Q_{22n} - Q_{22n}\,\mathrm{Re}\,Q_{32n},\\
\Delta T_{33n} &= Q_{32n}\,\mathrm{Re}\,Q_{23n} - Q_{22n}\,\mathrm{Re}\,Q_{33n},
\end{aligned} \tag{36b}$$

and

$$\Delta \equiv Q_{22n}Q_{33n} - Q_{32n}\,Q_{23n},$$

and preferably one uses $Q^0$ rather than $Q$.

For the sphere, where we obtain closed-form solutions of Eq. (18) [or (31)], the time-reversal requirement is met by inspection: In terms of the scattering matrix, we have

$$S^*S = (-Q^{-1}Q^*)^*(-Q^{-1}Q^*) = (Q^*)^{-1}QQ^{-1}Q^* = 1,$$

so that the $T$-matrix elements of Eq. (36b) must *automatically* satisfy

$$T^*T = -\mathrm{Re}\,T. \tag{36c}$$

All the conservation laws are thus met once $T$, as given in Eq. 36b, is verified to be symmetric.

Combining Eqs. (35b), (36a), and (36b), explicit results for the rigid sphere are

$$\begin{aligned}
f_{1n} &= -[\,j_n(\kappa a)/h_n(\kappa a)]\,a_{1n},\\
\delta_n f_{2n} &= [(\kappa aj_n)'(kah_n') - n(n+1)\,j_n(\kappa a)\,h_n(ka)]\,a_{2n}\\
&\quad + i[n(n+1)/(\kappa a)(ka)]^{1/2}a_{3n},\\
\delta_n f_{3n} &= i[n(n+1)/(\kappa a)(\kappa a)]^{1/2}a_{2n}\\
&\quad + [(kaj_n')(\kappa ah_n)' - n(n+1)\,j_n(ka)\,h_n(\kappa a)]\,a_{3n},
\end{aligned}$$

where

$$\delta_n \equiv -[(\kappa ah_n)'(kah_n') - n(n+1)\,h_n(\kappa a)\,h_n(ka)]. \tag{37}$$

By inspection $T$ is symmetric.

The above equations are appropriate for an arbitrary incident wave. For a plane wave, it suffices to consider incidence along the $z$ axis. The expansions of Eqs. (24a) and (24b) reduce to

$$\hat{x}\,e^{i\kappa z} = \sum (a_{1o1n}\,\mathrm{Re}\vec{\psi}_{1o1n} + a_{2e1n}\,\mathrm{Re}\vec{\psi}_{2e1n}), \tag{38a}$$

$$\hat{z}\,e^{ikz} = \sum a_{3e0n}\,\mathrm{Re}\vec{\psi}_{3e0n}, \tag{38b}$$

where

$$\begin{aligned}
a_{1o1n} &= (i)^n[2(2n+1)]^{1/2},\\
a_{2e1n} &= -(i)^{n+1}[2(2n+1)]^{1/2},\\
a_{3e0n} &= -2(i)^{n+1}(\kappa/k)^{3/2}(2n+1)^{1/2}.
\end{aligned} \tag{38c}$$

Equations (37) and (38a)–(38c) are in agreement with the earlier results of Ying and Truell, for longitudinal wave incidence,[5] and Einspruch *et al.* for transverse wave incidence,[6] and in addition show the underlying connection between the two problems through the conservation laws.

## IV. THE CAVITY

When the obstacle is a cavity, surface tractions must vanish on the boundary, i.e.,

$$\vec{t}_+ = 0 \quad \text{on } \sigma. \tag{39}$$

The curl and divergence of $\vec{u}$ may now be expressed, from Eqs. (13), as

$$\begin{aligned} n\times(\nabla\times\vec{u})_+ &= 2\hat{n}\times(\nabla_s\times\vec{u}_+) \\ (\nabla\cdot\vec{u})_+ &= 2(k/\kappa)^2(\nabla_s\cdot\vec{u}_+), \end{aligned} \tag{40}$$

and substituting these expressions in Eqs. (9) gives the general equations for the cavity. This time we assume expansions

$$\left.\begin{aligned} \vec{u}_{+\tan} &= \sum_{\tau=1,2} \alpha_n \operatorname{Re}\vec{\psi}_{n\tan} \\ \vec{u}_{+\text{norm}} &= \sum_{\tau=3} \alpha_n \operatorname{Re}\vec{\psi}_{n\,\text{norm}} \end{aligned}\right\} \quad \vec{r} \text{ on } \sigma \tag{41}$$

for the tangential and normal components of particle displacement at the surface. Taking the surface curl and divergence of these expressions, and putting everything in Eqs. (9) again leads to Eqs. (30) and (31), with the proviso that elements of $Q$ now take the form

$$\begin{aligned} Q_{\tau n\tau' n'(\tau\neq 3)} &= (\kappa/\pi)\int d\sigma\{2[\hat{n}\times(\nabla_s\times\operatorname{Re}\vec{\psi}_{\tau n\,\tan})]\cdot\vec{\psi}_{\tau' n'} \\ &\quad + (\hat{n}\times\operatorname{Re}\vec{\psi}_{\tau n\,\tan})\cdot\nabla\times\vec{\psi}_{\tau' n'} \\ &\quad - 2(\nabla_s\cdot\operatorname{Re}\vec{\psi}_{\tau n\,\tan})(\hat{n}\cdot\psi_{\tau' n'})\}, \\ Q_{3n\tau' n'} &= (\kappa/\pi)\int d\sigma\{2[\hat{n}\times(\nabla_s\times\operatorname{Re}\vec{\psi}_{3n\,\text{norm}})]\cdot\vec{\psi}_{\tau' n'} \\ &\quad - 2(\nabla_s\cdot\operatorname{Re}\vec{\psi}_{3n\,\text{norm}})(\hat{n}\cdot\vec{\psi}_{\tau' n'}) \\ &\quad + (\kappa/k)^2(\hat{n}\cdot\operatorname{Re}\vec{\psi}_{3n\,\text{norm}})(\nabla\cdot\vec{\psi}_{\tau' n'})\}. \end{aligned} \tag{42}$$

