

Matter in transition

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ABSTRACT: We explore a novel type of transition in certain 6D and 4D quantum field theories, in which the matter content of the theory changes while the gauge group and other parts of the spectrum remain invariant. Such transitions can occur, for example, for $SU(6)$ and $SU(7)$ gauge groups, where matter fields in a three-index antisymmetric representation and the fundamental representation are exchanged in the transition for matter in the two-index antisymmetric representation. These matter transitions are realized by passing through superconformal theories at the transition point. We explore these transitions in dual F-theory and heterotic descriptions, where a number of novel features arise. For example, in the heterotic description the relevant 6D $SU(7)$ theories are described by bundles on K3 surfaces where the geometry of the K3 is constrained in addition to the bundle structure. On the F-theory side, non-standard representations such as the three-index antisymmetric representation of $SU(N)$ require Weierstrass models that cannot be realized from the standard $SU(N)$ Tate form. We also briefly describe some other situations, with groups such as $Sp(3)$, $SO(12)$, and $SU(3)$, where analogous matter transitions can occur between different representations. For $SU(3)$, in particular, we find a matter transition between adjoint matter and matter in the symmetric representation, giving an explicit Weierstrass model for the F-theory description of the symmetric representation that complements another recent analogous construction.

KEYWORDS: F-Theory, Superstring Vacua, Superstrings and Heterotic Strings

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Contents

1	Introduction	2
2	SU(N) matter transitions in 6D supergravity	3
2.1	Anomaly-equivalent matter representations	3
2.2	SU(N) blocks in 6D supergravity	6
2.3	Higgsing processes	7
2.3.1	Higgsing fundamentals	7
2.3.2	Higgsing antisymmetric representations	8
2.4	Product groups and transitions	9
3	F-theory description of matter transitions	10
3.1	F-theory overview	10
3.1.1	Weierstrass models	10
3.1.2	Gauge groups in F-theory	12
3.1.3	Matter	13
3.1.4	Superconformal points and tensionless string transitions	14
3.2	F-theory tunings of SU(6)–SU(8) models	15
3.2.1	SU(6)	15
3.2.2	SU(7)	17
3.2.3	SU(8)	21
3.2.4	SU(9)	24
3.2.5	SU(10)	24
3.3	Realization of the transitions	25
3.4	Higgsing processes in F-theory	30
3.4.1	Higgsing on fundamental matter	31
3.4.2	Higgsing on two-index antisymmetric matter	33
3.4.3	Higgsing on three-index antisymmetric matter	36
4	Heterotic description of matter transitions	38
4.1	SU(N) theories in heterotic compactifications	39
4.1.1	SU(6)	39
4.1.2	SU(7)	40
4.1.3	SU(8) and beyond	41
4.2	Realization of the transitions	43
4.3	Higgsing processes	43
4.3.1	Higgsing on fundamental matter	44
4.3.2	Higgsing on two-index antisymmetric matter	45
4.3.3	Higgsing on three-index antisymmetric matter	47
4.4	Small Instanton transitions and Higgsing	50

5	Heterotic/F-theory duality	50
5.1	The stable degeneration limit	51
5.2	A very brief review of the spectral cover construction	52
5.3	Matter transitions in spectral covers	53
5.4	SU(6) heterotic/F-theory dictionary	55
5.5	SU(7) tuning	56
5.5.1	The heterotic $K3$ geometry	58
5.5.2	SU(7) symmetry and the spectral cover	59
5.5.3	SU(7) and Green-Schwarz massive U(1)s	61
5.5.4	SU(7) matter transitions	62
5.6	SU(8) tuning	63
5.7	Higgsing on antisymmetric matter	67
5.8	Comments on Green-Schwarz massive U(1)s	68
6	Matter transitions in other gauge groups	69
6.1	Sp(N) matter transitions	69
6.1.1	Field theory	69
6.1.2	F-theory	70
6.1.3	Heterotic description	70
6.2	SO(N) models	71
6.3	SU(3) with symmetric matter	73
7	Conclusions	78
7.1	Summary of results	78
7.2	Further directions	81
A	Addition of sections	83
B	Three stable degeneration limits for SU(8)	84
C	Stable degeneration limits and U(1) symmetries	85
D	Details of heterotic Higgsing analysis	88
D.1	Higgsing on fundamental matter	88
D.1.1	SU(7)	88
D.1.2	SU(8)	90
D.2	Higgsing on two-index antisymmetric matter	91
D.2.1	SU(6)	91
D.2.2	SU(8)	93
D.3	Higgsing on three-index antisymmetric matter	94
D.3.1	SU(6)	94
D.3.2	SU(7)	96

1 Introduction

A variety of different types of transitions can occur in physical theories in which the massless or light spectrum of the theory changes. For certain types of transitions, 6D supergravity forms a clear framework in which to classify and analyze the possible changes of spectrum; similar transitions occur in 4D supergravity theories, though the detailed description can involve more subtle issues. For 6D theories coupled to gravity, the different types of transitions can be characterized by the massless spectrum of the low-energy theory.

The most dramatic of these transitions in 6D theories are the *tensionless string* or *small instanton* transitions [1, 2], which involve a change in the number of tensor multiplets in the theory, accompanied by a corresponding change in the number of uncharged scalar hypermultiplets. These transitions are described in F-theory by blowing up or down points on the base manifold used for the F-theory compactification, and in the heterotic theory by shrinking an instanton to a point. In both pictures the resulting transition is fundamentally nonperturbative in nature, and in the low-energy theory it involves passing through a superconformal fixed point. Higgsing/unHiggsing transitions, on the other hand, leave the number of tensor multiplets unchanged but modify the gauge group of the theory and generally change the number of vector multiplets in addition to modifying the matter spectrum. Higgsing/unHiggsing type transitions have a simple description in both F-theory and heterotic pictures, in terms of a tuning of Weierstrass moduli on the one hand and tuning of bundle moduli on the other, and have a perturbative description in the low-energy theory.

In this paper we describe another type of transition, in which both the tensor and vector multiplet spectra remain unchanged, and only the representation content of the matter fields is modified. While the possibility of such transitions has been noted in the literature [3–6] these kinds of pure matter transitions have not been studied in depth, and we identify a number of new interesting transitions in this class here. We describe these transitions both from the F-theory point of view and in a dual heterotic picture. Because of the matter representations involved (frequently involving symmetry enhancement of the singular fibers associated to exceptional groups, etc.) these transitions are not accessible in a perturbative Type IIB description and can only be explored in F-theory. In the F-theory geometry these transitions can be realized by tuning Weierstrass moduli so that certain codimension two singularities coincide and then split into a distinct geometry. On the heterotic side these transitions arise when an instanton is shrunk and moved into a separate component of the bundle structure group in the same E_8 component. In both cases the transition can be described by moving along a one-parameter family of theories that passes through a strongly coupled superconformal fixed point, but does not move onto the tensor branch. In both the heterotic bundle and the resolved F-theory geometry these transitions are realizable as geometric transitions (i.e. topology changing transitions). Our description of these transitions in both F-theory and the heterotic theory is for the most part quite general, but for comparison of these perspectives we focus in particular on cases where the F-theory geometry is compactified on a $K3$ fibration over a base B and the heterotic geometry describes an elliptic fibration over the same base B .

While we primarily focus on compactifications to 6D to make the analysis completely concrete and precise, the transitions we study are local phenomena that will also arise in field theory without the supergravity coupling; these transitions should also arise in a similar fashion in 4D theories. Though some of the technical details and issues involved will be more subtle in 4D due to the presence of a superpotential and additional complexity in the theories with reduced supersymmetry, many aspects of the analysis carried out here, including the general forms of F-theory Weierstrass models and heterotic bundles, will hold in a large class of dual geometries for 4D compactifications. Only the details of the anomaly analysis and some specific features of the heterotic constructions on specific geometries will depend upon the dimensionality of the construction.

We begin in section 2 with a low-energy description of matter transitions in 6D theories with SU(6), SU(7), and SU(8) gauge groups. The strong constraints of anomaly cancellation dictate the transitions that can occur in matter content without a change in the gauge group of the theory. In section 3 we describe these transitions in F-theory using Weierstrass models, some of which do not have the standard Tate form for SU(N). In section 4 we describe the transitions from the heterotic point of view, where the transitions are manifested by instantons moving between factors in the structure group of a bundle within one E_8 factor. In section 5 we relate the F-theory and heterotic pictures using the spectral cover construction of the heterotic gauge bundle and explore novel forms of the stable degeneration limit of the F-theory compactification. In section 6 we briefly describe some examples of matter transitions in other groups, and section 7 contains some concluding comments. A variety of useful technical results are provided in the appendices.

2 SU(N) matter transitions in 6D supergravity

2.1 Anomaly-equivalent matter representations

In 6D supergravity, anomaly cancellation conditions strongly constrain the spectrum of massless matter fields that can be charged under a given gauge group. In some cases, however, there are multiple distinct types of matter that give equivalent solutions to the anomaly equations.

One of the simplest examples occurs for SU(6) and higher SU(N) gauge groups, with charged matter that transforms under the three-index antisymmetric representation, as described in [5, 6]. In 6D, the anomaly cancellation conditions for a matter spectrum containing x_R fields transforming in each representation R of an SU(N) gauge group are ([7, 8], as described in [9])

$$-a \cdot b = -\frac{1}{6} \left(A_{\text{adj}} - \sum_R x_R A_R \right) \tag{2.1}$$

$$0 = B_{\text{adj}}^i - \sum_R x_R B_R \tag{2.2}$$

$$b \cdot b = -\frac{1}{3} \left(C_{\text{adj}} - \sum_R x_R C_R \right) \tag{2.3}$$

where a, b are the coefficients of BR^2 and BF^2 Green-Schwarz terms, and are associated with vectors in the signature $(1, T)$ anomaly lattice of the 6D supergravity theory, so that $a \cdot b, b \cdot b$ are integers. The coefficients A_R, B_R, C_R are numerical constants associated with each representation computed from

$$\text{tr}_R F^2 = A_R \text{tr} F^2 \tag{2.4}$$

$$\text{tr}_R F^4 = B_R \text{tr} F^4 + C_R (\text{tr} F^2)^2. \tag{2.5}$$

When multiple gauge factors $G_i = \text{SU}(N_i)$ are involved, each gauge factor has an associated vector b_i in the anomaly lattice, and we have the further condition

$$b_i \cdot b_j = \sum_{R,S} x_{RS}^{ij} A_R^i A_S^j, \tag{2.6}$$

where x_{RS}^{ij} is the number of fields in the representation $R \otimes S$ of $G_i \times G_j$. In the simplest cases we are interested in here, these are bifundamental representations.

There is also a constraint that arises from the purely gravitational anomaly cancellation condition

$$n_H - n_V = 273 - 29n_T, \tag{2.7}$$

where n_H, n_V, n_T are the numbers of matter hypermultiplets, vector multiplets, and tensor multiplets in the theory respectively.

When the only types of matter that arise are in k -index antisymmetric (Λ^k) representations of $\text{SU}(N)$, we have, for some integer n ,

$$b \cdot b = n, \quad -a \cdot b = n + 2. \tag{2.8}$$

In the F-theory picture, these configurations come from gauge groups wrapped on rational (genus zero) curves of self-intersection n .

The dimension and coefficients $A_R - C_R$ for the adjoint, fundamental, symmetric, anti-symmetric, and three- and four-index antisymmetric representations of $\text{SU}(N)$ are listed in table 1 along with the ‘‘genus’’ contribution of each representation ([5, 12, 13]).

The following combinations of $\text{SU}(6)$ matter fields give equivalent contributions to each of the anomaly cancellation conditions (including the purely gravitational anomaly condition $n_H - n_V = 273 - 29n_T$).

$$\frac{1}{2} \mathbf{20} \left(\begin{array}{c} 1 \\ 2 \end{array} \square \right) + \mathbf{6} (\square) \leftrightarrow \mathbf{15} \left(\begin{array}{c} \square \\ \square \end{array} \right) + \mathbf{1}. \tag{2.9}$$

We refer to these combinations of matter fields as *anomaly equivalent* [6]. Note that we can have a half-hypermultiplet for the $\mathbf{20}$, since this is a self-conjugate (pseudoreal) representation of $\text{SU}(6)$. These combinations of representations can be seen to be equivalent by checking that the contribution to each of the terms $\sum_R x_R A_R, \sum_R x_R B_R, \sum_R x_R C_R$ are the same on both sides of (2.9). The equivalence of these representations under the anomaly conditions suggests that there is no obstruction to a transition between $\text{SU}(6)$ theories with the different matter representations. Explicit local models from F-theory realizing

Rep.	N	Dimension	A_R	B_R	C_R	g
Adj.	N	$N^2 - 1$	$2N$	$2N$	6	1
	6, 7, 8	35, 48, 63	12, 14, 16	12, 14, 16	6	1
\square	N	N	1	1	0	0
$\square\square$	N	$\frac{N(N+1)}{2}$	$N + 2$	$N + 8$	3	1
	6, 7, 8	21, 28, 36	8, 9, 10	14, 15, 16	3	1
$\begin{array}{ c } \hline \square \\ \hline \end{array}$	N	$\frac{N(N-1)}{2}$	$N - 2$	$N - 8$	3	0
	6, 7, 8	15, 21, 28	4, 5, 6	-2, -1, 0	3	0
$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$	N	$\frac{N(N-1)(N-2)}{6}$	$\frac{N^2-5N+6}{2}$	$\frac{N^2-17N+54}{2}$	$3N - 12$	0
	6, 7, 8	20[10], 35, 56	6[3], 10, 15	-6[-3], -8, -9	6, 9, 12	0
$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	N	$\frac{N(N-1)(N-2)(N-3)}{24}$	$\frac{(N-2)(N-3)(N-4)}{6}$	$\frac{(N-4)(N^2-23N+96)}{6}$	$\frac{3(N^2-9N+20)}{2}$	0
	8	70[35]	20[10]	-16[-8]	18[9]	0

Table 1. Values of the group-theoretic coefficients A_R, B_R, C_R , dimension and genus for some representations of $SU(N)$, $N \geq 4$, with specific values computed for convenience for $N = 6, 7, 8$. Values in brackets refer to half-hypermultiplets for self-conjugate representations.

transitions between these matter representations were identified in [5], as described in more detail in the following section. The main focus of this paper is the detailed analysis of this and related types of matter transitions in global models from the dual F-theory and heterotic perspectives.

For $SU(7)$, there is a similar type of anomaly equivalence and associated transition

$$35 \left(\begin{array}{|c|} \hline \square \\ \hline \end{array} \right) + 5 \times 7(\square) \leftrightarrow 3 \times 21 \left(\begin{array}{|c|} \hline \square \\ \hline \end{array} \right) + 7 \times \mathbf{1}. \tag{2.10}$$

For $SU(8)$, there are anomaly equivalent matter representations

$$56 \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right) + 9 \times 8(\square) \leftrightarrow 4 \times 28 \left(\begin{array}{|c|} \hline \square \\ \hline \end{array} \right) + 16 \times \mathbf{1}. \tag{2.11}$$

and

$$\frac{1}{2} 70 \left(\frac{1}{2} \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) + 8 \times 8(\square) \leftrightarrow 3 \times 28 \left(\begin{array}{|c|} \hline \square \\ \hline \end{array} \right) + 15 \times \mathbf{1}. \tag{2.12}$$

In the following sections, we describe the extent to which these equivalences correspond to transitions that have realizations in global F-theory and heterotic models. There is a similar anomaly equivalence for the 3-index antisymmetric (Λ^3) representation of $SU(9)$, which we discuss further below. Similar equivalences would seem at first to be possible for Λ^3 representations of $SU(10)$ and higher $SU(N)$ and for Λ^4 representations of $SU(9)$ and above, etc., but global considerations (discussed below) seem to rule out such models for all values of T .

2.2 SU(N) blocks in 6D supergravity

The anomaly conditions constrain the total matter content that can be charged under a given component SU(N) of the full 6D gauge group. For the most generic 6D SU(N) models with specific values of $b \cdot b = n$, a “genus” $g = 1 - (a \cdot b + b \cdot b)/2$, and $N \geq 4$, the matter content contains only adjoint, fundamental, and antisymmetric representations, and is given by

$$g \times (\text{adjoint}) + [16(1 - g) + (8 - N)n] \times \square + (n + 2 - 2g) \times \begin{array}{|c|} \hline \square \\ \hline \end{array}. \quad (2.13)$$

The sense in which this matter content is the most generic is that it corresponds to the theory with the largest number of uncharged scalar fields. Each of the anomaly-equivalent combinations involving a triple or quadruple-antisymmetric representation described above removes some number of scalar matter fields, corresponding to a more refined “tuning” of the low-energy field theory model.

For genus $g = 0$ models, we have the following spectra for SU(6), SU(7), SU(8), where $b \cdot b = n$

$$\text{SU}(6) : (16 + 2n + r) \times \mathbf{6}(\square) + (n + 2 - r) \times \mathbf{15} \left(\begin{array}{|c|} \hline \square \\ \hline \end{array} \right) + r \times \frac{1}{2} \mathbf{20} \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right). \quad (2.14)$$

$$\text{SU}(7) : (16 + n + 5r) \times \mathbf{7}(\square) + (n + 2 - 3r) \times \mathbf{21} \left(\begin{array}{|c|} \hline \square \\ \hline \end{array} \right) + r \times \mathbf{35} \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right). \quad (2.15)$$

$$\text{SU}(8) : (16 + 9r + 8r') \times \mathbf{8}(\square) + (n + 2 - 4r - 3r') \times \mathbf{28} \left(\begin{array}{|c|} \hline \square \\ \hline \end{array} \right) + r \times \mathbf{56} \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right) + r' \times \frac{1}{2} \mathbf{70} \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right). \quad (2.16)$$

The possible matter spectra for SU(6) factors was described in [14] and there also related to F-theory and heterotic theory. As we see in the later sections, both in F-theory and heterotic theory all of the SU(6) and SU(7) 6D models can be realized through global constructions, and there are matter transitions along paths in the space of theories that change between different values of r without changing the gauge group (i.e., without involving Higgsing processes). We have identified explicit constructions only for a subset of the possible SU(8) models, as discussed in further detail below, and it is less clear whether there is a UV-consistent description of the SU(8) transitions.

We briefly summarize the situation for SU(N) blocks with $N > 8$. For SU(9), the generic $g = 0$ model has $16 - n$ fundamentals (**9**'s) and $(n + 2)$ antisymmetrics (**36**'s). The SU(9) model with a Λ^3 representation and the smallest number of matter fields has $n = 3$ and at least 327 matter fields ($1 \times \mathbf{84} + 27 \times \mathbf{9}$). Thus, in the absence of other gauge factors, $n_H - n_V \geq 327 - 80 = 247$. SU(9) models with a Λ^3 representation appear to be consistent in $n_T = 0$ supergravity (with $n_H - n_V = 273$), though these models cannot have a heterotic description on a smooth K3. It seems just barely possible to construct SU(9) models with $n_T = 1$, where $n_H - n_V = 244$, which should in principle have an F-theory description and heterotic duals. For example, by adding an SU(3) factor, such as can be done in a heterotic

model with 12 ± 3 instantons in the two E_8 factors, corresponding to F-theory on \mathbb{F}_3 , there is just enough room to satisfy the gravitational anomaly bound. As we discuss in the following sections, however, it is not clear whether or how these models may be realized in either the F-theory or heterotic constructions. Similar considerations of the global gravitational anomaly condition show that $SU(9)$ models with Λ^4 or $SU(10)$ models with Λ^3 representations are not possible even with $n_T = 0$, assuming there are no further gauge factors. Note that this argument does not completely rule out 6D supergravity models in which the fundamentals of e.g. $SU(9)$ with a Λ^4 are also charged under additional gauge factors, effectively increasing n_V without changing n_H , or models where there is a second, non-Higgsable gauge factor, which contributes to n_V without a corresponding contribution to n_H . But such models seem very difficult to construct in a consistent fashion. Note also that the constraints just discussed rely on the purely gravitational anomaly cancellation condition, and do not in principle constrain the existence of e.g. $SU(10)$ models with Λ^3 representations in 6D field theory.

2.3 Higgsing processes

The different models with matter transitions that we consider are connected to one another, and to other related models, by a network of Higgsing transitions, many of which were also considered in [14]. In general, Higgsing a gauge group $SU(N)$ in a supersymmetric theory requires turning on a vacuum expectation value (VEV) for matter fields that transform in nontrivial representations. In the language of 4D $\mathcal{N} = 1$ theories, such expectation values must be turned on in such a way that the D-term constraints of the form $\sum_i \bar{\phi}_i T_A \phi_i = 0$ are satisfied for each generator T_A of the Lie algebra.

2.3.1 Higgsing fundamentals

The simplest Higgsing of $SU(N)$ is done by giving vacuum expectation values to two fundamental fields. Note that two fundamental fields must be given VEVs in order to satisfy the D-term constraints in the supersymmetric theory. Supersymmetric Higgsing on a single fundamental of $SU(N)$ is not possible. Recall that each full 6D $\mathcal{N} = 1$ hypermultiplet in a given representation R contains fields in both the representation R and its conjugate, so when Higgsing two fundamental fields we are really giving expectation values to a field component in the representation R and another field component in the conjugate representation \bar{R} , allowing the D-term constraint to be satisfied. For example, for the fundamental representation any VEV can be rotated into the canonical form $(0, 0, \dots, 0, v)$. This can be described in the language of Young tableaux by a single box containing the value $[N]$. When two fundamental fields are given VEVs in this way, the gauge group is broken down to $SU(N-1)$. The Goldstone bosons of the Higgsed matter fields are “eaten” by the broken gauge generators in the usual fashion. In 6D supergravity theories this matches with the gravitational anomaly cancellation condition $n_H - n_V = 273 - 29n_T$.

Explicitly, we can match the number of Goldstone bosons $2N - 1$ with the number of broken gauge generators $[N^2 - 1] - [(N - 1)^2 - 1] = 2N - 1$. After breaking $SU(N) \rightarrow$

$SU(N - 1)$, the other representations branch as follows:

$$\square_N \rightarrow \square_{N-1} + \mathbf{1} \quad (2.17)$$

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}_N \rightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}_{N-1} + \square_{N-1} \quad (2.18)$$

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}_N \rightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}_{N-1} + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}_{N-1} \quad (2.19)$$

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}_8 \rightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}_7 + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}_7 = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}_7 + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}_7. \quad (2.20)$$

This can easily be understood by looking at the Young tableaux for each representation; the k -antisymmetric representation with an N entry goes to a $(k - 1)$ -antisymmetric representation, and the tableaux without an N entry go to a k -antisymmetric representation of $SU(N - 1)$. The dimensions of these sets of representations match as, for example in the case of triple antisymmetrics,

$$\frac{N(N - 1)(N - 2)}{6} = \frac{(N - 1)(N - 2)(N - 3)}{6} + \frac{(N - 1)(N - 2)}{2}. \quad (2.21)$$

We can see that this Higgsing process takes an $SU(N)$ model with a specific value of n to an $SU(N - 1)$ model with the same value of n between the models (2.14), (2.15), (2.16). Higgsing the $SU(7)$ models with a value $r_7 > 0$ gives the corresponding $SU(6)$ model with twice that value of $r_6 = 2r_7$. Higgsing an $SU(8)$ model with $r_8 > 0, r'_8 \geq 0$ gives the $SU(7)$ model with $r_7 = r_8 + r'_8$.

In short, in breaking $SU(N)$ to $SU(N - 1)$ on a fundamental representation, we lose two full hypermultiplets in the $\mathbf{N} - \mathbf{1}$ (fundamental) representation of $SU(N - 1)$ and one singlet full hypermultiplet. This accounts for a reduction of $2N - 1$ full hypermultiplet degrees of freedom, one for each gauge boson that becomes massive, as required by the six-dimensional super-Higgs mechanism.

2.3.2 Higgsing antisymmetric representations

We can similarly Higgs two antisymmetric representations, giving VEVs to states with Young tableau entries $[N - 1, N]$. This breaks

$$SU(N) \rightarrow SU(N - 2) \times SU(2) \quad (2.22)$$

The two antisymmetrics that break the gauge group have $2(N - 2)$ Goldstone bosons each, corresponding to states with Young tableaux having entries $[i, j]$ with $i \leq N - 2, j \geq N - 1$, and the number of generators in the group is reduced by

$$[N^2 - 1] - [(N - 2)^2 - 1] - [3] = 4N - 7 = 2 \times (2N - 4) + 1. \quad (2.23)$$

Together with the loss of an additional singlet hypermultiplet (containing one goldstone boson and 3 degrees of freedom fixed by D-terms) the number of degrees of freedom match. The antisymmetric representations break down in the decomposition (2.22) as

$$\square_N = \square_{N-2} \times \mathbf{1} + \mathbf{1} \times \square_2 \quad (2.24)$$

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}_N = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}_{N-2} \times \mathbf{1} + \square_{N-2} \times \square_2 + \mathbf{1} \times \mathbf{1} \quad (2.25)$$

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}_N = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}_{N-2} \times \mathbf{1} + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}_{N-2} \times \square_2 + \square_{N-1} \times \mathbf{1} \quad (2.26)$$

For the two Λ^2 fields that take VEVs, one singlet is the VEV component, and the other singlet and the bifundamental degrees of freedom are the ones that are lost, so the number of antisymmetric representations stays unchanged in the breaking.

More generally, Higgsing two k -antisymmetric representations breaks

$$\mathrm{SU}(N) \rightarrow \mathrm{SU}(N - k) \times \mathrm{SU}(k). \quad (2.27)$$

There are $2k(N - k) + 1$ degrees of freedom eaten by the lost gauge bosons, and again the k -antisymmetrics are preserved under the breaking and the bifundamentals and singlets are lost.

2.4 Product groups and transitions

6D models with multiple $\mathrm{SU}(N)$ gauge factors can be constructed in close parallel to the single-block models with one $\mathrm{SU}(N)$ factor. We consider models where $\mathrm{SU}(N)$ and $\mathrm{SU}(M)$ are both realized as in section 2.2, with the same value of b in the anomaly lattice, so that $b_1 \cdot b_1 = b_2 \cdot b_2 = b_1 \cdot b_2 = n$. These are product group models with a smooth heterotic dual. Generically, the intersection condition (2.6) is satisfied by including n bifundamental (\mathbf{N}, \mathbf{M}) fields in the spectrum. For example, for $n \geq 0$, the spectrum for the generic $\mathrm{SU}(4) \times \mathrm{SU}(2)$ model is

$$(n + 2) \times \left(\begin{array}{c} \square \\ \square \end{array} \times \mathbf{1} \right) + n \times (\square_4 \times \square_2) + (16 + 2n) \times (\square_4 \times \mathbf{1}) + (16 + 2n) \times (\mathbf{1} \times \square_2). \quad (2.28)$$

Just as Higgsing $\mathrm{SU}(8)$ or $\mathrm{SU}(7)$ models on a pair of fundamentals relates models with a non-generic (e.g. Λ^3) representation to other such models, Higgsing on antisymmetric representations also gives rise to exotic matter structures for product groups. For example, consider breaking an $\mathrm{SU}(6)$ model with r half-hyper Λ^3 representations on a pair of antisymmetric (Λ^2) representations. Then, from (2.26), each of the $r \frac{1}{2}\mathbf{20}$'s breaks as

$$\frac{1}{2} \begin{array}{c} \square \\ \square \\ \square \end{array} \rightarrow (\square_4 \times \mathbf{1}) + \left(\frac{1}{2} \begin{array}{c} \square \\ \square \end{array} \times \square_2 \right). \quad (2.29)$$

In general, this gives an $\mathrm{SU}(4) \times \mathrm{SU}(2)$ model with spectrum

$$r \times \left(\frac{1}{2} \begin{array}{c} \square \\ \square \end{array} \times \square_2 \right) + (n + 2 - r) \times \left(\begin{array}{c} \square \\ \square \end{array} \times \mathbf{1} \right) + (n - r) \times (\square_4 \times \square_2) \quad (2.30)$$

$$+ (16 + 2n + 2r) \times (\square_4 \times \mathbf{1}) + (16 + 2n + r) \times (\mathbf{1} \times \square_2). \quad (2.31)$$

From this spectrum we see that there must be an anomaly equivalence

$$\left(\frac{1}{2} \begin{array}{c} \square \\ \square \end{array} \times \square_2 \right) + 2 \times (\square_4 \times \mathbf{1}) + (\mathbf{1} \times \square_2) \leftrightarrow \left(\begin{array}{c} \square \\ \square \end{array} \times \mathbf{1} \right) + (\square_4 \times \square_2) + 2 \times (\mathbf{1} \times \mathbf{1}). \quad (2.32)$$

Indeed, this relation also follows from a direct Higgsing of (2.9) under the breaking $\mathrm{SU}(6) \rightarrow \mathrm{SU}(4) \times \mathrm{SU}(2)$.

A similar transition can be found for $\mathrm{SU}(5) \times \mathrm{SU}(2)$ theories by breaking $\mathrm{SU}(7)$:

$$\left(\begin{array}{c} \square \\ \square \end{array} \times \square_2 \right) + 6 \times (\square_5 \times \mathbf{1}) + 5 \times (\mathbf{1} \times \square_2) \leftrightarrow 2 \times \left(\begin{array}{c} \square \\ \square \end{array} \times \mathbf{1} \right) + 3 \times (\square_5 \times \square_2) + 10 \times (\mathbf{1} \times \mathbf{1}). \quad (2.33)$$

And for $SU(6) \times SU(2)$ by breaking $SU(8)$ on a pair of antisymmetric representations:

$$(\square_6 \times \square_2) + 8 \times (\square_6 \times \mathbf{1}) + 9 \times (\mathbf{1} \times \square_2) \leftrightarrow 2 \times (\square_6 \times \mathbf{1}) + 4 \times (\square_6 \times \square_2) + 18 \times (\mathbf{1} \times \mathbf{1}). \quad (2.34)$$

These relations suggest that transitions between theories with these different matter contents should be possible; in the later sections of this paper we explicitly identify these transitions in F-theory and heterotic theories. While we find that the $SU(4) \times SU(2)$ and $SU(5) \times SU(2)$ transitions are described nicely in both F-theory and the heterotic theory, the $SU(6) \times SU(2)$ transition is less clear. The representation that transforms as an anti-symmetric field under $SU(N)$ and a fundamental under $SU(M)$ is an exotic representation, analogous in many ways to the triple-antisymmetric representation of $SU(N)$. Generalizing these constructions, we can Higgs on a pair of triple-antisymmetric representations; this gives interesting transitions in $SU(N) \times SU(3)$ theories, as we explore to some extent in the following sections.

It would be interesting to explore further product representations and associated transitions. For example, Higgsing the relation (2.12) on a pair of antisymmetric representations suggests that there should be a transition in $SU(6) \times SU(2)$ theories

$$\left(\frac{1}{2}\square_6 \times \square_2\right) + 8 \times (\square_6 \times \mathbf{1}) + 8 \times (\mathbf{1} \times \square_2) \leftrightarrow 2 \times (\square_6 \times \mathbf{1}) + 3 \times (\square_6 \times \square_2) + 16 \times (\mathbf{1} \times \mathbf{1}). \quad (2.35)$$

Since we have not identified an explicit realization of the Λ^4 representation of $SO(8)$ in either the F-theory or heterotic pictures, however, it is not clear whether this transition can be realized in consistent 6D supergravity theories. One could also use this approach to explicitly construct representations such as the trifundamental of $SU(2) \times SU(2) \times SU(2)$ by breaking the $(\mathbf{6}, \mathbf{2})$ of $SU(4) \times SU(2)$ to $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ by breaking $SU(4) \rightarrow SU(2) \times SU(2)$. And the relations (2.32)–(2.34) suggest that similar transitions may occur for higher groups such as $SU(7) \times SU(2)$. We leave further exploration of these possibilities to future work.

3 F-theory description of matter transitions

F-theory provides powerful methods for realizing stringy constructions of supergravity theories. In this section, we give F-theory realizations of the models with matter transitions described above, building on the work of [5]. These F-theory models provide further insights into the mechanisms behind the matter transitions.

