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MATRIX TRANSFORMATIONS IN THE SETS $\chi(\overline{N}_p\overline{N}_q)$ WHERE χ IS OF THE FORM s_ξ, OR $\overset{\circ}{s}_\xi$, OR $s_\xi^{(c)}$.

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Abstract. **In this paper we deal with matrix transformations mapping in either of the sets $s_\alpha(\overline{N}_q)$, $\overset{\circ}{s}_\alpha(\overline{N}_q)$ or $s_\alpha^{(c)}(\overline{N}_q)$. Then we study some properties of the sets $s_\alpha(\overline{N}_p\overline{N}_q)$ and $\overset{\circ}{s}_\alpha(\overline{N}_p\overline{N}_q)$ and give a characterization of matrix transformations in these spaces. These results generalize those given in** [11, 14, 16]**.**

1. Notations and preliminary results.

For a given infinite matrix $A = (a_{nm})_{n,m=1}^{\infty}$ we define the operators A_n for any integer $n \geq 1$, by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm}x_m \tag{1.1}$$

where $X = (x_n)_{n=1}^{\infty}$, the series being assumed convergent. So we are led to the study of the infinite linear system

$$A_n(X) = b_n \quad n = 1, 2, ... \tag{1.2}$$

where $B = (b_n)_{n=1}^{\infty}$ is a one-column matrix and X the unknown, see [5, 6, 7, 8, 9, 11]. The ssytem (1.2) can be written in the form $AX = B$, where $AX = (A_n(X))_{n=1}^{\infty}$. In this paper we shall also consider A as an operator from a sequence space into another sequence space.

A Banach space E of complex sequences with the norm $\|\cdot\|_E$ is a BK space if each projection $P_n : X \to P_nX = x_n$ is continuous. A BK space E is said to have AK if every sequence $X = (x_n)_{n=1}^{\infty} \in E$ has a unique representation

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$X=\sum_{n=1}^{\infty} x_n e_n$ where e_n is the sequence with 1 in the n-th position and 0 otherwise.

We write s for the set of all complex sequences, ℓ_∞, c, c_0 for the sets of bounded, convergent and null sequences, respectively. By cs and ℓ_1, we denote the sets of convergent and absolutely convergent series respectively. We use the set

$$U^+ = \{(u_n)_{n=1}^{\infty} \in s : u_n > 0 \text{ for all } n\}.$$

Using Wilansky's notations [16], given any sequence $\alpha = (\alpha_n)_{n=1}^{\infty} \in U^+$ and any subset E of s,we define the sets

$$(1/\alpha)^{-1} * E = \left\{(x_n)_{n=1}^{\infty} \in s : \left(\frac{x_n}{\alpha_n}\right)_{n=1}^{\infty} \in E\right\}.$$

Writing $\alpha * E = (1/\alpha)^{-1} * E$, we put

$$\alpha * E = \begin{cases} s_\alpha & \text{if } E = \ell_\infty\,, \\ \overset{\circ}{s}_\alpha & \text{if } E = c_0\,, \\ s_\alpha^{(c)} & \text{if } E = c\,; \end{cases}$$

we have for instance

$$\alpha * c_0 = \overset{\circ}{s}_\alpha = \{(x_n)_{n=1}^{\infty} \in s : x_n = o(\alpha_n) \quad (n \to \infty)\}. \tag{1.3}$$

Each of the spaces $\alpha * E$, where $E \in \{\ell_\infty, c_0, c\}$, is a BK space normed by

$$\|X\|_{s_\alpha} = \sup_{n \geq 1} \left(\frac{|x_n|}{\alpha_n}\right), \tag{1.4}$$

and $\overset{\circ}{s}_\alpha$ has AK, see [11].

Now let $\alpha = (\alpha_n)_{n=1}^{\infty}$ and $\beta = (\beta_n)_{n=1}^{\infty} \in U^+$. By $S_{\alpha,\beta}$ we denote the set of infinite matrices $A = (a_{nm})_{n,m=1}^{\infty}$ such that $\sup_{n \geq 1}(\sum_{m=1}^{\infty} |a_{nm}| \frac{\alpha_m}{\beta_n}) < \infty$; $S_{\alpha,\beta}$ is a Banach space normed by $\|A\|_{S_{\alpha,\beta}} = \sup_{n \geq 1}(\sum_{m=1}^{\infty} |a_{nm}| \frac{\alpha_m}{\beta_n})$. Let E and F be any subsets of s. When A maps E into F we write $A \in (E, F)$, see [4]. So $A \in (E, F)$ if and only if the series $y_n = \sum_{m=1}^{\infty} a_{nm} x_m$ converge for all n and all $X \in E$ and $AX = (y_n)_{n=1}^{\infty} \in F$ for all $X \in E$. It was proved in [14] that $A \in (s_\alpha, s_\beta)$ if and only if $A \in S_{\alpha,\beta}$. So we have $(s_\alpha, s_\beta) = S_{\alpha,\beta}$.

When $s_\alpha = s_\beta$ we obtain the Banach algebra with identity $S_{\alpha,\beta} = S_\alpha$, (see [6, 7, 8, 10, 11]) normed by $\|A\|_{S_\alpha} = \|A\|_{S_{\alpha,\alpha}}$.

If $\alpha = (r^n)_{n=1}^{\infty}$ for $r > 0$, then S_α, s_α, $\overset{\circ}{s}_\alpha$ and $s_\alpha^{(c)}$ are denoted by S_r, s_r, $\overset{\circ}{s}_r$ and $s_r^{(c)}$, respectively (see [5, 10]). When $r = 1$, we obtain $s_1 = \ell_\infty$, $\overset{\circ}{s}_1 = c_0$ and $s_1^{(c)} = c$, and putting $e = (1, 1, ...)$ we have $S_1 = S_e$.

For any subset E of s, we put $A(E) = \{Y : Y = AX \text{ for some } X \in E\}$. If F is a subset of s, then $F_A = \{X \in s : Y = AX \in F\}$ denotes the *matrix domain of A in X*.

2. The operator Δ mapping in the sets s_α, $\overset{\circ}{s}_\alpha$, or $s_\alpha^{(c)}$

Now recall that the operator of first difference [5], [7]–[12] is defined by $\Delta = (\nu_{nm})_{n,m\geq 1}$, with $\nu_{nn} = 1$ for all $n \geq 1$, $\nu_{n,n-1} = -1$ for all $n \geq 2$ and $\nu_{nm} = 0$ otherwise. An infinite matrix $T = (t_{nm})_{n,m=1}^{\infty}$ is said to be a triangle if $t_{nm} = 0$ for $m > n$ and $t_{nn} \neq 0$ for all n. If $\pounds$ is the set of all triangles, it can easily be seen that $\pounds$ is a group with respect to matrix multiplication. The infinite matrix $\Sigma = (\nu'_{nm})_{n,m=1}^{\infty}$ defined by $\nu'_{nm} = 1$ for all $m \leq n$ and $\nu'_{nm} = 0$ otherwise is the inverse of Δ in $\pounds$, and we may write $\Sigma = \Delta^{-1}$, see [3]. For any given sequence $\xi = (\xi_n)_{n=1}^{\infty}$, we put $D\xi = (\xi_n \delta_{nm})_{n,m=1}^{\infty}$, where $\delta_{nm} = 0$ if $m \neq n$ and $\delta_{nm} = 1$ for $m = n$. If U is the set of all sequences $X = (x_n)_{n=1}^{\infty}$ such that $x_n \neq 0$ for all n, we define the triangle $C(\lambda) = D_{\frac{1}{\lambda}}\Sigma = (c_{nm})_{n,m=1}^{\infty}$ for $\lambda = (\lambda_n)_{n=1}^{\infty} \in U$. We have $c_{nm} = 1/\lambda_n$ for $m \leq n$ and $c_{nm} = 0$ otherwise. Writing $C(\lambda)\lambda = ((\sum_{k=1}^{n}\lambda_k)/\lambda_n)_{n=1}^{\infty}$, we define the sets

$$\widehat{C_1} = \{\alpha \in U^+ : C(\alpha)\alpha \in \ell_\infty\}, \quad \widehat{C} = \{\alpha \in U^+ : C(\alpha)\alpha \in c\}$$

and

$$\Gamma = \left\{\alpha \in U^+ : \overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1\right\}.$$

Recall that $\alpha \in \Gamma$ if and only if there is an integer $q \geq 1$ such that $\gamma_q(\alpha) = \sup_{n\geq q+1}(\alpha_{n-1}/\alpha_n) < 1$ (see [7]). The following result was given in [10].

Lemma 2.1. *We have*

i) $s_\alpha(\Delta) = s_\alpha$ *if and only if* $\alpha \in \widehat{C_1}$*;*

ii) $\overset{\circ}{s}_\alpha(\Delta) = \overset{\circ}{s}_\alpha$ *if and only if* $\alpha \in \widehat{C_1}$*;*

iii) $s_\alpha^{(c)}(\Delta) = s_\alpha^{(c)}$ *if and only if* $\alpha \in \widehat{C}$*;*

iv) $\Delta_\alpha = D_{\frac{1}{\alpha}}\Delta D_\alpha$ *is bijective from* c *into itself with* $\lim X = \Delta_\alpha\text{-}\lim X$*, if and only if* $\alpha_{n-1}/\alpha_n \to 0$.

Let us put

$$\widehat{\Gamma} = \left\{\alpha \in U^+ : \overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1\right\}.$$

In the next proof we shall use the set $B(s_\alpha^{(c)})$ of all bounded linear operators mapping $s_\alpha^{(c)}$ into itself. Recall that since $s_\alpha^{(c)}$ is a Banach space with the norm $\|\cdot\|_{s_\alpha}$, the set $B(s_\alpha^{(c)})$ of all linear operators $A \in (s_\alpha^{(c)}, s_\alpha^{(c)})$ normed by

$$\|A\|_{B(s_\alpha^{(c)})} = \sup_{X\neq 0}\left(\frac{\|AX\|_{s_\alpha}}{\|X\|_{s_\alpha}}\right)$$

is the Banach algebra of all bounded linear operators that map $s_\alpha^{(c)}$ into itself, see [2].

Proposition 2.2. *We have* $\widehat{C} = \widehat{\Gamma} \subset \Gamma \subset \widehat{C_1}$.