As noted following Eq. (14b), some terms in the above integrands vanish identically. Just as with the rigid body, the equations must be solved numerically for cavities of arbitrary shape. The relationship with acoustics and electromagnetics is this time not quite so transparent, presumably because the requirement that the normal component of the stress tensor vanish is somewhat more subtle than a condition on scalar pressure, or even vector electromagnetic field. The relationship is still present, however, provided that in addition to resetting mode conversion elements to zero, one drops any terms in the remaining integrands containing a surface divergence or surface curl. The relationship this time, incidentally, is with the *soft* acoustic surface and the perfectly *magnetic* object ($\hat{n}\times\vec{H}_+=0$), the latter of course being the dual of the perfect electrically conducting object, with the two problems related through the transformation $(\vec{E},\vec{H})\to(\vec{H},-\vec{E})$.

For the spherical cavity, the Bessel functions are constant over the surface and thus, from the defining Eq.(5a), we need only know the surface divergence and curl of the vector spherical harmonics. These quantities are easily computed from their definitions; we find

$$\nabla_s\cdot\vec{A}_{\tau\sigma mn} = \begin{cases} 0 & (\tau=1) \\ -(1/a)[n(n+1)]^{1/2}Y_{\sigma mn} & (\tau=2) \\ (2/a)Y_{\sigma mn} & (\tau=3) \end{cases}$$

and (43)

$$\hat{n}\times(\nabla_s\times\vec{A}_{\tau\sigma mn}) = \begin{cases} -(1/a)\vec{A}_{1\sigma mn} & (\tau=1) \\ -(1/a)\vec{A}_{2\sigma mn} & (\tau=2) \\ (1/a)[n(n+1)]^{1/2}\vec{A}_{2\sigma mn} & (\tau=3). \end{cases}$$

Equation (42) can now be evaluated to get

$$\begin{aligned} Q_{11n} &= (\kappa a)^3 j_n(\kappa a)(h_n/\kappa a)', \\ Q_{22n} &= (1/\kappa a)(\kappa a j_n)'\{[2n(n+1)-(\kappa a)^2]h_n(\kappa a)-2(\kappa a h_n)'\}, \\ Q_{23n} &= 2[n(n+1)k/\kappa]^{1/2}(1/\kappa a)(\kappa a j_n)'(ka)^2(h_n/ka)', \\ Q_{32n} &= 2[n(n+1)\kappa/k]^{1/2} j_n'(ka)(ka)^2(h_n/\kappa a)', \\ Q_{33n} &= (k/\kappa)^2 j_n'(ka)\{[2n(n+1)-(\kappa a)^2]h_n(ka)-4(kah_n')\}, \\ Q_{\tau\tau' n} &= 0 \text{ otherwise.} \end{aligned} \tag{44a}$$

Dropping any common real factors in Eq. (31), elements of $Q$ take the simpler form

$$\begin{aligned} Q^0_{11n} &= (h_n/\kappa a)', \\ Q^0_{22n} &= [2n(n+1)-(\kappa a)^2]h_n(\kappa a)-2(\kappa a h_n)', \\ Q^0_{23n} &= 2[n(n+1)k/\kappa]^{1/2}(ka)^2(h_n/ka)', \\ Q^0_{32n} &= 2[n(n+1)\kappa/k]^{1/2}(\kappa a)^2(h_n/\kappa a)', \\ Q^0_{33n} &= [2n(n+1)-(\kappa a)^2]h_n(ka)-4(kah_n'), \\ Q^0_{\tau\tau' n} &= 0 \text{ otherwise.} \end{aligned} \tag{44b}$$

The explicit solution is obtained upon substituting Eq. (44b) in Eq. (36). (The reader can verify that Eq. (44a) leads to the same result.) The conservation laws are checked by looking for symmetry; making use of the Wronskian relation $j_n h_n' - j_n' h_n = (i/x^2)$ we easily find

$$\Delta T_{23n} \equiv \Delta T_{32n} = (2i/ka)[n(n+1)k/\kappa]^{1/2} \times[2(n+2)(n-1)-(\kappa a)^2]. \tag{45}$$

Comparing our results with the classical papers using separation of variables,[5,6] one finds exact agreement for

$$T_{11n} = -(j_n/\kappa a)'/(h_n/\kappa a)', \tag{46}$$

but discrepancies in the remaining elements. Both papers give a system of two equations in two unknowns to be solved for the $\tau=2,3$ mode coefficients, and in each case the matrix should be identical to ours [given by the last four of Eqs. (44b)] except for normalization. For the longitudinal incidence case,[5] multiplying each row and column of their matrix by appropriate normalizing constants one can bring complete agreement with Eq. (44b) except for an extra factor $\kappa/k$ in their $Q^0_{23n}$. This leads to violation of energy conservation and must be a misprint. For the transverse incidence case,[6] the matrix given is somewhat more complicated and we have not evaluated it completely; it appears to contain several misprints.