3.1 F-theory overview

Here, we briefly describe those aspects of F-theory that will be important in the upcoming analysis. More extensive reviews of F-theory can be found in [15–17].

3.1.1 Weierstrass models

F-theory [2, 18, 19] is a method for compactifying type IIB string theory in situations where the axiodilaton is allowed to vary over the compactification space. The axiodilaton and the corresponding $SL(2, \mathbb{Z})$ symmetry appear geometrically through an elliptic fibration over

the compactification base. Specifically, an F-theory compactification to $10 - 2d$ dimensions is given by an elliptic fibration over a base B of complex dimension d . These elliptic fibrations can be described using the Weierstrass form

$$y^2 = x^3 + fx + g. \tag{3.1}$$

Here, y and x are coordinates describing an elliptic curve, while f and g vary over the base B . To preserve supersymmetry, the total elliptic fibration must be a Calabi-Yau manifold. As a result, f and g must respectively be sections of $\mathcal{O}(-4K_B)$ and $\mathcal{O}(-6K_B)$, where K_B is the canonical class of the base B . Note that B does not necessarily need to be a Calabi-Yau manifold on its own.

The sections f and g can be described locally near a divisor Σ on B using the formalism of [20, 21]. Let Σ have an associated coordinate σ , so that $\Sigma = \{\sigma = 0\}$. We can then expand f and g in terms of σ as

$$f = f_0 + f_1\sigma + f_2\sigma^2 + \dots \tag{3.2}$$

and

$$g = g_0 + g_1\sigma + g_2\sigma^2 + \dots \tag{3.3}$$

The coefficients f_k and g_k are respectively sections of $\mathcal{O}(-4K_\Sigma + (4-k)N_\Sigma)$ and $\mathcal{O}(-6K_\Sigma + (6-k)N_\Sigma)$, where $-K_\Sigma$ and N_Σ are the anti-canonical and normal line bundles for Σ . Note that in general the coordinate σ is not globally defined, and only the first nonvanishing f_k, g_k is uniquely defined, as discussed in more detail in [21]. These subtleties are irrelevant for the discussion in this paper; for toric bases, in particular, σ can always be taken as a global toric coordinate.

While the analysis in this section is quite general, and applies to 4D as well as 6D F-theory models, in this paper we focus particularly on F-theory compactifications to 6D with heterotic duals, where the compactification base B is one of the Hirzebruch surfaces \mathbb{F}_n . For a description of Hirzebruch surfaces from an F-theory perspective, see [17, 19]. Hirzebruch surfaces are complex two-dimensional surfaces that can be described as \mathbb{P}_1 bundles over \mathbb{P}_1 . The divisors of \mathbb{F}_n have a basis consisting of divisors S and F . F and S span the cone of effective curves on \mathbb{F}_n . S is a section of the fibration, with a 1-1 map to the \mathbb{P}_1 base of \mathbb{F}_n , and has self-intersection number $-n$ (i.e. $S \cdot S = -n$). F meanwhile refers to the fiber \mathbb{P}_1 , and satisfies $S \cdot F = 1$ and $F \cdot F = 0$. Another important divisor class for \mathbb{F}_n is $\tilde{S} = S + nF$, which satisfies $S \cdot \tilde{S} = 0$ and $\tilde{S} \cdot \tilde{S} = n$. The anti-canonical class of \mathbb{F}_n is

$$-K_{\mathbb{F}_n} = 2S + (n+2)F, \tag{3.4}$$

so f and g are sections of $\mathcal{O}(8S + (4n+8)F)$ and $\mathcal{O}(12S + (6n+12)F)$. In most of our analysis we will tune the relevant gauge groups on \tilde{S} . We thus associate the coordinate σ with \tilde{S} and z with F , f and g are therefore polynomials of order 8 and 12 in σ , at

maximum. Using equations (3.2) and (3.3), we can expand f and g around \tilde{S} as

$$f = \sum_{k=0}^8 f_k \sigma^k,$$

$$g = \sum_{k=0}^{12} g_k \sigma^k.$$

f_k and g_k can be described in terms of a common line bundle on \mathbb{P}^1 , which we take to be $\mathcal{O}(1)$. $-K_{\tilde{S}}$ is then equivalent to 2, while $N_{\tilde{S}}$ is n . f_k and g_k are thus polynomials in z of order $8 + n(4 - k)$ and $12 + n(6 - k)$. For $n \geq 3$, some of the f_k or g_k may be ineffective for $k \leq 8$ or $k \leq 12$, meaning that higher order terms in σ must vanish. This signals the presence of a non-Higgsable cluster (described below) on S . The expressions for f and g given above are in fact the generic expansions near a $+n$ curve, so long as the limits of k are adjusted for the appropriate situation.

3.1.2 Gauge groups in F-theory

In F-theory, the total elliptically-fibered compactification space may admit certain types of singularities. Singularities of the fibration occur at loci where the discriminant Δ , defined as

$$\Delta = 4f^3 + 27g^2, \tag{3.5}$$

equals zero. Some of these singularities in the fibration give singularities in the total space as well. In general, not all singularities in the total space can be resolved to give a smooth Calabi-Yau manifold. Kodaira classified all of the resolvable singularities for situations where the singularity occurs along a codimension one locus on the base [22, 23]. As summarized in table 2, the resulting singularities can be described by the orders of vanishing of f , g , and Δ on the codimension one locus. If f and g vanish to orders 4 and 6 or greater on a codimension one locus, the resulting singularity has no Calabi-Yau resolution. For F-theory compactifications, singularities that arise over codimension one loci on the base can always be resolved to give a smooth Calabi-Yau manifold and obey the Kodaira classification.

When an F-theory compactification has such a codimension one singularity, the physical theory has a corresponding nonabelian gauge symmetry. Resolving the codimension one singularity will produce a set of 2-cycles whose intersection pattern can be mapped to a Dynkin diagram. The Dynkin diagram then identifies the algebra for the physical model's gauge symmetry. In this paper we generally describe theories using the Lie group, with the understanding that only the algebra is actually fixed definitively by F-theory, and that the given Lie group may be subject to a quotient by a finite discrete subgroup. Note that monodromy effects need to be considered when compactifying to 6D or 4D, meaning that the same Kodaira fiber type can give different gauge algebras. In such cases, determining the resulting gauge group requires a more in-depth analysis of f , g , and Δ (see, for example, [14, 24]). An example is the split condition, where an I_n fiber can correspond to either an \mathfrak{su} algebra or an \mathfrak{sp} algebra depending on the form of f_0 and g_0 .

Fiber Type	ord(f)	org(g)	ord(Δ)	Singularity Type	Gauge Algebra
I_0	0	0	1	none	none
I_n	0	0	n	A_{n-1}	$\mathfrak{su}(n)$ or $\mathfrak{sp}(\lfloor \frac{n}{2} \rfloor)$
II	≥ 1	1	2	none	none
III	1	≥ 2	3	A_1	$\mathfrak{su}(2)$
IV	≥ 2	2	4	A_2	$\mathfrak{su}(2)$ or $\mathfrak{su}(3)$
I_0^*	≥ 2	≥ 3	6	D_4	$\mathfrak{g}_2, \mathfrak{so}(7)$ or $\mathfrak{so}(8)$
I_n^*	2	3	$n+6$	D_{n+4}	$\mathfrak{so}(2n+7)$ or $\mathfrak{so}(2n+8)$
IV^*	≥ 3	4	8	E_6	\mathfrak{f}_4 or \mathfrak{e}_6
III^*	3	≥ 5	9	E_7	\mathfrak{e}_7
II^*	≥ 4	5	10	E_8	\mathfrak{e}_8

Table 2. Kodaira classification of codimension one singularities in elliptic fibrations and corresponding gauge algebras. When multiple gauge algebras are given, the gauge algebra is determined by monodromy conditions.

The coefficients in f and g can be tuned to special values that satisfy the conditions in table 2. In such cases, the gauge symmetry can be Higgsed by deforming the coefficients away from their special values. However, when constructing Weierstrass models on rigid divisors, some coefficients in the expansion of f and g around that divisor may be forced to vanish, giving a gauge symmetry that cannot be removed by altering coefficients. These gauge symmetries are known as non-Higgsable clusters and are discussed further in [21, 25]. Most of the examples we focus on here will involve tuned gauge symmetries rather than non-Higgsable clusters.

The gauge symmetry discussion has focused on local features of the Weierstrass model near a particular divisor. Local considerations are sufficient to determine the symmetry algebra of a nonabelian, continuous gauge symmetry. Producing abelian or some discrete symmetries involves creating an extra section in the elliptically fibered compactification space, which, however, requires an analysis of global behavior. Our F-theory analysis will mostly be concerned with nonabelian algebras, and local analyses involving expansions such as (3.2) and (3.3) will suffice.

3.1.3 Matter

If an F-theory compactification to $d \leq 6$ has a codimension one singularity, there generally will be codimension two loci within the codimension one locus where the singularity type is enhanced. These codimension two loci indicate that the model has matter charged under the corresponding gauge algebra. In some cases, it is easy to determine the representations of the charged matter. For example, if the singularity undergoes a rank-one enhancement to a standard Kodaira fiber, the resulting charged matter can be found using the Katz-Vafa method [26]. The enhanced singularity type has a corresponding gauge algebra; breaking the adjoint of this enhanced gauge algebra to the original gauge algebra gives the matter content. Importantly, the enhanced singularity does not represent an actual enhancement of the gauge group. Breaking the adjoint of the enhanced singularity's gauge algebra simply

provides a convenient way of determining the matter content. As an example, consider a situation where a codimension one A_{n-1} singularity enhances to an A_n singularity on a codimension two locus. The adjoint of A_n breaks as

$$\mathbf{Adj} \rightarrow \mathbf{Adj} + \mathbf{n} + \bar{\mathbf{n}} + \mathbf{1}. \tag{3.6}$$

The $\mathbf{n} + \bar{\mathbf{n}}$ term in the above breaking pattern represents the charged matter contributed by each A_n locus. In particular, matter in 6D F-theory models must come in quaternionic representations, and the \mathbf{n} and $\bar{\mathbf{n}}$ combine to form a full multiplet in the Λ^1 or \square representation. The Katz-Vafa analysis can be used for $A_{n-1} \rightarrow D_n$ and $A_6 \rightarrow E_7$ enhancements as well.

However, the Kodaira classification is strictly valid only for codimension one, and Kodaira codimension-one singularities can enhance to non-standard codimension-two singularities. Moreover, there can be codimension-two enhancements that do not enhance the rank by exactly one. Determining the resulting matter in these cases requires a more detailed analysis, as described in [5, 27–32]. Examples that will be of interest here are the $A_5 \rightarrow E_6$ and $A_7 \rightarrow E_8$ enhancements described in [5]. The work of [33] presents an alternative method of determining the matter content that does not require an explicit resolution of singularities.

The number of singlet hypermultiplets corresponds to (one more than) the $h^{2,1}$ of the compactification space. Alternatively, the number of singlets can be found by counting the number of complex degrees of freedom in the Weierstrass model. There is not a direct 1-1 equivalence between the Weierstrass degrees of freedom and the number of neutral hypermultiplets, as automorphisms on the base and the effects of -2 curves must be taken into account. The complete expression for relating the Weierstrass degrees of freedom to the number of neutral hypermultiplets is given in [34].

3.1.4 Superconformal points and tensionless string transitions

In some situations, a codimension two locus can have an enhanced singularity such that f and g vanish to orders 4 and 6. Resolving these codimension two singularities requires a blow-up on the base, as described in [2, 35]. In contrast, the codimension-two enhanced singularities giving matter can be resolved using blow-ups only on the elliptic fiber. From a field theory perspective, blowing up the base introduces an additional tensor multiplet, and the size of the new \mathbb{P}^1 corresponds to the expectation value $\langle S \rangle$ of the tensor multiplet's scalar. The original codimension-two locus can then be thought of as describing the limit where $\langle S \rangle$ approaches zero. $\langle S \rangle$ governs the tension of strings that couple to the tensor. In the limit where $\langle S \rangle$ goes to zero, these strings become tensionless [36]. The blow-up and blow-down processes associated with these codimension two loci are thus the tensionless string transitions described in the literature [1, 37]. $\langle S \rangle$ also controls the couplings of any gauge groups that appear after the blowup, and shrinking the exceptional curves to zero size takes any such gauge group to its strongly coupled limit. These loci are associated with 6D superconformal field theories [38] and have been the focus of much recent work [39–43]. For this reason, we will refer to these codimension two loci as superconformal points.

The tensionless string transitions provide a means of connecting F-theory compactifications on different bases, playing a similar role as the small instanton transitions of heterotic string theory [1, 2]. For instance, one can blow up \mathbb{F}_1 at a point and perform a subsequent blow-down to obtain an F-theory compactification on \mathbb{F}_2 . In fact, all 6D F-theory compactifications are connected into a single moduli space by such transitions [1, 9]. In some cases, the matter content can change during such a tensionless string transition, as in the chirality-changing 4D phase transitions studied in [3, 4, 10, 11]. Such transitions involve a change in the number of tensor multiplets. The transitions we consider here differ in that they also involve passing through a superconformal point in the moduli space of vacua, but do not involve a change in the number of tensor multiplets, and are also distinct from Higgsing/unHiggsing transitions, which always involve a change in the gauge field content of the theory and generally do not involve passing through a superconformal point.

3.2 F-theory tunings of SU(6)–SU(8) models

In this section, we derive the explicit local forms of the Weierstrass models for SU(6)–SU(8) that will be used to analyze the transitions. We mostly use the conventions of [5], where gauge groups were tuned on a divisor D with associated coordinate σ . We assume in particular that the divisor D is smooth and that we are working over a unique factorization domain (UFD). When D is singular there are more complicated ways of realizing SU(N) gauge groups, which generally involve representations other than the adjoint and k -index antisymmetric representations studied in this section; we describe one such example in section 6.3. This general analysis is valid for an arbitrary choice of F-theory base B , which could be of complex dimension two or three and an arbitrary smooth effective divisor D . For the rest of this F-theory section, $-K_B$ will refer to the anti-canonical class of the base, and $-K$ and N will refer to the anti-canonical class and the normal line bundle of D unless stated otherwise. Additionally, we will often refer to the situation where the gauge group is tuned on a $+n$ curve in a 6D F-theory model (such as \tilde{S} in \mathbb{F}_n); n will be this self-intersection number for the rest of the F-theory section, and in this case $-K = 2$ and $N = m$.

3.2.1 SU(6)

General tunings for SU models up to SU(6) were found in [5]. Here, we repeat the derivation of the SU(6) model as a warmup for subsequent tunings. As described in [5], the f and g for an SU(5) Weierstrass model are

$$f = \frac{-\phi_0^4}{48} - \frac{1}{6}\phi_0^2\phi_1\sigma + \left(\frac{\phi_0\psi_2}{2} - \frac{\phi_1^2}{3}\right)\sigma^2 + \mathcal{O}(\sigma^3), \tag{3.7}$$

$$g = \frac{\phi_0^6}{864} + \frac{1}{72}\phi_0^4\phi_1\sigma + \left(\frac{\phi_0^2\phi_1^2}{18} - \frac{\phi_0^3\psi_2}{24}\right)\sigma^2 + \frac{1}{108}\left(8\phi_1^3 - 18\phi_0\phi_1\psi_2 - 9\phi_0^2f_3\right)\sigma^3 + \frac{1}{12}\left(3\psi_2^2 - 4\phi_1f_3 - \phi_0^2f_4\right)\sigma^4 + \mathcal{O}(\sigma^5). \tag{3.8}$$

Here ϕ_0, ϕ_1, \dots are sections of certain line bundles over D , which can be thought of locally as polynomials in a local set of variables on D . For example, since f is a section of $\mathcal{O}(-4K_B)$,

ϕ_0 must be a section of $\mathcal{O}(-K_B)$, which descends to $\mathcal{O}(-K+N)$ on D . The discriminant of this SU(5) tuning vanishes to order σ^5 , as expected:

$$\Delta = \frac{\phi_0^4}{192} \left[12\phi_1\psi_2^2 - 12f_3\psi_2\phi_0 + \phi_0^2 \left(12g_5 + 4f_4\phi_1 + f_5\phi_0^2 \right) \right] \sigma^5 + \mathcal{O}(\sigma^6). \quad (3.9)$$

For the SU(6) tuning, the discriminant must vanish to $\mathcal{O}(\sigma^6)$, while f and g cannot be proportional to σ . These requirements demand that the term in square brackets in equation (3.9) vanishes. $\phi_1\psi_2^2$ must therefore be proportional to ϕ_0 , as every other term in the square brackets is at least first order in ϕ_0 . However, the various factors in ϕ_0 can be distributed in any way between ϕ_1 and ψ_2^2 . We can rewrite ϕ_0 , ϕ_1 and ψ_2 in a way that explicitly resolves this ambiguity. Defining α to be the GCD of ϕ_0 and ψ_2 , we have

$$\phi_0 = \alpha\beta, \quad (3.10)$$

$$\phi_1 = \beta\nu, \quad (3.11)$$

$$\psi_2 = -\frac{1}{3}\alpha\phi_2. \quad (3.12)$$

With these redefinitions, the discriminant now reads

$$\Delta = \frac{\alpha^6\beta^5}{576} \left[4\phi_2(3f_3 + \nu\phi_2) + 36\beta g_5 + 3\beta^2(4f_4\nu + \alpha^2\beta f_5) \right] \sigma^5 + \mathcal{O}(\sigma^6). \quad (3.13)$$

Removing the lowest order term in β requires that there exists a λ such that

$$f_3 = -\frac{1}{3}\nu\phi_2 - 3\beta\lambda. \quad (3.14)$$

Note that ϕ_2 does not share any factors with β , under the assumption that α is the GCD of ϕ_0 and ψ_2 . After tuning f_3 , all the terms under consideration are sixth-order in β , and we can solve for the g_5 that makes Δ vanish to $\mathcal{O}(\sigma^6)$.

$$g_5 = \lambda\phi_2 - \frac{1}{3}\beta\nu f_4 - \frac{1}{12}\alpha^2\beta^2 f_5. \quad (3.15)$$

The final f and g for the SU(6) tuning are

$$f = -\frac{\alpha^4\beta^4}{48} - \frac{1}{6}\alpha^2\beta^3\nu\sigma - \frac{\beta}{6}(\alpha^2\phi_2 + 2\beta\nu^2)\sigma^2 - \left(3\beta\lambda + \frac{\nu\phi_2}{3} \right) \sigma^3 + \mathcal{O}(\sigma^4) \quad (3.16)$$

and

$$\begin{aligned} g &= \frac{\alpha^6\beta^6}{864} + \frac{\alpha^4\beta^5}{72}\nu\sigma + \frac{\alpha^2\beta^3}{72}(4\beta\nu^2 + \alpha^2\phi_2)\sigma^2 + \frac{\beta^2}{108}(8\beta\nu^3 + 9\alpha^2\nu\phi_2 + 27\alpha^2\beta\lambda)\sigma^3 \\ &+ \frac{1}{36}(4\beta\nu^2\phi_2 + \alpha^2\phi_2^2 + 36\beta^2\nu\lambda - 3\alpha^2\beta^2 f_4)\sigma^4 \\ &+ \frac{1}{12}(12\lambda\phi_2 - 4\beta\nu f_4 - \alpha^2\beta^2 f_5)\sigma^5 + \mathcal{O}(\sigma^6) \end{aligned} \quad (3.17)$$

The SU(6) model has five free parameters (apart from the untuned f_k and g_k), which are summarized in table 3. Each parameter is a section of a line bundle over D . One

Parameter	Divisor Class	Order on $+n$ curve	Associated Matter
α	$-K + N - L$	$n + 2 - r$	\square (15)
β	L	r	$\frac{1}{2}\square$ ($\frac{1}{2}$ 20)
ν	$-2K + N - L$	$n + 4 - r$	—
ϕ_2	$-2K + L$	$4 + r$	—
λ	$-4K + N - L$	$n + 8 - r$	—

Table 3. Free parameters in the general SU(6) Weierstrass model. r must be greater than or equal to 0 and less than $n + 2$

parameter, which we have chosen to be the divisor class L associated with β , is independent. Once this divisor class is fixed, all the other divisor classes can be computed from the form of the expansion and the divisor classes of f, g . On the zeroes of α , (f, g, Δ) vanish to orders $(2, 3, 8)$, and the singularity type is enhanced to D_6 . Every zero of α therefore contributes a full multiplet in the \square representation. A zero of β meanwhile enhances the singularity type to E_6 , giving a half-multiplet in the $\frac{1}{2}\square$ representation. Fundamental matter comes from codimension two loci where (f, g, Δ) vanish to orders $(0, 0, 7)$ and the singularity type enhances to A_6 . Finally, the number of neutral multiplets can be found by counting the number of degrees of freedom, as described previously. The resulting multiplicities agree with the expected supergravity spectrum of equation (2.14).

3.2.2 SU(7)

Some aspects of SU(7) tunings were discussed in [5], but a general SU(7) tuning was not given there. Instead, tunings were presented for two limiting cases of the matter spectrum; the models lacked either \square (**21**) matter or $\frac{1}{2}\square$ (**35**) matter. We present a more general SU(7) tuning that can exhibit any of the antisymmetric matter spectra consistent with anomaly conditions.

The discriminant of the SU(6) model has the form

$$\Delta = \frac{\alpha^4 \beta^3}{432} \left[\alpha^2 \left(\phi_2^3 + 9\beta^2 \phi_2 f_4 + 27\beta^3 g_6 + 9\beta^4 f_5 \nu + 9 \frac{\alpha^2 \beta^5}{4} f_6 \right) - 3\beta (\nu \phi_2 - 9\beta \lambda)^2 \right] \sigma^6 + \mathcal{O}(\sigma^7). \tag{3.18}$$

For the SU(7) tuning, Δ must vanish to $\mathcal{O}(\sigma^7)$. As noted in [5], this demands that α^2 is proportional to β . We can implement this requirement by rewriting α and β as

$$\begin{aligned} \beta &= \gamma \delta^2, \\ \alpha &= \gamma \delta \xi. \end{aligned}$$

This decomposition is uniquely defined if we impose the condition that γ be square-free. These redefinitions would in turn require ν to be proportional to γ , so we will temporarily

write ν as $\gamma\zeta$. The term in square brackets in equation (3.18) would then be equivalent to

$$\gamma^2\delta^2 \left[\xi^2\gamma^2\delta^4 \left(9f_4\phi_2 + 27\gamma\delta^2g_6 + 9\gamma^3\delta^4f_5\zeta + \frac{9}{4}\xi^2\gamma^5\delta^8 \right) - 3\gamma \left(\zeta\phi_2 - 9\delta^2\lambda \right)^2 + \xi^2\phi_2^3 \right]. \quad (3.19)$$

ϕ_2 cannot share any factor with γ , since it shares no factors with β . ξ^2 must therefore be proportional to γ , and since γ is square-free, ξ must be proportional to γ . Then, the term lowest order in γ would be $3\gamma(\zeta\phi_2 - 9\delta^2\lambda)^2$, implying that $\zeta\phi_2 - 9\delta^2\lambda$ must also be proportional to γ . Then, $3\gamma(\zeta\phi_2 - 9\delta^2\lambda)^2$ would be order γ^3 , in turn demanding that ξ must be proportional to γ^2 . The tuning is stuck in a never-ending cycle where the two terms must be proportional to greater and greater powers of γ . The only way out of this cycle is if γ is a perfect square, which would violate the earlier square-free assumption. γ (and ζ) should therefore be ignored in the α and β redefinitions, leaving

$$\beta = \delta^2 \quad (3.20)$$

$$\alpha = \delta\xi \quad (3.21)$$

We will continue to refer to ν and will not use γ or ζ from this point.

Equations (3.20) and (3.21) could have been anticipated from field theory considerations alone. While each zero of β gives $\frac{1}{2}\square$ of SU(6) matter, matter in the SU(7) \square (35) representation must come in full multiplets. When the SU(7) model is Higgsed to SU(6), each **35** multiplet will give two $\frac{1}{2}\mathbf{20}$ s of SU(6). Any zeroes that produce **35**s in the SU(7) must therefore provide β with two identical factors when the Weierstrass model is deformed to an SU(6) model. In other words, β must be a perfect square, in agreement with equation (3.20). Each **35** also gives a full **15** multiplet. From (3.20), we expect that the zeroes of δ will give the **35**s of SU(7), so α must be proportional δ . The above redefinitions additionally imply that some SU(6) models cannot be enhanced to SU(7). Eq. (3.20) and (3.21) require the divisors $L' \sim \frac{1}{2}L$ and $-K + N - 3L'$ to be effective. For the $+n$ curve situation, only SU(6) models where r_6 is even and where $n + 2 \geq \frac{3}{2}r_6$ can be enhanced to SU(7). These restrictions follow naturally from the SU(7) \rightarrow SU(6) branching patterns.

With the redefinitions for α and β , the discriminant is now given by

$$\Delta = \frac{\delta^{12}\xi^4}{432} \left[\xi^2\delta^4 \left(9f_4\phi_2 + 27\delta^2g_6 + 9\delta^4f_5\nu + \frac{9}{4}\xi^2\delta^8 \right) - 3 \left(\nu\phi_2 - 9\delta^2\lambda \right)^2 + \xi^2\phi_2^3 \right]. \quad (3.22)$$

Considering the lowest order term in δ gives the constraint

$$\phi_2^2 \left(\xi^2\phi_2 - 3\nu^2 \right) \propto \delta^2. \quad (3.23)$$

To satisfy this condition, one could argue that when $\xi = 0$, from (3.22) ν should be proportional to δ^2 . This would suggest that we redefine ν as

$$\nu = \zeta_1\xi + \zeta_2\delta^2, \quad (3.24)$$

where ζ_1 and ζ_2 are independent, untuned polynomials. Note that we could have also considered the possibility that ξ and ν share some common factors with δ . However,

this possibility will end up being a special point in the moduli space of tunings found using (3.24). More specifically, the tuning involving common factors can be derived by taking the tuning found using (3.24) and further demanding that ξ , ν , and δ share common factors. Therefore, equation (3.24) will be sufficient to find a general tuning, and we do not need to consider the possibility of shared factors in this step.

Ultimately, we will want to solve for g_6 to make the sixth order term of Δ vanish. There are still lower order terms to deal with before we can solve for g_6 , but these terms can be removed using a set of standard polynomial redefinitions:

$$\phi_2 = 3\zeta_1^2 + \delta^2\omega, \quad (3.25)$$

$$\lambda = \frac{1}{3}\zeta_1^2\zeta_2 - \frac{1}{18}\zeta_1\xi\omega + \frac{1}{9}\zeta_2\delta^2\omega + \xi\delta^2\lambda_1, \quad (3.26)$$

$$f_4 = -6\zeta_1\lambda_1 - \frac{1}{12}\omega^2 + \psi_4\delta^2. \quad (3.27)$$

ω , λ_1 , and ψ_4 are free parameters. We can then solve for g_6 :

$$g_6 = \frac{-1}{108} \left[\omega^3 + 108\zeta_1(\lambda_1\omega + \zeta_1\psi_4) + 36\delta^2(\psi_4\omega - 27\lambda_1^2 + f_5(\zeta_1\xi + \zeta_2\delta^2) + \frac{\xi^2\delta^4}{4}f_6) \right]. \quad (3.28)$$

The final tunings are

$$\begin{aligned} f = & -\frac{\delta^{12}\xi^4}{48} - \frac{\delta^8\xi^2}{6}(\zeta_1\xi + \zeta_2\delta^2)\sigma - \frac{\delta^4}{6}(2\delta^4\zeta_2^2 + 4\delta^2\zeta_1\zeta_2\xi + \xi^2(5\zeta_1^2 + \delta^2\omega))\sigma^2 \\ & - \frac{1}{6}(4\delta^2(3\zeta_1^2\zeta_2 + \delta^2\zeta_2\omega) + \xi(6\zeta_1^3 + \delta^2\zeta_1\omega + 18\delta^4\lambda_1))\sigma^3 \\ & - \frac{1}{12}(\omega^2 + 72\zeta_1\lambda_1 - 12\delta^2\psi_4)\sigma^4 + \mathcal{O}(\sigma^5) \end{aligned} \quad (3.29)$$

and

$$\begin{aligned} g = & \frac{\delta^{18}\xi^6}{864} + \frac{\delta^{14}\xi^4}{72}(\zeta_1\xi + \zeta_2\delta^2)\sigma + \frac{\delta^{10}\xi^2}{72}(4\delta^4\zeta_2^2 + 8\delta^2\xi\zeta_1\zeta_2 + 7\zeta_1^2\xi^2 + \delta^2\xi^2\omega)\sigma^2 \\ & + \frac{\delta^6}{216}(16\delta^6\zeta_2^3 + 48\delta^4\xi\zeta_1\zeta_2^2 + 120\delta^2\zeta_1^2\zeta_2\xi^2 + 70\zeta_1^3\xi^3 \\ & + 54\delta^4\xi^3\lambda_1 + 24\delta^4\zeta_2\xi^2\omega + 15\delta^2\xi^3\zeta_1\omega)\sigma^3 \\ & + \frac{\delta^2}{144}(84\zeta_1^4\xi^2 + \delta^4(96\zeta_1^2\zeta_2^2 + 5\xi^2\omega^2 + 8\zeta_1\xi(27\lambda_1\xi + 5\zeta_2\omega)) \\ & + 16\delta^2\zeta_1^2\xi(9\zeta_1\zeta_2 + 2\xi\omega) + 4\delta^6(36\zeta_2\lambda_1\xi - 3\xi^2\psi_4 + 8\zeta_2^2\omega))\sigma^4 \\ & + \frac{1}{36}(2(3\zeta_1^2 + \delta^2\omega)(6\zeta_1^2\zeta_2 - \zeta_1\xi\omega + 2\delta^2(9\lambda_1\xi + \zeta_2\omega)) - 3\delta^6\xi^2f_5 \\ & - \delta^2(\delta^2\zeta_2 + \zeta_1\xi)(-72\zeta_1\lambda_1 + 12\delta^2\psi_4 - \omega^2))\sigma^5 \\ & - \left(\frac{\omega^3}{108} + \zeta_1(\lambda_1\omega + \zeta_1\psi_4) + \frac{\delta^2}{3}(\psi_4\omega - 27\lambda_1^2 + f_5\nu + \frac{\xi^2\delta^4}{4}f_6) \right)\sigma^6 + \mathcal{O}(\sigma^7). \end{aligned} \quad (3.30)$$

There are seven free polynomials apart from the untuned f_k and g_k , which are given in table 4. On the zeroes of ξ , the A_6 singularity is enhanced to an D_7 singularity, giving

Parameter	Divisor Class	Order on $+n$ curve	Associated Matter
δ	L'	r	$\begin{matrix} \square \\ \square \end{matrix} + \square$
ξ	$-K + N - 3L'$	$n + 2 - 3r$	$\begin{matrix} \square \\ \square \end{matrix}$
ζ_1	$-K + L'$	$r + 2$	—
ζ_2	$-2K + N - 4L'$	$n + 4 - 4r$	—
ω	$-2K$	4	—
λ_1	$-3K - L'$	$6 - r$	—
ψ_4	$-4K - 2L'$	$8 - 2r$	—

Table 4. Degrees of freedom in SU(7) Weierstrass model. r and n must satisfy $n + 2 \geq 3r$.

matter in the $\begin{matrix} \square \\ \square \end{matrix}$ (**21**). The zeroes of δ , meanwhile, enhance the singularity to E_7 ; decomposing the E_7 adjoint into A_6 representations shows that each zero of δ corresponds to one full $\begin{matrix} \square \\ \square \end{matrix}$ (**35**) multiplet and one full multiplet in the fundamental (**7**) representation. When the gauge group is tuned on a $+n$ curve, there are an additional $16 + n + 4r$ fundamentals coming from codimension two loci where the discriminant vanishes to order 8. The charged matter content agrees with the results from gauge anomaly cancellation conditions, as given in equation (2.15).