Proof. The inclusions $\widehat{C} \subset \widehat{\Gamma}$ and $\Gamma \subset \widehat{C_1}$ were shown [10] and [7], respectively. It remains to prove that $\widehat{\Gamma} \subset \widehat{C}$. Assume that $\alpha \in \widehat{\Gamma}$. Putting $D_{\frac{1}{\alpha}} \Delta D_\alpha = (\xi_{nm})_{n,m=1}^{\infty}$, we get $\xi_{nn} = 1$ for all n, $\xi_{n,n-1} = -\alpha_{n-1}/\alpha_n$ for all $n \geq 2$, and $\xi_{nm} = 0$ otherwise. Then from the characterization of (c, c) (cf. [14, 14 Theorem 1.36 p. 160]), the condition $D_{\frac{1}{\alpha}} \Delta D_\alpha \in (c, c)$ is equivalent to $(\alpha_{n-1}/\alpha_n)_{n \geq 2} \in c$. Let us show that Δ is invertible in $B(s_\alpha^{(c)})$. Consider the matrix

$$\Sigma^{(k)} = \begin{pmatrix} \left[\Delta^{(k)}\right]^{-1} & & O \\ & 1 & \\ O & & . \end{pmatrix} \text{ for any given integer } k \geq 1,$$

where $\Delta^{(k)}$ is the finite matrix whose entries are those of the k first rows and columns of Δ. We get $\Sigma^{(k)}\Delta = (a_{nm})_{n,m=1}^{\infty}$, with $a_{nn} = 1$ for all n; $a_{n,n-1} = -1$ for all $n \geq k+1$; and $a_{nm} = 0$ otherwise. We deduce that

$$\left\| I - \Sigma^{(k)}\Delta \right\|_{B(s_\alpha)} = \left\| I - \Sigma^{(k)}\Delta \right\|_{S_\alpha} = \sup_{k \geq k+1} \left(\frac{\alpha_{n-1}}{\alpha_n} \right).$$

So $\overline{\lim}_{n\to\infty}(\alpha_{n-1}/\alpha_n) = \lim_{n\to\infty}(\alpha_{n-1}/\alpha_n) < 1$ and $\|I - \Sigma^{(k)}\Delta\|_{B(s_\alpha)} < 1$ and we that $\Sigma^{(k)}\Delta$ is invertible in the Banach algebra $B(s_\alpha^{(c)})$ and $\Delta = (\Sigma^{(k)})^{-1}\Sigma^{(k)}$ is bijective from $s_\alpha^{(c)}$ into itself. Thus we have $\alpha \in \widehat{C}$ by Lemma 2.1 (ii) and we have shown that $\widehat{\Gamma} \subset \widehat{C}$. □

3. Sets of generalized weighted means and matrix transformations.

In this section we recall some results given in [15] and apply them to characterize matrix transformations in either of the sets $(\overline{N}, q)_\alpha$, $(\overline{N}, q)_\alpha^{\circ}$ or $(\overline{N}, q)_\alpha^{(c)}$. Then we give some properties of the identity $((\overline{N}, q)_\alpha, (\overline{N}, q)_\beta) = S_{\alpha', \beta'}$.

3.1. **Matrix transformations in the sets of weighted means.** Let u, $v \in U$ and $E \subset s$. Then we define

$$W(u, v; E) = v^{-1} * \left(u^{-1} * E\right)_\Sigma,$$

the set of generalized weighted means. Consider now the following conditions:

$$\sup_n \left(\sum_{m=1}^{\infty} \left| \frac{1}{u_m} \left(\frac{a_{nm}}{v_m} - \frac{a_{n,m+1}}{v_{m+1}} \right) \right| \right) < \infty; \tag{3.1}$$

(3.2)
$$\lim_{m\to\infty}\left(\frac{a_{nm}}{u_m v_m}\right)=0 \text{ for all } n;$$

(3.3)
$$\lim_{m\to\infty}\left(\frac{a_{nm}}{u_m v_m}\right)=l_n \text{ for all } n;$$

(3.4)
$$\sup_n |l_n|<\infty;$$

(3.5)
$$\sup_m\left(\left|\frac{a_{nm}}{u_m v_m}\right|\right)<\infty \text{ for each } n;$$

(3.6)
$$\lim_{n\to\infty}\left(\frac{1}{u_m}\left(\frac{a_{nm}}{v_m}-\frac{a_{n,m+1}}{v_{m+1}}\right)\right)=0 \text{ for each } m;$$

(3.7)
$$\lim_{n\to\infty}\left(\frac{1}{u_m}\left(\frac{a_{nm}}{v_m}-\frac{a_{n,m+1}}{v_{m+1}}\right)\right)=l'_m \text{ for each } m;$$

(3.8)
$$\lim_{n\to\infty}\left(\sum_{m=1}^{\infty}\frac{a_{nm}}{v_m}\left(\frac{1}{u_m}-\frac{1}{u_{m-1}}\right)\right)=0;$$

(3.9)
$$\lim_{n\to\infty}\left(\sum_{m=1}^{\infty}\frac{a_{nm}}{v_m}\left(\frac{1}{u_m}-\frac{1}{u_{m-1}}\right)\right)=L.$$

We have from [15, Theorem 3.3 p. 651]

Lemma 3.1. *We have*
(i) $A\in(W(u,v;\ell_\infty),\ell_\infty)$ *if and only if (3.1) and (3.2) hold;*
(ii) $A\in(W(u,v;c),\ell_\infty)$ *if and only if (3.1), (3.3) and (3.4) hold;*
(iii) $A\in(W(u,v;c_0),\ell_\infty)$ *if and only if (3.1) and (3.5) hold;*
(iv) $A\in(W(u,v;c_0),c_0)$ *if and only if (3.1), (3.5) and (3.6) hold;*
(v) $A\in(W(u,v;c_0),c)$ *if and only if (3.1), (3.5) and (3.7) hold;*
(vi) $A\in(W(u,v;c),c_0)$ *if and only if (3.1), (3.3), (3.4), (3.6) and (3.8) hold;*
(vii) $A\in(W(u,v;c),c)$ *if and only if (3.1), (3.3), (3.4), (3.7) and (3.9) hold.*

Then if $v=q=(q_n)_{n=1}^{\infty}\in U^+$ and $u=1/Q$ with $Q_n=\sum_{m=1}^{n}q_m$ $(n=1,2,...)$, we get $W(1/Q,q;\ell_\infty)=(\overline{N},q)_\infty$, $W(1/Q,q;c_0)=(\overline{N},q)_0$ and $W(1/Q,q;c)=(\overline{N},q)$. These sets are called *sets o weighted means that are bounded*, *convergent to zero* or *convergent*. We shall consider matrix transformations in the sets $(\overline{N},q)_\alpha=s_\alpha(\overline{N}_q)$, or $(\overline{N},q)_\alpha^\circ=s_\alpha^\circ(\overline{N}_q)$, or $(\overline{N},q)_\alpha^{(c)}=s_\alpha^{(c)}(\overline{N}_q)$, see [9].

We put

$$\gamma_{nm} = \left(\frac{a_{nm}}{q_m} - \frac{a_{n,m+1}}{q_{m+1}}\right)\frac{\alpha_m Q_m}{\beta_n} \quad \text{and}$$

$$\gamma'_{nm} = (\alpha_m Q_m - \alpha_{m-1}Q_{m-1})\frac{a_{nm}}{q_m\beta_n} \text{ for all } n, m$$

and consider the following conditions

$$\sup_n \left(\sum_{m=1}^{\infty} |\gamma_{nm}|\right) < \infty \tag{3.10}$$

$$\lim_{n\to\infty} \gamma_{nm} = 0 \text{ for all } m; \tag{3.11}$$

$$\lim_{n\to\infty} \gamma_{nm} = l_m \text{ for all } m; \tag{3.12}$$

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} \gamma'_{nm} = 0; \tag{3.13}$$

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} \gamma'_{nm} = L'. \tag{3.14}$$

We deduce the following

Proposition 3.2. *We have (i)* $A \in ((\overline{N}, q)_\alpha, s_\beta)$ *if and only if (3.10) holds and*

$$\lim_{m\to\infty}\left(a_{nm}\frac{\alpha_m Q_m}{q_m}\right) = 0 \quad \text{for all } n;$$

(ii) $A \in ((\overline{N}, q)^{(c)}_\alpha, s_\beta)$ *if and only if (3.10) holds,*

$$\lim_{m\to\infty}\left(a_{nm}\frac{\alpha_m Q_m}{q_m\beta_n}\right) = l'_n \text{ for all } n \text{ and } \sup_n(|l'_n|) < \infty; \tag{3.15}$$

(iii) $A \in ((\overline{N}, q)^\circ_\alpha, s_\beta)$ *if and only if (3.10) holds and*

$$\sup_m\left(|a_{nm}|\frac{\alpha_m Q_m}{q_m}\right) < \infty \text{ for all } n; \tag{3.16}$$

(iv) $A \in ((\overline{N}, q)^\circ_\alpha, s^\circ_\beta)$ *if and only if (3.10), (3.11) and (3.16) hold;*
(v) $A \in ((\overline{N}, q)^\circ_\alpha, s^{(c)}_\beta)$ *if and only if (3.10), (3.12) and (3.16) hold;*
(vi) $A \in ((\overline{N}, q)^{(c)}_\alpha, s^\circ_\beta)$ *if and only if (3.10), (3.11), (3.13) and (3.15) hold;*
(vii) $A \in ((\overline{N}, q)^{(c)}_\alpha, s^{(c)}_\beta)$ *if and only if (3.10), (3.12), (3.14) and (3.15) hold.*

Proof. Put $u = 1/\alpha Q$ and $v = q \in U^+$. Since $\Delta^{-1} = \Sigma \in \pounds$, we get

$$W(u,v;\ell_\infty) = W(1/\alpha Q, q;\ell_\infty) = (\overline{N},q)_\alpha = D_{\frac{1}{q}}\Delta D_{\alpha Q}\ell_\infty,$$

$W(1/\alpha Q, q; c_0) = (\overline{N},q)_\alpha^\circ$ and $W(1/\alpha Q, q; c) = (\overline{N},q)_\alpha^{(c)}$. Now the conclusion follows from Lemma 3.1 and the fact that, for any set of sequences E, the condition $A \in (E,F)$ for $F = s_\beta$, s_β° or $s_\beta^{(c)}$ is equivalent to $D_{\frac{1}{\beta}}A \in (E,G)$ where G is any of sets ℓ_∞, c_0 or c respectively. □

We shall use the following known result given by Malkowsky (cf. [13, Theorem 1].