For the hard and soft elastic sphere there is a third method of solution most direct of all, which might be called *vector separation of variables*: One writes down the total field using the expansions of Eq. 6 for the incident and scattered waves, then for the hard sphere equates to zero the coefficient of each vector spherical harmonic $\vec{A}_{\tau n}$, $\tau=1,2,3$, for $\vec{u}$ on the surface. This leads immediately to the $Q$ matrix of Eq. (35b). For the soft sphere, set the surface tractions to zero in Eqs. (13a)–(13b), then substitute in $\vec{u}$ and use Eqs. (43) to evaluate the surface derivatives. Now Eq. (13a) will contain only the angular functions $\vec{A}_{1n}$, $\vec{A}_{2n}$, and Eq. (13b) only $Y_{\sigma mn} = |\vec{A}_{3n}|$. Setting the coefficient of each to zero again gives the $Q$ matrix Eq. (44b) directly. Notice that this procedure is basically different from earlier methods, which worked with the $r, \vartheta, \varphi$ components of displacement and surface traction,[5,6] and does not require evaluating additional integrals of Legendre functions.

Returning to Eqs. (44b), we now find a remarkable equivalence between the mode coefficients for hard and soft spheres whenever the mode index is sufficiently greater than $\kappa a$. Suppose that

$$2n(n+1) \gg (\kappa a)^2, \tag{47}$$

so that the term containing $(\kappa a)^2$ can be neglected in both $Q^0_{22n}$ and $Q^0_{33n}$. (Of course we must be careful not to do this if $\kappa a$, or $ka$, is a root of either the real or imaginary part of the remaining expressions.) Considering only the $2\times 2$ matrix involving coupled modes, the reader can verify that $Q^0$, with the $(\kappa a)^2$ terms neglected, can be factored into the product $RQ^1$. Here $R$ is the real matrix

$$R = 2\begin{pmatrix} -1 & [n(n+1)k/\kappa]^{1/2} \\ [n(n+1)\,\kappa/k]^{1/2} & -2 \end{pmatrix}$$

which, except for the special case $n=1$ for which $\det(R)=0$, may be simply discarded in Eq. (31). The remaining matrix $Q^1$ is precisely the $2\times 2$ array of coupled mode elements for the rigid body, Eq. (35b)! Thus, for the coupled mode coefficients (but not the coefficients $Q_{11n}$ of the magnetic mode) the matrix elements of $Q$ and $T$, and also the scattering coefficients $f_{2n}$, $f_{3n}$, become equal for the rigid sphere and the spherical cavity, as the inequality Eq. (47) comes into force. This should serve as a useful check on both analytical and numerical computations.

The exceptional case $n=1$ is extremely important in the Rayleigh limit $\kappa a \ll 1$. In this limit the Bessel functions have the behavior $j_n(x) \doteq O(x^n)$ so that, from Eq. (44b),

$$\mathrm{Re}(Q^0_{22n}) \doteq [2n(n+1) - (\kappa a)^2](\kappa a)^n - 2(n+1)\,(\kappa a)^n,$$

$$\mathrm{Re}(Q^0_{33n}) \doteq [2n(n+1) - (\kappa a)^2]\,(ka)^n - 4n(ka)^n.$$

Notice that for $n=1$ the first and last terms cancel in both instances and the $(\kappa a)^2$ term can no longer be neglected. This cancellation changes the whole character of the scattering, and enables the cross section for the spherical cavity to take on the classical Rayleigh inverse fourth-power dependence on wavelength, in contrast to the anomalously large cross section displayed by the rigid sphere at low frequencies. Incidentally, a similar reduction in magnitude occurs for the magnetic mode [see numerator of Eq. (46)] so that our comments on cross section behavior apply for transverse as well as longitudinal wave incidence. Equivalence of all mode coefficients with $n>1$ should continue to hold in the Rayleigh limit, but these coefficients are no longer of much physical interest.

## V. THE FLUID-FILLED CAVITY

The situation is more complex with a fluid-filled cavity; for the first time we must deal with fields within the obstacle.

Let the fluid have density $\rho'$, propagation constant $k'$. Boundary conditions are now that the normal components of particle displacement be continuous across the interface (tangential components will not in general be continuous). In addition, the normal component of surface traction just outside the surface must be equal and opposite to the pressure $p_-$ just within. Finally, tangential components of surface traction must vanish just outside, as the fluid can support no shear stress. Respectively, one has

$$\begin{aligned} &\hat{n}\cdot\vec{u}_+ = \hat{n}\cdot\vec{u}_-, \\ &\hat{n}\cdot\vec{t}_+ = -p_- = (\omega^2\rho'/k'^2)(\nabla\cdot\vec{u})_-, \\ &\vec{t}_{+\tan} = 0. \end{aligned} \tag{48}$$

We begin by assuming for the longitudinal field, and its divergence, just inside the surface, the two independent expansions

$$\begin{aligned} &\vec{u}_- = \sum \alpha_{3n}\mathrm{Re}\vec{\psi}'_{3n} \\ &(\nabla\cdot\vec{u})_- = \sum \beta_{3n}\nabla\cdot\mathrm{Re}\vec{\psi}'_{3n} \qquad \vec{r} \text{ on } \sigma, \end{aligned} \tag{49a}$$

where $\vec{\psi}'_{3n}$ is obtained by replacing $k$ by $k'$ in the defining equation except in the factor $(k/\kappa)^{3/2}$, which must be left unchanged. Now by applying Huygens' principle (for irrotational waves) to the interior volume, rather than the exterior as was done in Eq. (4), an equation is obtained relating $\vec{u}$ and $(\nabla\cdot\vec{u})_-$. The necessary and sufficient condition that this equation be satisfied is then

$$\beta_{3n} = \alpha_{3n} \quad (\text{all } n). \tag{49b}$$

This technique has been employed earlier[12,13,26]; a detailed derivation was given for periodic surfaces.[27] We derive the analogous result for the general elastic obstacle in the following section. Of course for the spherical obstacle the expansion for $\vec{u}_-$ is convergent and differentiable everywhere inside, so that Eq. (49b) is self-evident.