As before, the number of neutral hypermultiplets corresponds to the total number of complex degrees of freedom. A naive counting gives more degrees of freedom than expected from the gravitational anomaly cancellation condition. However, the polynomials in table 4 can be redefined in the following way without changing f or g :

$$\begin{aligned}
 \zeta_1 &\rightarrow \zeta'_1 = \zeta_1 + \delta^2 \epsilon, \\
 \zeta_2 &\rightarrow \zeta'_2 = \zeta_2 - \xi \epsilon, \\
 \omega &\rightarrow \omega' = \omega - 6\zeta_1 \epsilon - 3\delta^2 \epsilon^2, \\
 \lambda_1 &\rightarrow \lambda'_1 = \lambda_1 + \frac{1}{6} \omega \epsilon - \frac{1}{2} \zeta_1 \epsilon^2 - \frac{1}{6} \delta^2 \epsilon^3, \\
 \psi_4 &\rightarrow \psi'_4 = \psi_4 + 6\lambda_1 \epsilon + \frac{1}{2} \omega \epsilon^2 - \zeta_1 \epsilon^3 - \frac{1}{4} \delta^2 \epsilon^4.
 \end{aligned}
 \tag{3.31}$$

ϵ is a section of $\mathcal{O}(-K - L')$, and on a $+n$ curve, ϵ is a polynomial of order $2 - r$. Therefore, $2 - r + 1$ of the complex degrees of freedom can be thought of as redundant and should not be considered when finding the number of neutral hypermultiplets. Subtracting these redundant degrees of freedom in fact leads to a number of neutral hypermultiplets consistent with the anomaly cancellation conditions. It is unclear whether the redundancies can be avoided in an alternative tuning or whether they are a necessary part of the tuning with some physical interpretation.

3.2.3 SU(8)

The SU(7) discriminant has the form

$$\Delta = \delta^8 \xi^4 \left(-\frac{1}{8} \zeta_1^7 \xi + \frac{1}{4} \zeta_2 \zeta_1^6 \delta^2 + \mathcal{O}(\delta^4, \xi \delta^2) \right) \sigma^7 + \mathcal{O}(\sigma^8) \quad (3.32)$$

In order for Δ vanish to $\mathcal{O}(\sigma^8)$, $\zeta_1^7 \xi$ must be proportional to δ^2 . If δ and ζ_1 share a common factor, the resulting Weierstrass model will have a codimension two (4,6) singularity. ξ must therefore be proportional to δ^2 , and we can rewrite ξ as

$$\xi = \delta^2 \tau. \quad (3.33)$$

Δ now reads

$$\Delta = \tau^4 \delta^{18} \left[\frac{\zeta_1^6}{4} \left(\zeta_2 - \frac{1}{2} \zeta_1 \tau \right) + \mathcal{O}(\delta^2) \right] \sigma^7 + \mathcal{O}(\sigma^8), \quad (3.34)$$

implying that

$$\zeta_2 = \frac{1}{2} \zeta_1 \tau + \zeta_3 \delta^2. \quad (3.35)$$

At this point, the leading order term in the discriminant is given by

$$\Delta = \tau^4 \delta^{20} \left[\frac{\zeta_1^5}{16} \left(4\zeta_1 \zeta_3 - \tau \omega \right) + \mathcal{O}(\delta^2) \right] \sigma^7 + \mathcal{O}(\sigma^8). \quad (3.36)$$

$4\zeta_1 \zeta_3 - \tau \omega$ must be therefore proportional to δ^2 . Following the same strategy as in the SU(7) tuning, we could argue that, when $\delta = 0$, ζ_1 is proportional to τ . ζ_1 would then decompose as

$$\zeta_1 = \tau \zeta_4 + \delta^2 \zeta_5. \quad (3.37)$$

However, equation (3.35) and the discussion of redundant degrees of freedom in the SU(7) tuning suggests that ζ_5 in the above decomposition could be absorbed into ζ_3 and other variables. Performing a full tuning with ζ_5 indeed shows that ζ_5 can be absorbed into other polynomials with removing any degrees of freedom. ζ_1 can therefore be redefined as

$$\zeta_1 = \zeta_4 \tau \quad (3.38)$$

without any loss of generality. Substituting this ζ_1 expression into (3.36) gives a solution for ω :

$$\omega = 4\zeta_3 \zeta_4 + \delta^2 \omega_1. \quad (3.39)$$

Eq. (3.36) now becomes

$$\begin{aligned} \Delta = \frac{1}{16} \delta^{22} \tau^4 & \left[-\zeta_4 \tau^5 \left(6\lambda_1 + \zeta_4 \tau \omega_1 \right) - 2\zeta_4^3 \tau^3 \left(\tau \psi_4 + 12\zeta_3 \lambda_1 + 2\zeta_3 \zeta_4 \tau \omega_1 \right) \delta^2 \right. \\ & - \zeta_4 \tau \left(4\zeta_3 \zeta_4 \tau \psi_4 - 18\lambda_1^2 \tau + \zeta_4^2 \tau^3 \omega_1^2 + 4\zeta_3^2 \zeta_4 (6\lambda_1 + \zeta_4 \tau \omega_1) \right) \delta^4 \\ & + \left(f_5 \zeta_4^2 \tau^4 - \zeta_4 \tau^2 \omega_1 (\psi_4 + 2\zeta_3 \zeta_4 \omega_1) + 6\lambda_1 (6\zeta_3 \lambda_1 + \tau \psi_4) \right) \delta^6 \\ & \left. + \tau \left(g_7 \tau + \frac{1}{4} \omega_1^2 (6\lambda_1 - \zeta_4 \tau \omega_1) \right) \delta^8 + \mathcal{O}(\tau^2 \delta^8) \right] \sigma^7 + \mathcal{O}(\sigma^8). \end{aligned}$$

Degree of Freedom	Line Bundle	Order on $+n$ curve	Associated Matter
δ	L'	r	$\begin{array}{ c } \hline \square \\ \hline \end{array} + \begin{array}{ c } \hline \square \\ \hline \end{array} + \square$
τ	$-K + N - 5L'$	$n + 2 - 5r$	$\begin{array}{ c } \hline \square \\ \hline \end{array}$
ζ_3	$-2K + N - 6L'$	$n + 4 - 6r$	—
ζ_4	$-N + 6L'$	$-n + 6r$	—
ω_1	$-2K - 2L'$	$4 - 2r$	—
λ_2	$-2K - N + 2L'$	$-n + 4 + 2r$	—
ϕ_4	$-4K - 6L'$	$8 - 6r$	—
ψ_5	$-4K - N - 2L'$	$-n + 8 - 2r$	—

Table 5. Degrees of freedom in SU(8) Weierstrass model, excluding untuned f_k and g_k . Note that r and n must satisfy $n + 2 \geq 5r$

Only standard redefinitions are needed from this point forward to get Δ in a form to solve for g_7 :

$$\lambda_1 = -\frac{1}{6}\zeta_4\omega_1\tau + \delta^2\tau\lambda_2, \tag{3.40}$$

$$\psi_4 = -3\zeta_4\lambda_2\tau^2 - \frac{1}{4}\delta^2\omega_1^2 - 6\zeta_3\lambda_2\delta^2 + \phi_4\delta^4, \tag{3.41}$$

$$f_5 = 2\zeta_4\phi_4 + \psi_5\delta^2. \tag{3.42}$$

Finally, we solve for g_7 :

$$g_7 = \frac{1}{12} \left(16\zeta_3\zeta_4^2\phi_4 - 12\zeta_4^2\tau^2\psi_5 - 16\delta^2\zeta_4(\zeta_3\psi_5 - \phi_4\omega_1) - 4\delta^4(18\lambda_2\phi_4 + \psi_5\omega_1) - 2\delta^4f_6(2\zeta_3\delta^2 + 3\zeta_4\tau^2) - f_7\delta^{10}\tau^2 \right). \tag{3.43}$$

For the sake of brevity, we do not rewrite the full f and g here.

The free polynomials for this SU(8) model are shown in table 5. When $\tau = 0$, the singularity type enhances from A_7 to D_8 , so each zero of τ contributes a full multiplet in the $\begin{array}{|c|} \hline \square \\ \hline \end{array}(\mathbf{28})$ representation. Meanwhile, the discriminant takes the form

$$\Delta = \delta^{12}\tau^4 \left(\frac{3}{16}\zeta_4^8\tau^8 + \mathcal{O}(\delta^2) \right) \sigma^8 + \mathcal{O}(\sigma^9). \tag{3.44}$$

The factor in parentheses represents the discriminant locus where the singularity is enhanced to A_8 . This discriminant locus is a section of $\mathcal{O}(-8K + 8L)$ and is of order $16 + 8r$. There are therefore $16 + 8r$ full multiplets in the fundamental $(\mathbf{8})$ representation coming from the divisor locus. On the zeroes of δ , the singularity type enhances to E_8 . As mentioned in [5], if the A_7 singularity structure is embedded in the standard E_8 Dynkin diagram, each zero of δ contributes one $\begin{array}{|c|} \hline \square \\ \hline \end{array}$ multiplet, one $\begin{array}{|c|} \hline \square \\ \hline \end{array}$ multiplet, and one \square multiplet. In principle, the A_7 singularity structure could also be embedded in the extended

Representation	Multiplicity (in full multiplets)
$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \end{array}$ (56)	r
$\begin{array}{ c } \hline \square \\ \hline \end{array}$ (28)	$n + 2 - 4r$
\square (8)	$16 + 9r$

Table 6. Matter content of the SU(8) F-theory model. Note that $n + 2 \geq 5r$. The SU(8) model we have constructed does not seem to give all of the possible spectra listed in equation (2.16).

E_8 Dynkin diagram in a different fashion, to give a half-multiplet in the $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$ representation and two full-multiplets in the $\begin{array}{|c|} \hline \square \\ \hline \end{array}$ representation. In this non-standard embedding, one of the roots of the A_7 is mapped to the “extra” root of the affine \hat{E}_8 . Either possibility is consistent with gauge anomaly cancellation conditions. Here, in the specific tunings we have identified, the A_7 is enhanced to an E_8 singularity, which we assume gives the most generic matter content, in the **56**, **28**, and **8** representations. The resulting charged matter content is summarized in table 6.

To find the number of neutral multiplets, we again count the number of complex degrees of freedom in the Weierstrass model and subtract any redundant degrees of freedom. As with the SU(7) model, the f and g expressions for the SU(8) tuning are invariant under the following transformations in the polynomials:

$$\begin{aligned}
 \zeta_3 &\rightarrow \zeta'_3 = \zeta_3 + \tau^2 \epsilon, \\
 \zeta_4 &\rightarrow \zeta'_4 = \zeta_4 - \frac{2}{3} \delta^2 \epsilon, \\
 \omega_1 &\rightarrow \omega'_1 = \omega_1 + \frac{8}{3} \zeta_3 \epsilon + \frac{4}{3} \tau^2 \epsilon^2, \\
 \lambda_2 &\rightarrow \lambda'_2 = \lambda_2 - \frac{18}{81} \omega_1 \epsilon - \frac{24}{81} \zeta_3 \epsilon^2 - \frac{8}{81} \tau^2 \epsilon^3, \\
 \psi_5 &\rightarrow \psi'_5 = \psi_5 + \frac{4}{3} \phi_4 \epsilon.
 \end{aligned} \tag{3.45}$$

In these transformations ϵ is a section of $\mathcal{O}(-N + 4L')$, so there are $-n + 4r + 1$ redundant degrees of freedom. After taking the transformations into account, the number of neutral hypermultiplets agrees with anomaly conditions only when δ gives $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$, $\begin{array}{|c|} \hline \square \\ \hline \end{array}$, and \square multiplets. This gives concrete evidence that the generic charged matter content of table 6 is correct for this form of E_8 singularity.

The tuning presented here does not seem to give matter in the $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$ (**70**) representation. As discussed earlier, the supergravity models with (**70**) matter seem to be consistent with the anomaly conditions, posing the question of whether these models have valid F-theory realizations. At this point, it is unclear if there is some F-theory constraint that forbids the Λ^4 models or if the tuning presented here can be extended to give 4-index antisymmetric matter. We will return to this issue in later sections.

There are also certain supergravity SU(8) models with \square matter that are not allowed in our F-theory construction. Equation (3.33) requires that $-K + N - 5L'$ be effective; for the $+n$ curve, $n + 2 \geq 5r$. The F-theory SU(8) models with $r > 0$ **56** multiplets therefore have at least one \square multiplet, even though models without \square multiplets can still be consistent with anomaly cancellation conditions. For instance, a model on a $+2$ curve with 1 **56** multiplet and 25 **8** multiplets is anomaly-free but cannot be realized in our F-theory constructions. This represents another potentially interesting point of disagreement between the low-energy anomaly analysis and what can be realized in F-theory. The restriction also implies that only some of the SU(7) models can be enhanced to SU(8). In particular, an SU(7) model cannot be enhanced unless it has at least two **21** multiplets for every **35** multiplet. These restrictions on enhancement are also consistent with what can be observed for models from a heterotic perspective (see, for example, table 24 in appendix D.1.2).

3.2.4 SU(9)

The SU(8) discriminant takes the form

$$\Delta = \frac{\delta^{12}\tau^4}{192} (36\zeta_4^8\tau^8 + \mathcal{O}(\delta^2)) \sigma^8 + \mathcal{O}(\sigma^9) \tag{3.46}$$

To tune an SU(9) singularity, Δ must vanish to order σ^9 , which would require $\zeta_4^8\tau^8$ to be proportional to δ^2 . But from the SU(8) tuning, δ cannot share factors with either τ or ζ_4 without introducing superconformal points where f and g vanish to order 4 and 6. The only way to tune an SU(9) gauge symmetry seems to be to have δ be a constant. If one proceeds with constant δ , the resulting SU(9) models have only \square , fundamental, and singlet matter.

The SU(9) tuning presents a similar challenge as the SU(8) models. From the anomaly conditions, there appear to be consistent SU(9) supergravity models with \square matter. Yet our tunings seem to forbid F-theory constructions of these models if one wishes to avoid superconformal points. Both situations also require the gauge group Dynkin diagram to be embedded in an extended Dynkin diagram. For instance, the Λ^3 representation of SU(9) comes from the enhancement $A_8 \rightarrow \hat{E}_8$. We will further discuss these missing cases later. Note that in [24], it is argued that all SU(N) models except those with $N = 6, 7, 8, 9$ can be put in Tate form, but that in these four cases there are non-Tate realizations. This suggests that a more sophisticated F-theory construction may indeed realize SU(9) models with exotic matter content, despite the analysis here.

3.2.5 SU(10)

There is no clear way in which a Λ^3 representation of SU(10) could be realized in F-theory, since the Dynkin diagram for A_9 does not embed in \hat{E}_8 . So there is no way to enhance the Kodaira A_9 singularity to an exceptional singularity that might carry exotic matter. Put differently, SU(10) is a rank 9 group and cannot possibly be a subgroup of E_8 . The impossibility of realizing theories with SU(10) or higher gauge groups and Λ^3 representations in

F-theory matches nicely with the low-energy theory where such models are essentially ruled out by anomaly cancellation. One puzzle here, however, is that the low-energy condition ruling out these models seems to rely on the pure gravitational anomaly cancellation condition, while the F-theory obstacle seems to arise from purely local considerations. Note that the absence of SU(10) models with Λ^3 representations is also consistent with the results of [24].

3.3 Realization of the transitions

The free polynomials for SU(6), SU(7), and SU(8) models have ambiguous divisor classes, as parametrized by the divisors L and L' . One can imagine a process where L or L' is allowed to change while keeping the gauge group and codimension one singularity structure fixed. The matter content will change as a result; in SU(6), for instance, parameters such as α and β that control the number of antisymmetric multiplets will have different classes if L changes. In fact, these transformations are the F-theory realization of the transitions described previously in the supergravity context.

As an example, consider the SU(6) model on a $+n$ curve. Different values of r parametrize the space of models with different L . There is a process by which we can transition between two models with different r . First, let α , ν , and λ develop a common factor a :

$$\begin{aligned} \alpha &\rightarrow a\alpha', \\ \nu &\rightarrow a\nu', \\ \lambda &\rightarrow a\lambda'. \end{aligned} \tag{3.47}$$

This can be done in particular by following a continuous family of models parameterized by a variable $\hat{\varepsilon} > 0$, with the factorization occurring as $\hat{\varepsilon} \rightarrow 0$. At this point in the transition, f and g vanish to orders 4 and 6 wherever $a = \sigma = 0$, indicating that a is a superconformal point. Then, we can regroup a into β and ϕ_2 :

$$\begin{aligned} a\beta &\rightarrow \beta', \\ a\phi_2 &\rightarrow \phi_2'. \end{aligned} \tag{3.48}$$

Note that regrouping this factor does not involve any change in the Weierstrass model, it is simply a new labeling of the factors in the Weierstrass model that leaves the individual terms in f, g such as $\alpha^4\beta^4$ unchanged, since e.g. $\alpha\beta = a\alpha'\beta = \alpha'\beta'$. When a is regrouped in this way, the new β' and ϕ_2' share a common factor, while the theory is still at the superconformal point. But β' and ϕ_2' are free parameters that can now be varied to remove the common factor. A complex structure deformation that “absorbs” a into β' (i.e. deforms this coefficient so that it no longer factors) can be realized for example by following a continuous family of models parameterized by the variable $\hat{\varepsilon} < 0$, with the superconformal point at $\hat{\varepsilon} = 0$. Once the common factor is removed, the model no longer has a superconformal point, and the transition is complete. If a is a polynomial of degree 1, then r has increased by 1 during this transition. The process can be reversed as well so

that r decreases by one; we simply let β and ϕ_2 develop a common factor and absorb the factor into α , ν and λ . We thus see that the transition process can be physically realized by a one-parameter family of Weierstrass models, with a superconformal field theory at $\hat{\varepsilon} = 0$ and theories with $SU(6)$ gauge group and two distinct matter contents for $\hat{\varepsilon} > 0$ and $\hat{\varepsilon} < 0$.

The one-parameter nature of the transition can be seen more directly by writing expressions for the parameters in terms of $\hat{\varepsilon}$. To illustrate the first step in the transition, where α , ν and λ obtain common factors, we could write

$$\alpha = \begin{cases} a\alpha' + \alpha''\hat{\varepsilon} & \hat{\varepsilon} > 0 \\ \alpha' & \hat{\varepsilon} \leq 0 \end{cases}, \tag{3.49}$$

with similar expressions for ν and λ . Such expressions show that the factorization step of equation (3.47) involves moving along a path of models parameterized by $\hat{\varepsilon}$. Likewise, we could describe the side as $\hat{\varepsilon}$ approaches zero from a negative value by defining β' as

$$\beta' = \begin{cases} \beta & \hat{\varepsilon} > 0 \\ a\beta + \beta''\hat{\varepsilon} & \hat{\varepsilon} \leq 0 \end{cases}. \tag{3.50}$$

As expected, equations (3.49) and (3.50) both lead to a common superconformal point when $\hat{\varepsilon}$ is taken to 0. With these expressions, the change in multiplicities can be seen directly. Consider the term $\beta^2\nu^2\sigma^2$ in equation (3.16); it vanishes on the \square loci but not on the \square loci and, along with σ^3 terms in g , distinguishes between the two matter possibilities. From the $\hat{\varepsilon}$ expressions, the $\beta^2\nu^2\sigma^2$ term could be written as

$$(a\beta\nu' + \hat{\varepsilon}\beta\nu''\Theta(\hat{\varepsilon}) + \hat{\varepsilon}\beta''\nu'\Theta(-\hat{\varepsilon}))^2\sigma^2 \tag{3.51}$$

where

$$\Theta(\hat{\varepsilon}) = \begin{cases} 1 & \hat{\varepsilon} > 0 \\ 0 & \hat{\varepsilon} \leq 0 \end{cases}. \tag{3.52}$$

Notice that this term vanishes on the locus $a\beta + \hat{\varepsilon}\beta'' = \sigma = 0$ locus for $\hat{\varepsilon} < 0$ but only on the smaller locus $\beta = \sigma = 0$ when $\hat{\varepsilon} > 0$, indicating that the number of \square multiplets has changed during the transition. Other terms in f and g can be shown to have similar behavior, although for brevity we will not write out the full f and g in terms of $\hat{\varepsilon}$. Nevertheless, the transition can be explicitly described as a single-parameter path through a family of models. It should be noted that in the resolved geometry associated to the two sides of the transition we would see that two smooth CY 3-folds with distinct topology could be tuned in their complex structure moduli spaces (i.e., three-cycles collapsed) to share a common singular locus (involving the superconformal point). We will return to the notion of these matter transitions as topology changing transitions in later sections when they are realized in the dual heterotic theories.

We can track the matter participating in the transition through each of these steps. When α , ν , and λ lose a factor of a , one \square multiplet is lost because of the change in the order of α . Three of the complex degrees of freedom in α , ν and λ are traded for one degree of freedom in a , so two neutral multiplets are lost. Finally, the number of fundamentals

is determined by the discriminant. Prior to the appearance of the shared factor, Δ takes the form

$$\Delta = \alpha^4 \beta^3 \Delta_6 \sigma^6 + \mathcal{O}(\sigma^7), \tag{3.53}$$

where the order of Δ_6 corresponds to the number of fundamental multiplets in the model. At the transition point, this expression becomes

$$\Delta = a^6 \alpha^4 \beta^3 \Delta'_6 \sigma^6 + \mathcal{O}(\sigma^7). \tag{3.54}$$

Two of the factors of a have come from Δ_6 , indicating that moving to the transition point causes two \square multiplets to disappear. The first step in the transition can therefore be thought of as one \square multiplet, two \square multiplets, and two singlets combining to form superconformal matter represented by a . Importantly, there are total of 29 multiplets participating in the transition, reminiscent of the appearance of an extra tensor multiplet in the tensionless string transitions. Thus, if we imagine going off onto the tensor branch of the theory by blowing up at the superconformal (4, 6) point, all the matter fields at that point would be absorbed in the associated transition.

When a is subsequently reabsorbed into β and ϕ_2 , a new $\frac{1}{2}\square$ multiplet comes from the now enlarged β . The single degree of freedom in a is traded for two new degrees of freedom in β and ϕ_2 , signaling the appearance of a new singlet. And as the discriminant returns to its previous form, three factors of a are absorbed into β , while the remaining three factors are absorbed into Δ'_6 . Three fundamentals have therefore appeared in the reabsorption step. The transition can be summarized as

$$\square + 2 \times \square + 2 \times \mathbf{1} \rightarrow \text{Superconformal Matter} \rightarrow \frac{1}{2}\square + 3 \times \square + \mathbf{1}. \tag{3.55}$$

The net change in matter content is

$$\square + \mathbf{1} \rightarrow \frac{1}{2}\square + \square, \tag{3.56}$$

in exact agreement with the expected transition from supergravity. Note that the total number of multiplets is the same before and after the transition.

The SU(7) transition happens in a similar fashion. First, ξ , ζ_2 , λ_1 , and ψ_4 develop common factors:

$$\begin{aligned} \xi &\rightarrow a^3 \xi' \\ \zeta_2 &\rightarrow a^4 \zeta'_2 \\ \lambda_1 &\rightarrow a \lambda'_1 \\ \psi_4 &\rightarrow a^2 \psi'_4. \end{aligned}$$

a is once again a superconformal point where (f, g) vanish to order (4, 6). The common factor is then absorbed into δ and ζ_1 :

$$\begin{aligned} a\delta &\rightarrow \delta' \\ a\zeta_1 &\rightarrow \zeta'_1. \end{aligned}$$

We can once again track the matter in the transitions through a procedure similar to that of the SU(6) transition, although we will only summarize the results here. To produce a in the first step, $3 \square$ multiplets, 3 fundamentals, and 8 singlets disappear. Note that the redundant degrees of freedom need to be considered when counting singlets. a subsequently breaks into a \square multiplet, 8 fundamentals, and one singlet. The SU(7) transition is therefore

$$3 \times \square + 3 \times \square + 8 \times \mathbf{1} \rightarrow \text{Superconformal Matter} \rightarrow \square + 8 \times \square + \mathbf{1}, \quad (3.57)$$

with a net change in matter content of

$$3 \times \square + 7 \times \mathbf{1} \rightarrow \square + 5 \times \square. \quad (3.58)$$

The net matter change agrees exactly with the supergravity expectations. Just as with SU(6), the transition can occur in reverse as well.

A total of 92 multiplets participate in the transition, which is not a multiple of 29. While the transition does not explicitly require a blowup on the base, the general wisdom of superconformal points and tensionless string transitions would suggest that the multiplets in the transition should somehow fit into new tensor multiplets. To see the source of the mismatch, we can move to the transition point and resolve a using blow-ups on the base. The blow-ups are performed using the procedure of [2], but we will not go through the details of the blow-up process here. In the end, a total of three blow-ups are required to resolve a , leading to a situation illustrated in figure 1. One of the three exceptional curves carries an I_2 singularity, signaling the presence of a new SU(2) gauge algebra with 4 fundamentals. From anomaly considerations, the change in matter content due to the blowups should satisfy

$$\delta n_H - \delta n_V = -29\delta n_T. \quad (3.59)$$

87 multiplets are traded for the three tensor multiplets, while a net of 5 multiplets are needed to create the 4 fundamentals and 3 vector multiplets of the SU(2) gauge algebra. This adds up to a total of 92 multiplets, in exact agreement with (3.57). In the limit where the new exceptional curves shrink to zero size, any gauge groups on the exceptional divisors become strongly coupled. Hence, the transition point a for SU(7) should involve a superconformal field theory with three tensor multiplets and a strongly coupled SU(2) gauge symmetry.

Finally, we turn to the SU(8) transition. To convert \square matter to \square matter, we first let the following parameters obtain common factors:

$$\begin{aligned} \tau &\rightarrow a^5 \tau, \\ \zeta_3 &\rightarrow a^6 \zeta_3, \\ \omega_1 &\rightarrow a^2 \omega_1, \\ \phi_4 &\rightarrow a^6 \phi_4, \\ \psi_5 &\rightarrow a^2 \psi_5. \end{aligned}$$

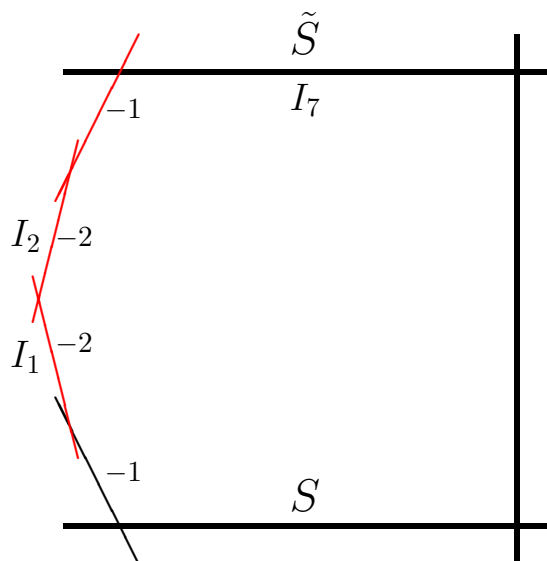


Figure 1. SU(7) transition point when blown up. Here, the compactification base was taken to be \mathbb{F}_n , while the original SU(7) gauge group was tuned on \tilde{S} . The blow-up procedure introduces three exceptional curves shown in red, one of which has an I_2 singularity. The I_2 singularity indicates there is a strongly coupled SU(2) at the transition point.

5 \square multiplets, 10 fundamentals, and 24 singlets disappear to form a . a is once again a superconformal point, but (f, g) vanish to orders (6, 8). a is then reabsorbed into δ , ζ_4 , and λ_2 :

$$\begin{aligned} a\delta &\rightarrow \delta \\ a^6\zeta_4 &\rightarrow \zeta_4 \\ a^2\lambda_2 &\rightarrow \lambda_2. \end{aligned}$$

Once δ , ζ_4 , and λ_2 are allowed to vary independently, a breaks into a \square multiplet, a \square multiplet, 19 fundamentals, and 8 singlets. The complete transition is therefore

$$5 \times \square + 10 \times \square + 24 \times \mathbf{1} \rightarrow \text{Superconformal Matter} \rightarrow \square + \square + 19 \times \square + 8 \times \mathbf{1}, \quad (3.60)$$

with a corresponding net matter change of

$$4 \times \square + 16 \times \mathbf{1} \rightarrow \square + 9 \times \square. \quad (3.61)$$

At the transition point, however, the codimension two singularity at $a = \sigma = 0$ does not seem to be resolvable even with blowups on the base. If one tries to perform the resolution, there will be a codimension one singularity along one of the exceptional curves where f and g vanish to orders 4 and 6. This result suggests that the SU(8) transition may not be valid. However, it is unclear whether our SU(8) tuning is completely general, and a different tuning may admit a resolvable transition point. We leave the question of whether the SU(8) transition is valid for future work.

But the SU(6) and SU(7) transitions do seem to be valid. The analysis here does not give a full description of the mechanism behind these transitions. Superconformal points seem to be key to the actual transition. Importantly, it does not seem necessary to have a strongly coupled gauge group associated with the superconformal point. While the SU(7) transition does include a strongly coupled SU(2), the SU(6) transition seems to not require any additional gauge group. A better understanding of the superconformal points could perhaps lead to a more complete picture of the transition mechanism.

3.4 Higgsing processes in F-theory

Here, we examine how the SU(6)-SU(8) F-theory models fit into the Higgsing structure discussed earlier in the supergravity context. In F-theory, Higgs transitions occur when the coefficients in the Weierstrass model are deformed from particular values, breaking a gauge symmetry in the process. Such deformations were recently explored in [33, 44]. We discuss the F-theory deformations that represent Higgsing processes where fundamental, \square , and \square multiplets obtain VEVs. We also examine how Higgsing affects the SU(6) – SU(8) matter transitions. In particular, some the Higgsed models will have transitions where the product-group representations change, as discussed previously in the supergravity section.

There will often be several parameters in the Weierstrass model that can be deformed to give the same Higgsing process. We can therefore associate each Higgsing process with a set of deformations, with each deformation corresponding to a particular degree of freedom in the Weierstrass model. Suppose that the number of deformations for a particular Higgsing process is n_D . From anomaly cancellation, we know that $n_H - n_V$ stays unchanged in the Higgsing process. This corresponds to the fact that Δn_V of Goldstone bosons are eaten by the gauge field, so from a $\Delta n_V + 1$ dimensional space of deformations related by gauge symmetry, only one remains as a deformation. The remaining $n_D - 1$ deformation directions arise when originally charged fields become singlets after the reduction in gauge group. In the cases considered here, this seems to occur in a similar way with a simple characterization. From the field theory perspective, Higgsing occurs when a certain number of multiplets in a particular representation R obtain expectation values. Let us say that the unHiggsed model has n_R multiplets in this representation and that n_{vev} of them need to obtain expectation values. For the Higgsing processes we consider in this paper, the number of deformations n_D seems to be related to n_R :

$$n_D = n_R - n_{vev} + 1. \tag{3.62}$$

Most of the Higgsing processes we examine will involve two multiplets obtaining VEVs, so that $n_{vev} = 2$. In these cases, we will find that there is one fewer deformation than the number of multiplets that can obtain VEVs. For most of the Higgsing processes in this paper, this holds since each singlet in the Higgsed model arises from a single non-Higgsed field in the representation R . But it is unclear if (3.62) should hold more generally, and we do not present a more general proof of this conjecture here.

3.4.1 Higgsing on fundamental matter

As described earlier, an $SU(N)$ symmetry can be broken to $SU(N - 1)$ by giving VEVs to two fundamental multiplets. Because the tuning process of section 3.2 proceeds through each $SU(N)$ algebra sequentially, the Higgsing deformations can be read off directly from the individual tuning steps.