Lemma 3.3. *Let $T \in \pounds$. Then, for arbitrary subsets E and F of s, $A \in (E, F_T)$ if and only if $TA \in (E,F)$.*

Consider now the following conditions.

(3.17)
$$\sup_n \left(\sum_{m=1}^{\infty} \left| \frac{1}{P_n\beta_n} \sum_{k=1}^{n} p_k \left(\frac{a_{km}}{q_m} - \frac{a_{k,m+1}}{q_{m+1}} \right) \right| \alpha_m Q_m \right) < \infty;$$

(3.18)
$$\lim_{n\to\infty} \left[\frac{1}{P_n\beta_n} \sum_{k=1}^{n} p_k \left(\frac{a_{km}}{q_m} - \frac{a_{k,m+1}}{q_{m+1}} \right) \alpha_m Q_m \right] = 0 \quad \text{for all } m = 1,2,...;$$

(3.19)
$$\lim_{m\to\infty} \left[\frac{\alpha_m Q_m}{q_m} \left(\sum_{k=1}^{n} \frac{p_k a_{km}}{P_n} \right) \right] = \xi'_n \quad \text{for all } n = 1,2,...;$$

(3.20)
$$\sup_n |\xi'_n| < \infty;$$

(3.21)
$$\sup_m \left[\frac{\alpha_m Q_m}{\beta_n q_m} \left(\sum_{k=1}^{n} \frac{p_k a_{km}}{P_n} \right) \right] < \infty \quad \text{for all } n = 1,2,...;$$

(3.22)
$$\lim_{n\to\infty} \frac{1}{P_n\beta_n} \sum_{m=1}^{\infty} \left[\left(\frac{\alpha_m Q_m - \alpha_{m-1}Q_{m-1}}{q_m} \right) \sum_{k=1}^{n} p_k a_{km} \right] = 0;$$

(3.23)

$$\lim_{n\to\infty}\frac{1}{P_n\beta_n}\sum_{m=1}^{\infty}\left[\left(\frac{\alpha_m Q_m-\alpha_{m-1}Q_{m-1}}{q_m}\right)\sum_{k=1}^{n}p_k a_{km}\right]=L'.$$

Proposition 3.4. *We have*
(i) $A\in((\overline{N},q)_\alpha,(\overline{N},p)_\beta)$ *if and only if (3.17) holds and*

$$\lim_{m\to\infty}\left[\frac{\alpha_m Q_m}{q_m}\left(\sum_{k=1}^{n}\frac{p_k a_{km}}{P_n}\right)\right]=0\quad \text{for all } n=1,2,...;$$

(ii) $A\in((\overline{N},q)_\alpha^{(c)},(\overline{N},p)_\beta)$ *if and only if (3.17), (3.19) and (3.20) hold;*
(iii) $A\in((\overline{N},q)_\alpha^{\circ},(\overline{N},p)_\beta)$ *if and only if (3.17) and (3.21) hold;*
(iv) $A\in((\overline{N},q)_\alpha^{\circ},(\overline{N},p)_\beta^{\circ})$ *if and only if (3.17), (3.18) and (3.21) hold;*
(v) $A\in((\overline{N},q)_\alpha^{\circ},(\overline{N},p)_\beta^{(c)})$ *if and only if (3.17), (3.21) hold and*

(3.24)

$$\lim_{n\to\infty}\left[\frac{1}{P_n\beta_n}\sum_{k=1}^{n}p_k\left(\frac{a_{km}}{q_m}-\frac{a_{k,m+1}}{q_{m+1}}\right)\alpha_m Q_m\right]=l_m\quad \text{for all } m=1,2,...;$$

(vi) $A\in((\overline{N},q)_\alpha^{(c)},(\overline{N},p)_\beta^{\circ})$ *if and only if (3.17), (3.18), (3.19) and (3.22) hold;*
(vii) $A\in((\overline{N},q)_\alpha^{(c)},(\overline{N},p)_\beta^{(c)})$ *if and only if (3.17), (3.23), (3.24) and (3.19) hold.*

Proof. These results are a direct consequence of Proposition 3.2 and Lemma 3.3. Indeed, for (i) we have $A\in((\overline{N},q)_\alpha,(\overline{N},p)_\beta)$ if and only if $\overline{N}_pA\in((\overline{N},q)_\alpha,s_\beta)$, where

$$\overline{N}_pA=\left(\sum_{k=1}^{n}\frac{p_k a_{km}}{P_n}\right)_{n,m=1}^{\infty}.$$

Then it is enough to replace the entries of A by those of $\overline{N}_pA$ in Proposition 3.2 (i). The remaining parts can be shown in the same way. □

3.2. **Properties of matrix transformations between sets of weighted means.** First we need some additional results on the set $S_{\alpha,\beta}$. Recall that, for any subsets E and F of s, $E*F$ is the set of all products $XY=(x_ny_n)_{n=1}^{\infty}$, where $X=(x_n)_{n=1}^{\infty}\in E$ and $Y=(y_n)_{n=1}^{\infty}\in F$. We can state the following results.

Theorem 3.5. *Let* $\alpha,\beta,\alpha',\beta'\in U^+$. *Then*

(i) $\alpha_n = O(\beta_n)$ $(n \to \infty)$ *if and only if* $s_\alpha \subset s_\beta$*;*
(ii) $\alpha_n = O(\beta_n)$ *and* $\beta_n = O(\alpha_n)$ $(n \to \infty)$ *if and only if* $s_\alpha = s_\beta$*;*
(iii) $s_\alpha = s_\beta$ *if and only if there exist* K_1 *and* $K_2 > 0$ *such that* $K_1\alpha_n \leq \beta_n \leq K_2\alpha_n$ *for all* n*;*
(iv) *(a)* $s_\alpha = s_\beta$ *if and only if* $\overset{\circ}{s}_\alpha = \overset{\circ}{s}_\beta$*;*
(b) $\alpha_n/\beta_n \to l \neq 0$ *if and only if* $s_\alpha^{(c)} = s_\beta^{(c)}$*;*
(c) $s_\alpha^{(c)} = s_\beta^{(c)}$ *implies* $s_\alpha = s_\beta$ *and* $\overset{\circ}{s}_\alpha = \overset{\circ}{s}_\beta$*;*
(v) the identity $S_{\alpha,\beta} = S_{\alpha',\beta'}$ *is equivalent to* $s_\alpha = s_{\alpha'}$ *and* $s_\beta = s_{\beta'}$*.*
(vi) *(a) The identity* $(\overset{\circ}{s}_\alpha, s_\beta) = (\overset{\circ}{s}_{\alpha'}, s_{\beta'})$ *is equivalent to* $s_\alpha = s_{\alpha'}$ *and* $s_\beta = s_{\beta'}$*,*
(b) the identity $(s_\alpha^{(c)}, s_\beta) = (s_{\alpha'}^{(c)}, s_{\beta'})$ *is equivalent to* $s_\alpha = s_{\alpha'}$ *and* $s_\beta = s_{\beta'}$*.*
(vii) $s_{\alpha\beta} = s_\alpha * s_\beta$*,* $\overset{\circ}{s}_{\alpha\beta} = \overset{\circ}{s}_\alpha * \overset{\circ}{s}_\beta$ *and* $s_{\alpha\beta}^{(c)} = s_\alpha^{(c)} * s_\beta^{(c)}$*.*

Proof. (i) Assume that $\alpha_n = O(\beta_n)$ $(n \to \infty)$. If $X = (x_n)_{n=1}^\infty \in s_\alpha$, then we have

$$\frac{x_n}{\beta_n} = \frac{x_n}{\alpha_n}\frac{\alpha_n}{\beta_n} = O(1)\ (n \to \infty)$$

and $X \in s_\beta$, hence $s_\alpha \subset s_\beta$. Conversely, $\alpha \in s_\alpha \subset s_\beta$ implies $\alpha_n/\beta_n = O(1)$ and $\alpha_n = O(\beta_n)$ $(n \to \infty)$.

(ii) is obvious.

(iii) The conditions $s_\alpha \subset s_\beta$ and $s_\beta \subset s_\alpha$ are equivalent to $\alpha_n = O(\beta_n)$ and $\beta_n = O(\alpha_n)$ $(n \to \infty)$. This shows (iii).

(iv) (a) The identity $\overset{\circ}{s}_\alpha = \overset{\circ}{s}_\beta$ is equivalent to $I \in (\overset{\circ}{s}_\alpha, \overset{\circ}{s}_\beta)$ and $I \in (\overset{\circ}{s}_\beta, \overset{\circ}{s}_\alpha)$. This means $D_{\alpha/\beta}, D_{\beta/\alpha} \in (c_0, c_0)$. From the characterization of the class (c_0, c_0), we conclude $\alpha_n/\beta_n = O(1)$ and $\beta_n/\alpha_n = O(1)$ $(n \to \infty)$, that is $s_\alpha = s_\beta$.

(b) Similarly the identity $s_\alpha^{(c)} = s_\beta^{(c)}$ is equivalent to $D_{\alpha/\beta}, D_{\beta/\alpha} \in (c, c)$. So $s_\alpha^{(c)} = s_\beta^{(c)}$ is equivalent to the following conditions: $\alpha_n/\beta_n \to l$, $\beta_n/\alpha_n \to l'$, $\alpha_n/\beta_n = O(1)$ and $\beta_n/\alpha_n = O(1)$ $(n \to \infty)$.

(v) The sufficiency being obvious, we study the necessity.