We also assume, for the tangential components of displacement just outside the surface, the expansion

$$(\vec{u}_+)_{\tan} = \sum_{\tau=1,2} \alpha_n \mathrm{Re}\vec{\psi}_{n\,\tan}, \quad \vec{r} \text{ on } \sigma. \tag{49c}$$

Now from Eq. (13a) and the third boundary condition, the tangential curl of $\vec{u}$ just outside is given by

$$\hat{n}\times(\nabla\times\vec{u})_+ = 2\hat{n}\times(\nabla_s\times\vec{u}). \tag{50a}$$

Similarly, the divergence of $\vec{u}$ just outside becomes

$$(\nabla\cdot\vec{u})_+ = 2(k/\kappa)^2(\nabla_s\cdot\vec{u}) + (\rho' k^2/\rho k'^2)(\nabla\cdot\vec{u})_- \tag{50b}$$

from Eq. (13b) and the second boundary condition.

Equations (48)–(50) specify all surface field quantities needed in Eqs. (9); substituting in, we find again Eqs. (30) and (31), with elements of $Q$ given by

$$Q_{\tau n\tau' n'\,(\tau\neq 3)} = (\kappa/\pi)\int d\sigma\{2[\hat{n}\times(\nabla_s\times \mathrm{Re}\vec{\psi}_{\tau n\,\tan})]\cdot\vec{\psi}_{\tau' n'} + (\hat{n}\times \mathrm{Re}\vec{\psi}_{\tau n\,\tan})\cdot\nabla\times\vec{\psi}_{\tau' n'} - 2(\nabla_s\cdot\mathrm{Re}\vec{\psi}_{\tau n\,\tan})(\hat{n}\cdot\vec{\psi}_{\tau' n'})\},$$
$$Q_{3n\tau' n'} = (\kappa/\pi)\int d\sigma\{2[\hat{n}\times(\nabla_s\times\mathrm{Re}\vec{\psi}'_{3n\,\mathrm{norm}})]\cdot\vec{\psi}_{\tau' n'} - [2(\nabla_s\cdot\mathrm{Re}\vec{\psi}'_{3n\,\mathrm{norm}}) + (\rho'\kappa^2/\rho k'^2)(\nabla\cdot\mathrm{Re}\vec{\psi}'_{3n})](\hat{n}\cdot\vec{\psi}_{\tau' n'})$$
$$+ (\kappa/k)^2(\hat{n}\cdot\mathrm{Re}\vec{\psi}'_{3n\,\mathrm{norm}})(\nabla\cdot\vec{\psi}_{\tau' n'})\}. \tag{51}$$

Notice that this readily reduces to the $Q$ matrix of the empty cavity upon setting $k'=k$, then letting $\rho'/\rho\to 0$. If one were to reset mode coupling coefficients to zero, the electromagnetic analog would be exactly as for the empty cavity, while the acoustic analog (i.e., the elements $Q_{3n3n'}$ with surface derivatives dropped) is a fluid obstacle embedded in a fluid medium.

Specializing to the fluid sphere, we get

$$\begin{aligned}
Q_{11n} &= (\kappa a)^3 j_n(\kappa a)\,(h_n/\kappa a)',\\
Q_{22n} &= (1/\kappa a)\,(\kappa a j_n)'\{[2n(n+1)-(\kappa a)^2]\,h_n(\kappa a) - 2(\kappa a h_n)'\},\\
Q_{23n} &= 2\,[n(n+1)\,k/\kappa]^{1/2}(1/\kappa a)(\kappa a j_n)'(ka)^2(h_n/ka)',\\
Q_{32n} &= [n(n+1)\kappa/k]^{1/2}\{2j_n'(k'a)(ka)^2(h_n/\kappa a)' + (\rho' k^2/\rho k'^2)(k'aj_n)\,h_n(\kappa a)\},\\
Q_{33n} &= (k/\kappa)^2 j_n'(k'a)\{[2n(n+1)-(\kappa a)^2]\,h_n(ka) - 4(kah_n')\} + (\rho' k^2/\rho k'^2)(k'aj_n)(kah_n'),\\
Q_{\tau\tau' n} &= 0 \text{ otherwise.}
\end{aligned}\tag{52a}$$

For computing scattering (but not surface fields), this reduces further to

$$\begin{aligned}
Q^0_{11n} &= (h_n/\kappa a)',\\
Q^0_{22n} &= [2n(n+1)-(\kappa a)^2]\,h_n(\kappa a) - 2(\kappa a h_n)',\\
Q^0_{23n} &= 2[n(n+1)k/\kappa]^{1/2}(ka)^2(h_n/ka)',\\
Q^0_{32n} &= 2\,[n(n+1)\kappa/k]^{1/2}\{j_n'(k'a)\,(\kappa a)^2(h_n/\kappa a)' + (\rho'\kappa^2/2\rho k'^2)\,(k'aj_n)\,h_n(\kappa a)\},\\
Q^0_{33n} &= j_n'(k'a)\{[2n(n+1)-(\kappa a)^2]\,h_n(ka) - 4(kah_n')\} + (\rho'\kappa^2/\rho k'^2)\,(k'aj_n)\,(kah_n'),\\
Q^0_{\tau\tau' n} &= 0 \text{ otherwise.}
\end{aligned}\tag{52b}$$