The $SU(6)$ model can be Higgsed to $SU(5)$ through a set of possible deformations contained in $\tilde{\phi}_1, \tilde{\psi}_2, \tilde{f}_3$, and \tilde{g}_5 :

$$\phi_1 = (\beta + \tilde{\phi}_1)\nu \tag{3.63}$$

$$\psi_2 = -\frac{1}{3}(\alpha + \tilde{\psi}_2)\phi_2 \tag{3.64}$$

$$f_3 = -\frac{1}{3}\nu\phi_2 - 3(\beta + \tilde{f}_3)\lambda \tag{3.65}$$

$$g_5 = -\frac{1}{12}\alpha^2\beta^2 - \frac{1}{3}f_4\beta\nu + \lambda\phi_2 + \tilde{g}_5. \tag{3.66}$$

Each of the parameters in the four polynomials is an independent deformation that can be adjusted separately; as long as at least one parameter is non-zero, the $SU(6)$ gauge symmetry will be Higgsed to $SU(5)$. $\tilde{\phi}_1$ and \tilde{f}_3 are sections of $\mathcal{O}(L)$, $\tilde{\psi}_2$ is a section of $\mathcal{O}(-K + N - L)$, and \tilde{g}_5 is a section of $\mathcal{O}(-6K + N)$. When the $SU(6)$ is tuned on a $+n$ curve, there are a combined $2n + 15 + r$ complex degrees of freedom, one fewer than the number of fundamental multiplets in the $SU(6)$ model. This fits with the expectation that (3.62) is satisfied for the $SU(6) \rightarrow SU(5)$ Higgsing process. Note that in 4D, this set of deformations is not in general complete since, for example, quantities such as ϕ_1 and ψ_2 need not factorize.

From the branching patterns, all of the possible $SU(6)$ models on a $+n$ curve should give the same $SU(5)$ charged matter content when Higgsed. To see whether this is true in F-theory, we can plug (3.63)–(3.66) into the $SU(5)$ model of (3.7) and (3.8) (while setting $\phi_0 = \alpha\beta$) and examine the resulting matter. We can consider a maximally-deformed model in which all of the deformations are turned on. In this case, f and g vanish to orders 2 and 3 when either $\alpha = \sigma = 0$ or when $\beta = \sigma = 0$, so both α and β will give **10** matter in the resulting $SU(5)$ model. The discriminant meanwhile takes the form

$$\Delta = \alpha^4\beta^4\Delta_5\sigma^5 + \mathcal{O}(\sigma^6). \tag{3.67}$$

The $SU(5)$ model thus has a total of $n + 2$ **10** multiplets and $3n + 16$ fundamental multiplets regardless of the initial $SU(6)$ model, indicating all of the $SU(6)$ models have Higgsed to the same $SU(5)$ model. If some of the deformations are turned off, the resulting $SU(5)$ model may have codimension-two loci with rank-two singularity enhancements. These loci may contribute more than one multiplet of charged matter, making the analysis more involved. While we do not go through all of the possible situations, all of these specialized situations should give the same charged matter content.

The $SU(7) \rightarrow SU(6)$ Higgsing pattern has a more interesting structure, as a given $SU(7)$ model can Higgs only to a particular $SU(6)$ model. Moreover, only a subset of the

SU(6) models can be reached by Higgsing an SU(7) model, as discussed previously. In F-theory, the corresponding deformations are

$$\beta = (\delta + \tilde{\beta})\delta, \tag{3.68}$$

$$\alpha = (\delta + \tilde{\alpha})\xi, \tag{3.69}$$

$$\phi_2 = 3\zeta_1^2 + (\delta^2 + \tilde{\phi}_2)\omega, \tag{3.70}$$

$$\lambda = \frac{1}{3}\zeta_1^2\zeta_2 - \frac{1}{18}\zeta_1\xi + \frac{1}{9}\delta^2\zeta_2\omega + (\xi\delta^2 + \tilde{\lambda})\lambda_1, \tag{3.71}$$

$$f_4 = -6\zeta_1\lambda_1 - \frac{1}{12}\omega^2 + (\delta^2 + \tilde{f}_4)\psi_4, \tag{3.72}$$

along with a deformation of g_6 by adding \tilde{g}_6 to (3.28). The resulting set of deformations has $n + 16 + 5r$ complex degrees of freedom, one fewer than the number of **7** multiplets for SU(7). Note that there is no deformation associated with the ν redefinition (3.24). The ν redefinition does not seem to remove degrees of freedom like the other tuning steps; instead, the step simply reorganizes ν to have a particular structure. A ν deformation could be included, but we would then need to account for the redundant degrees of freedom described earlier in the counting. Once the redundant degrees of freedom are subtracted off, the total number of deformations is the same as before. We therefore do not include a ν deformation in the above set of deformations.

ξ will give **15** matter in the SU(6) model, as (f, g) vanish to orders (2, 3) on $\xi = \sigma = 0$ regardless of which deformations are turned on. When considering the maximally deformed situation, the loci where $\delta + \tilde{\alpha} = \sigma = 0$ also contribute **15** matter. Finally, the singularity type enhances to an incompletely resolved E_6 when $\delta = \sigma = 0$ or when $\delta + \tilde{\beta} = \sigma = 0$, giving $\frac{1}{2}\mathbf{20}$ matter. In total, the resulting SU(6) model has $2r \frac{1}{2}\mathbf{20}$ multiplets, $n + 2 - 2r$ **15** multiplets, and $16 + 2n + 2r$ fundamentals, so each SU(7) model is Higgsed to the SU(6) model with $r_6 = 2r_7$.

Higgsing relates the SU(6) and SU(7) transitions in a non-trivial way. Directly applying the SU(7) \rightarrow SU(6) branching patterns to the SU(7) transition (2.10) would seem to imply that the SU(6) theory undergoes minimal transitions of the form

$$\mathbf{20} \left(\begin{array}{c} \square \\ \square \end{array} \right) + 2 \times \mathbf{6} (\square) \leftrightarrow 2 \times \mathbf{15} \left(\begin{array}{c} \square \\ \square \end{array} \right) + 2 \times \mathbf{1}. \tag{3.73}$$

However, equation (2.9) and the F-theory SU(6) model both suggest the minimal SU(6) transition should involve half this amount of matter. This discrepancy reflects the fact mentioned in sections 2.3.1 and 3.2.2 that only SU(6) models with an even number of $\frac{1}{2}\mathbf{20}$ multiplets can be unHiggsed to SU(7). Applying the transition (2.9) only once could produce an SU(6) spectrum with an odd number $\frac{1}{2}\mathbf{20}$ multiplets that cannot be unHiggsed to SU(7). Thus, (2.9) should not be directly visible from the SU(7) model. Instead, (2.9) must be applied twice to move between two SU(6) vacua that can be enhanced to SU(7), as reflected in 3.73. Said differently, the smallest SU(7) transition changes r_7 by 1, which becomes a $\Delta r_6 = 2$ change in the resulting SU(6) model. The heterotic analogue of this phenomenon will be discussed in section 4.4.

For the $SU(8) \rightarrow SU(7)$ Higgsing process, the corresponding deformations are

$$\xi = (\delta^2 + \tilde{\xi})\tau, \tag{3.74a}$$

$$\zeta_2 = \frac{1}{2}\zeta_4\tau^2 + (\delta^2 + \tilde{\zeta}_2)\zeta_3, \tag{3.74b}$$

$$\omega = 4\zeta_3\zeta_4 + (\delta^2 + \tilde{\omega})\omega_1, \tag{3.74c}$$

$$\lambda_1 = -\frac{1}{6}\zeta_4\tau\omega_1 + (\tau\delta^2 + \tilde{\lambda}_1)\lambda_2, \tag{3.74d}$$

$$\psi_4 = -3\zeta_4\lambda_2 - \frac{1}{4}\omega_1^2\delta^2 - 6\zeta_3\lambda_2\delta^2 + (\delta^4 + \tilde{\psi}_4)\phi_4, \tag{3.74e}$$

$$f_5 = 2\zeta_4\phi_4 + (\delta^2 + \tilde{f}_5)\psi_5, \tag{3.74f}$$

along with a deformation of g_7 by adding \tilde{g}_7 to equation (3.43). This leads to a total of $15 + 9r$ possible deformations, which is one fewer than the number of **8** multiplets. Modifications to the ζ_1 redefinition were not considered due to the redundant degrees of freedom in the tuning. In the Higgsed model, there are two codimension two loci, $\tau = \sigma = 0$ and $\delta^2 + \tilde{\xi} = \sigma = 0$, where the singularity type enhances to D_7 ; these loci give **21** matter. Loci where $\delta = \sigma = 0$ have an enhanced E_7 singularity and contribute one **35** multiplet and one **7** multiplet. The Higgsed model therefore has r **35** multiplets, $n + 2 - 3r$ **21** multiplets, and $n + 2 + 5r$ fundamental multiplets, indicating that each $SU(8)$ model is Higgsed to the $SU(7)$ model with $r_7 = r_8$. Once again, the restriction that $5r \leq n + 2$ for $SU(8)$ means that only some $SU(7)$ models can be reached by Higgsing an $SU(8)$ model.

3.4.2 Higgsing on two-index antisymmetric matter

From field theory, an $SU(N)$ gauge symmetry can be broken to $SU(N - 2) \times SU(2)$ by giving expectation values to two \square multiplets. For $N = 4$ and $N = 5$, such Higgsing processes can be realized in F-theory by tuning an $SU(N - 2)$ gauge symmetry on $\sigma = 0$ and tuning an $SU(2)$ gauge symmetry on $\sigma - \epsilon = 0$. Here, ϵ , like σ , is a section of the line bundle $\mathcal{O}(D)$ on B . When considering a compactification base \mathbb{F}_n with the coordinate σ associated with \tilde{S} , ϵ will be a polynomial in the coordinate associated with F of order n ; the rest of the discussion in this F-theory Higgsing section will focus mostly on this particular setup. Note that in this discussion we treat all the coefficients in the expansion of f, g as sections of line bundles over B and do not pull them back to a given divisor. This may be subtle in general circumstances of is clear in the toric context at least where all these coefficients can be expanded in a local coordinate system. In the situation just described, the discriminant will then be proportional to $\sigma^{N-2}(\sigma - \epsilon)^2$. In the limit where ϵ goes to zero, the discriminant becomes proportional to σ^N , unHiggsing the gauge symmetry to $SU(N)$. ϵ therefore parametrizes the set of possible deformations corresponding to this Higgsing process. Physically, the Higgsing process occurs by separating the stack of coincident branes forming the $SU(N)$ singularity into two distinct sets, with ϵ representing the separation between the two new sets.

Identifying the deformations requires tuning singularities on $\sigma = 0$ and $\sigma = \epsilon$ simultaneously, which is difficult when f and g are expanded in σ alone. The tuning process is

easier when f and g are expanded in $\sigma(\sigma - \epsilon)$. Specifically, we write

$$f = F_0 + F_1\sigma(\sigma - \epsilon) + F_2\sigma^2(\sigma - \epsilon)^2 + \dots \tag{3.75}$$

$$g = G_0 + G_1\sigma(\sigma - \epsilon) + G_1\sigma^2(\sigma - \epsilon)^2 + \dots, \tag{3.76}$$

where

$$F_i = f_{2i} + f_{2i+1}\sigma \tag{3.77}$$

$$G_i = g_{2i} + g_{2i+1}\sigma. \tag{3.78}$$

With this expansion, the $SU(N - 2)$ and $SU(2)$ symmetries can be tuned through a process similar to that of tuning $SU(N)$ on σ . Some terms of the $SU(N)$ tuning process will obtain modifications proportional to ϵ when tuning $SU(N - 2) \times SU(2)$. These additional terms represent the Higgsing deformations.

ϵ is the only free parameter in the $SU(N - 2) \times SU(2)$ not present in the unHiggsed $SU(N)$ model. As an order n polynomial, ϵ has $n + 1$ degrees of freedom, whereas the unHiggsed $SU(4)$ and $SU(5)$ models both have $n + 2$ \square multiplets. Once again, the number of deformation parameters is one fewer than the number of matter multiplets that can obtain VEVs for this Higgsing pattern. Since ϵ will be ineffective unless $n \geq 0$, the unHiggsed $SU(4)$ or $SU(5)$ model must have at least two \square multiplets. Moreover, the n zeroes of ϵ are the only source of bifundamental matter in the $SU(N - 2) \times SU(2)$ model, reflecting the fact that two bifundamental multiplets are eaten during Higgsing.

A similar story holds when Higgsing $SU(6)$ to $SU(4) \times SU(2)$. As noted earlier, the $SU(4) \times SU(2)$ model itself has a transition where the product-group representations are allowed to change. Such transitions can be achieved in F-theory by having ϵ depend on other parameters in the Weierstrass model. We tune $SU(4)$ on $\sigma = 0$ and tune $SU(2)$ on $\sigma - \beta\epsilon_1 = 0$. The overall tuning process is similar to that of $SU(6)$, except that some of the tuned parameters may obtain additional terms proportional to ϵ_1 . When ϵ_1 is taken to zero, we recover the general $SU(6)$ model; ϵ_1 therefore represents the set of deformations corresponding to the Higgsing process. Note that ϵ_1 is a polynomial of order $n - r$ in our standard \mathbb{F}_n . There are $n + 2 - r$ $\mathbf{15}$ multiplets in the $SU(6)$ model, so we once again have a number of deformations that is one fewer than the number of Higgsable multiplets. As before, the original $SU(6)$ model must have at least 2 $\mathbf{15}$ multiplets for ϵ_1 to be effective and for the $SU(6)$ to be Higgsable.

The zeroes of ϵ_1 contribute bifundamental $(\mathbf{4}, \mathbf{2})$ matter, while the zeroes of α give $(\mathbf{6}, \mathbf{1})$ matter. When $\beta = 0$, the singularity type enhances to D_5 , indicating every zero of β contributes a half multiplet of $(\mathbf{6}, \mathbf{2})$ matter. The total matter content from the F-theory model agrees exactly with the spectra derived from the $SU(6)$ branching patterns. Once again, the fact that ϵ_1 is of order $n - r$ indicates that two bifundamental multiplets are eaten in the Higgsing process. The transition between product-group representations occurs in a

similar fashion as the SU(6) transition. First, α , ϵ_1 , ν and λ obtain common factors:

$$\begin{aligned}\alpha &\rightarrow a\alpha', \\ \epsilon_1 &\rightarrow a\epsilon'_1, \\ \nu &\rightarrow a\nu', \\ \lambda &\rightarrow a\lambda'.\end{aligned}$$

The common factor a is once again a superconformal point; (f, g) vanish to order $(4, 6)$ on the locus $a = \sigma = 0$. a is then absorbed into β and ϕ_2 as before:

$$\begin{aligned}a\beta &\rightarrow \beta', \\ a\phi_2 &\rightarrow \phi'_2.\end{aligned}$$

The transition can be summarized as

$$(\mathbf{6}, \mathbf{1}) + (\mathbf{4}, \mathbf{2}) + 2 \times (\mathbf{4}, \mathbf{1}) + 2 \times (\mathbf{1}, \mathbf{2}) + 3 \times (\mathbf{1}, \mathbf{1}) \rightarrow \frac{1}{2} \times (\mathbf{6}, \mathbf{2}) + 4 \times (\mathbf{4}, \mathbf{1}) + 3 \times (\mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}), \quad (3.79)$$

which is the Higgsed version of (3.55). Just as in the SU(6) transition, there are a total of 29 multiplets participating in the transition. The net change in matter content is

$$(\mathbf{6}, \mathbf{1}) + (\mathbf{4}, \mathbf{2}) + 2 \times (\mathbf{1}, \mathbf{1}) \rightarrow \frac{1}{2} \times (\mathbf{6}, \mathbf{2}) + 2 \times (\mathbf{4}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}), \quad (3.80)$$

as expected from supergravity. The transition can be reversed as well by inverting the steps.

To realize the SU(7) \rightarrow SU(5) \times SU(2) Higgsing process, we tune the SU(5) symmetry on $\sigma = 0$ and the SU(2) symmetry on $\sigma - \delta^3 \epsilon_2 = 0$. While β is still defined to be δ^2 , α is redefined as

$$\alpha = \xi\delta - \zeta_1 \epsilon_2. \quad (3.81)$$

The rest of the tuning process is similar to that of SU(7), but the parameter redefinitions may have additional terms proportional to ϵ_2 . ϵ_2 is of order $n - 3r$; again, there is one fewer deformation parameter than the number of $\mathbf{21}$'s of SU(7). Taking the $\epsilon_2 \rightarrow 0$ limit gives the SU(7) tuning, and the zeroes ϵ_2 therefore contribute bifundamental $(\mathbf{5}, \mathbf{2})$ matter. $(\mathbf{10}, \mathbf{1})$ matter comes from the codimension two locus where $\xi\delta - \zeta_1 \epsilon_2 = \sigma = 0$ with $\delta \neq 0$, so there are a total of $n + 2 - 3r$ $(\mathbf{10}, \mathbf{1})$ multiplets. Finally, the singularity type enhances to E_6 on the $\delta = \sigma = 0$ loci; each zero of δ therefore contributes a $(\mathbf{10}, \mathbf{2})$ multiplet and a $(\mathbf{5}, \mathbf{1})$ multiplet. The resulting charged matter content agrees exactly with that from field theory considerations.

The SU(5) \times SU(2) model inherits the SU(7) transition. The steps in the transition are the same as those for the SU(7) transition, only ϵ_2 develops the common factor a along with ξ , ζ_2 , λ_1 , and ψ_4 :

$$\epsilon_2 \rightarrow a^3 \epsilon'_2. \quad (3.82)$$

The complete transition is therefore

$$\begin{aligned}3 \times (\mathbf{10}, \mathbf{1}) + 3 \times (\mathbf{5}, \mathbf{2}) + 3 \times (\mathbf{5}, \mathbf{1}) + 3 \times (\mathbf{2}, \mathbf{1}) + 11 \times (\mathbf{1}, \mathbf{1}) \\ \rightarrow (\mathbf{10}, \mathbf{2}) + (\mathbf{10}, \mathbf{1}) + 9 \times (\mathbf{5}, \mathbf{1}) + 8 \times (\mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}),\end{aligned} \quad (3.83)$$

which is the Higgsed version of the SU(7) transition. The corresponding net change in matter content is

$$2 \times (\mathbf{10}, \mathbf{1}) + 3 \times (\mathbf{5}, \mathbf{2}) + 10 \times (\mathbf{1}, \mathbf{1}) \rightarrow (\mathbf{10}, \mathbf{2}) + 6 \times (\mathbf{5}, \mathbf{1}) + 5 \times (\mathbf{1}, \mathbf{2}). \quad (3.84)$$

For SU(8) \rightarrow SU(6) \times SU(2), we enhance the symmetry tuned on $\sigma = 0$ to SU(6) while having the SU(2) occur on $\sigma - \delta^4 \epsilon_3 = 0$. ξ is redefined to be

$$\xi = \delta^2 - \zeta_1 \epsilon_3, \quad (3.85)$$

while some of the other SU(8) parameter redefinitions get additional terms proportional to ϵ_3 . ϵ_3 contains the set of deformations corresponding to the Higgsing process, and taking ϵ_3 to zero recovers the SU(8) model.

ϵ_3 contributes bifundamental $(\mathbf{6}, \mathbf{2})$ matter, and τ gives $(\mathbf{15}, \mathbf{1})$ matter. When $\delta = \sigma = 0$, the singularity type enhances to E_7 , so each zero of δ gives a $(\mathbf{15}, \mathbf{2})$ multiplet, a $(\mathbf{15}, \mathbf{1})$ multiplet, and a $(\mathbf{1}, \mathbf{2})$ multiplet. Finally, the singularity type enhances to E_6 on the locus $\delta^2 - 2\epsilon_3 \zeta_4$, contributing r $(\mathbf{20}, \mathbf{1})$ multiplets. There does not seem to be a locus that gives $(\mathbf{20}, \mathbf{2})$ matter, which can only come from $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$ matter in the SU(8) model. This is further evidence that our SU(8) tuning does not have 4-index antisymmetric matter.

The SU(6) \times SU(2) model has a transition similar to the SU(8) transition. The only difference is that ϵ_3 participates in the first step along with τ , shedding four factors of a :

$$\epsilon_3 \rightarrow a^4 \epsilon_3. \quad (3.86)$$

In terms of the matter content, the transition is

$$\begin{aligned} &5 \times (\mathbf{15}, \mathbf{1}) + 4 \times (\mathbf{6}, \mathbf{2}) + 10 \times (\mathbf{6}, \mathbf{1}) + 10 \times (\mathbf{1}, \mathbf{2}) + 28 \times (\mathbf{1}, \mathbf{1}) \\ &\rightarrow (\mathbf{20}, \mathbf{1}) + (\mathbf{15}, \mathbf{2}) + (\mathbf{15}, \mathbf{1}) + 20 \times (\mathbf{6}, \mathbf{1}) + 19 \times (\mathbf{1}, \mathbf{2}) + 8 \times (\mathbf{1}, \mathbf{1}). \end{aligned} \quad (3.87)$$

This is not the Higgsed version of the SU(8) transition, as there is a missing $(\mathbf{6}, \mathbf{2})$ multiplet in the transition. However, just as with the SU(8) transition, the transition point does not seem to be resolvable even with blow-ups on the base.

3.4.3 Higgsing on three-index antisymmetric matter

We can find deformations for the SU(N) \rightarrow SU(N - 3) \times SU(3) Higgsing process using a similar strategy; we tune an SU(N - 3) symmetry on $\sigma = 0$ and an SU(3) symmetry on $\sigma - \epsilon = 0$. As in the $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$ Higgsing process, ϵ will take particular forms when breaking SU(6) through SU(8) in order to accommodate the matter transitions. Because the $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$ representation appears only at SU(6) and above, the $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$ Higgsing processes will exclusively involve situations where ϵ has a specialized form.

To find the deformations that break SU(6) to SU(3) \times SU(3), we tune one SU(3) algebra on $\sigma = 0$ and the other on $\sigma - \alpha \epsilon_1 = 0$. ϵ_1 is a polynomial of order $r - 2$. Performing this tuning requires a modified redefinition of ϕ_0 :

$$\phi_0 = \alpha \beta - \epsilon_1 \nu.$$

With this redefinition, the split condition is satisfied for both of the codimension-one singularities. The other steps of this tuning are similar to those of the $SU(6)$ tuning, but some parameter redefinitions may involve additional terms proportional to ϵ_1 . The new terms dependent on ϵ_1 are the Higgsing deformations, and ϵ_1 thus parametrizes the possible deformations. As expected, there is one fewer deformation than the number of $\frac{1}{2}\mathbf{20}$ multiplets in the $SU(6)$ model. Moreover, there must be at least two $\frac{1}{2}\mathbf{20}$ multiplets for ϵ_1 to be effective and for the deformations to be possible.

α and ϵ_1 both contribute bifundamental matter in the $SU(3) \times SU(3)$ model, while β does not contribute any matter. However, there are two ways to form bifundamental matter in the $SU(3) \times SU(3)$ model, as described in [5]; in half-hypermultiplets, bifundamental matter can be in the form $(\mathbf{3}, \mathbf{3}) + (\mathbf{3}, \mathbf{3})$ or in the form $(\mathbf{3}, \mathbf{3}) + (\mathbf{3}, \bar{\mathbf{3}})$. From the field theory perspective, $(\mathbf{3}, \bar{\mathbf{3}}) + (\bar{\mathbf{3}}, \mathbf{3})$ matter should come from the $\frac{1}{2}\mathbf{20}$ multiplets of the $SU(6)$ model, whereas the $(\mathbf{3}, \mathbf{3}) + (\bar{\mathbf{3}}, \bar{\mathbf{3}})$ matter should come from the $\mathbf{15}$ multiplets. For this reason, ϵ_1 , which represents bifundamentals originating from $\mathbf{10}$ matter, should give $(\mathbf{3}, \bar{\mathbf{3}}) + (\mathbf{3}, \bar{\mathbf{3}})$ bifundamental matter (which we will refer to as a $(\mathbf{3}, \bar{\mathbf{3}})$ full multiplet). α meanwhile should contribute $(\mathbf{3}, \mathbf{3}) + (\bar{\mathbf{3}}, \bar{\mathbf{3}})$ matter (or a full $(\mathbf{3}, \mathbf{3})$ multiplet). These two realizations are physically indistinguishable in the $SU(3) \times SU(3)$ model, so the distinction is somewhat arbitrary. However, similar types of distinctions will be important in tunings considered later.

The $SU(3) \times SU(3)$ model has a transition, although the transition does not have as interesting a change in the representations. α , ν , and λ obtain a common factor a just as in the $SU(6)$ transition, while β , ϕ_2 and ϵ_1 each absorb one factor of a . a is once again a superconformal point. The transition is therefore

$$(\mathbf{3}, \mathbf{3}) + 3 \times (\mathbf{3}, \mathbf{1}) + 3 \times (\mathbf{1}, \mathbf{3}) + 2 \times (\mathbf{1}, \mathbf{1}) \rightarrow (\mathbf{3}, \bar{\mathbf{3}}) + 3 \times (\mathbf{3}, \mathbf{1}) + 3 \times (\mathbf{1}, \mathbf{3}) + 2 \times (\mathbf{1}, \mathbf{1}), \quad (3.88)$$

which is the Higgsed version of the $SU(6)$ transition. The only net effect of this transition is to exchange the $(\mathbf{3}, \mathbf{3})$ and $(\mathbf{3}, \bar{\mathbf{3}})$ representations. Since the two bifundamental representations are essentially equivalent, the transition may not seem to be as interesting as the other transitions. Nevertheless, the structure of the transition is the same as that of the other transitions, and the $SU(3) \times SU(3)$ transition will be important when analyzing transitions in other models.

Enhancing the $SU(3)$ algebra on σ to $SU(4)$ allows us to find the $SU(7) \rightarrow SU(4) \times SU(3)$ Higgsing deformations. The $SU(3)$ singularity, which was previously on $\sigma - \alpha\epsilon_1 = 0$, now occurs on the locus $\sigma - \xi\delta^2\epsilon_2 = 0$. β is no longer forced to be a perfect square, as it is redefined as

$$\beta = \delta^2 + \epsilon_2\zeta_1. \quad (3.89)$$

The rest of the $SU(4) \times SU(3)$ tuning process is similar to that of $SU(7)$, except that the ϕ_2 and g_6 redefinitions of equations (3.25) and (3.28) obtain additional terms proportional to ϵ_2 . ϵ_2 , a polynomial of order $r - 2$, parametrizes the set of deformations.

There is a matter-changing transition in this $SU(4) \times SU(3)$ model. The steps in the transition are nearly identical to the $SU(7)$ transition, only ϵ_2 absorbs a single factor of a along with δ and ζ_1 . Both ϵ_2 and ξ contribute bifundamentals, while each zero of δ

gives a $(\mathbf{6}, \mathbf{3})$ multiplet and a $(\mathbf{1}, \mathbf{3})$. On the locus $\xi - \epsilon_2 \zeta_2 = \sigma = 0$, the singularity type enhances to $D_4 \times A_2$, giving $(\mathbf{6}, \mathbf{1})$ matter. Including fundamentals and singlets, the matter transition is

$$\begin{aligned} & 3 \times (\mathbf{6}, \mathbf{1}) + 3 \times (\mathbf{4}, \mathbf{3}) + 3 \times (\mathbf{4}, \mathbf{1}) + 6 \times (\mathbf{1}, \mathbf{3}) + 8 \times (\mathbf{1}, \mathbf{1}) \\ & \rightarrow (\mathbf{6}, \mathbf{3}) + (\mathbf{4}, \mathbf{3}) + 9 \times (\mathbf{4}, \mathbf{1}) + 8 \times (\mathbf{1}, \mathbf{3}) + 2 \times (\mathbf{1}, \mathbf{1}), \end{aligned} \quad (3.90)$$

leading to a net matter change of

$$3 \times (\mathbf{6}, \mathbf{1}) + 2 \times (\mathbf{4}, \mathbf{3}) + 6 \times (\mathbf{1}, \mathbf{1}) \rightarrow (\mathbf{6}, \mathbf{3}) + 6 \times (\mathbf{4}, \mathbf{1}) + 2 \times (\mathbf{1}, \mathbf{3}). \quad (3.91)$$

This transition is the Higgsed version of the $SU(7)$ transition.

For the $SU(8) \rightarrow SU(5) \times SU(3)$ Higgsing process, we tune the $SU(5)$ symmetry on σ and the $SU(3)$ symmetry on $\sigma - \delta^4 \tau \epsilon_3$. While we are able to find an explicit F-theory realization of this Higgsing process, there are tight constraints on the possible $SU(8)$ models that can be Higgsed on $\mathbf{56}$ matter. For ϵ_3 to be effective and for this Higgsing process to be possible, the original $SU(8)$ model must have at least 2 $\mathbf{56}$ multiplets. From anomaly cancellations alone, this requires $n \geq 6$; our F-theory tunings support two $\mathbf{56}$ multiplets only when $n \geq 8$. All of these models oversaturate the gravitational anomaly bound on their own, and it is necessary to include a second gauge symmetry. For instance, when the singularity is tuned on \tilde{S} with compactification base \mathbb{F}_n , a non-Higgsable cluster on S can allow the global model to satisfy the gravitational anomaly bound. In F-theory, we are able to find the deformations for $SU(8)$ models on \tilde{S} with explicit tunings given earlier. These models have $n \geq 8$ and only two $\mathbf{56}$ multiplets. The resulting $SU(5) \times SU(3)$ matter spectrum agrees exactly with that expected from the branching patterns. However, we cannot realize $SU(8)$ models with more than three $\mathbf{56}$ multiplets on \tilde{S} using our constructions, as our F-theory tunings will not support three $\mathbf{56}$ multiplets for $n \leq 12$. We therefore cannot see $SU(5) \times SU(3)$ transitions in the \tilde{S} tunings.

4 Heterotic description of matter transitions

In this section we describe the $SU(N)$ theories with exotic matter, as well as the Higgsing and small instanton transitions of interest in this paper in terms of the bundle geometry on the heterotic side of the duality. We begin in the next subsection by considering the construction of $SU(N)$ theories where $N = 6, 7, 8$. In subsection 4.2 we describe the small instanton transitions that are possible in such theories. Finally, in subsection 4.3, we describe how the Higgsing processes are realized at the level of bundle geometry. Some of the results presented here are, of course, not new and can be found in [14], for example. We find it useful to present them again here, however, in the same language that we use to describe previously unstudied cases.

4.1 SU(N) theories in heterotic compactifications

4.1.1 SU(6)

The relevant group theory for studying a compactification of the $E_8 \times E_8$ heterotic string with an SU(6) gauge group in six dimensions is as follows.

$$\begin{aligned}
 E_8 &\supset \text{SU}(6) \times \text{SU}(3) \times \text{SU}(2) & (4.1) \\
 \mathbf{248} &= (\mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{8}, \mathbf{1}) + (\mathbf{35}, \mathbf{1}, \mathbf{1}) + (\overline{\mathbf{15}}, \mathbf{3}, \mathbf{1}) + (\mathbf{15}, \overline{\mathbf{3}}, \mathbf{1}) + (\mathbf{6}, \mathbf{3}, \mathbf{2}) + (\overline{\mathbf{6}}, \overline{\mathbf{3}}, \mathbf{2}) + (\mathbf{20}, \mathbf{1}, \mathbf{2})
 \end{aligned}$$

From this we see that we need a bundle with structure group $\text{SU}(3) \times \text{SU}(2)$ to obtain an unbroken gauge group of SU(6). We take the gauge bundle to be

$$\mathcal{V} = \mathcal{V}_3 \oplus \mathcal{V}_2 \tag{4.2}$$

where \mathcal{V}_3 has structure group SU(3) and \mathcal{V}_2 has structure group SU(2).

The matter content resulting from a sum of vector bundles such as (4.2) can be computed in terms of the first cohomology groups of combinations of those objects and their wedge powers. This can either be seen in terms of dimensional reduction of gaugino degrees of freedom in ten dimensions to give fermionic matter [45], or in terms of dimensional reduction of bosonic degrees of freedom. The bosonic components of the low-energy theory all descend from adjoint valued gauge fields in ten dimensions. We consider first cohomologies because these are associated with one forms - which can be used to account for the space-time index of the gauge field leading to scalar degrees of freedom in six dimensions. The particular combination of bundles that one considers is determined by the decomposition (4.1). For example, we can see from the final term in the second line of (4.1) that if we want to obtain matter transforming in the **20** representation of SU(6), then the relevant one forms in the dimensional reduction must carry an index in the fundamental of SU(2), leading us to consider $H^1(\mathcal{V}_2)$.