Suppose that $S_{\alpha,\beta} = S_{\alpha',\beta'}$. First, we prove that $S_{\alpha,\beta} = S_{\alpha',\beta}$. For this, denote by $\widetilde{c_1} = (c_{nm})_{n,m=1}^\infty$ the infinite matrix defined by $c_{n1} = \beta_n/\alpha_1$ for all $n \geq 1$ and $c_{nm} = 0$ otherwise. We immediatly see that $\widetilde{c_1} \in S_{\alpha,\beta}$ and since $S_{\alpha,\beta} = S_{\alpha',\beta'}$, we get $\widetilde{c_1} \in S_{\alpha',\beta'}$. So $\widetilde{c_1}\alpha' = ((\beta_n/\alpha_1)\,\alpha_1')_{n=1}^\infty \in s_{\beta'}$, that is

$$\beta_n = \beta_n' O(1)\ (n \to \infty),$$

and we conclude from (i) that $s_\beta \subset s_{\beta'}$. By a similar argument, taking $\widetilde{c_1'} = (c'_{nm})_{n,m=1}^\infty$ with $c'_{n1} = \beta_n'/\alpha_1'$ for all $n \geq 1$ and $c'_{nm} = 0$ otherwise, we get $\widetilde{c_1'}\alpha = ((\beta_n'/\alpha_1')\,\alpha_1)_{n=1}^\infty \in s_\beta$ and $s_{\beta'} \subset s_\beta$. Thus we have shown

$s_\beta = s_{\beta'}$, so $S_{\alpha,\beta} = S_{\alpha',\beta'}$ implies $S_{\alpha,\beta} = S_{\alpha',\beta}$. It remains to show that the equality $S_{\alpha,\beta} = S_{\alpha',\beta}$ implies $s_\alpha = s_{\alpha'}$. For this, we consider the matrix $D_{\frac{\beta}{\alpha}} \in S_{\alpha,\beta}$. Since $S_{\alpha,\beta} = S_{\alpha',\beta}$, we deduce that

$$D_{\frac{\beta}{\alpha}} s_{\alpha'} = s_{\frac{\beta}{\alpha}\alpha'} \subset s_\beta \tag{3.25}$$

and $\alpha'_n/\alpha_n = O(1)$ $(n \to \infty)$. So we have $s_\alpha \subset s_{\alpha'}$ by (i). Similarly, since $D_{\frac{\beta}{\alpha'}} \in S_{\alpha',\beta} = S_{\alpha,\beta}$, we get

$$D_{\frac{\beta}{\alpha'}} s_\alpha = s_{\frac{\beta}{\alpha'}\alpha} \subset s_\beta. \tag{3.26}$$

So we have $\alpha_n = O(\alpha'_n)$ and $s_{\alpha'} \subset s_\alpha$. Now we conclude $s_\alpha = s_{\alpha'}$ and (v) is proved.

(vi) (a) Since $(c_0, \ell_\infty) = S_1$, we easily deduce $(\overset{\circ}{s}_\alpha, s_\beta) = S_{\alpha,\beta}$ and $(\overset{\circ}{s}_{\alpha'}, s_{\beta'}) = S_{\alpha',\beta'}$. Then, by (v), the condition $(\overset{\circ}{s}_\alpha, s_\beta) = (\overset{\circ}{s}_{\alpha'}, s_{\beta'})$ implies $s_\alpha = s_{\alpha'}$ and $s_\beta = s_{\beta'}$.

Part (b) can be obtained by a similar argument using the fact that $(c, \ell_\infty) = S_1$.

(vii) Let $Z = (z_n)_{n=1}^\infty \in s_\alpha * s_\beta$. There are $X = (x_n)_{n=1}^\infty \in s_\alpha$ and $Y = (y_n)_{n=1}^\infty \in s_\beta$ such that $Z = XY \in s_\alpha * s_\beta$.Then $z_n = x_n y_n = \alpha_n O(1)\beta_n O(1) = \alpha_n\beta_n O(1)$ $(n \to \infty)$ and $Z \in s_{\alpha\beta}$. So we have shown $s_\alpha * s_\beta \subset s_{\alpha\beta}$. Conversely if $Z \in s_{\alpha\beta}$, there is a sequence $h = (h_n)_{n=1}^\infty \in \ell_\infty$, such that $z_n = \alpha_n\beta_n h_n$ and since $\alpha \in s_\alpha$ and $\beta h \in s_\beta$, we conclude $Z \in s_\alpha * s_\beta$. So we have shown $s_{\alpha\beta} \subset s_\alpha * s_\beta$. We conclude $s_\alpha * s_\beta = s_{\alpha\beta}$. Let us show $\overset{\circ}{s}_{\alpha\beta} = \overset{\circ}{s}_\alpha * \overset{\circ}{s}_\beta$. If $Z = (z_n)_{n=1}^\infty \in \overset{\circ}{s}_\alpha * \overset{\circ}{s}_\beta$ then $z_n = \alpha_n o(1)\beta_n o(1) = \alpha_n\beta_n o(1)$ $(n \to \infty)$ and $Z = (z_n)_{n=1}^\infty \in \overset{\circ}{s}_{\alpha\beta}$. Thus we have $\overset{\circ}{s}_\alpha * \overset{\circ}{s}_\beta \subset \overset{\circ}{s}_{\alpha\beta}$. Conversely let $Z \in \overset{\circ}{s}_{\alpha\beta}$. Then there exists a sequence $\varepsilon = (\varepsilon_n)_{n=1}^\infty \in c_0$ such that $z_n = \alpha_n\beta_n\varepsilon_n = \alpha_n\sqrt{|\varepsilon_n|}\beta_n\sqrt{|\varepsilon_n|}k_n$, with $|k_n| = 1$. This proves $Z \in \overset{\circ}{s}_\alpha * \overset{\circ}{s}_\beta$ and $\overset{\circ}{s}_{\alpha\beta} \subset \overset{\circ}{s}_\alpha * \overset{\circ}{s}_\beta$. So we have shown $\overset{\circ}{s}_{\alpha\beta} = \overset{\circ}{s}_\alpha * \overset{\circ}{s}_\beta$. The last case can be shown in a similar way. $\square$

Remark 3.6. *It can be easily seen that for any given sequences $\alpha, \beta \in U^+$, the property $\alpha_n/\beta_n \to l \neq 0$ implies $s_\alpha = s_\beta$, $\overset{\circ}{s}_\alpha = \overset{\circ}{s}_\beta$ and $s_\alpha^{(c)} = s_\beta^{(c)}$.*

Remark 3.7. *We can see from Theorem 3.5 (iii) that if we define the relation $\alpha R \beta$ if and only if $s_\alpha = s_\beta$ for any given $\alpha, \beta \in U^+$, then R is an equivalence relation. Note that we also have $\alpha R \beta$ if and only if $1/\alpha R 1/\beta$, and for any sequence $\gamma \in U^+$, $\alpha R \beta$ is equivalent to $(\alpha\gamma)R(\beta\gamma)$.*

Theorem 3.8. *Let $\alpha, \alpha', \beta, \beta' \in U^+$ and assume that $\alpha Q, \beta P \in \widehat{C}_1$. Then we have*
(i) $((\overline{N}, q)_\alpha, (\overline{N}, p)_\beta) = ((\overline{N}, q)_\alpha^\circ, (\overline{N}, p)_\beta) = S_{\alpha Q/q, \beta P/p}$;
(ii) $((\overline{N}, q)_\alpha, (\overline{N}, p)_\beta) = S_{\alpha',\beta'}$ if and only if $s_{\alpha'/\alpha} = s_{Q/q}$ and $s_{\beta'/\beta} = s_{P/p}$.

(iii) Assume that $\alpha'/\alpha, \beta'/\beta \in \ell_\infty$. *Then* $((\overline{N},q)_\alpha, (\overline{N},p)_\beta) = S_{\alpha',\beta'}$ *implies* $p, q \in \widehat{C}_1$.
(iv) Assume that $s_\alpha = s_{\alpha'}$ *and* $s_\beta = s_{\beta'}$. *Then* $((\overline{N},q)_\alpha, (\overline{N},p)_\beta) = S_{\alpha',\beta'}$ *if and only if* $p, q \in \widehat{C}_1$.

Proof. (i) The conditions $\alpha Q, \beta P \in \widehat{C}_1$ imply $(\overline{N},q)_\alpha = (\overline{N},q)^\circ_\alpha = s_{\alpha Q/q}$ and $(\overline{N},p)_\beta = (\overline{N},p)^\circ_\beta = s_{\beta P/p}$. Indeed, we have $(\overline{N},q)_\alpha = D_{\frac{1}{q}}\Delta D_Q s_\alpha = D_{\frac{1}{q}}\Delta s_{\alpha Q}$ and by Lemma 2.1, $\alpha Q \in \widehat{C}_1$ implies that $\Delta \in \pounds$ is bijective from $s_{\alpha Q}$ into itself and $\Delta s_{\alpha Q} = s_{\alpha Q}$. So we have $(\overline{N},q)_\alpha = s_{\alpha Q/q}$. By a similar argument, we get $(\overline{N},q)^\circ_\alpha = s^\circ_{\alpha Q/q}$. Furthermore $\beta P \in \widehat{C}_1$ implies $(\overline{N},p)_\beta = s_{\beta P/p}$ and $(\overline{N},p)^\circ_\beta = s^\circ_{\beta P/p}$. Then we have

$$\left(\left(\overline{N},q\right)_\alpha, \left(\overline{N},p\right)_\beta\right) = \left(s_{\alpha Q/q}, s_{\beta P/p}\right) = \left(s^\circ_{\alpha Q/q}, s_{\beta P/p}\right)$$
$$= \left(\left(\overline{N},q\right)^\circ_\alpha, \left(\overline{N},p\right)_\beta\right),$$

and the conclusion follows from the identity $(s_{\alpha Q/q}, s_{\beta P/p}) = S_{\alpha Q/q, \beta P/p}$.

(ii) By Theorem 3.5 (iii), the identity $((\overline{N},q)_\alpha, (\overline{N},p)_\beta) = S_{\alpha',\beta'}$ is equivalent to $s_{\frac{\alpha Q}{q}} = s_{\alpha'}$ and $s_{\frac{\beta P}{p}} = s_{\beta'}$. Therefore we have $s_{\alpha'} * s_{1/\alpha} = s_{\alpha Q/q} * s_{1/\alpha} = s_{Q/q}$ and also $s_{\beta'/\beta} = s_{P/p}$. This shows (ii).

(iii) Using Theorem 3.5 (iii), we have $s_{\frac{\alpha Q}{q}} = s_{\alpha'}$, and $s_{\frac{\beta P}{p}} = s_{\beta'}$ imply together that there are constants K_1 and K_2 such that

(3.27)
$$\frac{Q_n}{q_n} \le K_1 \frac{\alpha'_n}{\alpha_n} = O(1) \quad \text{and} \quad \frac{P_n}{p_n} \le K_2 \frac{\beta'_n}{\beta_n} = O(1) \quad \text{for all } n.$$

Then we have $p, q \in \widehat{C}_1$.