The scattered waves are obtained upon inserting the last expressions in Eqs. (36a)–(36b); conservation laws are verified by computing that

$$\Delta T_{23n} \equiv \Delta T_{32n} = (2i/ka)[n(n+1)\,k/\kappa]^{1/2}\{[2(n+2)\,(n-1)-(\kappa a)^2]j_n'(k'a) + (\rho'\kappa^2/\rho k'^2)(k'aj_n)\}. \tag{53}$$

As was noted by Einspruch *et al.*, the scattering coefficients for the magnetic modes are unchanged from the cavity case.[6] Fluid motions are coupled to the exterior through their normal displacement and pressure; for the sphere, the magnetic modes involve neither of these. Comparing other modes with the published results, we find that separation of variables leads to a system of three equations in three unknowns,[6] in contrast to the explicit results given above. We will comment on this difference in structure of the results when we examine the elastic obstacle (where differences are even more pronounced). A partial comparison can be made by noting that one of the three Einspruch *et al.* equations is basically identical with one of their equations for the cavity. In the present context the equation in question is in accord with $Q^0_{22n}$ and $Q^0_{23n}$ (which also are unaffected by the presence of the fluid), except for the misprints mentioned previously.

## VI. THE GENERAL ELASTIC BODY

For the final case of interest, consider the elastic obstacle having material parameters $\lambda'$, $\mu'$, $\rho'$ all of which may differ from those of the host medium. Both longitudinal and transverse waves will be excited in the interior, with propagation constants $k'$, $\kappa'$, respectively. We suppose the objects' surface to be in intimate contact with its surroundings; boundary conditions are then that particle displacements, as well as surface tractions, be continuous across the interface, i.e.,

$$\vec{u}_+ = \vec{u}_-,\quad \vec{r} \text{ on } \sigma \tag{54a}$$
$$\vec{t}_+ = \vec{t}_-,\quad \vec{r} \text{ on } \sigma. \tag{54b}$$

At this stage we are confronted with essentially three sets of equations [Eq. (9b)] for six unknowns (the scalar components of $\vec{u}$ and $\vec{t}$). Fortunately, however, the six interior surface fields are not independent, but must satisfy constraints imposed by Huygens' principle for the interior. Introduce a Green's dyadic $\mathcal{G}'$ and wave functions $\vec{\psi}_n'$ appropriate to the object interior, by replacing $\kappa$, $k$ by $\kappa'$, $k'$ throughout the defining equations. (Note that this differs slightly from the preceding section where the factor $(k/\kappa)^{3/2}$ was left unchanged; for the fluid medium, the quantity $\kappa'$ is undefined.) Applying the divergence theorem to the interior now gives[10,11]

$$-\kappa'\int d\sigma'\{\mathcal{G}'\cdot[\hat{n}'\times(\nabla\times\vec{u})_-] + (\hat{n}'\times\vec{u}_-)\cdot(\nabla'\times\mathcal{G}')$$
$$-(\kappa'/k')^2[(\hat{n}'\cdot\mathcal{G}')(\nabla'\cdot\vec{u})_- - (\nabla'\cdot\mathcal{G}')(\hat{n}'\cdot\vec{u}_-)\}$$
$$= \begin{cases} 0, & \vec{r} \text{ outside } \sigma\\ \vec{u}(\vec{r}), & \vec{r} \text{ inside } \sigma. \end{cases} \tag{55}$$

In contrast to Eq. 4 the incident wave no longer appears explicitly, there is a sign change on the integral (because we continue to use the outward-pointing normal), and the left side of the equation now vanishes for field points *outside* $\sigma$.

We suppose the interior field to be representable, at least within the inscribed sphere of Fig. 1 within the body, by

$$\vec{u}(\vec{r}) = \sum \alpha_n \operatorname{Re}\vec{\psi}_n'(\vec{r}), \quad r < r_{\min \text{ on } \sigma}, \tag{56}$$

where the expansion coefficients $\alpha_n$ are of course presently unknown. Now, using orthogonality of the wave functions as before, Eq. (55) reduces to

$$0 = -(i\kappa'/\pi)\int d\sigma\{(\operatorname{Re}\vec{\psi}_n')\cdot[\hat{n}\times(\nabla\times\vec{u})_-] + (\nabla\times\operatorname{Re}\vec{\psi}_n')\cdot(\hat{n}\times\vec{u}_-) - (\kappa'/k')^2[(\hat{n}\cdot\operatorname{Re}\vec{\psi}_n')(\nabla\cdot\vec{u})_- - (\nabla\cdot\operatorname{Re}\vec{\psi}_n')(\hat{n}\cdot\vec{u}_-)]\}, \quad n=1,2,\ldots, \tag{57a}$$

$$\alpha_n = -(i\kappa'/\pi)\int d\sigma\{\vec{\psi}_n'\cdot[\hat{n}\times(\nabla\times\vec{u})_-] + (\nabla\times\vec{\psi}_n')\cdot(\hat{n}\times\vec{u}_-) - (\kappa'/k')^2[(\hat{n}\cdot\vec{\psi}_n')(\nabla\cdot\vec{u})_- - (\nabla\cdot\vec{\psi}_n')(\hat{n}\cdot\vec{u}_-)]\}, \quad n=1,2,\ldots. \tag{57b}$$

Equation (57a) constitutes three sets of constraining equations on the six interior surface fields, whereas Eq. (57b) give a prescription for finding the field within the inscribed sphere once the surface fields have been obtained.