The different first cohomology groups of interest can be computed, in the case of K3 compactifications, by using the Hirzebruch-Riemann-Roch theorem [46]. Using the fact that bundles in a heterotic compactification are required to be slope stable, the statement of this theorem can be reduced to the following formula for the dimension of the first cohomology of a bundle \mathcal{F} in such a situation.

$$-h^1(\mathcal{F}) = \int_{K3} \text{ch}(\mathcal{F}) \wedge \left(1 + \frac{c_2(K3)}{12} \right) \tag{4.3}$$

Applying this formula to the bundles that are relevant given the decomposition (4.1) gives rise to the results given in table 7.

In the SU(6) case we are considering, the numbers of vector and tensor multiplets are $n_V = 35$ and $n_T = 1$ respectively. The number of half-hypermultiplets associated to the other (“hidden sector”) E_8 bundle is $h^1(\text{End}_0 \mathcal{V}_{E_8}) = c_2(\text{End}_0 \mathcal{V}_{E_8}) - 496$. We also have an additional 20 hypermultiplets from the metric moduli of K3. This information, together with the matter content given in table 7 can be substituted into the six-dimensional

Representation	Cohomology	Multiplicity
15	$H^1(\mathcal{V}_3^\vee)$	$c_2(\mathcal{V}_3) - 6$
$\overline{\mathbf{15}}$	$H^1(\mathcal{V}_3)$	$c_2(\mathcal{V}_3) - 6$
6	$H^1(\mathcal{V}_3 \otimes \mathcal{V}_2)$	$2c_2(\mathcal{V}_3) + 3c_2(\mathcal{V}_2) - 12$
$\overline{\mathbf{6}}$	$H^1(\mathcal{V}_3^\vee \otimes \mathcal{V}_2)$	$2c_2(\mathcal{V}_3) + 3c_2(\mathcal{V}_2) - 12$
20	$H^1(\mathcal{V}_2)$	$c_2(\mathcal{V}_2) - 4$
1	$H^1(\text{End}_0(\mathcal{V}_3)) \oplus H^1(\text{End}_0(\mathcal{V}_2))$	$(4c_2(\mathcal{V}_2) - 6) + (6c_2(\mathcal{V}_3) - 16)$

Table 7. The cohomology and multiplicity associated to each representation of the low-energy gauge group $SU(6)$.

anomaly cancelation condition to give the following.

$$n_H + 29n_T - n_V = 273 \tag{4.4}$$

$$\Rightarrow c_2(\mathcal{V}_3) + c_2(\mathcal{V}_2) + \frac{1}{60}c_2(\text{End}_0(\mathcal{V}_{E_8})) = 24 \tag{4.5}$$

This is precisely the ten-dimensional anomaly cancelation condition as expected.

The matter content outlined in table 7 takes the form described in equation (2.14), where we have

$$n = c_2(\mathcal{V}_3) + c_2(\mathcal{V}_2) - 12 \text{ and } r = c_2(\mathcal{V}_2) - 4. \tag{4.6}$$

4.1.2 $SU(7)$

The relevant group theory in this case is

$$E_8 \supset SU(7) \times SU(2) \times U(1) \tag{4.7}$$

$$\begin{aligned} \mathbf{248} = & (\mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3})_0 + (\mathbf{7}, \mathbf{2})_9 + (\overline{\mathbf{7}}, \mathbf{2})_{-9} + (\mathbf{48}, \mathbf{1})_0 + (\mathbf{7}, \mathbf{1})_{-12} + (\mathbf{21}, \mathbf{2})_{-3} + (\mathbf{35}, \mathbf{1})_6 \\ & + (\overline{\mathbf{7}}, \mathbf{1})_{12} + (\overline{\mathbf{21}}, \mathbf{2})_3 + (\overline{\mathbf{35}}, \mathbf{1})_{-6}. \end{aligned} \tag{4.8}$$

To obtain a bundle with structure group $SU(2) \times U(1) \cong S(U(2) \times U(1))$ embedded inside E_8 , we take the gauge bundle to be

$$\mathcal{V} = \mathcal{V}_2 \oplus \mathcal{L}. \tag{4.9}$$

Here \mathcal{V}_2 is a $U(2)$ bundle, \mathcal{L} is a line bundle and $c_1(\mathcal{V}_2) = -c_1(\mathcal{L})$. The $U(1)$ factor above appears both in the structure group and also in its commutant inside E_8 . Naively, therefore, one might expect this Abelian group to be unbroken. However, as is well documented in this context [47–52], this $U(1)$ gains a mass through the Green-Schwarz mechanism. In this process the $U(1)$ gauge boson is made massive and one entire hypermultiplet from the $K3$ metric moduli is removed from the low energy spectrum. One of the degrees of freedom in the hypermultiplet is eaten by the gauge boson and the remaining three are made massive by the triplet of D-terms of the six-dimensional $\mathcal{N} = 1$ theory.

The matter content that results from the bundle (4.9) is given in table 8.

Representation	Cohomology	Multiplicity
1	$H^1(\text{End}_0(\mathcal{V}_2))$	$4c_2(\mathcal{V}_2) - c_1(\mathcal{L})^2 - 6$
7	$H^1(\mathcal{V}_2^\vee \otimes \mathcal{L}) \oplus H^1(\mathcal{L}^{\vee 2})$	$(c_2(\mathcal{V}_2) - \frac{5}{2}c_1(\mathcal{L})^2 - 4) + (-2c_1(\mathcal{L})^2 - 2)$
$\bar{\mathbf{7}}$	$H^1(\mathcal{V}_2 \otimes \mathcal{L}^\vee) \oplus H^1(\mathcal{L}^2)$	$(c_2(\mathcal{V}_2) - \frac{5}{2}c_1(\mathcal{L})^2 - 4) + (-2c_1(\mathcal{L})^2 - 2)$
35	$H^1(\mathcal{L})$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$
$\bar{\mathbf{35}}$	$H^1(\mathcal{L}^\vee)$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$
21	$H^1(\mathcal{V}_2)$	$c_2(\mathcal{V}_2) - \frac{1}{2}c_1(\mathcal{L})^2 - 4$
$\bar{\mathbf{21}}$	$H^1(\mathcal{V}_2^\vee)$	$c_2(\mathcal{V}_2) - \frac{1}{2}c_1(\mathcal{L})^2 - 4$

Table 8. The cohomology associated to each representation of the low-energy gauge group SU(7).

Following a similar procedure to that discussed in the SU(6) case, table (8) leads to the following anomaly cancelation constraint

$$\begin{aligned}
 n_H + 29n_T - n_V &= 273 & (4.10) \\
 \Rightarrow c_2(\mathcal{V}_2) - 2c_1(\mathcal{L})^2 + \frac{1}{60}c_2(\text{End}_0(\mathcal{V}_{E_8})) &= 24
 \end{aligned}$$

Naively there is a mismatch in the second equation, which is corrected by including an additional vector multiplet, beyond those associated to SU(7), to account for the Green-Schwarz massive U(1). Alternatively, considering the theory below the mass scale associated to the Abelian factor, we can drop both the number of vectors and the number of metric moduli hypermultiplets by one to account for the effects of the Higgs process described above.

The matter content outlined in table 8 takes the form described in equation (2.15) where we have

$$n = c_2(\mathcal{V}_2) - 2c_1(\mathcal{L})^2 - 12 \quad \text{and} \quad r = -\frac{1}{2}c_1(\mathcal{L})^2 - 2. \quad (4.11)$$

4.1.3 SU(8) and beyond

The relevant group theory in this case is,

$$E_8 \supset \text{SU}(8) \times \text{U}(1) \quad (4.12)$$

$$\mathbf{248} = \mathbf{1}_0 + \mathbf{8}_9 + \bar{\mathbf{8}}_{-9} + \mathbf{28}_{-6} + \bar{\mathbf{28}}_6 + \mathbf{56}_3 + \bar{\mathbf{56}}_{-3} + \mathbf{63}_0. \quad (4.13)$$

To embed a bundle with U(1) = S(U(1) × U(1)) structure group inside E_8 we write the following.

$$\mathcal{V} = \mathcal{L} \oplus \mathcal{L}^\vee \quad (4.14)$$

Here \mathcal{L} is a simple line bundle. Once again computing the spectrum we obtain the result given in table 9.

Anomaly cancelation in this case is as follows.

$$\begin{aligned}
 n_H + 29n_T - n_V &= 273 & (4.15) \\
 \Rightarrow -4c_1(\mathcal{L})^2 + \frac{1}{60}c_2(\text{End}_0(\mathcal{V}_{E_8})) &= 24
 \end{aligned}$$

Representation	Cohomology	Multiplicity
8	$H^1(\mathcal{L}^3)$	$-\frac{9}{2}c_1(\mathcal{L})^2 - 2$
$\bar{\mathbf{8}}$	$H^1(\mathcal{L}^{\vee 3})$	$-\frac{9}{2}c_1(\mathcal{L})^2 - 2$
28	$H^1(\mathcal{L}^{\vee 2})$	$-2c_1(\mathcal{L})^2 - 2$
$\bar{\mathbf{28}}$	$H^1(\mathcal{L}^2)$	$-2c_1(\mathcal{L})^2 - 2$
56	$H^1(\mathcal{L})$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$
$\bar{\mathbf{56}}$	$H^1(\mathcal{L}^{\vee})$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$

Table 9. The cohomology associated to each representation of the low-energy gauge group SU(8).

Here, again, there is a massive U(1), which can be thought of as reducing the number of massless metric moduli on the K3 by 1, so only 19 K3 moduli are included in the anomaly matching condition.

The matter content given in table 9 takes the form described in equation (2.16) where we have

$$n = -4c_1(\mathcal{L})^2 - 12 = 8r + 4 \quad , \quad r = -\frac{1}{2}c_1(\mathcal{L})^2 - 2 \quad \text{and} \quad r' = 0 \quad (4.16)$$

Note that we get an extremely non-generic spectrum in this case from a six-dimensional field theory point of view. The spectrum of the SU(8) charged matter in the heterotic compactification is controlled by a single integer rather than by 3 as in (2.16). In fact, we see from the expression for n in (4.16), together with the topological fact that $c_1(\mathcal{L})^2$ is even on K3 for any \mathcal{L} , that $c_1(\mathcal{L})^2 \leq -4$. Studying equation (4.15), we see that the smallest $c_1(\mathcal{L})^2$ can be is -6 . Thus the two possibilities are $n = 4$ and $n = 12$. Neither of these two possibilities leaves a large enough second Chern class available for the hidden sector bundle to completely break the E_8 in the other sector. We will have at minimum an SO(8) and E_8 “hidden” sector gauge group respectively in these two cases [20]. These match the non-Higgsable clusters that one would expect in a dual compactification of F-theory on an elliptic fibration over F_4 and F_{12} . As in the F-theory analysis of section 3.2.3, we obtain no matter in the **70** representation of SU(8) in these perturbative heterotic compactifications.

The gauge group SU(8) is on the edge of what can be achieved perturbatively in compactifications of the $E_8 \times E_8$ heterotic string. SU(9) is a subgroup of E_8 but its commutant inside E_8 is empty, meaning that it can not be achieved as the unbroken commutant of some continuous bundle structure group (although it can be achieved on singular K3 manifolds by using a \mathbb{Z}_3 structure group [53]). The groups SU(10) and higher are simply not subgroups of E_8 and thus can't be achieved as gauge groups in the case of perturbative compactifications. It is interesting that the non-genericity of spectrum described in the preceding paragraph arises in the boundary SU(N) case of the largest possible N .

4.2 Realization of the transitions

The matter transitions described from a field theory perspective in section 2, and from an F-theory perspective in section 3.3 are also realized concretely in these heterotic compactifications.

In the $SU(6)$ case of section 4.1.1 we can utilize small instanton transitions to swap second Chern class contributions between \mathcal{V}_3 and \mathcal{V}_2 , subject to the overall constraint (4.5). Note that we could also swap second Chern class with the “hidden sector” bundle but this would lead to an intermediate stage involving an increase in the number of tensor multiplets.

Studying table 7, we see that lowering $c_2(\mathcal{V}_2)$ by 1 and raising $c_2(\mathcal{V}_3)$ by 1 causes the number of **6** and $\overline{\mathbf{6}}$'s to go down by a single unit and the number of **20**'s also to lower by 1. Conversely we gain one **15**, one $\overline{\mathbf{15}}$ and two singlets. This is precisely a transition of the form described in equation (2.9) from a field theory perspective and in equation (3.56) in an F-theory context (it is important in making this comparison to realize that the multiplicities given in tables such as table 7 count half-hypermultiplets).

In the $SU(7)$ case of section 4.1.2, small instanton transitions within one E_8 factor swap contributions to $c_2(\mathcal{V}_2)$ with $c_1(\mathcal{L})^2$ in such a manner as to preserve equation (4.10). Note that not only the second, but also the first Chern class of \mathcal{V}_2 changes under such a transition.

We see from equation (4.10) that if we increase $c_1(\mathcal{L})^2$ by 2^1 in table 8 we must also increase $c_2(\mathcal{V}_1)$ by 4. Studying table 8, we see that, under such a transition, the number of **35** and $\overline{\mathbf{35}}$ half hypermultiplets lowers by 1, the number of **7** and $\overline{\mathbf{7}}$ half hypermultiplets lowers by 5, the number of **21** and $\overline{\mathbf{21}}$ half hypermultiplets increases by 3 and the number of singlet half hypermultiplets increases by 14. This is precisely a transition of the form described in equation (2.10) from a field theory perspective and in equation (3.58) in an F-theory context.

The $SU(8)$ case of section 4.1.3 mirrors what was found in an F-theory context in section 3.3. No small instanton transitions purely in one E_8 factor, giving rise to either of the forms (2.11) or (2.12) seen in our field theory discussion, is possible, however, due to the constraint given in equation (4.15). Small instanton transitions are of course possible if one allows a modification of the second E_8 bundle and it is also possible that more generic results from a field theoretic point of view could be obtained on a singular $K3$, where non-perturbative contributions to the gauge charged sector are possible.

4.3 Higgsing processes

In heterotic compactification, Higgsing processes in the low-energy field theory have a clear interpretation in terms of deformations of the gauge bundle. Here, we describe one example each of the deformations associated to Higgsing on fundamental, double antisymmetric and triple antisymmetric matter, together with a table detailing some of the key information in the other cases. The full analysis of the remaining examples can be found in appendix D.

¹This is the minimum possible change given the expression for the number of 35's. In fact, that $c_1(\mathcal{L})^2$ is even for an arbitrary line bundle \mathcal{L} is enforced by the topology of $K3$.

Representation	Cohomology	Multiplicity
5	$H^1(\wedge^2 \mathcal{V})$	$3c_2(\mathcal{V}) - 20$
$\bar{\mathbf{5}}$	$H^1(\wedge^2 \mathcal{V}^\vee)$	$3c_2(\mathcal{V}) - 20$
10	$H^1(\mathcal{V}^\vee)$	$c_2(\mathcal{V}) - 10$
$\bar{\mathbf{10}}$	$H^1(\mathcal{V})$	$c_2(\mathcal{V}) - 10$
1	$H^1(\text{End}_0(\mathcal{V}))$	$10c_2(\mathcal{V}) - 48$

Table 10. The cohomology associated to each representation of the low-energy gauge group $SU(5)$.

4.3.1 Higgsing on fundamental matter

Let us start by considering what happens to the bundle as we Higgs from $SU(6)$ to $SU(5)$. The relevant group theory in this case is

$$SU(6) \rightarrow SU(5) \times U(1) \tag{4.17}$$

$$\mathbf{6} = \mathbf{1}_{-5} + \mathbf{5}_1 \tag{4.18}$$

$$\mathbf{15} = \mathbf{5}_{-4} + \mathbf{10}_2 \tag{4.19}$$

$$\mathbf{20} = \mathbf{10}_{-3} + \bar{\mathbf{10}}_3 . \tag{4.20}$$

Clearly, we wish to turn on the singlet of $SU(5)$ inside the fundamental of $SU(6)$ to achieve the Higgsing. We see from table 7 that the $\mathbf{6}$ and $\bar{\mathbf{6}}$ half hypermultiplets, which combine to form a single hypermultiplet, are given by the cohomologies $H^1(\mathcal{V}_3 \otimes \mathcal{V}_2)$ and $H^1(\mathcal{V}_3^\vee \otimes \mathcal{V}_2)$ respectively.

In terms of bundle geometry, turning on fields descending from these cohomology groups corresponds to forming the following extension.

$$0 \rightarrow \mathcal{V}_2 \rightarrow \mathcal{V} \rightarrow \mathcal{V}_3 \rightarrow 0 \tag{4.21}$$

In fact, the bundle that we form is a deformation of this extension and its dual, as described in the work of Li and Yau [54].

The bundle \mathcal{V} in (4.21) has structure group $SU(5)$, which is the relevant case to arrive at an $SU(5)$ low-energy symmetry. One can check how the matter that one obtains from \mathcal{V} compares to that which follows from a simple decomposition of the multiplets in the original $SU(6)$ theory using the branching rules in (4.17).

We start by computing the matter content associated to an $SU(5)$ bundle, with the general result being given in table 10. In computing this table we have used the usual decomposition of the adjoint representation of E_8 ,

$$E_8 \supset SU(5) \times SU(5) \tag{4.22}$$

$$\mathbf{248} = (\mathbf{5}, \mathbf{10}) + (\bar{\mathbf{10}}, \mathbf{5}) + (\mathbf{10}, \bar{\mathbf{5}}) + (\bar{\mathbf{5}}, \bar{\mathbf{10}}) + (\mathbf{24}, \mathbf{1}) + (\mathbf{1}, \mathbf{24}) \tag{4.23}$$

together with Hirzebruch-Riemann-Roch.

SU(5) Representation	# from SU(6) multiplet decomposition	# found after transition
1	$10c_2(\mathcal{V}_3) + 10c_2(\mathcal{V}_2) - 46$	$10c_2(\mathcal{V}_3) + 10c_2(\mathcal{V}_2) - 48$
5	$3c_2(\mathcal{V}_3) + 3c_2(\mathcal{V}_2) - 18$	$3c_2(\mathcal{V}_3) + 3c_2(\mathcal{V}_2) - 20$
$\bar{\mathbf{5}}$	$3c_2(\mathcal{V}_3) + 3c_2(\mathcal{V}_2) - 18$	$3c_2(\mathcal{V}_3) + 3c_2(\mathcal{V}_2) - 20$
10	$c_2(\mathcal{V}_3) + c_2(\mathcal{V}_2) - 10$	$c_2(\mathcal{V}_3) + c_2(\mathcal{V}_2) - 10$
$\bar{\mathbf{10}}$	$c_2(\mathcal{V}_3) + c_2(\mathcal{V}_2) - 10$	$c_2(\mathcal{V}_3) + c_2(\mathcal{V}_2) - 10$

Table 11. Matter content after Higgsing an SU(6) to an SU(5) theory, both via a naive decomposition of the initial SU(6) multiplets and via a direct computation from the resulting SU(5) bundle.

Next we note that the second Chern class of the \mathcal{V} resulting from a Higgsing transition such as that being considered in equation (4.21) is given by,

$$c_2(\mathcal{V}) = c_2(\mathcal{V}_3) + c_2(\mathcal{V}_2) . \tag{4.24}$$

Using these results we can compile table 11, which compares the matter content associated to the bundle \mathcal{V} in (4.21) with a naive decomposition of the original SU(6) matter, as given in table 7, using the branching rules (4.17).

The differences in the last two columns of table 11 consist of two full fundamental hypermultiplets and one scalar hypermultiplet, and arise naturally due to degrees of freedom being absorbed by massive gauge bosons, or being given a mass by D-terms, in the Higgsing process. We can now confirm that this result matches the field theory analysis given in section 2.3.1.

Similar results are found by Higgsing SU(7) and SU(8) on their fundamental representations. The details of these computations can be found in appendix D. The SU(7) case in particular has some different structure in that there are two different sources of 7 representations in terms of cohomology, corresponding to two different bundles that are being deformed during the transition. Here we content ourselves with a presentation of the relevant bundle deformations in table 12.

4.3.2 Higgsing on two-index antisymmetric matter

In this case we will give the example of Higgsing an SU(7) gauge group on the **21** dimensional representation. The relevant group theory in this case is the following.

$$\begin{aligned}
 \text{SU}(7) &\supset \text{SU}(5) \times \text{SU}(2) \times \text{U}(1) & (4.25) \\
 \mathbf{21} &= (\mathbf{1}, \mathbf{1})_{-10} + (\mathbf{5}, \mathbf{2})_{-3} + (\mathbf{10}, \mathbf{1})_4 \\
 \mathbf{7} &= (\mathbf{1}, \mathbf{2})_{-5} + (\mathbf{5}, \mathbf{1})_2 \\
 \mathbf{35} &= (\mathbf{5}, \mathbf{1})_{-8} + (\bar{\mathbf{10}}, \mathbf{1})_6 + (\mathbf{10}, \mathbf{2})_{-1}
 \end{aligned}$$

We see that giving a VEV to a **21** $\bar{\mathbf{21}}$ pair will break SU(7) down to SU(5) \times SU(2) with a Green-Schwarz massive U(1) also being present (this Green-Schwarz U(1) is the one originally present in the heterotic SU(7) model and does not correspond to the U(1)

Group transition	Bundle transition	Fields Gaining Vev
$SU(6) \rightarrow SU(5)$	$\mathcal{V}_{SU(2)} \oplus \mathcal{V}_{SU(3)} \rightarrow \tilde{\mathcal{V}}_{SU(5)}$ where $0 \rightarrow \mathcal{V}_{SU(2)} \rightarrow \mathcal{V}_{SU(5)} \rightarrow \mathcal{V}_{SU(3)} \rightarrow 0$	$H^1(\mathcal{V}_{SU(2)} \otimes \mathcal{V}_{SU(3)})$
$SU(7) \rightarrow SU(6)$	$\mathcal{V}_{U(2)} \oplus \mathcal{L} \rightarrow \tilde{\mathcal{V}}_{SU(3)} \oplus \tilde{\mathcal{V}}_{SU(2)}$ where $0 \rightarrow \mathcal{L}^\vee \rightarrow \mathcal{V}_{SU(2)} \rightarrow \mathcal{L} \rightarrow 0$ and $0 \rightarrow \mathcal{L} \rightarrow \mathcal{V}_{SU(3)} \rightarrow \mathcal{V}_{U(2)} \rightarrow 0$	$H^1(\mathcal{V}_{U(2)}^\vee \otimes \mathcal{L})$ $H^1(\mathcal{L}^{\vee 2})$
$SU(8) \rightarrow SU(7)$	$\mathcal{L} \oplus \mathcal{L}^\vee \rightarrow \tilde{\mathcal{V}}_{U(2)} \oplus \mathcal{L}$ where $0 \rightarrow \mathcal{L}^{\vee 2} \rightarrow \mathcal{V}_{U(2)} \rightarrow \mathcal{L} \rightarrow 0$	$H^1(\mathcal{L}^3)$

Table 12. Higgsing on the fundamental in various heterotic theories, and the resulting deformation of the gauge bundle. The tildes over some bundles in the second column indicate a Li-Yau type deformation of the untilded object and its dual [54].

in (4.25)). The $\mathbf{21}$'s, according to table 8, lie in the cohomology $H^1(\mathcal{V}_2)$. In terms of bundle topology, giving an expectation value to such a field corresponds to forming the following bundle.

$$\mathcal{V} = \mathcal{Q} \oplus \mathcal{L} \tag{4.26}$$

where $0 \rightarrow \mathcal{V}_2 \rightarrow \mathcal{Q} \rightarrow \mathcal{O} \rightarrow 0$

As in previous cases, one should really think of \mathcal{Q} as being a deformation of this extension and its dual, a la Li-Yau [54]. Here \mathcal{Q} is a $U(3)$ bundle and the line bundle \mathcal{L} is unaffected by the transition. The overall structure group is $S(U(3) \times U(1))$ which does indeed break E_8 to $SU(5) \times SU(2) \times U(1)$ (where the last factor is a common Green-Schwarz anomalous factor between the structure group and the visible gauge group).

In order to compare the matter content before and after such a Higgsing transition, we must first compute the matter content in the $SU(5) \times SU(2)$ theory. The group theory for a general heterotic $SU(5) \times SU(2)$ case is as follows.

$$E_8 \supset SU(5) \times SU(2) \times SU(3) \times U(1) \tag{4.27}$$

$$\begin{aligned} \mathbf{248} = & (\mathbf{5}, \mathbf{1}, \mathbf{1})_{-6} + (\mathbf{5}, \mathbf{1}, \bar{\mathbf{3}})_4 + (\mathbf{5}, \mathbf{2}, \mathbf{3})_{-1} + (\bar{\mathbf{10}}, \mathbf{2}, \mathbf{1})_{-3} + (\bar{\mathbf{10}}, \mathbf{1}, \mathbf{3})_2 \\ & + (\mathbf{10}, \mathbf{2}, \mathbf{1})_3 + (\mathbf{10}, \mathbf{1}, \bar{\mathbf{3}})_{-2} + (\bar{\mathbf{5}}, \mathbf{1}, \mathbf{1})_6 + (\bar{\mathbf{5}}, \mathbf{1}, \mathbf{3})_{-4} + (\bar{\mathbf{5}}, \mathbf{2}, \bar{\mathbf{3}})_1 \\ & + (\mathbf{24}, \mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{2}, \mathbf{3})_5 + (\mathbf{1}, \mathbf{2}, \bar{\mathbf{3}})_{-5} + (\mathbf{1}, \mathbf{1}, \mathbf{8})_0 \end{aligned} \tag{4.28}$$

This leads us to the matter content given in table 13 for such a theory.

For the particular case of an $S(U(3) \times U(1))$ bundle formed by a transition of the form given in equation (4.26) we have:

$$c_2(\mathcal{Q}) = c_2(\mathcal{V}_2) \tag{4.29}$$

$$c_1(\mathcal{L}) = c_1(\mathcal{L}) . \tag{4.30}$$

Given this we can form the same table as we did in the case of Higgsing on the fundamental: comparing a direct decomposition of the $SU(7)$ multiplets under the symmetry breaking

Representation	Cohomology	Multiplicity
$(\mathbf{5}, \mathbf{1})$	$H^1(\mathcal{L}^{\vee 2}) \oplus H^1(\mathcal{Q}^{\vee} \otimes \mathcal{L})$	$c_2(\mathcal{Q}) - 5c_1(\mathcal{L})^2 - 8$
$(\mathbf{5}, \mathbf{2})$	$H^1(\mathcal{Q})$	$c_2(\mathcal{Q}) - \frac{1}{2}c_1(\mathcal{L})^2 - 6$
$(\overline{\mathbf{10}}, \mathbf{2})$	$H^1(\mathcal{L}^{\vee})$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$
$(\overline{\mathbf{10}}, \mathbf{1})$	$H^1(\mathcal{Q} \otimes \mathcal{L})$	$c_2(\mathcal{Q}) - c_1(\mathcal{L})^2 - 6$
$(\mathbf{1}, \mathbf{1})$	$H^1(\text{End}_0(\mathcal{Q}))$	$6c_2(\mathcal{Q}) - 2c_1(\mathcal{L})^2 - 16$
$(\mathbf{1}, \mathbf{2})$	$H^1(\mathcal{Q} \otimes \mathcal{L}^2)$	$c_2(\mathcal{Q}) - \frac{9}{2}c_1(\mathcal{L})^2 - 6$

Table 13. The cohomology associated to each representation of the low-energy gauge group $SU(5) \times SU(2) \times U(1)$.

$SU(5) \times SU(2)$ Representation	# from $SU(7)$ multiplet decomposition	# found after transition
$(\mathbf{1}, \mathbf{1})$	$6c_2(\mathcal{V}_2) - 2c_1(\mathcal{L})^2 - 14$	$6c_2(\mathcal{V}_2) - 2c_1(\mathcal{L})^2 - 16$
$(\mathbf{1}, \mathbf{2})$	$c_2(\mathcal{V}_2) - \frac{9}{2}c_1(\mathcal{L})^2 - 6$	$c_2(\mathcal{V}_2) - \frac{9}{2}c_1(\mathcal{L})^2 - 6$
$(\mathbf{5}, \mathbf{1})$	$c_2(\mathcal{V}_2) - 5c_1(\mathcal{L})^2 - 8$	$c_2(\mathcal{V}_2) - 5c_1(\mathcal{L})^2 - 8$
$(\mathbf{5}, \mathbf{2})$	$c_2(\mathcal{V}_2) - \frac{1}{2}c_1(\mathcal{L})^2 - 4$	$c_2(\mathcal{V}_2) - \frac{1}{2}c_1(\mathcal{L})^2 - 6$
$(\overline{\mathbf{10}}, \mathbf{1})$	$c_2(\mathcal{V}_2) - c_1(\mathcal{L})^2 - 6$	$c_2(\mathcal{V}_2) - c_1(\mathcal{L})^2 - 6$
$(\overline{\mathbf{10}}, \mathbf{2})$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$

Table 14. Matter content after Higgsing an $SU(7)$ to an $S(U(5) \times U(2))$ theory, both via a naive decomposition of the initial $SU(7)$ multiplets and via a direct computation from the resulting $S(U(5) \times U(2))$ bundle.

with the spectrum of the bundle after the transition. This is given in table 14. Once more, the differences between the second and third columns in table 14 precisely match what we would expect from an analysis of the Higgs mechanism in such a situation. This Higgsing is precisely of the form described in a field theory context in section 2.3.2.

Similar results are found by Higgsing $SU(6)$ and $SU(8)$ on their two-index antisymmetric representations. The details of these computations can be found in appendix D. Here we content ourselves with a presentation of the relevant bundle deformations in table 15.

4.3.3 Higgsing on three-index antisymmetric matter

To illustrating Higgsing on three-index antisymmetric matter we will consider the example of $SU(8)$. The relevant group theory in this case is as follows.

$$\begin{aligned}
 SU(8) &\supset SU(5) \times SU(3) \times U(1) \\
 \mathbf{56} &= (\mathbf{1}, \mathbf{1})_{-15} + (\mathbf{5}, \overline{\mathbf{3}})_{-7} + (\overline{\mathbf{10}}, \mathbf{1})_9 + (\mathbf{10}, \mathbf{3})_1 \\
 \mathbf{8} &= (\mathbf{1}, \mathbf{3})_{-5} + (\mathbf{5}, \mathbf{1})_3 \\
 \mathbf{28} &= (\mathbf{1}, \overline{\mathbf{3}})_{-10} + (\mathbf{5}, \mathbf{3})_{-2} + (\mathbf{10}, \mathbf{1})_6
 \end{aligned}$$

Giving a VEV to the $SU(5)$ singlets in a $\mathbf{56}, \overline{\mathbf{56}}$ pair will therefore break $SU(8) \rightarrow SU(5) \times SU(3)$ with a Green-Schwarz massive $U(1)$ also being present. The $\mathbf{56}$'s, according

Group transition	Bundle transition	Fields Gaining VEV
$SU(6) \rightarrow SU(4) \times SU(2)$	$\mathcal{V}_{SU(2)} \oplus \mathcal{V}_{SU(3)} \rightarrow \mathcal{V}_{SU(2)} \oplus \tilde{\mathcal{V}}_{SU(4)}$ where $0 \rightarrow \mathcal{V}_{SU(3)} \rightarrow \mathcal{V}_{SU(4)} \rightarrow \mathcal{O} \rightarrow 0$	$H^1(\mathcal{V}_{SU(3)})$
$SU(7) \rightarrow SU(5) \times SU(2)$	$\mathcal{V}_{U(2)} \oplus \mathcal{L} \rightarrow \tilde{\mathcal{V}}_{U(3)} \oplus \mathcal{L}$ where $0 \rightarrow \mathcal{V}_{U(2)} \rightarrow \mathcal{V}_{U(3)} \rightarrow \mathcal{O} \rightarrow 0$	$H^1(\mathcal{V}_{U(2)})$
$SU(8) \rightarrow SU(6) \times SU(2)$	$\mathcal{L} \oplus \mathcal{L}^\vee \rightarrow \tilde{\mathcal{V}}_{SU(2)} \oplus \mathcal{L} \oplus \mathcal{L}^\vee$ where $0 \rightarrow \mathcal{L}^\vee \rightarrow \mathcal{V}_{SU(2)} \rightarrow \mathcal{L} \rightarrow 0$	$H^1(\mathcal{L}^{\vee 2})$

Table 15. Higgsing on the two-index antisymmetric representation in various heterotic theories, and the resulting deformation of the gauge bundle. The tildes over some bundles in the second column indicate a Li-Yau type deformation of the untilded object and its dual [54].