(iv) The necessity comes from (ii). For the sufficiency, we assume $s_\alpha = s_{\alpha'}$ and $s_\beta = s_{\beta'}$. Then there are constants $M_1, M_2 > 0$ such that $\alpha'_n/\alpha_n \ge M_1$ and $\beta'_n/\beta_n \ge M_2$ for all n. Now $p, q \in \widehat{C}_1$ imply that there are constants $M'_1, M'_2 > 0$ such that

$$\frac{1}{M'_1}\frac{Q_n}{q_n} \le 1 \le \frac{Q_n}{q_n} \quad \text{and} \quad \frac{1}{M'_2}\frac{P_n}{p_n} \le 1 \le \frac{P_n}{p_n} \quad \text{for all } n.$$

So $s_{\alpha/\alpha'} = l_\infty = s_{Q/q}$ and $s_{\beta/\beta'} = \ell_\infty = s_{P/p}$, and we have shown

$$((\overline{N},q)_\alpha, (\overline{N},p)_\beta) = S_{\alpha',\beta'}.$$

□

Remark 3.9. *Reasoning as above it can easily be shown that the conditions* $\alpha Q \in \Gamma$ *and* $\beta P \in \widehat{C}_1$, *imply together* $((\overline{N},q)^{(c)}_\alpha, (\overline{N},p)_\beta) = S_{\alpha Q/q, \beta P/p}$.

Corollary 3.10. *Assume* $\alpha Q, \beta P \in \widehat{C}_1$ *and consider the following hypotheses:*

(i) $((\overline{N},q)_\alpha, (\overline{N},p)_\beta) = S_{\alpha,\beta}$*;*
(ii) $p,\ q \in \widehat{C}_1$*;*
(iii) there are $K, K' > 0$ *and* $\gamma, \mu > 1$ *such that*

$$p_n \geq K\gamma^n \text{ and } q_n \geq K'\mu^n \text{ for all } n;$$

(iv) $((\overline{N},q)_\alpha, (\overline{N},p)_\beta) = S_1$*;*
(v) $s_{\alpha Q} = s_q, s_{\beta P} = s_p$*;*
(vi) $q/\alpha \notin c_0$ *or* $p/\beta \notin c_0$*;*
(vii) there are constants $K_1,\ K_2 > 0$ *such that*

$$K_1 \tfrac{\beta_n}{\alpha_n} \leq \tfrac{Q_n}{P_n}\tfrac{p_n}{q_n} \leq K_2 \tfrac{\beta_n}{\alpha_n} \quad \text{for all } n.$$

Then (i) and (ii) are equivalent, (i) implies (iii), (iv) is equivalent to (v), and (iv) implies (vi) and (vii).

Proof. By Theorem 3.8 (iv), conditions (i) and (ii) are equivalent.

Let us show that (ii) implies (iii). First, $p \in \widehat{C}_1$ implies that there exists a real $M > 1$ such that

$$[C(p)p]_n = \frac{P_n}{P_n - P_{n-1}} \leq M \text{ for all } n.$$

So $P_n \geq (M/(M-1))P_{n-1}$ and $P_n \geq (M/(M-1))^{n-1}p_1$ for all n. Therefore we conclude from

$$\frac{p_1}{p_n}\left(\frac{M}{M-1}\right)^{n-1} \leq [C(p)p]_n = \frac{P_n}{p_n} \leq M,$$

that $p_n \geq K\gamma^n$ for all n, with $K = (M-1)p_1/M^2$ and $\gamma = M/(M-1) > 1$. We get the same result for q. Since (ii) implies (iii) and (i) implies (ii) we conclude that (i) implies (iii).

By Remark 3.7, the conditions $s_{\frac{\alpha Q}{q}} = s_1$ and $s_{\frac{\beta P}{p}} = s_1$ are equivalent to $s_{\frac{\alpha Q}{q}} * s_q = s_{\alpha Q} = s_q$ and $s_{\frac{\alpha P}{p}} * s_p = s_{\alpha P} = s_p$ and then (iv) is equivalent to (v).

Let us show that (iv) implies (vi). Condition (iv) implies $s_{\frac{\alpha Q}{q}} = s_1$ and $s_{\frac{\beta P}{p}} = s_1$. Then there are constants $K_1, K_2 > 0$ such that $K_1 \leq \alpha Q/q \leq K_2$ and

$$0 < \frac{Q_1}{K_2} \leq \frac{Q_n}{K_2} \leq \frac{q_n}{\alpha_n} \quad \text{for all } n.$$

So $q/\alpha \notin c_0$. Similarly we obtain that (iv) implies $p/\beta \notin c_0$. Condition (iv) implies that $s_\alpha = s_{q/Q}$ and $s_\beta = s_{p/P}$ and since $s_{1/\alpha} = s_{Q/q}$ we deduce from

Theorem 3.5 (vii) that $s_{1/\alpha} * s_\beta = s_{\beta/\alpha} = s_{Q/q} * s_{p/P} = s_{\frac{Qp}{Pq}}$. So we have shown that (iv) implies (vi) and (vii). □

Remark 3.11. *It is easy to show that if* $\alpha,\ \beta \in \Gamma$, *then* $((\overline{N},q)_\alpha, (\overline{N},p)_\beta) = S_{Q,P}$ *if and only if* $s_\alpha = s_q$ *and* $s_\beta = s_p$. *This result comes from the identities* $s_{\frac{\alpha Q}{q}} = s_Q$ *and* $s_{\frac{\beta P}{p}} = s_P$.
Note also that $\alpha, \beta \in \Gamma$ *implies* $\alpha Q, \beta P \in \Gamma$. *Then* $\alpha, \beta \in \Gamma$ *implies that* $((\overline{N},q)_\alpha, (\overline{N},p)_\beta) = S_{\alpha,\beta}$ *if and only if* $p, q \in \widehat{C}_1$.

Remark 3.12. *If* $\beta/\alpha \in c_0$, $\frac{q}{p}P \in l_\infty$ *and* $\alpha Q, \beta P \in \widehat{C}_1$ *then* $((\overline{N},q)_\alpha, (\overline{N},p)_\beta) \neq S_1$. *Indeed, suppose that* $((\overline{N},q)_\alpha, (\overline{N},p)_\beta) = S_1$. *Then, since (iv) implies (vii) in Corollary 3.10,* $\beta/\alpha \in c_0$ *implies* $(Q_n/P_n)(p_n/q_n) = o(1)$ $(n \to \infty)$, *and since* $qP/p \in \ell_\infty$, *we ahve* $Q_n = (P_n q_n/p_n)o(1) = o(1)$ $(n \to \infty)$. *This is contradictory because* $Q_n \geq q_1 > 0$ *for all* n *and so* $((\overline{N},q)_\alpha, (\overline{N},p)_\beta) \neq S_1$.
On the other hand it can easily be shown that if $\beta/\alpha \notin \ell_\infty$ *and* $\alpha Q, \beta P \in \widehat{C}_1$, *then*

$$\left(\left(\overline{N},q\right)_\alpha, \left(\overline{N},p\right)_\beta\right) = S_1 \quad \textit{implies} \quad Q/q \notin \ell_\infty.$$

Indeed, if $\beta/\alpha \notin \ell_\infty$ *then there is a nondecreasing sequence* $(n_i)_{i=1}^\infty$ *of integers tending to infinity such that* $\beta_{n_i}/\alpha_{n_i} \to \infty$, *and since (iv) implies (vii) in Corollary 3.10, we have* $Q_{n_i}p_{n_i}/P_{n_i}q_{n_i} \to \infty$. *From the inequality* $Q_{n_i}/q_{n_i} \geq Q_{n_i}p_{n_i}/q_{n_i}P_{n_i}$, *we conclude* $Q/q \notin \ell_\infty$.

4. Matrix transformations in the sets $s_\alpha(\overline{N}_p\overline{N}_q)$, $\overset{\circ}{s}_\alpha(\overline{N}_p\overline{N}_q)$ and $s_\alpha^{(c)}(\overline{N}_p\overline{N}_q)$.

In this section, we study some properties of the sets $s_\alpha(\overline{N}_p\overline{N}_q)$, $\overset{\circ}{s}_\alpha(\overline{N}_p\overline{N}_q)$ and $s_\alpha^{(c)}(\overline{N}_p\overline{N}_q)$ and give a characterization of matrix transformations mapping in either of the sets $s_\alpha(\overline{N}_p\overline{N}_q)$, $\overset{\circ}{s}_\alpha(\overline{N}_p\overline{N}_q)$, or $s_\alpha^{(c)}(\overline{N}_p\overline{N}_q)$.

4.1. A study of the equation $(\overline{N}_p\overline{N}_q)X = B$.

Proposition 4.1. *(i) Let* B *be any given sequence. Then the equation* $(\overline{N}_p\overline{N}_q)X = B$ *is equivalent to the infinite linear system*

$$\frac{1}{P_n}\sum_{m=1}^{n}\left(\sum_{k=m}^{n}\frac{p_k}{Q_k}\right)q_m x_m = b_n \qquad (n = 1, 2, \dots).$$

(ii) Assume that $\alpha, \alpha/p \in \Gamma$. *Then, for any given* $B \in s_\alpha$, *(resp.* $B \in \overset{\circ}{s}_\alpha$*), the equation* $(\overline{N}_p\overline{N}_q)X = B$ *admits in* $s_{\alpha\frac{PQ}{pq}}$, *(resp.* $\overset{\circ}{s}_{\alpha\frac{PQ}{pq}}$*) the unique*

solution $X = \overline{N}_q^{-1}\overline{N}_p^{-1}B$ *given by*

$$x_n = \frac{Q_{n-1}}{p_{n-1}}\frac{P_{n-2}}{q_n}b_{n-2} - \frac{P_{n-1}}{q_n}\left(\frac{Q_{n-1}}{p_{n-1}} + d\frac{Q_n}{p_n}\right)b_{n-1} + \frac{Q_n}{q_n}\frac{P_n}{p_n}b_n \quad \text{for } n = 1, 2, ..., \tag{4.1}$$

with the convention $b_n = 0$ *for* $n \leq 0$.

(iii) If $\alpha P, \alpha P/p \in \widehat{\Gamma}$, *then, for any given* $B \in s_\alpha^{(c)}$, *the equation* $(\overline{N}_p\overline{N}_q)X = B$ *admits in* $s^{(c)}_{\alpha\frac{PQ}{pq}}$ *a unique solution given by (32).*

Proof. (i) We have $\overline{N}_p\overline{N}_q = D_{\frac{1}{P}}\Sigma D_p D_{\frac{1}{Q}}\Sigma D_q = D_{\frac{1}{P}}(\Sigma D_{\frac{p}{Q}}\Sigma)D_q$; and putting $\Sigma D_{\frac{p}{Q}}\Sigma = (\sigma_{nm})_{n,m=1}^{\infty}$, we get $\sigma_{nm} = \sum_{k=m}^{n}(p_k/Q_k)$ for $m \leq n$ and $\sigma_{nm} = 0$ otherwise. This shows (i).