At this point we notice that by choosing expansion functions appropriately we can cause the matrix $C$, discussed earlier, to appear in these equations. Of course $C$ this time will depend on $\kappa'$ and $k'$ rather than $\kappa$ and $k$. This does not affect Eq. (14c), however; it will remain true that $-iC'(\kappa', k') =$ Identity matrix.

The choice is clear. Assume for the interior surface fields the expansions

$$\begin{aligned} \vec{u}_- &= \sum_{\tau=1,2,3} \alpha_n' \operatorname{Re}\vec{\psi}_n', \\ \hat{n}\times(\nabla\times\vec{u})_- &= \sum_{\tau=1,2} \alpha_n''\, \hat{n}\times(\nabla\times\operatorname{Re}\vec{\psi}_n'), \\ (\nabla\cdot\vec{u})_- &= \sum_{\tau=3} \alpha_n''(\nabla\cdot\operatorname{Re}\vec{\psi}_n') \quad \vec{r} \text{ on } \sigma. \end{aligned} \tag{58}$$

When we substitute these formulas in Eq. (57), also writing $\alpha'' = \alpha' + (\alpha'' - \alpha')$, we get

$$0 = -i\operatorname{Re}(C')\alpha' - i\operatorname{Re}(Q'_{\text{rigid}})(\alpha'' - \alpha'),$$

$$\alpha = -iC'\alpha' - iQ'_{\text{rigid}}(\alpha'' - \alpha'),$$

where $C$ is the matrix defined in Eq. (14b), and $Q_{\text{rigid}}$ is the $Q$ matrix for the rigid body, defined in Eq. (29), except that in both cases $\kappa$, $k$ are replaced by $\kappa'$, $k'$. Now using Eq. (14c) the above expressions take the final form

$$\operatorname{Re}(Q'_{\text{rigid}})(\alpha'' - \alpha') = 0, \tag{59a}$$

$$Q'_{\text{rigid}}(\alpha'' - \alpha') = i(\alpha - \alpha'). \tag{59b}$$

At this point we have the classical Fredholm alternative. Suppose first that the determinant of Re $Q'_{\text{rigid}}$ does not vanish. Then Eq. (59a) is a necessary and sufficient condition for the vanishing of $\alpha'' - \alpha'$, i.e.,

$$\alpha'' \equiv \alpha'. \tag{60a}$$

It then follows from Eq. (59b) that

$$\alpha' \equiv \alpha. \tag{60b}$$

In short, under the condition stated *Huygens' principle is a necessary and sufficient condition for the convergence and differentiability of the expansion Eq.* (56), *not just within the inscribed sphere, but throughout the interior volume of the obstacle, including the surface approached from the inside.*

The alternative, of course, is that

$$\det(\operatorname{Re} Q'_{\text{rigid}}) = 0. \tag{61}$$

Equation (59a) may then have a nontrivial solution, in which event we are dealing with fields in the interior of the object, satisfying the fixed surface boundary condition Eq. (26), which [see Eq. (30a)] *do not radiate.* These fields are the interior resonant cavity modes for the rigid boundary, and Eq. (61) constitutes the secular equation for determination of the discrete frequencies (values of $\kappa'$, $k'$) at which they occur. Similarly, Eq. (61), using the $Q$ matrix Eq. (42) for the cavity, is the secular equation for resonant modes in the free-surface case. Once eigenfrequencies have been obtained, the fields themselves are found by solving the homogeneous form of Eq. (30a) obtained by setting $f=0$. For the exterior scattering problem of present interest, the equalities of Eqs. (60a) and (60b) still hold, provided we follow the usual procedure of orthogonalizing our field $\alpha'$, $\alpha''$ to any resonant modes that might be present.

One clarifying comment is in order. In going from $Q$ [Eq. (35a)] to $Q^0$ [Eq. (35b)] for the rigid sphere, the common real factors that were dropped involve Bessel functions that vanish at an eigenfrequency of the interior resonant *free-surface* sphere problem. Similarly, the Bessel function factors dropped from Eq. (44a) in the spherical cavity problem vanish at an eigenfrequency of the *rigid* sphere. The eigenfunctions of the free-surface resonant body of course must be included when working with the rigid body, and vice versa, and for the sphere there was no problem because we effectively were able to use L'Hospital's rule. For the nonspherical body, however, it is not clear but what numerical difficulties may arise. Although no numerical problems of this sort were ever encountered in the electromagnetic case (possibly because we orthogonalized $Q$ and used Eq. (21) rather than inverting), further investigation seems indicated here.