Representation	Cohomology	Multiplicity
$(\mathbf{5}, \mathbf{1})$	$H^1(\mathcal{L}^{\vee 3})$	$-\frac{9}{2}c_1(\mathcal{L})^2 - 2$
$(\mathbf{5}, \mathbf{3})$	$H^1(\mathcal{L}^2)$	$-2c_1(\mathcal{L})^2 - 2$
$(\mathbf{5}, \bar{\mathbf{3}})$	$H^1(\mathcal{Q}^\vee)$	$c_2(\mathcal{Q}) - \frac{1}{2}c_1(\mathcal{L})^2 - 4$
$(\mathbf{10}, \mathbf{1})$	$H^1(\mathcal{Q}^\vee \otimes \mathcal{L}^2)$	$c_2(\mathcal{Q}) - \frac{5}{2}c_1(\mathcal{L})^2 - 4$
$(\bar{\mathbf{10}}, \bar{\mathbf{3}})$	$H^1(\mathcal{L})$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$
$(\mathbf{1}, \mathbf{1})$	$H^1(\text{End}_0(\mathcal{Q}))$	$4c_2(\mathcal{Q}) - c_1(\mathcal{L})^2 - 6$
$(\mathbf{1}, \bar{\mathbf{3}})$	$H^1(\mathcal{Q} \otimes \mathcal{L}^2)$	$c_2(\mathcal{Q}) - \frac{13}{2}c_1(\mathcal{L})^2 - 4$

Table 16. The cohomology associated to each representation of the low-energy gauge group $SU(5) \times SU(3)$.

to table 9, lie in the cohomology $H^1(\mathcal{L})$. In terms of bundle topology, giving an expectation value to such a field corresponds to forming the following bundle.

$$\mathcal{V} = \mathcal{Q} \oplus \mathcal{L}^\vee \tag{4.31}$$

$$\text{where } 0 \rightarrow \mathcal{L} \rightarrow \mathcal{Q} \rightarrow \mathcal{O} \rightarrow 0 \tag{4.32}$$

Here \mathcal{V} is an $S(U(2) \times U(1))$ bundle. The correct embedding of $S(U(2) \times U(1))$ does indeed break E_8 to $SU(5) \times SU(3) \times SU(2) \times U(1)$.

The group theory for the $SU(5) \times SU(3)$ case is as follows.

$$\begin{aligned}
 E_8 \supset & SU(5) \times SU(3) \times SU(2) \times U(1) \tag{4.33} \\
 \mathbf{248} = & (\mathbf{5}, \mathbf{1}, \mathbf{1})_{-6} + (\mathbf{5}, \mathbf{3}, \mathbf{1})_4 + (\mathbf{5}, \bar{\mathbf{3}}, \mathbf{2})_{-1} + (\bar{\mathbf{10}}, \mathbf{1}, \mathbf{2})_{-3} + (\bar{\mathbf{10}}, \bar{\mathbf{3}}, \mathbf{1})_2 \\
 & + (\mathbf{10}, \mathbf{1}, \mathbf{2})_3 + (\mathbf{10}, \mathbf{3}, \mathbf{1})_{-2} + (\bar{\mathbf{5}}, \mathbf{1}, \mathbf{1})_6 + (\bar{\mathbf{5}}, \bar{\mathbf{3}}, \mathbf{1})_{-4} + (\bar{\mathbf{5}}, \mathbf{3}, \mathbf{2})_1 \\
 & + (\mathbf{24}, \mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{2})_5 + (\mathbf{1}, \mathbf{3}, \mathbf{2})_{-5} + (\mathbf{1}, \mathbf{8}, \mathbf{1})_0
 \end{aligned}$$

This leads to a spectrum for such a heterotic compactification as given in table 16.

SU(5) × SU(3) Representation	# from SU(8) multiplet decomposition	# found after transition
(5, 1)	$-\frac{9}{2}c_1(\mathcal{L})^2 - 2$	$-\frac{9}{2}c_1(\mathcal{L})^2 - 2$
(5, 3)	$-2c_1(\mathcal{L})^2 - 2$	$-2c_1(\mathcal{L})^2 - 2$
(5, $\bar{3}$)	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 4$
(10, 1)	$-\frac{5}{2}c_1(\mathcal{L})^2 - 4$	$-\frac{5}{2}c_1(\mathcal{L})^2 - 4$
($\bar{10}$, $\bar{3}$)	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$
(1, 1)	$-c_1(\mathcal{L})^2 - 4$	$-c_1(\mathcal{L})^2 - 6$
(1, $\bar{3}$)	$-\frac{13}{2}c_1(\mathcal{L})^2 - 4$	$-\frac{13}{2}c_1(\mathcal{L})^2 - 4$

Table 17. Matter content after Higgsing an SU(8) to an SU(5) × SU(3) theory, both via a naive decomposition of the initial SU(8) multiplets and via a direct computation from the resulting S(U(2) × U(1)) bundle.

Group transition	Bundle transition	Fields Gaining VEV
SU(6) → SU(3) × SU(3)	$\mathcal{V}_{\text{SU}(2)} \oplus \mathcal{V}_{\text{SU}(3)} \rightarrow \tilde{\mathcal{V}}'_{\text{SU}(3)} \oplus \tilde{\mathcal{V}}_{\text{SU}(3)}$ where $0 \rightarrow \mathcal{V}_{\text{SU}(2)} \rightarrow \mathcal{V}'_{\text{SU}(3)} \rightarrow \mathcal{O} \rightarrow 0$	$H^1(\mathcal{V}_{\text{SU}(2)})$
SU(7) → SU(4) × SU(3)	$\mathcal{V}_{\text{U}(2)} \oplus \mathcal{L} \rightarrow \tilde{\mathcal{V}}'_{\text{U}(2)} \oplus \mathcal{V}_{\text{U}(2)}$ where $0 \rightarrow \mathcal{L} \rightarrow \mathcal{V}'_{\text{U}(2)} \rightarrow \mathcal{O} \rightarrow 0$	$H^1(\mathcal{L})$
SU(8) → SU(5) × SU(3)	$\mathcal{L} \oplus \mathcal{L}^\vee \rightarrow \tilde{\mathcal{V}}_{\text{U}(2)} \oplus \mathcal{L}^\vee$ where $0 \rightarrow \mathcal{L} \rightarrow \mathcal{V}_{\text{U}(2)} \rightarrow \mathcal{O} \rightarrow 0$	$H^1(\mathcal{L})$

Table 18. Higgsing on the three-index antisymmetric representation in various heterotic theories, and the resulting deformation of the gauge bundle. The tildes over some bundles in the second column indicate a Li-Yau type deformation of the untilded object and its dual [54].

For the particular S(U(2) × U(1)) bundle that we achieve after transition, as given in equation (4.31), we have that,

$$c_2(\mathcal{Q}) = 0. \tag{4.34}$$

With this information we can finally construct the table comparing the break up of the SU(8) multiplets with the directly computed matter spectrum after the bundle transition, as we did in the previous cases. This is found in table 17.

As in all of the previous cases, this result is in perfect agreement with the field theory expectations given in section 2.3.2.

Similar results are found by Higgsing SU(6) and SU(7) on their three-index antisymmetric representations. The details of these computations can be found in appendix D. Here we content ourselves with a presentation of the relevant bundle deformations in table 18.

It should be noted that, due to the lack of **70** (quadruple antisymmetric) representations in the perturbative heterotic spectrum, one can not break to SU(4) × SU(4) × U(1) in this context and so our analysis of the possible Higgsing transitions terminates here. Once again, it is this boundary case of what is possible in perturbative heterotic theory that fails to reproduce the most general possibility from a field theory perspective.

4.4 Small Instanton transitions and Higgsing

It is interesting to note that the processes of Higgsing and undergoing small instanton transitions need not commute. We illustrate this here with the case of small instanton transitions before and after Higgsing SU(7) to SU(6).

As described in section 4.2, before the Higgsing a minimal small instanton transition results in the following change in spectrum.

$$1 \times (\mathbf{35} + \overline{\mathbf{35}}) + 5 \times (\mathbf{7} + \overline{\mathbf{7}}) \leftrightarrow 3 \times (\mathbf{21} + \overline{\mathbf{21}}) + 14 \times (\mathbf{1}) \quad (4.35)$$

The group theory governing a Higgsing of SU(7) on the fundamental representation is as follows.

$$\begin{aligned} \text{SU}(7) &\rightarrow \text{SU}(6) \times \text{U}(1) & (4.36) \\ \mathbf{7} &= \mathbf{1}_{-6} + \mathbf{6}_1 \\ \mathbf{21} &= \mathbf{6}_{-5} + \mathbf{15}_2 \\ \mathbf{35} &= \mathbf{15}_{-4} + \mathbf{20}_3 \end{aligned}$$

Application of these branching rules to the transition (4.35) results in the following SU(6) transition

$$2 \times (\mathbf{20}) + 2 \times (\mathbf{6} + \overline{\mathbf{6}}) \leftrightarrow 4 \times (\mathbf{1}) + 2 \times (\mathbf{15} + \overline{\mathbf{15}}) \quad (4.37)$$

Purely from the point of view of an SU(6) bundle as in section 4.1.1, after Higgsing, we know that one can have the following small instanton transition

$$(\mathbf{20}) + (\mathbf{6} + \overline{\mathbf{6}}) \leftrightarrow 2 \times (\mathbf{1}) + (\mathbf{15} + \overline{\mathbf{15}}) \quad (4.38)$$

which is more minimal than the one in (4.37) above.

If an SU(6) theory is obtained by Higgsing SU(7), however, then the bundle of structure group SU(2) will have the special form described in table 12:

$$0 \rightarrow \mathcal{L}^\vee \rightarrow \mathcal{V}_{\text{SU}(2)} \rightarrow \mathcal{L} \rightarrow 0. \quad (4.39)$$

Such a bundle has $c_1(\mathcal{V}_{\text{SU}(2)}) = -c_1(\mathcal{L})^2$. The quantity $c_1(\mathcal{L})^2$ for line bundles on $K3$ is always even and thus small instanton transitions involving an exchange of second Chern class between SU(3) and SU(2) bundles of the form (4.39) always results in a non-minimal transition of the type given in (4.37) in the SU(6) theory.

After an SU(7) Higgsing, a subsequent small instanton transition in the SU(6) theory can change the bundle associated to the SU(6) theory such that its SU(2) valued component is not of the form (4.39). Such a transition could be of the more minimal type (4.38) and the resulting SU(6) theory then could not be obtained by Higgsing any SU(7) model.

5 Heterotic/F-theory duality

The solutions presented in previous sections provide an interesting playground for heterotic/F-theory duality since they correspond to generically *reducible* vector bundles

in the heterotic theory. Such reducible vector bundles give rise to many interesting features not yet fully explored in even the 6-dimensional duality, including small instanton transitions on a single E_8 fixed plane, the intricate intersection structure of reducible spectral covers and the presence of generically massive Green-Schwarz $U(1)$ symmetries.

In this section we consider the geometry of heterotic/F-theory dual pairs to be as follows [2, 18, 19]:

$$\text{Heterotic on } \pi_h : X_n \xrightarrow{\mathbb{E}} B_{n-1} \Leftrightarrow \text{F-theory on } \pi_f : Y_{n+1} \xrightarrow{K3} B_{n-1} \quad (5.1)$$

where X_n is elliptically fibered over B_{n-1} and the $K3$ -fibered manifold Y_{n+1} admits a more detailed description as an elliptically-fibered Calabi-Yau $(n + 1)$ -fold with section over a base \mathcal{B}_n which is itself \mathbb{P}^1 fibered over B_{n-1} .

5.1 The stable degeneration limit

To begin, we review briefly the standard arguments of heterotic/F-theory duality [2, 14, 19, 55]. As discussed in the Introduction, in the case that the heterotic geometry is elliptically fibered and the F-theory geometry is $K3$ fibered, there exists a weakly coupled limit of both theories.

As is well-known in the literature (see [20, 55, 56] for reviews), this limit in parameter space corresponds to the large volume and weak coupling regime in the heterotic theory and is realized geometrically in the F-theory geometry via the following log semi-stable degeneration of the Calabi-Yau manifold, Y :

$$Y_{n+1} \longrightarrow Y_{n+1}^{(1)} \cup_D Y_{n+1}^{(2)} \quad (5.2)$$

where $Y^{(1)}$ and $Y^{(2)}$ are non-CY, dP_9 fibered $(n + 1)$ -folds, glued along a common divisor D [56–58]. In the case of heterotic duality, $D = X_n$ is a CY variety of one lower dimension than Y_{n+1} and forms the background of the heterotic geometry.

Now, the heterotic/F-theory dictionary says that if the $E_8 \times E_8$ heterotic theory is compactified on $D = X_n$ with vector bundles $\mathcal{V}_1, \mathcal{V}_2$ over X_n , the spacetime symmetries and matter spectrum should match that of F-theory compactified on $Y^{(1)} \cup_D Y^{(2)}$. That is, for singularities leading to symmetries G_i on $Y^{(i)}$ ($i = 1, 2$), we expect structure groups H_i for \mathcal{V}_i where $G_i \subset E_8$ is the commutant of H_i . Moreover, the full degrees of freedom of the theory can be counted and found to match (see table 19 for a schematic review of the matching of the geometric degrees of freedom in the 6-dimensional effective theories).

Practically, to take the stable degeneration limit of the F-theory geometry we must consider scaling the coefficients of f, g such that the $(n + 1)$ -dimensional F-theory $K3$ fibration degenerates into a fiber product of dP_9 s. In terms of the Weierstrass model itself, if $\sigma = 0$ and $\sigma = \infty$ are chosen to define the loci where the symmetries arise in each E_8 factor of the heterotic dual (i.e., the two sections defining the base B_n of the \mathbb{P}^1 fibration), then we have

$$f \sim \sum_{i=0}^8 f_i \sigma^i, \quad g \sim \sum_{j=0}^{12} g_j \sigma^j \quad (5.3)$$

Het/Bundle	Het/Spec. Cov.	F-theory
$H^1(\text{End}_0(\mathcal{V}_i))$	$\text{Def}(\mathcal{S})$	$H^{2,1}(Y^{(i)})$
	$\text{Jac}(\mathcal{S})$	$H^3(Y^{(i)}, \mathbb{R}/\mathbb{Z})$
	Discrete data of $\mathcal{L}_{\mathcal{S}}$	$H^{2,2}(Y^{(i)}, \mathbb{Z})$

Table 19. A schematic matching of the duality in 6-dimensions: heterotic vector bundle moduli, encoded as spectral data $(\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ are matched to geometric moduli of the (resolved) F-theory dP_9 -fibered geometry in the stable degeneration limit [55–57].

and in the stable degeneration limit, it is possible to choose a scaling in which

$$\begin{aligned} f_i &\text{ scales as } \epsilon^{(i-4)} & (5.4) \\ g_j &\text{ scales as } \epsilon^{(j-6)} . \end{aligned}$$

In the limit that $\epsilon \rightarrow 0$, it is clear that the zero locus of the discriminant $\Delta = 4f^3 + 27g^2$ can be divided in “half” with nonabelian symmetry potentially present on each pole ($\sigma = 0$ and $\sigma = \infty$) of the \mathbb{P}^1 fiber. In particular, the limit $\epsilon \rightarrow 0$ divides the $K3$ fiber into two rationally elliptically fibered surfaces (i.e. dP_9 surfaces) [55, 57]. This well-known limit is straightforward to apply for all Weierstrass models with $SU(N)$ symmetry with $N \leq 6$. In these cases, if the complex structure is tuned to induce an $SU(5)$ symmetry on the $\sigma = 0$ locus for example, the stable degeneration limit isolates the $SU(5) \subset E_8$ inside a single E_8 factor and effectively “separates” all f_i with $i > 4$ and g_j with $j > 6$ which correspond to the second E_8 factor (fully broken in this case). However, for $SU(7), SU(8)$ as we will see below, this limit can be somewhat subtle to take, since some f_i, g_j with $i \leq 4, j \leq 6$ are determined in terms of f_{i+n}, g_{j+m} for some n, m such that $i + n \geq 4, j + m \geq 6$ (See for example the Weierstrass coefficients in (3.29) and (3.30) in section 3.2). In these cases, care must be taken with the powers of ϵ in each term. We will return to this issue in the following sections.

For now, we will begin our investigation of heterotic/F-theory dual solutions, as well as the “Higgsing chains” linking them, by reviewing a particular representation of the geometry of principal bundles in heterotic compactifications — the so-called spectral cover construction [59, 60].

5.2 A very brief review of the spectral cover construction

In order to match the degrees of freedom in the heterotic geometry $(X_n, \pi : \mathcal{V} \rightarrow X_n)$ with that in the F-theory $(n + 1)$ -fold (i.e. Y_{n+1} in (5.1)), it is necessary to present the data of the heterotic bundle as a spectral cover [59–61] or more generally, a cameral cover [61–63]. For a review of this bundle construction see [60]. For now, we will focus on the simple case of $SU(N)$ structure groups and spectral covers. It will suffice to recall that under the Fourier-Mukai transform [59], a vector bundle with structure group $SU(N)$ is equivalent to a pair $(\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ where \mathcal{S} is a divisor in the elliptically fibered heterotic CY geometry and $\mathcal{L}_{\mathcal{S}}$ is a rank 1 sheaf over \mathcal{S} . The heterotic fibration can be presented in Weierstrass form as

$$Y^2 = X^3 + f_4X + g_6 . \tag{5.5}$$

Then, the divisor \mathcal{S} is called the “spectral cover” and can be represented as the zero locus of a polynomial constraint of the form

$$a_0 Z^N + a_2 X Z^{N-2} + a_3 Y Z^{N-3} + \dots = 0 \tag{5.6}$$

ending in $a_N X^{\frac{N}{2}}$ for N even and $a_N X^{\frac{N-3}{2}} Y$ for N odd [55]. The coefficients a_j are sections of line bundles over the base B , of $\pi : X_n \xrightarrow{\mathbb{E}} B$, given by

$$a_j \in H^0(B, K_B^{\otimes j} \otimes \mathcal{O}(\eta)) \tag{5.7}$$

where in 6-dimensional compactifications, $\eta = c_2(V)$ and in 4-dimensional compactifications $c_2(V) = S_0 \wedge \eta + \zeta^{2,2}$ with S_0 the form dual the zero section and $\zeta^{2,2}$ the pullback of a $\{2, 2\}$ form on B . The class of \mathcal{S} is determined to be $[\mathcal{S}] = N[S_0] + \pi^*(\eta)$. The class of the rank one sheaf on the spectral cover $\mathcal{L}_{\mathcal{S}}$ is also determined entirely by the topology of \mathcal{V} (see, for example, [20] for a review). With this in hand we turn now to the matter transitions explored in sections 4, viewed now through the lens of the spectral cover construction.

5.3 Matter transitions in spectral covers

To begin, let us consider briefly the form that the matter transitions described in sections 3–4 will take in the context of heterotic/F-theory duality. Examples of other related transitions have been explored in the heterotic literature (see [64, 65] for representative examples). As described in those works and in section 4, small instanton transitions on a single E_8 fixed plane can only arise when the vector bundle geometry — an H -principal bundle on X_n with $H \subset E_8$ — is reducible. That is, when $H = H_1 \otimes H_2$ and the associated vector bundle decomposes as a direct sum $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$. In this case, the small instanton transition takes the form

$$\mathcal{V}_1 \oplus \mathcal{V}_2 \longrightarrow \mathcal{V}'_1 \oplus \mathcal{V}_2 \oplus \mathcal{I} \longrightarrow \mathcal{V}'_1 \oplus \mathcal{V}'_2 \tag{5.8}$$

where \mathcal{I} is a sky-scraper sheaf supported on the codimension 2 locus wrapped by $M5/N5$ -branes in X_n . In the language of spectral covers then, we consider the simplest case where $\mathcal{V}_1, \mathcal{V}_2$ are both $SU(N)$ bundles for some N . In particular, the situation in which instanton number is removed from \mathcal{V}_1 and then added to \mathcal{V}_2 can be summarized as

$$(\mathcal{S}_1)(\mathcal{S}_2) = 0 \longrightarrow a(w)(\mathcal{S}'_1)(\mathcal{S}_2) = 0 \longrightarrow (\mathcal{S}'_1)(\mathcal{S}'_2) = 0 \tag{5.9}$$

where w represents coordinates on the base B of the heterotic elliptic fibration. Here $a(w)$ is a so-called “vertical component” of the spectral cover — corresponding to a small instanton in the heterotic theory [66]. The function $a(w)$ corresponds exactly to the function (also called “ a ”) describing the F-theory matter transitions given in section 3.3.

As a concrete example, consider an $SO(12)$ theory. The commutant of this symmetry within E_8 is $SU(2) \times SU(2)$ which leads to a heterotic bundle geometry consisting of a sum of two $SU(2)$ vector bundles $\mathcal{V}_1 \oplus \mathcal{V}_2$. For such reducible vector bundle, the spectral cover takes the form of a product

$$\mathcal{S}_{\mathcal{V}} = (a_0 Z^2 + a_2 X)(b_0 Z^2 + b_2 X) = 0 \tag{5.10}$$

In the SO(12) theory, the matter consists of localized **32** and **32'** multiplets, located at the zeros of a_2 and b_2 , respectively, and fundamental matter (**12s**) associated with the intersection $\mathcal{S}_1 \cap \mathcal{S}_2$ of the two SU(2) components of the spectral cover. The simplest transitions then take the form of separating out a common factor of (for example) a_2, a_0 and then “absorbing” it into b_0, b_2 .

To illustrate this more concretely, consider 6-dimensional heterotic/F-theory geometry with the spectral covers given above corresponding to curves inside a $K3$ surface. Since the minimal second Chern class for an SU(2) bundle on $K3$ is $c_2(\mathcal{V}) = 4$, we will parameterize the topology of the pair $\mathcal{V}_1, \mathcal{V}_2$ as $c_2(\mathcal{V}_1) = 4 + r, c_2(\mathcal{V}_2) = 8 + n - r$ [14]. It follows that the degrees of the functions in (5.10) are

$$a_0 \sim 4 + r \qquad b_0 \sim 8 + n - r \qquad (5.11)$$

$$a_2 \sim r \qquad b_2 \sim 4 + n - r \qquad (5.12)$$

With this in mind, we can denote an arbitrary function of degree k on \mathbb{P}^1 via the subscript f_k . Then, as in (5.9), a transition removing a small instanton from \mathcal{V}_1 and merging it into the bundle \mathcal{V}_2 takes the form

$$\begin{aligned} & (e_{4+r}Z^2 + f_r X)(g_{8+n-r}Z^2 + h_{4+n-r}X) \\ & \longrightarrow (a_p)(e'_{4+r-p}Z^2 + f'_{r-p}X)(g_{8+n-r}Z^2 + h_{4+n-r}X) \\ & \longrightarrow (e'_{4+r-p}Z^2 + f'_{r-p}X)(g'_{8+n-r+p}Z^2 + h'_{4+n-r+p}X) \end{aligned} \qquad (5.13)$$

where $p \leq r$. Note that in the process of realizing the transition in the spectral cover ((tuning a common factor in \mathcal{S}_1) \rightarrow (identifying it as an overall factor of \mathcal{S}_2) \rightarrow (deforming \mathcal{S}_2 away from factorized form)), the topology of the bundles \mathcal{V}_1 and \mathcal{V}_2 has changed since $c_2(\mathcal{V}'_1) = 4 + r - p$ and $c_2(\mathcal{V}'_2) = 8 + n - r + p$. In this case the spectrum changes only between the **32** and **32'** fields:²

$$(r) \frac{1}{2} \mathbf{32}'_s + (4 + n - r) \frac{1}{2} \mathbf{32}'_s \rightarrow (r - p) \frac{1}{2} \mathbf{32}'_s + (4 + n - r + p) \frac{1}{2} \mathbf{32}'_s \qquad (5.14)$$

As a concluding observation, it should be noted that while we illustrated this example in 6 dimensions for simplicity, entirely analogous structure exists in 4 dimensions. There the spectral covers are complex surfaces inside Calabi-Yau 3 folds and once again, co-dimension two components (here curves) can be separated off one component and the deformed smoothly into the other. One remarkable difference in the 4-dimensional theory however is that in the $\mathcal{N} = 1$ theory, such transitions can also be chirality changing [64, 65]. We will return to this point later. For now, we continue towards understanding the basic structure of the transitions described in sections 3–4, and the Higgsing chains linking them, beginning with the SU(6) theory.

²Note that the number of **12**'s is unchanged in this correspondence since it is determined solely by the geometric intersection number $[\mathcal{S}_1] \cdot [\mathcal{S}_2] \subset K3$. For the geometry above this gives $2(8 + n)$ points which is independent of the transition given in (5.13).

5.4 SU(6) heterotic/F-theory dictionary

As described in section 3.2.1, and equations (3.16) and (3.17), the SU(6) F-theory geometry is parameterized by 5 functions

$$\phi_2, \alpha, \beta, \nu, \lambda \tag{5.15}$$

associated, respectively with the divisors

$$-2K + L, -K + N - L, L, -2K + N - L, -4K + N - L \tag{5.16}$$

The matching of the heterotic and F-theory effective physics can be made readily in both 6- and 4-dimensions. For concreteness, in the following paragraphs we will make the explicit correspondence between the degrees of freedom in the 6-dimensional theory, but it should be kept in mind that the general structure is equally applicable to 4-dimensions.

From section 4.1.1, it is clear that the heterotic dual SU(6) theory on K3 has commutant SU(2) × SU(3) inside E₈. The bundle \mathcal{V} , with $c_2(\mathcal{V}) = 12 + n$ can be denoted as

$$\mathcal{V} = \mathcal{V}_2 \oplus \mathcal{V}_3 \tag{5.17}$$

As above, let \mathcal{V}_2 have structure group SU(2) and $c_2(\mathcal{V}_2) = 4 + r$, while \mathcal{V}_3 has structure group SU(3) and $c_2(\mathcal{V}_3) = 8 + n - r$.

Using the spectral cover construction as described in the previous section, the vector bundle \mathcal{V} in (5.17) can be described as a reducible spectral cover, $\mathcal{S}_V = \mathcal{S}_1 \cup \mathcal{S}_2$ of the form given in (5.6), inside the elliptically fibered K3 surface. Explicitly,

$$\mathcal{S}_V = (a_0 Z^2 + a_2 X)(b_0 Z^3 + b_2 XZ + b_3 Y) = 0 \tag{5.18}$$

where the degrees of the functions a_i and b_j (over the \mathbb{P}^1 base) are

$$a_0 \sim 4 + r \qquad b_0 \sim 8 + n - r \tag{5.19}$$

$$a_2 \sim r \qquad b_2 \sim 4 + n - r \tag{5.20}$$

$$b_3 \sim 2 + n - r \tag{5.21}$$

A key to matching these 5 functions with those in (5.15) is to note that the **20**'s of SU(6) are located at the zeros of a_2 and the **15**'s at the zeros of b_3 . It follows by inspection of (3.16) and (3.17) then, that the parameter matching takes the form

	a_0	a_2	b_0	b_2	b_3
\sim	ϕ_2	β	λ	ν	α

(5.22)

This matches with the association made in table 3.

Re-writing the spectral cover in this notation we have:

$$S_V = (\phi_2 Z^2 + \beta X)(\lambda Z^3 + \nu XZ + \alpha Y) = 0 \tag{5.23}$$

It is important to recall here that the matching in (5.22) is only up to proportionality. There may be constants/normalization in this matching that could be significant in understanding the dual pair; we return to this topic below.

Finally, as observed above, it can be verified that the charged matter spectrum of the SU(6) theory is readily understood in terms of the spectral cover as the loci where $\alpha = 0$ (**15** representations), $\beta = 0$ (**20s**) and the points where $S_1 \cap S_2$ (**6s**). The matter transitions corresponding to (5.8) in this case exactly match those found in the F-theory geometry given in section 3. Recall, from section 3.3 the transition was realized via

$$\begin{aligned} \alpha &\rightarrow a\alpha', \\ \nu &\rightarrow a\nu', \\ \lambda &\rightarrow a\lambda'. \end{aligned} \tag{5.24}$$

which corresponds to the separation of a small instanton from the SU(3) bundle \mathcal{V}_3 in (5.23). This was followed by a deformation which deformed the instanton into a smooth SU(2) bundle \mathcal{V}'_2 :

$$\begin{aligned} a\beta &\rightarrow \beta', \\ a\phi_2 &\rightarrow \phi'_2. \end{aligned} \tag{5.25}$$

exactly as expected from (5.23). Given this exact matching between the parameter spaces of the heterotic and F-theory descriptions, we can clearly follow the heterotic transition just as the F-theory transition along a one-parameter family of theories with the parameter $\hat{\varepsilon} = 0$ at the superconformal point and taking positive and negative signs for the theories with two different matter contents.

5.5 SU(7) tuning

In this section we explore heterotic/F-theory dual pairs in the SU(7) theory. Unlike the case of SU(6) given above, here the spectral cover is not a simple factorization of the form (5.10) and (5.18), but a more interesting and complex object. To begin, we will compare the tunings/symmetry enhancements described in sections 3 and 4 for SU(7) gauge symmetry. Recall that from (3.29) and (3.30), the tuning taking SU(6) \rightarrow SU(7) is given by

$$\begin{aligned} \phi_2 &= 3\zeta_1^2 + \omega\delta^2 \\ \beta &= \delta^2 \\ \lambda &= \frac{1}{3}\zeta_1^2\zeta_2 - \frac{1}{18}\zeta_1\xi\omega + \frac{1}{9}\delta^2\zeta_2\omega + \lambda_1\delta^2\xi \\ \nu &= \zeta_2\delta^2 + \zeta_1\xi \\ \alpha &= \delta\xi \end{aligned} \tag{5.26}$$

and the “middle” coefficients that will be related (in the stable degeneration limit) to the heterotic Weierstrass model, (5.5) take the form:

$$\begin{aligned} f_4 &= -6\zeta_1\lambda_1 - \frac{1}{12}\omega^2 + \psi_4\delta^2 \\ g_6 &= \frac{1}{108}(-36f_5\delta^4\zeta_2 + 972\delta^2\lambda_1^2 - 36f_5\delta^2\zeta_1\xi - 9f_6\delta^6\xi^2 - 108\zeta_1^2\psi_4 - 108\zeta_1\lambda_1\omega - 36\delta^2\psi_4\omega - \omega^3) \end{aligned} \tag{5.27}$$

The key question that must be addressed is whether or not this choice of complex structure corresponds in the heterotic theory to the deformation (i.e. splitting) of vector bundles required by this symmetry enhancement described in section 4.1.2?