(ii) Consider the case when $B \in s_\alpha$. First, since P is nondecreasing and

$$\overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n-1}}{\alpha_n}\frac{P_{n-1}}{P_n}\right) \leq \overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n-1}}{\alpha_n}\right)\overline{\lim_{n\to\infty}}\left(\frac{P_{n-1}}{P_n}\right),$$

we deduce that $\alpha \in \Gamma$ implies $\alpha P \in \Gamma$. Then the operator represented by $\overline{N}_p^{-1} = D_{\frac{1}{p}}\Delta D_P$ is bijective from s_α into $s_{\alpha\frac{P}{p}}$. Now, from the inequality

$$\overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n-1}}{\alpha_n p_{n-1}}\frac{p_n}{p_{n-1}}\frac{P_{n-1}Q_{n-1}}{P_nQ_n}\right) \leq \overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n-1}}{\alpha_n}\frac{1}{\frac{p_{n-1}}{p_n}}\right),$$

the condition $\alpha/p \in \Gamma$ implies that $\alpha PQ/p \in \Gamma$ and $\overline{N}_q^{-1} = D_{\frac{1}{q}}\Delta D_Q$ is also bijective from $s_{\alpha\frac{P}{p}}$ into $s_{\alpha\frac{PQ}{pq}}$. We conclude that $\overline{N}_p\overline{N}_q$ is bijective from $s_{\alpha\frac{PQ}{pq}}$ into s_α. To obtain (4.1), we need to explicitly obtain the matrix $(\overline{N}_p\overline{N}_q)^{-1}$. We have $\overline{N}_q^{-1}\overline{N}_p^{-1} = D_{\frac{1}{q}}\Delta D_Q D_{\frac{1}{p}}\Delta D_P = D_{\frac{1}{q}}\Delta D_u\Delta D_P$, with $u = Q/p$ and $\Delta' = D_u\Delta D_P = (\eta_{nm})_{n,m=1}^{\infty}$, where

$$\eta_{nm} = \begin{cases} u_nP_n & \text{for } m = n, \\ -\dfrac{Q_n}{p_n}P_{n-1} & \text{for } m = n-1, \quad n \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that $\overline{N}_q^{-1}\overline{N}_p^{-1} = D_{\frac{1}{q}}\Delta\Delta' = (\eta'_{nm})_{n,m=1}^{\infty}$, with

$$\eta'_{nm} = \begin{cases} \dfrac{Q_n}{q_n}\dfrac{P_n}{p_n} & \text{for } m = n, \\ \dfrac{P_{n-1}}{q_n}\left(\dfrac{Q_{n-1}}{p_{n-1}} + \dfrac{Q_n}{p_n}\right) & \text{for } m = n-1, \quad n \geq 2, \\ \dfrac{Q_{n-1}}{p_{n-1}}\dfrac{P_{n-2}}{q_n} & \text{for } m = n-2, \quad n \geq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Part (iii) can be shown similarly. □

It follows from Part (i) in the previous theorem that

$$s_\alpha(\overline{N}_p\overline{N}_q) = \left\{X \in s : \frac{1}{P_n}\sum_{m=1}^{n}\left(\sum_{k=m}^{n}\frac{p_k}{Q_k}\right)q_m x_m = \alpha_n O(1)\ (n \to \infty)\right\}.$$

We have the following result.

Proposition 4.2. *We have*

(i) $s_\alpha(\overline{N}_p\overline{N}_q) = s_{\alpha\frac{P}{p}}(\overline{N}_q)$ *if and only if* $\alpha P \in \widehat{C_1}$;

(ii) $s^\circ_\alpha(\overline{N}_p\overline{N}_q) = s^\circ_{\alpha\frac{P}{p}}(\overline{N}_q)$ *if and only if* $\alpha P \in \widehat{C_1}$;

(iii) $s^{(c)}_\alpha(\overline{N}_p\overline{N}_q) = s^{(c)}_{\alpha\frac{P}{p}}(\overline{N}_q)$ *if and only if* $\alpha P \in \widehat{\Gamma}$;

(iv) $\alpha \in \Gamma$ *implies* $s_\alpha(\overline{N}_p\overline{N}_q) = s_{\alpha\frac{P}{p}}(\overline{N}_q)$ *and* $s^\circ_\alpha(\overline{N}_p\overline{N}_q) = s^\circ_{\alpha\frac{P}{p}}(\overline{N}_q)$.

(v) Assume that $\alpha \in \Gamma$. *Then*

(a) $s_\alpha(\overline{N}_p\overline{N}_q) = s_{\alpha\frac{PQ}{pq}}$ *if and only if* $\alpha\frac{PQ}{p} \in \widehat{C_1}$;

(b) $s^\circ_\alpha(\overline{N}_p\overline{N}_q) = s^\circ_{\alpha\frac{PQ}{pq}}$ *if and only if* $\alpha\frac{PQ}{p} \in \widehat{C_1}$;

(vi) Let $\alpha P \in \widehat{\Gamma}$. *Then* $s^{(c)}_\alpha(\overline{N}_p\overline{N}_q) = s^{(c)}_{\alpha\frac{PQ}{pq}}$ *if and only if* $\alpha\frac{PQ}{p} \in \widehat{\Gamma}$.

Proof. (i) First we have $s_{\alpha\frac{P}{p}}(\overline{N}_q) = \overline{N}_q^{-1}s_{\alpha\frac{P}{p}} = D_{\frac{1}{q}}\Delta s_{\alpha\frac{PQ}{p}}$. Then $s_{\alpha\frac{P}{p}}(\overline{N}_q) = \overline{N}_q^{-1}\overline{N}_p^{-1}s_\alpha = D_{\frac{1}{q}}\Delta D_Q D_{\frac{1}{p}}\Delta D_P s_\alpha$ if and only if $s_{\alpha\frac{PQ}{p}} = D_{\frac{Q}{q}}\Delta s_{\alpha P}$ and $s_{\alpha P} = \Delta s_{\alpha P}$. The last identity means $\alpha P \in \widehat{C_1}$.

Parts (ii) and (iii) can shown similarly.

(iv) As we have seen in the proof of Proposition 4.1 (ii), the condition $\alpha \in \Gamma$ implies $\alpha P \in \widehat{C_1}$ and the conclusion follows from (i) and (ii).

(v)(a) As we have seen in (i), the identity $\overline{N}_q^{-1}\overline{N}_p^{-1}s_\alpha = s_{\alpha\frac{PQ}{pq}}$ is equivalent to

$$D_{\frac{1}{q}}\Delta D_Q D_{\frac{1}{p}}\Delta D_P s_\alpha = s_{\alpha\frac{PQ}{pq}}. \tag{4.2}$$

Since $\alpha \in \Gamma$, we have $\alpha P \in \Gamma$ and ,by Proposition 2.2, $\alpha P \in \widehat{C_1}$. So $\Delta s_{\alpha P} = s_{\alpha P}$ and identity (4.2) is equivalent to $\Delta D_{Q/p} \Delta s_{\alpha P} = \Delta D_{Q/p} s_{\alpha P} = \Delta s_{\frac{Q}{p}\alpha P} = s_{\alpha \frac{PQ}{p}}$ which in turn is equivalent to $\alpha PQ/p \in \widehat{C_1}$.
Assertions (v) (b) and (v) (c) can be shown similarly.

(vi) Reasoning as above we get $D_{\frac{1}{q}} \Delta D_Q D_{\frac{1}{p}} \Delta s^{(c)}_{\alpha P} = s^{(c)}_{\alpha \frac{PQ}{pq}}$ if and only if $\Delta s^{(c)}_{\alpha \frac{PQ}{p}} = s^{(c)}_{\alpha \frac{PQ}{p}}$. This means $\alpha PQ/p \in \widehat{\Gamma}$, and we have shown (vi). □

4.2. **Matrix transformations between $\chi(\overline{N}_p \overline{N}_q)$ and $\chi'(\overline{N}_r \overline{N}_s)$, where χ and χ' are of the form s_ξ, $\overset{\circ}{s}_\xi$ or $s^{(c)}_\xi$.** In this section, among other things, we study matrix transformations between $\chi(\overline{N}_q)$ and $\chi'(\overline{N}_r \overline{N}_s)$, where χ and χ' are of the form s_ξ, $\overset{\circ}{s}_\xi$ or $s^{(c)}_\xi$ for $\xi \in U^+$. We also consider the case when a matrix transformation maps $\chi(\overline{N}_p \overline{N}_q)$ into $\chi'(\overline{N}_r \overline{N}_s)$ where χ and χ' are of the form s_ξ, $\overset{\circ}{s}_\xi$ or $s^{(c)}_\xi$. Note that until now there is no characterization of the sets $(\chi(\overline{N}_p \overline{N}_q), \chi')$ where χ' is s_α, $\overset{\circ}{s}_\alpha$ or $s^{(c)}_\alpha$.