We are now in position to invoke the basic moment Eq. (9b) of the *exterior* Huygens' principle. Writing down Eqs. (13a)–(13b) separately for the interior and exterior and eliminating the surface tractions, which are identical due to the boundary condition Eq. (54b), one gets

$$\begin{aligned} \hat{n}\times(\nabla\times\vec{u})_+ &= (\mu'/\mu)\,\hat{n}\times(\nabla\times\vec{u})_- - 2(\mu'/\mu - 1)\,\hat{n}\times(\nabla_s\times\vec{u}), \\ (\nabla\cdot\vec{u})_+ &= (\rho' k^2/\rho k'^2)(\nabla\cdot\vec{u})_- - 2(\mu'/\mu - 1)(k/\kappa)^2(\nabla_s\cdot\vec{u}). \end{aligned} \tag{62}$$

Using these expressions, along with the first boundary condition Eq. (54a) and the surface field expansions

Eqs. (58) (with all primes omitted on the $\alpha$'s), we come once again to the standard Eqs. (30) and (31), with $Q$ this time defined by

$$
\begin{aligned}
Q_{nn'} = (\kappa/\pi)\int d\sigma\{[(\mu'/\mu)\,\hat{n}\times(\nabla\times\mathrm{Re}\vec{\psi}_n') - 2(\mu'/\mu-1)\,\hat{n}\times(\nabla_s\times\mathrm{Re}\vec{\psi}_n')]\cdot\vec{\psi}_{n'} \\
+(\hat{n}\times\mathrm{Re}\vec{\psi}_n')\cdot(\nabla\times\vec{\psi}_{n'}) + (\kappa/k)^2(\hat{n}\cdot\mathrm{Re}\vec{\psi}_n')(\nabla\cdot\vec{\psi}_{n'}) - [\rho'\kappa^2/\rho k'^2)(\nabla\cdot\mathrm{Re}\vec{\psi}_n') - 2(\mu'/\mu-1)(\nabla_s\cdot\mathrm{Re}\vec{\psi}_n')]\,(\hat{n}\cdot\vec{\psi}_{n'})\},
\end{aligned}
\tag{63}
$$

where two terms of the integrand vanish identically for each choice of $\tau, \tau'$.

The trivial case is easily checked; letting $\lambda', \mu', \rho', \kappa', k'$ equal $\lambda, \mu, \rho, \kappa, k$, one sees that $Q$ reduces to $C$ of Eq. (14b), $\alpha = a$ and there is no scattering, as it should be. Resetting the elements concerned with mode conversion to zero, and dropping terms in the integrand containing surface derivatives (notice that the latter step would be accomplished automatically if one specialized to the case of equal shear moduli, $\mu' = \mu$), Eq. (63) goes over exactly to describe the independent electromagnetic/acoustic scattering by general penetrable objects.

For the elastic sphere, integrations are carried out as before to get

$$
\begin{aligned}
&Q_{11n} = (\kappa a)^3[\,j_n(\kappa' a)\,(h_n/\kappa a)' - (\rho'/\rho)\,(j_n/\kappa' a)' h_n(\kappa a)], \\
&Q_{22n} = -[(\kappa/\kappa')(\kappa' a j_n)' + 2(\mu'/\mu-1)(\kappa'/\kappa)\,n(n+1)(j_n/\kappa' a)']\,(\kappa a h_n) \\
&\qquad + \{(\mu'/\mu)\,(\kappa' a j_n) + 2(\mu'/\mu-1)\,(1/\kappa' a)[(\kappa' a j_n)' - n(n+1)\,j_n(\kappa' a)]\}\,(\kappa a h_n)', \\
&[n(n+1)\,k/\kappa]^{-1/2}\,Q_{23n} = [(\rho'/\rho-1)\,(\kappa a)^2 - 2(\mu'/\mu-1)\,(n+2)\,(n-1)]\,(j_n/\kappa' a)\,h_n(ka) - 2(\mu'/\mu-1)\,(\kappa' a)\,(j_n/\kappa' a)'(ka)^2(h_n/ka)', \\
&[n(n+1)\,(k'/\kappa')^3]^{-1/2} Q_{32n} = -2(\mu'/\mu-1)\,(k'a)\,(j_n/k'a)'(\kappa a h_n)' \\
&\qquad + \{(\rho'\rho-1)\,(\kappa a)^2(j_n/k'a) + 2(\mu'/\mu-1)\,[2j_n'(k'a) - n(n+1)\,(j_n/k'a)]\}\,h_n(\kappa a), \\
&[\kappa\kappa'^3/kk'^3]^{1/2} Q_{33n} = -[(\kappa a)^2 j_n'(k'a) + 2(\mu'/\mu-1)\,n\,(n+1)\,(k'a)\,(j_n/k'a)']\,h_n(ka) \\
&\qquad + \{(\rho'\kappa^2/\rho k'^2)(k' a j_n) + 2(\mu'/\mu-1)\,[2j_n'(k'a) - n(n+1)\,(j_n/k'a)]\}(kah_n'), \\
&Q_{\tau\tau' n} = 0 \quad \text{otherwise.}
\end{aligned}
\tag{64}
$$

For general values of the constitutive parameters no further simplification of these equations is possible. The scattering is again given by substituting in Eqs. (36a) and (36b); straightforward evaluation shows that $T_{23n} = T_{32n}$, so all conservation laws are satisfied.

Comparison with the separation of variables analysis for plane-wave incidence[5,6] reveals a surprising difference. Aside from

$$
\begin{aligned}
\alpha_{1n} &= (1/Q_{11n})\,i a_{1n}, \\
f_{1n} &= -(1/Q_{11n})\,\mathrm{Re}(Q_{11n})\,a_{1n},
\end{aligned}
\tag{65}
$$

which are in agreement, separation of variables leads to a system of four coupled equations in the four unknowns $\alpha_{2n}$, $\alpha_{3n}$, $f_{2n}$, $f_{3n}$ which must be solved numerically. The present solution appears superior for two reasons: First, the conservation laws can be verified *a priori*. Second, dealing with the system of four equations presents problems; the system apparently becomes ill-conditioned in some cases, requiring that iterative correction techniques be employed.[8] Other authors have expressed doubts over the validity of certain of their numerical results for the same reason.[9] Numerical comparison of results obtained from Eq. (64) with the existing literature seems strongly indicated here.