First, it is useful to briefly review the enhancement of symmetry from the point of view of vector bundle geometry in the heterotic theory. As in (5.17), the reducible bundle with structure group $SU(2) \times SU(3)$ leading to commutant $SU(6) \subset E_8$ is

$$\mathcal{V} = \mathcal{V}_2 \oplus \mathcal{V}_3 \tag{5.28}$$

From section 4, we recall that in the transition from $SU(6) \rightarrow SU(7)$ the heterotic bundle structure group must reduce from $SU(2) \times SU(3)$ to $S[U(2) \times U(1)]$ via the “splitting” of the bundles $\mathcal{V}_2, \mathcal{V}_3$:

$$\mathcal{V}_2 \rightarrow \mathcal{L} \oplus \mathcal{L}^\vee \tag{5.29}$$

$$\mathcal{V}_3 \rightarrow \mathcal{L}^\vee \oplus \mathcal{U}_2 \tag{5.30}$$

where \mathcal{L} is a line bundle and $c_1(\mathcal{U}_2) = c_1(\mathcal{L})$ (i.e. \mathcal{U}_2 is a $U(2)$ bundle). The commutant of $SU(7) \subset E_8$ is $SU(2) \times U(1)$ and the underlying (fundamental) bundle geometry is $\mathcal{L}^\vee \oplus \mathcal{U}_2$ (the decomposition of \mathcal{V}_2 above is simply auxiliary information in this case).

To match the tuning in the heterotic and F-theory descriptions, the first step is to consider how such a decomposition is manifest when the bundle $\mathcal{V} = \mathcal{V}_2 \oplus \mathcal{V}_3$ is described via a spectral cover as in (5.23). Can we match this decomposition to the enhancement given in (5.26)? For a bundle described via a smooth spectral cover,³ decomposition of the vector bundle into a direct sum usually corresponds to factorization of the spectral cover (i.e. the spectral cover becomes reducible). In the case above, it is clear from (5.29) that the $SU(2)$ portion of the bundle must split into a sum of two line bundles. As it turns out, this decomposition into Abelian components is a particularly subtle process from the point of view of a spectral cover description [55, 60, 68]. A well-behaved (i.e. smooth) spectral cover (corresponding to a rank N vector bundle) intersects each fiber at exactly N points. Thus, by definition, a smooth spectral cover associated to a line bundle must be a 1-sheeted cover of the base, intersecting each fiber exactly once. This however, is a familiar object in the fibration geometry: such a 1-sheeted cover is in fact *a section of the elliptic fibration*. It is important to note that it is *not* the case that any line bundle produces a section of the fibration under Fourier-Mukai transform. Instead, generic line bundles lead to singular/vertical spectral covers.⁴ However, it is the case that any smooth 1-sheeted cover of the base (describing a line bundle) is also a section to the elliptic fibration.

Returning again to the expected geometry, it is clear from (5.29) that upon tuning $SU(6) \rightarrow SU(7)$ the $SU(2)$ bundle should decompose as a sum of a line bundle and its dual. As a result, if such a bundle can be described as a smooth spectral cover, this should in turn *correspond to an extra section appearing in the K3 geometry*. In this vein, a sum of the form $(\mathcal{L} \oplus \mathcal{L}^\vee)$ should correspond to a section and the “inverse” of that section (i.e. two sections leading to marked points on each elliptic fiber with coordinates $[X', Y', Z]$ and $[X'', Y'', -Z]$ which sum to the zero section under the addition law of the elliptic curve).

³I.e. for so-called “regular” bundles [60, 67].

⁴In some cases these can correspond to so-called “T-brane” solutions [69]. See also section 7.9 of [20].

5.5.1 The heterotic $K3$ geometry

As described above, an $SU(7)$ gauge symmetry must correspond to the reduction of an $SU(2)$ bundle structure group to $S[U(1) \times U(1)]$, which after Fourier-Mukai transform will correspond to 1-sheeted spectral covers of the base $\mathbb{P}^1 \subset K3$. As a result, an $SU(7)$ symmetry in the heterotic theory does not allow the form of the spectral cover and that of the $K3$ surface to be considered separately. They are intrinsically correlated and a generic $SU(7)$ tuned F-theory geometry (with a heterotic dual) should lead to an enhanced Mordell-Weil group in the dual $K3$ geometry. To investigate this expectation we first consider the form of the Weierstrass coefficients f_4 and g_6 from (5.27). If this $K3$ geometry has an additional rational section (i.e. Mordell-Weil rank 1), then as derived in [70] it must be possible to write it in the following general form (see eq. (B.19) of [70]):

$$Y^2 = X^3 + \left(c_1 c_3 - b^2 c_0 - \frac{c_2^2}{3} \right) XZ^4 + \left(c_0 c_3^2 - \frac{1}{3} c_1 c_2 c_3 + \frac{2}{27} c_2^3 - \frac{2}{3} b^2 c_0 c_2 + \frac{b^2 c_1^2}{4} \right) Z^6 \tag{5.31}$$

for some functions c_i, b .

However, before comparing to (5.27), care must be taken with the stable degeneration limit. In matching the degrees of freedom in the heterotic and F-theory geometries, it is necessary to take the limit described above in (5.2) and (5.4) in which the F-theory geometry undergoes stable degeneration. In this case we hope to compare the coefficients of f_i with $0 \leq i \leq 3$ and g_j with $0 \leq j \leq 5$ with the data of the spectral cover in (5.23) (subject to the tuning in (5.26)) and (5.5) with the coefficients of f_4 and g_6 given in (5.27). However, as described in section 5.1, we must consider the powers of ϵ present in each term in the Weierstrass model in the limit that $\epsilon \rightarrow 0$. In fact, for the $SU(7)$ solution, g_6 (5.27) has been tuned in terms of f_5 and f_6 , both of which carry additional powers of ϵ . As a result, this dependence must be taken into account in the ϵ limits, where the terms dependent on f_5 and f_6 vanish. Here the stable degeneration limit leads to a modified form for the “middle” coefficients in (5.27):

$$f_4 \rightarrow -6\zeta_1 \lambda_1 - \frac{1}{12} \omega^2 + \psi_4 \delta^2 \tag{5.32}$$

$$g_6 \rightarrow \frac{1}{108} (972\delta^2 \lambda_1^2 - 108\zeta_1^2 \psi_4 - 108\zeta_1 \lambda_1 \omega - 36\delta^2 \psi_4 \omega - \omega^3) \tag{5.33}$$

It is these values that we must take as defining the heterotic base geometry (5.5) in the stable degeneration limit. That is, this elliptically fibered $K3$ forms the divisor D along which Y_1 and Y_2 are glued in (5.2).

Remarkably, an inspection of (5.33) shows that it is of *precisely the form required by (5.31)*. The exact matching to the general two-section Weierstrass model is

$$\begin{aligned} c_0 &= \psi_4 \\ c_1 &= (6i)\lambda_1 \\ c_2 &= -(1/2)\omega \\ c_3 &= (i)\zeta_1 \\ b &= \pm(i)\delta \end{aligned} \tag{5.34}$$

Choosing the sign identification of $b = -i\delta$, then the fiber coordinate of the new section in the $K3$ geometry is

$$[X, Y, Z] = \left[-\frac{1}{3} (3\zeta_1^2 + \delta^2\omega), i \left(\zeta_1^3 + \frac{1}{2}\zeta_1\delta^2\omega - 3\lambda_1\delta^4 \right), -i\delta \right] \quad (5.35)$$

With these observations in hand, it is at last possible to compare the tuning given in (5.26) to the spectral cover and the expected bundle geometry in (5.29) and (5.30).

5.5.2 SU(7) symmetry and the spectral cover

We must begin by substituting the tuning (5.26) into the general SU(6) spectral cover in (5.23). But first, it will be convenient for to choose the constants of proportionality that were left free in (5.22) through specific normalization (this normalization choice is not physically significant, but will merely serve to provide the clearest interpretation of the degrees of freedom in the dual theories):

$$\begin{aligned} a_0 &= \phi_2 & (5.36) \\ a_2 &= -3\beta \\ b_0 &= -3\lambda \\ b_2 &= \nu \\ b_3 &= -\alpha \end{aligned}$$

With these choices, the tuning of the complex structure in F-theory given by (5.26) can be substituted into the heterotic spectral cover in (5.23), to yield a new spectral cover:

$$\begin{aligned} &((3\zeta_1^2 + \delta^2\omega)Z^2 - 3\delta^2X) \times & (5.37) \\ &\left(-3 \left(\frac{1}{3}\zeta_1^2\zeta_2 - \frac{1}{18}\zeta_1\xi\omega + \frac{1}{9}\delta^2\zeta_2\omega + \lambda_1\delta^2\xi \right) Z^3 + (\zeta_2\delta^2 + \zeta_1\xi)XZ - \delta\xi Y \right) = 0 \end{aligned}$$

Which can be re-written as

$$\begin{aligned} &((3\zeta_1^2 + \delta^2\omega)Z^2 - 3\delta^2X) \times & (5.38) \\ &\left(-\frac{\zeta_2 Z}{3} ((3\zeta_1^2 + \delta^2\omega)Z^2 - 3\delta^2X) + \xi \left[-3 \left(-\frac{1}{18}\zeta_1\omega + \delta^2\lambda_1 \right) Z^3 + \zeta_1 XZ - \delta Y \right] \right) = 0 \end{aligned}$$

If the degrees of freedom have been paired correctly in the dual theories, this new spectral cover must be exactly the Fourier-Mukai transform of the reduced bundle geometry given in (5.29) and (5.30).

To verify this, consider first the SU(2) component of the spectral cover. This vanishes on the locus

$$X = \frac{1}{3\delta^2} (3\zeta_1^2 + \delta^2\omega)Z^2 \quad (5.39)$$

But by inspection, this is exactly the constraint yielding the additional rational sections given in (5.35)! As expected, it is clear that over every point on the base, the two roots of the SU(2) spectral cover sweep out precisely the new section and its “inverse” under the addition law of the elliptic curve (replacing $Z = -i\delta$ with $Z = +i\delta$). Now, since the new

sections are “horizontal” in the K3 elliptic fibration, they intersect each fiber exactly once and at equal and opposite points on the elliptic curve relative to the zero section (located at $Z = 0$). As a result, in the Fourier-Mukai transform this is exactly a pair of line bundles of the form

$$\mathcal{L} \oplus \mathcal{L}^\vee \tag{5.40}$$

Thus, the tuning of (5.26), the enhanced rational sections of the elliptically fibered K3 and the SU(2) component of the spectral cover exactly match the expectation of the bundle decomposition given in (5.29). All that remains then, is to examine the SU(3) component of the spectral cover.

Not only must the SU(3) component also decompose, it is clear by inspection of (5.29) and (5.30) that one line bundle factor (\mathcal{L}^\vee) is repeated in both the SU(2) and SU(3) components of the reducible bundle. In the spectral cover description, we have seen that the two roots of the SU(2) spectral cover have formed into a pair (corresponding to a line bundle and its dual) and now it must be verified that at least one of those roots has *also become a root of the SU(3) spectral cover*.

Inspecting (5.38), it is clear that the form of the SU(3) component on the locus where (5.39) is satisfied (i.e. the roots of the SU(2) factor) reduces to

$$\xi \left[-3 \left(-\frac{1}{18} \zeta_1 \omega + \delta^2 \lambda_1 \right) Z^3 + \zeta_1 XZ - \delta Y \right] = 0 \tag{5.41}$$

We can now ask, does this vanish along either of the roots of (5.39)? Substituting the new section of the K3 fibration (5.35) (i.e. the root with Z coordinate $Z = -i\delta$) into the remaining expression in (5.41), we find that it vanishes identically. Thus, our expectations are fully verified and a single SU(2) root is now overlapping with one root of the SU(3) spectral cover. Moreover it is easy to verify that this expression does not vanish on the other SU(2) root (with $Z = +i\delta$) as required. The remaining two roots of the SU(3) component are distinct and correspond to the expected rank 2 bundle in the heterotic geometry given in (5.30).

Thus far, on this locus, the correspondence is perfect between the F-theory Weierstrass data, the heterotic bundle geometry, and the spectral cover description. There is only one remaining element to be considered and this is the presence of a Green-Schwarz massive U(1) in the heterotic effective theory.

From group theory alone, the reducible bundle $\mathcal{V} = \mathcal{L}^\vee + \mathcal{U}_2$ with $c_1(\mathcal{U}_2) = c_1(\mathcal{L})$ given in (5.30) has structure group $S(U(2) \times U(1))$ and within E_8 this gives rise to a commutant $SU(7) \times U(1)$. That is, at the level of group theory, any SU(7) symmetry arising in a heterotic theory *must* be accompanied by an additional abelian gauge symmetry. Generically, by the Green-Schwarz mechanism, this enhanced U(1) couples to the Kähler axions of the base CY geometry (which transforms via shift symmetries) and becomes massive (see [47–50, 52]). Since the presence of Green-Schwarz massive U(1)s in F-theory has been the topic of much recent interest (see for example [71–78]), we turn now to the appearance of this enhanced U(1) in heterotic/F-theory pair described above.

5.5.3 SU(7) and Green-Schwarz massive U(1)s

As described in the previous subsection, the presence of an enhanced U(1) symmetry in the heterotic theory is unavoidable (by group theory within E_8). How then, are we to understand this U(1) from the point of view of F-theory? To explore this, consider the stable degeneration limit $Y \rightarrow Y^{(1)} \cup_D Y^{(2)}$ as described in section 5.1 and (5.2), and the Weierstrass model of $Y^{(1)}$ corresponding to the physics of a single E_8 factor of the heterotic effective theory. In this limit, the Weierstrass coefficients of $Y^{(1)}$ are given by

$$\begin{aligned}
 f \sim & -\frac{\delta^{12}\xi^4}{48} + \sigma \left(-\frac{1}{6}\delta^{10}\zeta_2\xi^2 - \frac{1}{6}\delta^8\zeta_1\xi^3 \right) \\
 & + \frac{1}{6}\sigma^2 \left(-2\delta^8\zeta_2^2 - 4\delta^6\zeta_1\zeta_2\xi - \delta^6\xi^2\omega - 5\delta^4\zeta_1^2\xi^2 \right) \\
 & + \sigma^3 \left(-\frac{2}{3}\delta^4\zeta_2\omega - 3\delta^4\lambda_1\xi - 2\delta^2\zeta_1^2\zeta_2 - \frac{1}{6}\delta^2\zeta_1\xi\omega - \zeta_1^3\xi \right) \\
 & + \sigma^4 \left(\delta^2\psi_4 - 6\zeta_1\lambda_1 - \frac{\omega^2}{12} \right)
 \end{aligned} \tag{5.42}$$

and

$$\begin{aligned}
 g \sim & \frac{\delta^{18}\xi^6}{864} + \frac{1}{72}\sigma \left(\delta^{16}\zeta_2\xi^4 + \delta^{14}\zeta_1\xi^5 \right) \\
 & + \frac{1}{72}\sigma^2 \left(4\delta^{14}\zeta_2^2\xi^2 + 8\delta^{12}\zeta_1\zeta_2\xi^3 + \delta^{12}\xi^4\omega + 7\delta^{10}\zeta_1^2\xi^4 \right) \\
 & + \frac{1}{216}\sigma^3 \left(16\delta^{12}\zeta_2^3 + 48\delta^{10}\zeta_1\zeta_2^2\xi + 24\delta^{10}\zeta_2\xi^2\omega + 54\delta^{10}\lambda_1\xi^3 + 120\delta^8\zeta_1^2\zeta_2\xi^2 + 15\delta^8\zeta_1\xi^3\omega + 70\delta^6\zeta_1^3\xi^3 \right) \\
 & + \frac{1}{144}\sigma^4 \left(32\delta^8\zeta_2^2\omega + 144\delta^8\zeta_2\lambda_1\xi - 12\delta^8\xi^2\psi_4 + 96\delta^6\zeta_1^2\zeta_2^2 + 40\delta^6\zeta_1\zeta_2\xi\omega + 216\delta^6\zeta_1\lambda_1\xi^2 + 5\delta^6\xi^2\omega^2 \right. \\
 & \left. + 144\delta^4\zeta_1^3\zeta_2\xi + 32\delta^4\zeta_1^2\xi^2\omega + 84\delta^2\zeta_1^4\xi^2 \right) \\
 & + \frac{1}{36}\sigma^5 \left(-12\delta^6\zeta_2\psi_4 + 72\delta^4\zeta_1\zeta_2\lambda_1 - 12\delta^4\zeta_1\xi\psi_4 + 5\delta^4\zeta_2\omega^2 + 36\delta^4\lambda_1\xi\omega + 24\delta^2\zeta_1^2\zeta_2\omega + 180\delta^2\zeta_1^2\lambda_1\xi \right. \\
 & \left. - \delta^2\zeta_1\xi\omega^2 + 36\zeta_1^4\zeta_2 - 6\zeta_1^3\xi\omega \right) \\
 & + \frac{1}{108}\sigma^6 \left(972\delta^2\lambda_1^2 - 36\delta^2\psi_4\omega - 108\zeta_1^2\psi_4 - 108\zeta_1\lambda_1\omega - \omega^3 \right)
 \end{aligned} \tag{5.43}$$

where $\sigma = 0$ defines a section of the \mathbb{P}^1 fiber of the F-theory base geometry (recall $\sigma = 0$ and $\sigma = \infty$ mark the locations of the symmetry groups arising from each E_8 factor).

By group theory alone, the U(1) accompanying the SU(7) in the heterotic theory should be visible in this limit. How then do we see it? Remarkably, we find that the U(1) is very much present in this limit! Let us compare once more to the generic two-section Weierstrass model given in (5.31). We will see that in fact an additional section, and hence U(1) symmetry has become visible in the entire Weierstrass model of $Y^{(1)}$.

As in section 5.5.1, we must establish the dictionary that puts the coefficients given in (5.42) and (5.43) into the form necessary for a generic two-section model, (5.31). Here

we find that this correspondence can be achieved by one of two choices. First,

$$c_0 = -\frac{1}{4}\delta^6\zeta_2^2\sigma^2 + (-\zeta_1^2\zeta_2 - \frac{1}{2}\delta^2\zeta_2\omega)\sigma^3 + \psi_4\sigma^4 \quad (5.44)$$

$$c_1 = \frac{1}{2}i\delta^6\zeta_2\xi\sigma + i(\delta^2\zeta_1\zeta_2 + \zeta_1^2\xi + \frac{1}{2}\delta^2\xi\omega)\sigma^2 + 6i\lambda_1\sigma^3 \quad (5.45)$$

$$c_2 = \frac{1}{4}\delta^6\xi^2 + (-\frac{1}{2}\delta^4\zeta_2 + \delta^2\zeta_1\xi)\sigma - \frac{1}{2}\omega\sigma^2 \quad (5.46)$$

$$c_3 = \frac{i}{2}\delta^4\xi + i\zeta_1\sigma \quad (5.47)$$

$$b = -i\delta \quad (5.48)$$

The second choice for a variable change arises from the freedom associated to a Weierstrass model with Mordell-Weil rank 2 (i.e. the freedom to define which, of the section S_1 and its inverse $-S_1$, we choose to call “positive” in the elliptic addition law). The other solution is

$$c_0 = \left(-\frac{1}{4}\delta^6\zeta_2^2 - \delta^4\zeta_1\zeta_2\xi - \delta^2\zeta_1^2\xi^2\right)\sigma^2 + \left(-\zeta_1^2\zeta_2 - \frac{1}{2}\delta^2\zeta_2\omega - 6\delta^2\lambda_1\xi\right)\sigma^3 + \psi_4\sigma^4 \quad (5.49)$$

$$c_1 = i\left(\frac{1}{2}\delta^6\zeta_2\xi + \delta^4\zeta_1\xi^2\right)\sigma + i\left(\delta^2\zeta_1\zeta_2 + \zeta_1^2\xi - \frac{1}{2}\delta^2\xi\omega\right)\sigma^2 + 6i\lambda_1\sigma^3 \quad (5.50)$$

$$c_2 = \frac{1}{4}\delta^6\xi^2 + \left(-\frac{1}{2}\delta^4\zeta_2 - 2\delta^2\zeta_1\xi\right)\sigma - \frac{1}{2}\omega\sigma^2 \quad (5.51)$$

$$c_3 = -\frac{i}{2}\delta^4\xi + i\zeta_1\sigma \quad (5.52)$$

$$b = -i\delta \quad (5.53)$$

Either of these two solutions makes it clear that in the stable degeneration limit, $Y^{(1)}$ has a non-trivial Mordell-Weil group and a new rational section. Once again, in the stable degeneration limit and from the section addition law of the elliptic fiber of the dP_9 fiber of $Y^{(1)}$, this $U(1)$ symmetry was expected to arise and it is gratifying to see it manifestly appear. It only remains then to understand in the F-theory geometry how to understand the Green-Schwarz massive nature of this Abelian symmetry. We will return to this question in a following section.

5.5.4 SU(7) matter transitions

For SU(7) it is possible to once again compare the realization of F-theory matter transitions with heterotic small instanton transitions. From the F-theory geometry in section 3.3, a matter transition of the form (3.57):

$$3 \times \square + 3 \times \square + 8 \times \mathbf{1} \rightarrow \text{Superconformal Matter} \rightarrow \square + 8 \times \square + \mathbf{1}, \quad (5.54)$$

is realized by first deforming

$$\begin{aligned} \xi &\rightarrow a^3\xi' \\ \zeta_2 &\rightarrow a^4\zeta_2' \\ \lambda_1 &\rightarrow a\lambda_1' \\ \psi_4 &\rightarrow a^2\psi_4' \end{aligned} \quad (5.55)$$

and then tuning further so that the common factor is then absorbed into δ and ζ_1 :

$$\begin{aligned} a\delta &\rightarrow \delta' \\ a\zeta_1 &\rightarrow \zeta'_1 \end{aligned} \tag{5.56}$$

In the heterotic geometry this should correspond to removing m point-like instantons (taking a to be a degree m polynomial over \mathbb{P}^1 above) from either factor of the reducible bundle $\mathcal{L} \oplus \mathcal{U}_2$ in (5.30) and then smoothing them back into a new sum $\mathcal{L}' \oplus \mathcal{U}'_2$. Let us consider this from the point of view of the spectral cover given in (5.37):

$$\begin{aligned} &((3\zeta_1^2 + \delta^2\omega)Z^2 - 3\delta^2X) \times \\ &\left(-3\left(\frac{1}{3}\zeta_1^2\zeta_2 - \frac{1}{18}\zeta_1\xi\omega + \frac{1}{9}\delta^2\zeta_2\omega + \lambda_1\delta^2\xi\right)Z^3 + (\zeta_2\delta^2 + \zeta_1\xi)XZ - \delta\xi Y\right) = 0 \end{aligned} \tag{5.57}$$

Recall that this corresponds to the bundle geometry $(\mathcal{L} \oplus \mathcal{L}^\vee) \oplus (\mathcal{L}^\vee \oplus \mathcal{U}_2)$. Substituting the tuning of (5.55) into (5.57) above we find a transitional spectral cover

$$\begin{aligned} &((3\zeta_1^2 + \delta^2\omega)Z^2 - 3\delta^2X) \times (a^2) \times \\ &\left(-3\left(\frac{1}{3}(a^2)\zeta_1^2\zeta_2 - \frac{1}{18}(a)\zeta_1\xi\omega + \frac{1}{9}(a^2)\delta^2\zeta_2\omega + \lambda_1(a^2)\delta^2\xi\right)Z^3 \right. \\ &\quad \left. + (\zeta_2(a^2)\delta^2 + (a)\zeta_1\xi)XZ - (a)\delta\xi Y\right) = 0 \end{aligned} \tag{5.58}$$

This is precisely of the form required to consistently allow δ and ζ_1 to each absorb a factor of a (as in (5.56)) and return the spectral cover to its canonical form of (5.37) but with the degrees of the relevant functions shifted.

The remarkable observation to be made is that it is clear from the quadratic terms in this spectral cover that the only transitions possible are ones which involve a perfect square vertical factor (a^2) as above. That is, the only consistent transitions must involve an *even number of point-like instantons*. This exactly matches the observations made in section 4.2 (and under (4.39)) on the smooth heterotic geometry, even-ness of Chern classes and the particle content of the $SU(7)$ heterotic theory on $K3$. Finally it should be noted that the transition above clearly deforms the dP_9 -fibered 3-fold $Y^{(1)}$ as well as the class of the enhanced sections to the elliptic fibrations (with non-trivial Mordell-Weil group) of the elliptically fibered $K3$ surface. As a result, its impact on the dual heterotic/F-theory pair is more substantial than in the case of $SU(6)$ theories.

5.6 $SU(8)$ tuning

In this section, we will repeat the symmetry enhancement analysis described above for the tuning of $SU(7) \rightarrow SU(8)$. Once again, we return to the F-theory tuning described in section 3 for $SU(8)$ gauge symmetry.

Recall that from section 3.2.3, the tuning taking $SU(7) \rightarrow SU(8)$ is given by

$$\begin{aligned}
 \xi &= \delta^2 \tau & (5.59) \\
 \zeta_2 &= \delta^2 \zeta_3 + \frac{\zeta_4 \tau^2}{2} \\
 \zeta_1 &= \zeta_4 \tau \\
 \omega &= \delta^2 \omega_1 + 4 \zeta_3 \zeta_4 \\
 \lambda_1 &= \delta^2 \lambda_2 \tau - \frac{\zeta_4 \tau \omega_1}{6} \\
 \psi_4 &= \delta^4 \phi_4 - 6 \delta^2 \zeta_3 \lambda_2 - \frac{\delta^2 \omega_1^2}{4} - 3 \zeta_4 \lambda_2 \tau^2 \\
 f_5 &= \delta^2 \psi_5 + 2 \zeta_4 \phi_4 \\
 g_7 &= \frac{1}{12} (-72 \delta^4 \lambda_2 \phi_4 - 4 \delta^4 \psi_5 \omega_1 - 16 \delta^2 \zeta_3 \zeta_4 \psi_5 + 16 \delta^2 \zeta_4 \omega_1 \phi_4 + 16 \zeta_3 \zeta_4^2 \phi_4 - 12 \zeta_4^2 \tau^2 \psi_5 \\
 &\quad - 4 \delta^6 \zeta_3 f_6 - 6 \delta^4 \zeta_4 f_6 \tau^2 + \delta^{10} (-f_7) \tau^2)
 \end{aligned}$$

and again the “middle” coefficients (f_4, g_6) will be related (in the stable degeneration limit) to the heterotic Weierstrass model.

As described in section 5.1, it is clear that the $SU(8)$ symmetry depends on a Weierstrass model with structure that is spread across both “halves” of the F-theory base geometry and not easily localized on a single patch of the \mathbb{P}^1 fiber. Unlike in $SU(N)$ heterotic/F-theory dual pairs with $N \leq 5$, care must be taken in the stable degeneration limit.

In order to take the stable degeneration limit described by (5.4), specifying the overall powers of ϵ in each Weierstrass coefficient f_i, g_j is not sufficient. Since f_5, g_7 are determined by the functions $\xi, \zeta_2, \zeta_1, \dots$ above, it must further be specified how these functions are chosen to scale so that $f_5, g_7 \rightarrow 0$ in the $\epsilon \rightarrow 0$ limit.

A Groebner-basis calculation (using [79]) demonstrates that there are three possible paths to the stable degeneration limit in this case corresponding to

1. $\psi_5, \phi_4 \rightarrow 0$
2. $\zeta_4, \delta \rightarrow 0$
3. $(\zeta_3 \psi_5^2 \phi_4 - 24 \lambda_1 \phi_4^3 - \tau^2 \psi_5^3 - 4 \psi_5 \omega_1 \phi_4^2) \rightarrow 0$

As argued in appendix B, in fact only the first of these paths leads to a smooth $K3$ surface in the dual heterotic theory.

Thus, the appropriate limit to take which leads to a smooth, weakly coupled (perturbative) heterotic theory with $SU(8)$ symmetry is

$$\psi_5 \rightarrow 0, \quad f_6 \rightarrow 0, \quad \phi_4 \rightarrow 0, \quad f_7 \rightarrow 0 \tag{5.60}$$

In this case then, the form of the SU(8) Weierstrass model of $Y^{(1)}$ is given by

$$f \sim -\frac{\delta^{20}\tau^4}{48} + \sigma \left(-\frac{1}{6}\delta^{16}\zeta_3\tau^2 - \frac{1}{4}\delta^{14}\zeta_4\tau^4 \right) \quad (5.61)$$

$$+ \frac{1}{12}\sigma^2 \left(-4\delta^{12}\zeta_3^2 - 2\delta^{12}\tau^2\omega_1 - 20\delta^{10}\zeta_3\zeta_4\tau^2 - 15\delta^8\zeta_4^2\tau^4 \right) \quad (5.62)$$

$$+ \sigma^3 \left(-\frac{2}{3}\delta^8\zeta_3\omega_1 - 3\delta^8\lambda_2\tau^2 - \frac{8}{3}\delta^6\zeta_3^2\zeta_4 - 4\delta^4\zeta_3\zeta_4^2\tau^2 - 2\delta^2\zeta_4^3\tau^4 \right) \quad (5.63)$$

$$+ \sigma^4 \left(-6\delta^4\zeta_3\lambda_2 - \frac{\delta^4\omega_1^2}{3} - \frac{2}{3}\delta^2\zeta_3\zeta_4\omega_1 - 9\delta^2\zeta_4\lambda_2\tau^2 - \frac{4\zeta_3^2\zeta_4^2}{3} + \zeta_4^2\tau^2\omega_1 \right) \quad (5.64)$$

and

$$\begin{aligned} g \sim & \frac{\delta^{30}\tau^6}{864} + \sigma \left(\frac{1}{72}\delta^{26}\zeta_3\tau^4 + \frac{1}{48}\delta^{24}\zeta_4\tau^6 \right) + \frac{1}{72}\sigma^2 \left(4\delta^{22}\zeta_3^2\tau^2 + \delta^{22}\tau^4\omega_1 + 16\delta^{20}\zeta_3\zeta_4\tau^4 + 12\delta^{18}\zeta_4^2\tau^6 \right) \\ & + \frac{1}{108}\sigma^3 \left(8\delta^{18}\zeta_3^3 + 12\delta^{18}\zeta_3\tau^2\omega_1 + 27\delta^{18}\lambda_2\tau^4 + 84\delta^{16}\zeta_3^2\zeta_4\tau^2 + 9\delta^{16}\zeta_4\tau^4\omega_1 \right. \\ & + 144\delta^{14}\zeta_3\zeta_4^2\tau^4 + 72\delta^{12}\zeta_4^3\tau^6 \left. \right) \\ & + \frac{1}{36}\sigma^4 \left(8\delta^{14}\zeta_3^2\omega_1 + 54\delta^{14}\zeta_3\lambda_2\tau^2 + 2\delta^{14}\tau^2\omega_1^2 + 32\delta^{12}\zeta_3^3\zeta_4 + 22\delta^{12}\zeta_3\zeta_4\tau^2\omega_1 + 81\delta^{12}\zeta_4\lambda_2\tau^4 \right. \\ & + 116\delta^{10}\zeta_3^2\zeta_4^2\tau^2 + 3\delta^{10}\zeta_4^2\tau^4\omega_1 + 120\delta^8\zeta_3\zeta_4^3\tau^4 + 45\delta^6\zeta_4^4\tau^6 \left. \right) \\ & + \frac{1}{36}\sigma^5 \left(72\delta^{10}\zeta_3^2\lambda_2 + 8\delta^{10}\zeta_3\omega_1^2 + 36\delta^{10}\lambda_2\tau^2\omega_1 + 40\delta^8\zeta_3^2\zeta_4\omega_1 + 360\delta^8\zeta_3\zeta_4\lambda_2\tau^2 + 80\delta^6\zeta_3^3\zeta_4^2 \right. \\ & + 270\delta^6\zeta_4^2\lambda_2\tau^4 + 120\delta^4\zeta_3^2\zeta_4^3\tau^2 - 30\delta^4\zeta_4^3\tau^4\omega_1 + 60\delta^2\zeta_3\zeta_4^4\tau^4 + 18\zeta_4^5\tau^6 \left. \right) \\ & + \frac{1}{108}\sigma^6 \left(216\delta^6\zeta_3\lambda_2\omega_1 + 972\delta^6\lambda_2^2\tau^2 + 8\delta^6\omega_1^3 + 864\delta^4\zeta_3^2\zeta_4\lambda_2 + 24\delta^4\zeta_3\zeta_4\omega_1^2 \right. \\ & - 324\delta^4\zeta_4\lambda_2\tau^2\omega_1 - 48\delta^2\zeta_3^2\zeta_4^2\omega_1 + 648\delta^2\zeta_3\zeta_4^2\lambda_2\tau^2 + 72\delta^2\zeta_4^2\tau^2\omega_1^2 - 64\zeta_3^3\zeta_4^3 \\ & + 72\zeta_3\zeta_4^3\tau^2\omega_1 + 324\zeta_4^3\lambda_2\tau^4 \left. \right) \end{aligned} \quad (5.65)$$

A remarkable observation can be made at this stage by considering the coefficients of σ^4 in f and σ^6 in g above. In this limit, the heterotic $K3$ surface is defined by

$$f_4 = \left(-6\delta^4\zeta_3\lambda_2 - \frac{\delta^4\omega_1^2}{3} - \frac{2}{3}\delta^2\zeta_3\zeta_4\omega_1 - 9\delta^2\zeta_4\lambda_2\tau^2 - \frac{4\zeta_3^2\zeta_4^2}{3} + \zeta_4^2\tau^2\omega_1 \right) \quad (5.66)$$

$$\begin{aligned} g_6 = & \frac{1}{108} \left(216\delta^6\zeta_3\lambda_2\omega_1 + 972\delta^6\lambda_2^2\tau^2 + 8\delta^6\omega_1^3 + 864\delta^4\zeta_3^2\zeta_4\lambda_2 + 24\delta^4\zeta_3\zeta_4\omega_1^2 \right. \\ & - 324\delta^4\zeta_4\lambda_2\tau^2\omega_1 - 48\delta^2\zeta_3^2\zeta_4^2\omega_1 + 648\delta^2\zeta_3\zeta_4^2\lambda_2\tau^2 + 72\delta^2\zeta_4^2\tau^2\omega_1^2 - 64\zeta_3^3\zeta_4^3 \\ & + 72\zeta_3\zeta_4^3\tau^2\omega_1 + 324\zeta_4^3\lambda_2\tau^4 \left. \right) \end{aligned} \quad (5.67)$$

which leads to a discriminant locus for the $K3$ of the form

$$\begin{aligned} \Delta_{K3} = & - \left(-3\delta^6\lambda_2 + \delta^4\zeta_4\omega_1 + 2\delta^2\zeta_3\zeta_4^2 + \zeta_4^3\tau^2 \right)^2 \\ & \times \left(96\zeta_3^3\lambda_2 + 4\zeta_3^2\omega_1^2 - 108\zeta_3\lambda_2\tau^2\omega_1 - 243\lambda_2^2\tau^4 - 4\tau^2\omega_1^3 \right) \end{aligned} \quad (5.68)$$

Inspection of this discriminant leads us to an immediate and important observation. As we have constructed this SU(8) solution thus far, it is clear that the discriminant in (5.68)

is *generically singular*. As a result, the generic heterotic dual of the SU(8) F-theory Weierstrass model defined in (5.59) must be generically non-perturbative. But from section 4.1.3 it is clear that *perturbative* SU(8) heterotic theories do exist. It is natural to inquire then, under what circumstances could we find a perturbative heterotic dual to (5.59)? By inspection of (5.68) it is clear that the $K3$ surface will be singular unless the first quadratic factor in the determinant is in fact a constant. Consulting table 5, the degrees of the functions appearing in this term are given by

	δ	λ_2	ζ_4	ω_1	ζ_3	τ
degree	$\frac{r}{2}$	$4 + r - n$	$3r - n$	$4 - r$	$n + 4 - 3r$	$n + 2 - \frac{5r}{2}$

(5.69)

Thus, the quadratic factor $(-3\delta^6\lambda_2 + \delta^4\zeta_4\omega_1 + 2\delta^2\zeta_3\zeta_4^2 + \zeta_4^3\tau^2)^2$ in (5.68) will be a (generically non-zero) constant if $4(r + 1) = n$. In this section $r = 2r_8$ and thus the non-trivial condition on the spectrum is that

$$4(2r_8 + 1) = n \tag{5.70}$$

which is precisely the integer restriction seen to determine the topology of the bundles $\mathcal{V} = \mathcal{L} \oplus \mathcal{L}^\vee$ in section 4.1.3! (See (4.15) and the following equations for the restricted spectrum.) As a result, we see a perfect correspondence between those SU(8) F-theory solutions which have a perturbative heterotic dual, as well as the origin of generically non-perturbative heterotic duals with singular $K3$ surfaces.