In this part, we use the sequences $r = (r_n)_{n=1}^\infty$, $s = (s_n)_{n=1}^\infty \in U^+$, $R = (R_n)_{n=1}^\infty$, $S = (S_n)_{n=1}^\infty$, with $R_n =_{k=1}^n r_k$ and $S_n = \sum_{k=1}^n s_k$. From the previous results, we deduce the following

Proposition 4.3. *We have*

(i) $(s_\alpha, s_\beta(\overline{N}_r \overline{N}_s)) = (\overset{\circ}{s}_\alpha, s_\beta(\overline{N}_r \overline{N}_s)) = (s^{(c)}_\alpha, s_\beta(\overline{N}_r \overline{N}_s))$

and $A \in (s_\alpha, s_\beta(\overline{N}_r \overline{N}_s))$

if and only if

$$\sup_{n \geq 1} \left(\frac{1}{\beta_n} \sum_{m=1}^\infty \left| \sum_{k=1}^\infty \frac{a_{nk}}{R_k} \left(\sum_{i=m}^k \frac{r_i}{S_i} \right) s_m \right| \alpha_m \right) < \infty; \tag{4.3}$$

(ii) $A \in (\overset{\circ}{s}_\alpha, \overset{\circ}{s}_\beta(\overline{N}_r \overline{N}_s))$ *if and only if (34) holds and*

(4.4)
$$\lim_{n \to \infty} \frac{1}{\beta_n} \left[\sum_{k=1}^\infty \frac{a_{nk}}{R_k} \left(\sum_{i=m}^k \frac{r_i}{S_i} \right) s_m \alpha_m \right] = 0 \textit{ for all } m = 1, 2, ...;$$

(iii) $A \in (s^{(c)}_\alpha, \overset{\circ}{s}_\beta(\overline{N}_r \overline{N}_s))$ *if and only if (4.3) and (4.4) hold and*

$$\lim_{n \to \infty} \frac{1}{\beta_n} \sum_{m=1}^\infty \left[\sum_{k=1}^\infty \frac{a_{nk}}{R_k} \left(\sum_{i=m}^k \frac{r_i}{S_i} \right) s_m \alpha_m \right] = 0 \textit{ for all } m = 1, 2, ...,$$

(iv) $A \in (s_\alpha^\circ, s_\beta^{(c)}(\overline{N}_r\overline{N}_s))$ *if and only if (4.3) holds and*

(4.5)
$$\lim_{n\to\infty}\frac{1}{\beta_n}\left[\sum_{k=1}^{\infty}\frac{a_{nk}}{R_k}\left(\sum_{i=m}^{k}\frac{r_i}{S_i}\right)s_m\alpha_m\right] = l_m \text{ for all } m = 1, 2, ...;$$

(v) $A \in (s_\alpha^{(c)}, s_\beta^{(c)}(\overline{N}_r\overline{N}_s))$ *if and only if (4.3), (4.5) hold and*

$$\lim_{n\to\infty}\frac{1}{\beta_n}\sum_{m=1}^{\infty}\left[\sum_{k=1}^{\infty}\frac{a_{nk}}{R_k}\left(\sum_{i=m}^{k}\frac{r_i}{S_i}\right)s_m\alpha_m\right] = l.$$

Proof. A short computation yields $\overline{N}_r\overline{N}_sA = (\kappa_{nm})_{n,m=1}^{\infty}$ with

$$\kappa_{nm} = \sum_{k=1}^{\infty}a_{nk}\frac{1}{R_k}\left(\sum_{i=m}^{k}\frac{r_i}{S_i}\right)s_m. \tag{4.6}$$

By Lemma 3.3, we have $A \in (s_\alpha, s_\beta(\overline{N}_r\overline{N}_s))$ if and only if $\overline{N}_r\overline{N}_sA \in S_{\alpha,\beta}$, and we have shown (i).

Parts (ii) and (iii) follow in a similar way using the characterizations of (c_0, c_0) and (c, c), (cf. [14, Theorem 1.36, p.160]). □

We also have the following

Corollary 4.4. *Let* $\alpha, \beta \in U^+$. *Then* $A \in (s_\alpha(\overline{N}_q), s_\beta(\overline{N}_r\overline{N}_s))$ *if and only if*

$$\sup_{n\geq 1}\left(\frac{1}{\beta_n}\sum_{m=1}^{\infty}\left|\sum_{k=1}^{\infty}\frac{a_{nk}}{R_k}\left[\frac{r_ms_m}{q_mS_m} + \left(\frac{s_m}{q_m} - \frac{s_{m+1}}{q_{m+1}}\right)\left(\sum_{i=m+1}^{k}\frac{r_i}{S_i}\right)\right]\right| \alpha_mQ_m\right) < \infty \tag{4.7}$$

and

(4.8)
$$\lim_{m\to\infty}\left[\sum_{k=1}^{\infty}\frac{a_{nk}}{R_k}\left(\sum_{i=m}^{k}\frac{r_i}{S_i}\right)\frac{s_m\alpha_mQ_m}{q_m}\right] = 0 \text{ for all } n = 1, 2, ...$$

Proof. Now $A \in (s_\alpha(\overline{N}_q), s_\beta(\overline{N}_r\overline{N}_s)$ if and only if $\overline{N}_r\overline{N}_sA \in (s_\alpha(\overline{N}_q), s_\beta)$, and applying Lemma 3.3 (i), we get (4.7) and (4.8). □

Remark 4.5. *Reasoning as in the proof of Corollary 4.4 and using Propsition 3.2 and Lemma 3.3, we easily get the characterizations of the sets* (E, F), *where* E *is any of the sets* $s_\alpha(\overline{N}_q)$, $s_\alpha^\circ(\overline{N}_q)$ *or* $s_\alpha^{(c)}(\overline{N}_q)$, *and* F *is any of the sets* $s_\beta(\overline{N}_r\overline{N}_s)$, $s_\beta^\circ(\overline{N}_r\overline{N}_s)$ *or* $s_\beta^{(c)}(\overline{N}_r\overline{N}_s)$. *So we have for instance* $A \in$

$(s_\alpha^\circ(\overline{N}_q), s_\beta^{(c)}(\overline{N}_r\overline{N}_s))$ *if and only if* $\overline{N}_r\overline{N}_s A \in (s_\alpha^\circ(\overline{N}_q), s_\beta^{(c)})$, *that is if and only if (4.7) holds and*

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{a_{nk}}{R_k}\left[\frac{r_m s_m}{q_m S_m}+\left(\frac{s_m}{q_m}-\frac{s_{m+1}}{q_{m+1}}\right)\left(\sum_{i=m+1}^{k}\frac{r_i}{S_i}\right)\right]\frac{\alpha_m Q_m}{\beta_n}=0 \text{ for all } n.$$

Proposition 4.6. *(i) Assume that* $\alpha \in \Gamma$.

(a) Then $A \in (s_\alpha(\overline{N}_p\overline{N}_q), s_\beta)$ *if and only if*

(4.9) $$\sup_{n\geq 1}\left[\frac{1}{\beta_n}\left(\sum_{m=1}^{\infty}\left|\frac{a_{nm}}{q_m}-\frac{a_{n,m+1}}{q_{m+1}}\right|\frac{\alpha_m P_m Q_m}{p_m}\right)\right]<\infty$$

and

(4.10)

$$\lim m\to\infty\left(a_{nm}\frac{\alpha_m P_m Q_m}{p_m q_m}\right)=0 \quad \text{for all } n=1,2,\ldots$$

(b) $A \in (s_\alpha^\circ(\overline{N}_p\overline{N}_q), s_\beta)$ *if and only if (4.9) holds and*

$$\sup_{m\geq 1}\left(|a_{nm}|\frac{\alpha_m P_m Q_m}{p_m q_m}\right)<\infty$$

(ii) If $\alpha P \in \widehat{\Gamma}$, *then* $A \in (s_\alpha^{(c)}(\overline{N}_p\overline{N}_q), s_\beta)$ *if and only if (4.9) holds,*

$$\lim_{m\to\infty}\left(a_{nm}\frac{\alpha_m P_m Q_m}{p_m q_m \beta_n}\right)=\xi_n \quad \text{for all } n=1,2,\ldots \text{ and } \sup_{n\geq 1}|\xi_n|<\infty.$$

(iii) (a) Assume that $\alpha, \alpha/p \in \Gamma$. *Then* $(s_\alpha(\overline{N}_p\overline{N}_q), s_\beta) = (s_\alpha^\circ(\overline{N}_p\overline{N}_q), s_\beta)$ *and* $A \in (s_\alpha(\overline{N}_p\overline{N}_q), s_\beta)$ *if and only if*

(4.11) $$\sup_{n\geq 1}\left(\frac{1}{\beta_n}\sum_{m=1}^{\infty}|a_{nm}|\frac{\alpha_m P_m Q_m}{p_m q_m}\right)<\infty.$$

(b) If $\alpha P, \alpha PQ/p \in \widehat{\Gamma}$, *then* $A \in (s_\alpha^{(c)}(\overline{N}_p\overline{N}_q), s_\beta)$ *if and only if (4.11) holds.*

Proof. (i) (a) As we have seen in the proof of Proposition 4.1, $\alpha \in \Gamma$ implies $\alpha P \in \Gamma$ and $\Delta s_{\alpha P} = s_{\alpha P}$. Thus we have $s_\alpha(\overline{N}_p\overline{N}_q) = \overline{N}_q^{-1}\overline{N}_p^{-1}s_\alpha = \overline{N}_q^{-1}D_{\frac{1}{p}}\Delta s_{\alpha P} = \overline{N}_q^{-1}s_{\alpha\frac{P}{p}} = s_{\alpha\frac{P}{p}}(\overline{N}_q)$ and $A \in (s_\alpha(\overline{N}_p\overline{N}_q), s_\beta)$ if and only if $A \in (s_{\alpha\frac{P}{p}}(\overline{N}_q), s_\beta)$. Then it is enough to apply Propositon 3.2 (i).
Part (b) can be shown similarly.

(ii) The condition $\alpha P \in \widehat{\Gamma}$ implies $s_\alpha^{(c)}(\overline{N}_p\overline{N}_q) = s_{\alpha P/p}^{(c)}(\overline{N}_q)$. Then $A \in (s_\alpha^{(c)}(\overline{N}_p\overline{N}_q), s_\beta)$ if and only if $A \in (s_{\alpha P/p}^{(c)}(\overline{N}_q), s_\beta)$, and the conclusion follows from Proposition 3.2 (ii).