The simpler nature of our results can be attributed to the integral equation nature of Huygens' principle, as opposed to the field approach taken by separation of variables. A similar simplification occurs in electromagnetics and acoustics, as we now show by reducing Eq. (64) to the electromagnetic/acoustic case. For simplicity let $\mu'/\mu = 1$ (dielectric sphere with relative magnetic permeability of unity) and set the mode conversion coefficients $Q_{23n}$, $Q_{32n}$ to zero. $Q$ then becomes diagonal with elements

$$
\begin{aligned}
Q_{11n} &= (\kappa a)^2[\,j_n(\kappa' a)\,h_n'(\kappa a) - (\kappa'/\kappa)\,j_n'(\kappa' a)\,h_n(\kappa a)], \\
Q_{22n} &= -(\kappa/\kappa')(\kappa a)\,[(\kappa' a j_n)' h_n(\kappa a) \\
&\qquad - (\kappa'/\kappa)^2 j_n(\kappa' a)\,(\kappa a h_n)'], \\
Q_{33n} &= -[\kappa^3 k'/\kappa'^3 k]^{1/2}(ka)^2[(k'/k)\,j_n'(k'a)\,h_n(ka) \\
&\qquad - (\rho'/\rho)\,j_n(k'a)\,h_n'(ka)],
\end{aligned}
\tag{66a}
$$

and the scattering coefficients are simply

$$
f_{\tau n} = -(1/Q_{\tau\tau n})\,\mathrm{Re}(Q_{\tau\tau n})\,a_{\tau n}, \quad \tau = 1, 2, 3. \tag{66b}
$$

$f_{1n}$ and $f_{2n}$ are precisely the Mie theory magnetic and electric mode scattering coefficients for a dielectric sphere[28]; the $f_{3n}$ give the separation-of-variables solution for an acoustic sphere having disparities in both density and compressibility.[29] In contrast to Eq. (66a) and (66b), however, separation of variables leads in all three cases to a pair of coupled equations that must be solved simultaneously for the scattered wave and internal field coefficients.

Reduction of number of equations and unknowns by a factor of two is of course hardly significant in Eqs. (66a) and (66b). For nonspherical objects, however, where the scattering coefficients of different radial function index are coupled and $T$ is no longer diagonal, the present theory probably yields about an order of magnitude reduction in numerical computation over any method that must deal simultaneously with fields inside and outside the object's boundary.

Throughout the discussion we have considered obstacles with no dissipation. This restriction can be removed, however. One simply reinterprets Re $Q$ to mean the "regular part of $Q$," i.e., *replace Hankel functions by Bessel functions wherever they occur*, rather than the "real part of $Q$." Thus Re $h_n(k'a)$ becomes $j_n(k'a)$, even though $k'$ may be complex. The conservation laws must also be reconsidered. We conjecture that $S$ and $T$ remain symmetric, due to a reciprocity principle, although we know of no proof of this for elastic waves. Time-reversal invariance is lost, however, and the unitary property requires modification, so our Eq. (21), for example, cannot be employed as it stands. An excellent discussion of these questions has been given by Saxon for the electromagnetic case.[30]

## VII. DISCUSSION

The principal goal of this paper was to set up matrix equations for the scattering of elastic waves under a variety of boundary conditions, in a form most suited to efficient numerical computation. Only time, of course, will reveal to what extent we have succeeded in this endeavor.

Further analytical study of the equations should be profitable. High- and low-frequency limiting cases can be investigated choosing $ka$ and $\kappa a$ to be very small, or very large compared to unity; a good start in this direction was made by the group at Brown University.[5,6] An interesting discussion of the Rayleigh limit has also been given by Miles.[31] Other limiting cases involve the constitutive parameters. For example, letting the shear modulus vanish in the host medium, the equations of the previous section should go over to describe scattering behavior of an elastic obstacle in a *fluid* medium.

Another boundary value problem of practical interest involves the elastic object with "slip" boundary, for which only normal components of displacement and surface traction are required to be continuous. From comments made earlier in discussing the fluid-filled cavity, one infers that at least for spheres no magnetic modes would be generated in the interior in this case; electric and acoustic modes would, however. Offhand, such a boundary appears easier to fabricate in the laboratory for experimental observation than the "welded" boundary of the previous section, and may also occur frequently in nature.

For spheres in particular, some effort is called for to sort out possible discrepancies between the present equations and results obtained in the literature using separation of variables. Discrepancies may exist due either to the misprints noted, or because of numerical precision problems associated with solving a system of simultaneous equations.

In connection with the latter, we point out that analogous systems of equations arise when separation of variables is applied to circular cylindrical obstacles.[4] Use of Huygens' principle in the cylindrical case would probably lead to corresponding simplifications. Note that the cylindrical cavity is of interest for oilwell diagnostics. It is also a convenient boundary to achieve in the laboratory; recent results in this area are discussed by Sachse and Pao.[32]

The programming and numerical solution of the matrix equations for non-spherical shapes is of course not a trivial exercise. Success already achieved in the simpler but otherwise analogous acoustic and electromagnetic cases demonstrates the soundness of the approach, however. Nearly all the numerical techniques necessary to the elastic case have been documented.[23] The unification with electromagnetics/acoustics should be very helpful. One question remaining is to express the surface curl and divergence in their most convenient forms for carrying out the numerical quadratures. In this regard, Weatherburn,[20] and Van Bladel[21] give Gauss, Green, and Stokes theorem analogs involving the surface derivatives which should be useful.

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