(iii) (a) The condition $\alpha, \alpha/p \in \Gamma$ implies $s_\alpha(\overline{N}_p\overline{N}_q) = s_{\alpha PQ/pq}$ and $\overset{\circ}{s}_\alpha(\overline{N}_p\overline{N}_q) = \overset{\circ}{s}_{\alpha PQ/pq}$. So we have $(s_\alpha(\overline{N}_p\overline{N}_q), s_\beta) = (\overset{\circ}{s}_\alpha(\overline{N}_p\overline{N}_q), s_\beta) = S_{\alpha PQ/pq\beta}$.
Part (iii) (b) follows from Proposition 4.2 (vi). □

Remark 4.7. *Reasoning as in Proposition 4.6, we get the characterizations of the sets (E, F), where E is any of the sets $s_\alpha(\overline{N}_p\overline{N}_q)$, $\overset{\circ}{s}_\alpha(\overline{N}_p\overline{N}_q)$ or $s_\alpha^{(c)}(\overline{N}_p\overline{N}_q)$, and F is any of the sets s_β, $\overset{\circ}{s}_\beta$ or $s_\beta^{(c)}$.*

Proposition 4.8. *(i) (a) Assume that $\alpha, \alpha/p \in \Gamma$. Then we have*

$$\left(s_\alpha\left(\overline{N}_p\overline{N}_q\right), s_\beta\left(\overline{N}_r\overline{N}_s\right)\right) = \left(\overset{\circ}{s}_\alpha\left(\overline{N}_p\overline{N}_q\right), s_\beta\left(\overline{N}_r\overline{N}_s\right)\right)$$

and $A \in (s_\alpha(\overline{N}_p\overline{N}_q), s_\beta(\overline{N}_r\overline{N}_s))$ if and only if

(4.12)
$$\sup_{n\geq 1}\left[\frac{1}{\beta_n}\sum_{m=1}^{\infty}\left|\sum_{k=1}^{\infty}\frac{a_{nk}}{R_k}\left(\sum_{i=m}^{k}\frac{r_i}{S_i}\right)s_m\right|\frac{\alpha_m P_m Q_m}{p_m q_m}\right] < \infty.$$

(b) If $\alpha P \in \Gamma, \alpha PQ/p \in \widehat{\Gamma}$, then $A \in (s_\alpha^{(c)}(\overline{N}_p\overline{N}_q), s_\beta(\overline{N}_r\overline{N}_s))$ if and only if (4.12) holds.

(ii) (a) Assume that $\alpha \in \Gamma$. Then $A \in (s_\alpha(\overline{N}_p\overline{N}_q), s_\beta(\overline{N}_r\overline{N}_s))$ if and only if

(4.13)
$$\sup_{n\geq 1}\left[\frac{1}{\beta_n}\sum_{m=1}^{\infty}\left|\sum_{k=1}^{\infty}\frac{a_{nk}}{R_k}\left[\frac{r_m s_m}{q_m S_m} + \left(\frac{s_m}{q_m} - \frac{s_{m+1}}{q_{m+1}}\right)\left(\sum_{i=m+1}^{k}\frac{r_i}{S_i}\right)\right]\right| \frac{\alpha_m P_m Q_m}{p_m}\right] < \infty$$

and

$$\lim_{m\to\infty}\left[\sum_{k=1}^{\infty}\left(\frac{a_{nk}}{R_k}\left(\sum_{i=m}^{k}\frac{r_i}{S_i}\right)\right)\frac{\alpha_m s_m P_m Q_m}{p_m q_m}\right] = 0 \quad \textit{for all } n.$$

(b) $A \in (\overset{\circ}{s}_\alpha(\overline{N}_p\overline{N}_q), s_\beta(\overline{N}_r\overline{N}_s))$ if and only if (4.13) holds and

(4.14)
$$\sup_{m\geq 1}\left|\sum_{k=1}^{\infty}\left(\frac{a_{nk}}{R_k}\left(\sum_{i=m}^{k}\frac{r_i}{S_i}\right)\right)\frac{\alpha_m s_m P_m Q_m}{p_m q_m}\right| < \infty \quad \textit{for all } n.$$

(iii) If $\alpha P \in \widehat{\Gamma}$, then $A \in (s_\alpha^{(c)}(\overline{N}_p \overline{N}_q), s_\beta(\overline{N}_r \overline{N}_s))$ if and only if (4.13) holds and

$$\lim_{m\to\infty}\left[\frac{1}{\beta_n}\sum_{k=1}^{\infty}\left(\frac{a_{nk}}{R_k}\left(\sum_{i=m}^{k}\frac{r_i}{S_i}\right)\right)\frac{\alpha_m s_m P_m Q_m}{p_m q_m}\right] = \zeta_n$$

$$\text{for all } n \text{ and } \sup_{n\geq 1}|\zeta_n| < \infty.$$

Proof. (i) As we have seen in the proof of Proposition 4.1, the condition $\alpha/p \in \Gamma$ implies that $\alpha PQ/p \in \Gamma$. So $\alpha, \alpha/p \in \Gamma$ together imply $s_\alpha(\overline{N}_p \overline{N}_q) = s_{\alpha\frac{PQ}{pq}}$. Thus $A \in (s_\alpha(\overline{N}_p \overline{N}_q), s_\beta(\overline{N}_r \overline{N}_s))$ if and only if $\overline{N}_r \overline{N}_s A \in (s_{\alpha\frac{PQ}{pq}}, s_\beta) = S_{\alpha\frac{PQ}{pq},\beta}$. Now the conclusion follows from (4.5) and

$$\left(s_{\alpha\frac{PQ}{pq}}, s_\beta\right) = \left(\overset{\circ}{s}_{\alpha\frac{PQ}{pq}}, s_\beta\right) = \left(s^{(c)}_{\alpha\frac{PQ}{pq}}, s_\beta\right).$$

(i) (b) By Proposition 4.2 (vi), the conditions $\alpha P \in \widehat{\Gamma}$ and $\alpha PQ/p \in \widehat{\Gamma}$ together imply $s_\alpha^{(c)}(\overline{N}_p \overline{N}_q) = s^{(c)}_{\alpha\frac{PQ}{pq}}$. Thus $A \in (s_\alpha(\overline{N}_p \overline{N}_q), s_\beta(\overline{N}_r \overline{N}_s))$ is equivalent to $\overline{N}_r \overline{N}_s A \in (s^{(c)}_{\alpha\frac{PQ}{pq}}, s_\beta) = S_{\alpha\frac{PQ}{pq},\beta}$.

(ii) (a) Reasoning as in Proposition 4.6 (i) (a), we get that $\alpha \in \Gamma$ implies $s_\alpha(\overline{N}_p \overline{N}_q) = s_{\alpha\frac{P}{p}}(\overline{N}_q)$. So $A \in (s_\alpha(\overline{N}_p \overline{N}_q), s_\beta(\overline{N}_r \overline{N}_s))$ if and only if $\overline{N}_r \overline{N}_s A \in (s_{\alpha\frac{P}{p}}(\overline{N}_q), s_\beta)$, and the conclusion follows from Proposition 3.2 (i).

(ii) (b) Since we have $(\overset{\circ}{s}_\alpha(\overline{N}_p \overline{N}_q), s_\beta(\overline{N}_r \overline{N}_s)) = (\overset{\circ}{s}_{\alpha\frac{P}{p}}(\overline{N}_q), s_\beta(\overline{N}_r \overline{N}_s))$ and $A \in (\overset{\circ}{s}_{\alpha\frac{P}{p}}(\overline{N}_q), s_\beta(\overline{N}_r \overline{N}_s))$ if and only if $\overline{N}_r \overline{N}_s A \in (\overset{\circ}{s}_{\alpha\frac{P}{p}}(\overline{N}_q), s_\beta)$, the conclusion follows by Proposition 3.2 (iii).

(iii) By Proposition 4.2 (iii), the condition $\alpha P \in \widehat{\Gamma}$ implies $s_\alpha^{(c)}(\overline{N}_p \overline{N}_q) = s^{(c)}_{\alpha P/p}(\overline{N}_q)$ and, as above, $A \in (s_\alpha^{(c)}(\overline{N}_p \overline{N}_q), s_\beta(\overline{N}_r \overline{N}_s))$ if and only if $\overline{N}_r \overline{N}_s A \in (s^{(c)}_{\alpha\frac{P}{p}}(\overline{N}_q, s_\beta)$. Now the conclusion follows from Proposition 3.2 (ii). □

Proposition 4.9. *(i) Assume that $\alpha, \beta \in \Gamma$.*

(a) Then $A \in (s_\alpha(\overline{N}_p \overline{N}_q), s_\beta(\overline{N}_r \overline{N}_s))$ if and only if

(4.15)
$$\sup_{n\geq 1}\left[\sum_{m=1}^{\infty}\left|\frac{s_n}{\beta_n S_n R_n}\sum_{k=1}^{n}s_k\left(\frac{a_{km}}{q_m}-\frac{a_{k,m+1}}{q_{m+1}}\right)\right|\frac{\alpha_m P_m Q_m}{p_m}\right] < \infty.$$

and

(4.16)

$$\lim_{m\to\infty}\left[\frac{\alpha_m P_m Q_m}{p_m q_m}\frac{s_n}{\beta_n R_n}\left(\sum_{k=1}^{n}\frac{p_k a_{km}}{P_n}\right)\right]=0 \quad \text{for all } n.$$

(b) Then $A\in(s\alpha^{\circ}(\overline{N}_p\overline{N}_q),s_\beta(\overline{N}_r\overline{N}_s))$ *if and only if (4.15) holds and*

$$\sup_{m\geq 1}\left[\frac{\alpha_m P_m Q_m}{p_m q_m}\left(\sum_{k=1}^{n}\frac{s_k a_{km}}{S_n}\right)\right]<\infty. \tag{4.17}$$

(ii) If $\alpha\in\widehat{\Gamma}$ *and* $\beta\in\Gamma$*, then* $A\in(s_\alpha^{(c)}(\overline{N}_p\overline{N}_q),s_\beta(\overline{N}_r\overline{N}_s))$ *if and only if (4.15) holds and*

(4.18)

$$\lim_{m\to\infty}\left[\frac{\alpha_m P_m Q_m}{p_m q_m\beta_n}\left(\sum_{k=1}^{n}\frac{s_k a_{km}}{S_n}\right)\right]=\xi'_n \text{ for all } n \quad \text{and} \sup_{n\geq 1}|\xi'_n|<\infty.$$

Proof. The condition $\alpha,\beta\in\Gamma$ implies

$$(s_\alpha(\overline{N}_p\overline{N}_q),s_\beta(\overline{N}_r\overline{N}_s))=(s_{\alpha\frac{P}{p}}(\overline{N}_q),s_{\beta\frac{R}{s}}(\overline{N}_s)).$$

Now the conclusiom follows from Proposition 3.4 (i).
The statements (i) (b) and (ii) can be shown similarly. □

Remark 4.10. *Reasoning as in the previous corollaries we can easily get the characterizations of the sets* (E,F)*, where* E *is any of the sets* $s_\alpha(\overline{N}_p\overline{N}_q)$, $s_\alpha^{\circ}(\overline{N}_p\overline{N}_q)$ *or* $s_\alpha^{(c)}(\overline{N}_p\overline{N}_q)$ *and* F *is any of the sets* $s_\beta(\overline{N}_r\overline{N}_s)$, $s_\beta^{\circ}(\overline{N}_p\overline{N}_q)$ *or* $s_\beta^{(c)}(\overline{N}_p\overline{N}_q)$.

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