All that remains is to match the spectral cover associated to the tuned complex structure in (5.59) to the bundle geometry given in section 4.1.3. What do we expect to happen to the spectral cover in this case? From the bundle analysis in section 4.1.3, we expect the $S[U(2) \times U(1)]$ bundle from (5.30) to decompose further into a sum of line bundles with structure group $S[U(1) \times U(1)]$ as

$$\mathcal{L} \oplus \mathcal{U}_2 \rightarrow \mathcal{L} \oplus (\mathcal{L} \oplus \mathcal{L}^{\vee \otimes 2}) \tag{5.71}$$

In the spectral cover then, one must once again consider the possible overlapping of roots for the order 2 component. Under the tuning given in (5.59) the SU(7) spectral cover of (5.38) specializes further to

$$\begin{aligned} & ((3(\zeta_4\tau)^2 + \delta^2(\delta^2\omega_1 + 4\zeta_3\zeta_4))Z^2 - 3\delta^2X) \times \\ & \left(\frac{1}{3} \left(\delta^2\zeta_3 + \frac{\zeta_4\tau^2}{2} \right) Z ((3\zeta_4\tau^2 + \delta^2(\delta^2\omega_1 + 4\zeta_3\zeta_4))Z^2 - 3\delta^2X) \right. \\ & \left. + \delta^2\tau \left[-3 \left(-\frac{1}{18}(\zeta_4\tau)(\delta^2\omega_1 + 4\zeta_3\zeta_4) + \delta^2 \left(\delta^2\lambda_2\tau - \frac{\zeta_4\tau\omega_1}{6} \right) \right) Z^3 + \zeta_4\tau XZ - \delta Y \right] \right) = 0 \end{aligned} \tag{5.72}$$

The essential structure of the enhanced Mordell-Weil group of both the heterotic $K3$ surface and $Y^{(1)}$ are both preserved in the present case. For the $K3$ geometry, on the patch

$Z = 1$, the rational section is now defined by

$$\left[\begin{aligned} X &\rightarrow \frac{\delta^2 (\delta^2 \omega_1 + 4\zeta_3 \zeta_4) + 3\zeta_4^2 \tau^2}{3\delta^2}, \\ Y &\rightarrow -\frac{\frac{1}{2}\delta^2 \zeta_4 \tau (\delta^2 \omega_1 + 4\zeta_3 \zeta_4) - 3\delta^4 \left(\delta^2 \lambda_2 \tau - \frac{\zeta_4 \tau \omega_1}{6} \right) + \zeta_4^3 \tau^3}{\delta^3}, Z \rightarrow 1 \end{aligned} \right] \quad (5.73)$$

In order to make the correspondence between (5.71) and (5.72), we need to demonstrate that if the line bundle \mathcal{L} corresponds to the rational section S_1 , then $-2S_1$ (corresponding to $\mathcal{L}^{\vee \otimes 2}$) is now a root of (5.71). Given the rational section, S_1 in (5.73), we need first to determine the coordinates of $-2S_1$ under the addition law of the elliptic fiber. This addition is reviewed in appendix A. Using these standard techniques, $-2S_1$ can be found to correspond to the points on the fiber given by

$$\left[\begin{aligned} X &\rightarrow \frac{\delta^2 \zeta_3^2}{\tau^2} + \frac{\zeta_4^2 \tau^2}{4\delta^2} - \frac{1}{3} 2\delta^2 \omega_1 + \frac{\zeta_3 \zeta_4}{3}, \\ Y &\rightarrow \frac{8\delta^6 (\zeta_3^3 - \zeta_3 \tau^2 \omega_1 - 3\lambda_2 \tau^4) + 4\delta^4 \zeta_4 \tau^2 (\zeta_3^2 - \tau^2 \omega_1) - 2\delta^2 \zeta_3 \zeta_4^2 \tau^4 - \zeta_4^3 \tau^6}{8\delta^3 \tau^3}, \\ Z &\rightarrow 1 \end{aligned} \right] \quad (5.74)$$

Finally, then it is possible to substitute this point into the spectral cover given by (5.72) and verify that it vanishes identically. Thus, as expected, the vector bundle has reduced to two copies of the rational section S_1 and one of $-2S_1$ which corresponds exactly to the required bundle geometry: $\mathcal{L} \oplus (\mathcal{L} \oplus \mathcal{L}^{\vee \otimes 2})$.

As pointed out in section 4.2, due to the restricted spectrum imposed by the condition for a smooth $K3$ manifold, (5.70) matter transitions are not possible in this stable degeneration limit. If we allow for singular $K3$ surfaces the analysis is of the same form as those for $SU(7)$ given in section 5.5.4 above.

We turn now to other Higgsing transitions in the language of spectral covers.

5.7 Higgsing on antisymmetric matter

In this section we briefly review the deformations of spectral covers corresponding to Higgsing on antisymmetric matter in the $SU(6), SU(7)$ and $SU(8)$ theories described in the previous sections. Such Higgsing chains are surprisingly simple in the language of spectral covers

Consider a heterotic gauge bundle $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ with reducible structure group $G_1 \times G_2$. As discussed above, this can be described via a (possibly further reducible) spectral cover with structure group $SU(n) \times SU(m)$:

$$\mathcal{S}_1 \cup \mathcal{S}_2 = (a_0 Z^n + \dots a_n X^{\frac{n}{2}})(b_0 Z^m + \dots b_m X^{\frac{m-3}{2}} Y) \quad (5.75)$$

(illustrating here the case where n is even and m is odd).

Then Higgsing on an antisymmetric tensor field corresponds to non-trivial deformations of

either

$$\mathcal{V}_1 \oplus \mathcal{O}_{K3} \rightarrow \mathcal{V}' \tag{5.76}$$

or

$$\mathcal{V}_2 \oplus \mathcal{O}_{K3} \rightarrow V' \tag{5.77}$$

In the language of spectral covers, these deformations correspond to

$$\mathcal{S}_1 \rightarrow (a_0 Z^{n+1} \oplus \dots \oplus a_n Y + a_{n+1} X^{\frac{n-2}{2}} Y) \quad \text{or} \quad \mathcal{S}_2 \rightarrow (b_0 Z^{m+1} \oplus \dots \oplus b_m Y + b_{m+1} X^{\frac{m+1}{2}}) \tag{5.78}$$

controlled by the coefficients a_{n+1} and b_{m+1} respectively.

5.8 Comments on Green-Schwarz massive U(1)s

The nature of the Green-Schwarz massive U(1) symmetries of the previous section provides an intriguing puzzle in heterotic/F-theory duality. In the heterotic theory, the U(1) symmetries are required by the group theory of E_8 subgroups alone. The fact that they generically become massive arises from a separate field theory mechanism: namely the transformation of Kähler axions under U(1) shift symmetries. Since the U(1) symmetry is clearly visible in the Weierstrass model of $Y^{(1)}$ in the stable degeneration limit (5.2) it is natural to ask: how can we understand the origin of its mass term in F-theory? While we leave a systematic study of this question to future work, for now we simply raise two possibilities:

1. In 6-dimensional compactifications of F-theory the presence of massless U(1) symmetries is controlled by the structure of the Mordell-Weil group of Y , the CY 3-fold. Since in all the cases studied here the U(1) is present only in “half” the geometry ($Y^{(1)}$), one should not view this as generating a massless U(1). This agrees with the analysis of [78]. In particular only in the limit where $\epsilon = 0$ would this become truly massless. To some extent this agrees with the expectation in the heterotic theory since the non-trivial mass terms scale as $1/(\text{Kähler modulus})$ [49] and in the limit of strictly infinite volume these would vanish.
2. The geometric origin of the heterotic U(1) mass term is separate from the group theory of the elliptic fibration. Thus, it is possible that in F-theory the mass originates (even in 6-dimensions) from a source entirely separate to the holomorphic geometry of Y (i.e. a “Stueckelberg” mechanism [71, 73–75]).

The clarification of these possibilities is an intriguing area to study further. For now, though we make one final observation: it should be noted that the presence of U(1) symmetries in the stable degeneration limit is itself a subtle thing.⁵ We give a brief illustration of some of the uncertainty that may arise in appendix C.

⁵While this paper was in the final stages of preparation we became aware of the work of [78] which is also focused on the question of massless and Green-Schwarz massive U(1) symmetries in heterotic/F-theory duality and has some overlap with the content of this section and appendix C.

6 Matter transitions in other gauge groups

In the preceding sections we have focused on matter transitions involving gauge groups $SU(N)$ with $N = 6, 7, 8$ and 3-antisymmetric (Λ^3) representations. There are a variety of other situations where similar transitions can occur. Here we explore a few such cases, particularly those related to the $SU(N)$ Λ^3 matter transitions. These include models with $Sp(3)$ and $SO(12)$ gauge groups, and a class of $SU(3)$ transitions involving matter in the symmetric representation, where there is an intricate structure to the Weierstrass model similar to that found recently in [80]. The existence of distinct families of $Sp(3)$, $SO(12)$, and $SU(3)$ models with varying matter content was recognized in [14]; we explore here the explicit connection of these models through matter transitions and comment on generalizations to other related models.

6.1 $Sp(N)$ matter transitions

6.1.1 Field theory

While breaking an $SU(N)$ theory on a pair of k -index antisymmetric representations gives a theory with gauge group $SU(N - k) \times SU(k)$, it is also possible to Higgs $SU(2k)$ on a single Λ^2 representation, giving the breaking

$$SU(2k) \rightarrow Sp(k). \tag{6.1}$$

In this case, the Higgs field takes the VEV $\Phi = J$, where J is the antisymmetric matrix defining $Sp(N)$ through $[h, J] = 0$ for $h \in SU(N)$. For $SU(6)$, this breaking gives a branching of representations $\mathbf{20} \rightarrow \mathbf{14}' + \mathbf{6}$ and $\mathbf{15} \rightarrow \mathbf{14} + \mathbf{1}$. The $SU(6)$ blocks (2.14) thus are Higgsed to $Sp(3)$ blocks

$$(16 + 2n + 3r/2) \times \mathbf{6} + (r/2) \times \mathbf{14}' + (1 + n - r) \times \mathbf{14}. \tag{6.2}$$

The transition (2.9) becomes a transition between $Sp(3)$ representations

$$\frac{1}{2}\mathbf{14}' + \frac{3}{2}\mathbf{6} \leftrightarrow \mathbf{14} + 2 \times \mathbf{1}. \tag{6.3}$$

Similarly, for $SU(8)$ there is a breaking

$$SU(8) \rightarrow Sp(4) \tag{6.4}$$

$$\mathbf{56} \rightarrow \mathbf{48} + \mathbf{8} \tag{6.5}$$

$$\mathbf{28} \rightarrow \mathbf{27} + \mathbf{1} \tag{6.6}$$

$$\mathbf{8} \rightarrow \mathbf{81}, \tag{6.7}$$

from which we expect an $Sp(4)$ transition that follows from (2.11)

$$\mathbf{48} + 10 \times \mathbf{8} \leftrightarrow 4 \times \mathbf{27} + 20 \times \mathbf{1}. \tag{6.8}$$

Note that the $\mathbf{70}$ of $SU(8)$ branches to $\mathbf{42} + \mathbf{27} + \mathbf{1}$, so we would expect $SU(8)$ with a Λ^4 representation to break to an $Sp(4)$ with a matter field in the $\mathbf{42}$ representation.

Representation	Cohomology	Multiplicity
14'	$H^1(\mathcal{V}_2)$	$c_2(\mathcal{V}_2) - 4$
6	$H^1(\mathcal{V}_2 \otimes \mathcal{V}'_2)$	$7c_2(\mathcal{V}_2) + 2c_2(\mathcal{V}'_2) - 28$
14	$H^1(\mathcal{V}'_2)$	$c_2(\mathcal{V}'_2) - 14$
1	$H^1(\text{End}_0(\mathcal{V}_2)) \oplus H^1(\text{End}_0(\mathcal{V}'_2))$	$(4c_2(\mathcal{V}_2) - 6) + (4c_2(\mathcal{V}'_2) - 28)$

Table 20. The cohomology associated to each representation of the low-energy gauge group $\text{Sp}(3)$.

6.1.2 F-theory

From the F-theory point of view, the $\text{Sp}(k)$ and $\text{SU}(2k)$ models are very closely related. Both come from a Kodaira type I_k singularity, the only difference is whether it is a “split” type singularity or not. At the level of the Weierstrass model, the Higgsing is achieved by allowing ϕ_0^2 to deform into an irreducible polynomial ϕ in the tunings of section 3.2. This deformation must be associated with an analogous deformation of α, β in (3.10). Specifically, α^2 is allowed to deform into the irreducible polynomial h , while β is unchanged. With this modification, all the tuning, transition, and Higgsing analysis for $\text{SU}(6)$ goes through unchanged for $\text{Sp}(3)$. The story would be more complicated for $\text{SU}(8)$, where α and β themselves are decomposed into further components.

6.1.3 Heterotic description

From the heterotic point of view the $\text{Sp}(N)$ matter transitions can be computed using techniques similar to those in section 4. Looking at the $\text{Sp}(3)$ case as an example, the relevant group theory is as follows.

$$E_8 \supset \text{Sp}(3) \times \text{SU}(2) \times G_2 \quad (6.9)$$

$$\mathbf{248} = (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{21}, \mathbf{1}, \mathbf{1}) + (\mathbf{14}', \mathbf{2}, \mathbf{1}) + (\mathbf{6}, \mathbf{2}, \mathbf{7}) + (\mathbf{14}, \mathbf{1}, \mathbf{7}) + (\mathbf{1}, \mathbf{1}, \mathbf{14}) \quad (6.10)$$

We denote the $\text{SU}(2)$ gauge bundle by \mathcal{V}_2 and the G_2 gauge bundle by \mathcal{V}'_2 . We then have the correspondence between representations and cohomologies, and derive from this the multiplicities of matter representations, as given in table 20.

Computing the total number of hyper multiplets from the data in table 20 we obtain the familiar anomaly cancelation condition.

$$\begin{aligned} n_H + 29n_T - n_V &= 273 \quad (6.11) \\ \Rightarrow c_2(\mathcal{V}_2) + \frac{1}{2}c_2(\mathcal{V}'_2) + \frac{1}{60}c_2(\text{End}_0(\mathcal{V}_{E_8})) &= 24 \end{aligned}$$

From equation (6.11) and table 20 we see that the following matter transition between full hypermultiplets is possible upon small instanton transition (recalling that all of the representations involved are real).

$$\frac{1}{2} \times \mathbf{14}' + \frac{3}{2} \times \mathbf{6} \leftrightarrow \times \mathbf{14} + 2 \times \mathbf{1} \quad (6.12)$$

This is precisely of the form (6.3) expected from field theory considerations.

6.2 $SO(N)$ models

There are two ways in which $SO(N)$ models may exhibit transitions like those described for $SU(N)$ groups in the earlier sections. One is for $SO(N)$ models with matter in the analogue of the Λ^3 representation. For example, for $SO(12)$, which has the self-conjugate spinorial representations $\mathbf{32}$ and $\mathbf{32}'$, there are blocks

$$(r/2) \times \mathbf{32} + \frac{4+n-r}{2} \times \mathbf{32}' + (n+8) \times \mathbf{12}, \quad (6.13)$$

and associated transitions

$$\frac{1}{2}\mathbf{32} \leftrightarrow \frac{1}{2}\mathbf{32}'. \quad (6.14)$$

These representations branch to the representations in the $SU(6)$ transition (2.9) under the embedding $SU(6) \subset SO(12)$. In the F-theory picture this transition follows much like the $SU(6)$ transition, and can be constructed by taking $\alpha \rightarrow 0$ in the general $SU(6)$ tuning (3.16), (3.17). The $\frac{1}{2}\mathbf{32}$ and $\frac{1}{2}\mathbf{32}'$ representations are respectively given by the loci $\beta = \sigma = 0$ and $\nu = \sigma = 0$, and the transition involves transferring factors between ν, λ and β, ϕ_2 . Like the previous transitions considered, the $SO(12)$ transition involves passing through a superconformal point. In the heterotic picture the $SO(12)$ models can be constructed by taking an E_8 bundle with structure group $SU(2) \times SU(2)$, as explained in section 5.3. These models, and similar constructions for $SO(14)$, etc. are similar in principle to the models already considered, with similar features on both the F-theory and heterotic sides when applicable.

The heterotic picture, however, suggests another kind of $SO(N)$ transition that may occur when we construct bundles with a product structure group for the $SO(32)$ theory. For example, consider a $SO(32)$ heterotic compactification on $K3$ in the presence of a gauge bundle with $SO(6) \times SU(2)$ structure group. First we need the appropriate group theory.

$$\begin{aligned} SO(32) \supset SO(22) \times SU(2) \times SO(6) \times SU(2) & \quad (6.15) \\ \mathbf{496} = (\mathbf{231}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{22}, \mathbf{1}, \mathbf{6}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{15}, \mathbf{1}) + (\mathbf{22}, \mathbf{2}, \mathbf{1}, \mathbf{2}) \\ & + (\mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{2}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) \end{aligned}$$

We will take the last two factors above to be those associated to the gauge bundle. In particular we will describe the situation in terms of a $SU(4)$ vector bundle \mathcal{V}_1 and an $SU(2)$ bundle \mathcal{V}_2 . We are using an $SU(4)$ rather than $SO(6)$ structure group for ease of description. We must account for this, however, in matching cohomologies to representations in the decomposition (6.15) with the relevant cohomologies for determining matter multiplicities, which are presented in table 21.

The anomaly cancellation condition $n_H + 29n_T - n_V = 273$ results in the following condition in this case

$$c_2(\mathcal{V}_1) + c_2(\mathcal{V}_2) = 24. \quad (6.16)$$

Equation (6.16) is of course simply the 10D anomaly cancellation condition as it should be.

Representation	Cohomology	Multiplicity
$(\mathbf{22}, \mathbf{1})$	$H^1(\wedge^2 \mathcal{V}_1)$	$2c_2(\mathcal{V}_1) - 12$
$(\mathbf{22}, \mathbf{2})$	$H^1(\mathcal{V}_2)$	$c_2(\mathcal{V}_2) - 4$
$(\mathbf{1}, \mathbf{2})$	$H^1(\wedge^2 \mathcal{V}_1 \otimes \mathcal{V}_2)$	$4c_2(\mathcal{V}_1) + 6c_2(\mathcal{V}_2) - 24$
$(\mathbf{1}, \mathbf{1})$	$H^1(\text{End}_0(\mathcal{V}_1)) \oplus H^1(\text{End}_0(\mathcal{V}_2))$	$8c_2(\mathcal{V}_1) - 30 + 4c_2(\mathcal{V}_2) - 6$

Table 21. The cohomology associated to each representation of the low-energy gauge group $\text{SO}(22) \times \text{SU}(2)$.

Using (6.16) and table 21, we see that, in terms of full multiplets, the following kind of matter change can be implemented by small instanton transitions.

$$(\mathbf{22}, \mathbf{1}) + 2 \times (\mathbf{1}, \mathbf{1}) \leftrightarrow \frac{1}{2}(\mathbf{22}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}) \tag{6.17}$$

But small instantons behave differently in $\text{SO}(32)$ theories than in $\text{E}_8 \times \text{E}_8$ theories. Unlike the superconformal points of the $\text{E}_8 \times \text{E}_8$ models discussed earlier, the $\text{SO}(32)$ small instanton point leads to a new $\text{SU}(2)$ symmetry that can be analyzed with field theory [66]. The $\text{SO}(22) \times \text{SU}(2)$ transition can therefore be understood completely in terms of Higgsing and unHiggsing transitions (see also [81, 82] for matter transitions of a similar nature), as can be seen in its F-theory realization.

The matter content of table 21 suggests an F-theory compactification with base \mathbb{F}_4 where the $\text{SO}(22)$ symmetry is tuned on a curve $u = 0$ of divisor class S . In fact, F-theory models dual to $\text{SO}(32)$ heterotic string theory can only be realized on \mathbb{F}_4 [83]. Meanwhile, the $\text{SU}(2)$ symmetry should be tuned on a curve $\sigma = 0$ of divisor class $S + (4 + r)F$, with $r = c_2(\mathcal{V}_2) - 4$ denoting the number of $(\mathbf{22}, \mathbf{2})$ half-multiplets. The global Weierstrass model is then

$$y^2 = x^3 + \left(-\frac{1}{48}\Phi^2 u^2 + F_1 \sigma u^6 \right) x + \left(\frac{1}{864}\Phi^3 u^3 - \frac{1}{12}F_1 \Phi \sigma u^7 + \gamma_2^2 \sigma^2 u^{10} \right). \tag{6.18}$$

Φ , F_1 , and γ_2 are respectively sections of $\mathcal{O}(3S + 12F)$, $\mathcal{O}(S + (20 - r)F)$, and $\mathcal{O}(14 - r)F$.

For a transition where r increases, F_1 and γ_2 develop a common factor a that is a section of $\mathcal{O}(F)$ ($F_1 \rightarrow aF_1, \gamma_2 \rightarrow a\gamma_2$). a is then absorbed into σ ($a\sigma \rightarrow \sigma'$), and the divisor class of $\sigma = 0$ changes from $\tilde{S} + rF$ to $\tilde{S} + (r + 1)F$. Immediately after absorbing a , σ is a reducible curve. σ can then be deformed into a non-reducible curve, completing the transition. To perform the transition in the reverse direction, we let σ become $a\sigma$ and reabsorb a into F_1 and γ_2 (i.e. $aF_1 \rightarrow F_1'$ and $a\gamma_2 \rightarrow \gamma_2'$).

At the transition point, there is an additional I_2 singularity on the $a = 0$ locus, signaling the expected appearance of a new $\text{SU}(2)$ symmetry. We will refer to this new symmetry as $\text{SU}(2)_a$ to distinguish it from the original $\text{SU}(2)$ tuned on $\sigma = 0$. There is also matter charged under $\text{SU}(2)_a$, as $a = 0$ intersects the curves $u = 0$ and $\sigma = 0$ once. In terms of the $\text{SO}(22) \times \text{SU}(2) \times \text{SU}(2)_a$ representations, this charged matter consists of a half-multiplet of $(\mathbf{22}, \mathbf{1}, \mathbf{2}_a)$ matter and a full $(\mathbf{1}, \mathbf{2}, \mathbf{2}_a)$ multiplet; the a subscript is used

to identify the $SU(2)_a$ representations. The two sides of the transition correspond to the two ways of Higgsing $SU(2)_a$. Giving a VEV to the $(\mathbf{1}, \mathbf{2}, \mathbf{2}_a)$ multiplet merges to the two $SU(2)$ symmetries into a single, diagonal $SU(2)$, and the $(\mathbf{22}, \mathbf{1}, \mathbf{2}_a)$ half-multiplet reduces to a half-multiplet of bifundamental $(\mathbf{22}, \mathbf{2})$ matter. But when the $(\mathbf{22}, \mathbf{1}, \mathbf{2}_a)$ half-multiplet is given a VEV, $SU(2)_a$ is Higgsed separately from the original $SU(2)$. As a result, the $(\mathbf{22}, \mathbf{1}, \mathbf{2}_a)$ half-multiplet is left as a full $(\mathbf{22}, \mathbf{1})$ multiplet. Tracking the matter fully, the total transition can be summarized as

$$(\mathbf{22}, \mathbf{1}) + 2 \times (\mathbf{1}, \mathbf{2}) + 3 \times (\mathbf{1}, \mathbf{1}) \leftrightarrow \frac{1}{2}(\mathbf{22}, \mathbf{2}) + 3 \times (\mathbf{1}, \mathbf{2}) + 1 \times (\mathbf{1}, \mathbf{1}), \quad (6.19)$$

reproducing the net matter change of equation (6.17). Even though the transition consists only of Higgsing and unHiggsing transitions, the fact that 29 multiplets participate in the transition suggests a parallel structure to the matter transitions mediated through superconformal points, which may reflect the underlying small-instanton behavior.

A plethora of exotic transitions of this type are possible in different compactifications of the $SO(32)$ heterotic string. As in the $SO(22) \times SU(2)$ model, these transitions should be described by phenomena accessible from field theory.

6.3 $SU(3)$ with symmetric matter

A particularly interesting set of exotic matter representations are the representations of $SU(N)$ that have Young diagrams with more than one column. As described in [13], there is a natural quantity g associated with any representation R of a simple Lie group G that plays the role of a “genus” of the representation. The geometric interpretation of this quantity is conjectured to be that when $g > 0$, for any representation other than the adjoint, the representation R is realized in F-theory through a Kodaira singularity on a divisor D that is itself singular, where g represents the arithmetic genus contribution of the singularity to the curve D in the 6D case. This correspondence works most simply for the symmetric representation of $SU(N)$, which has $g = 1$, and which can be realized on a double point singularity of D as first suggested by Sadvov [84], described further in [5], and recently confirmed through an explicit F-theory construction [80]. The explicit F-theory construction of the symmetric matter representation of $SU(3)$ has the unusual feature that the Weierstrass model cannot be built from a standard generic Tate $SU(3)$ construction; rather, the vanishing of the discriminant to order 3 follows from a nontrivial cancellation that involves the explicit algebraic structure of the singular divisor locus carrying the $SU(3)$ gauge group. Understanding matter transitions in this context gives further insight into this story.

We can realize a symmetric representation of $SU(3)$ by breaking $Sp(3)$ into $SU(3)$, or more directly by breaking $SU(6)$ into $SU(3) \times SU(3)$ and then breaking

$$SU(3) \times SU(3) \rightarrow SU(3) \quad (6.20)$$

by Higgsing a bifundamental field. The anomaly equivalent matter representations in the resulting theories exchange an adjoint (plus a singlet) with a symmetric and an antisymmetric matter representation

$$\mathbf{8} + \mathbf{1} \leftrightarrow \mathbf{6} + \mathbf{3}. \quad (6.21)$$

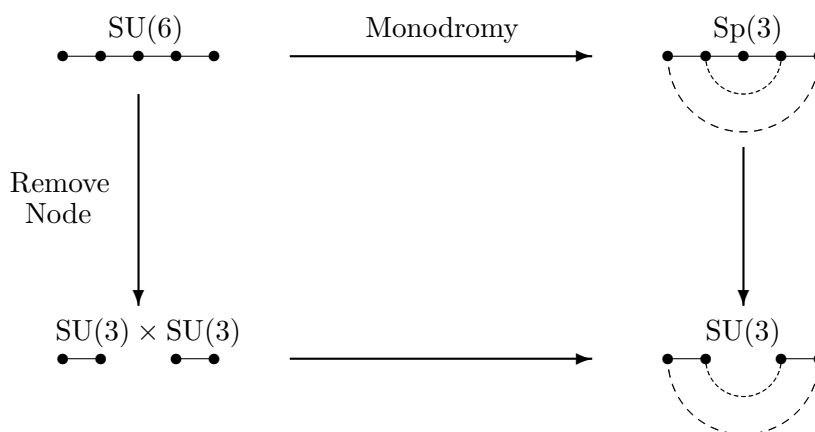


Figure 2. Higgsing chain involving $SU(6)$, $Sp(3)$, $SU(3) \times SU(3)$, and $SU(3)$ along with associated Dynkin diagrams. Dotted lines indicate nodes exchanged under monodromy. The Higgsing chain involves two types of F-theory deformations that either introduce monodromy or remove the central node in the Dynkin diagrams. The $SU(3)$ singularity occurs when both types of deformations are performed.

Thus, the generic $SU(3)$ model with g adjoints and $b \cdot b = n$ (2.13) has anomaly equivalent variations

$$(g - r)\mathbf{8} + r\mathbf{6} + (18 + 6n - 18g + r)\mathbf{3}. \quad (6.22)$$

In F-theory,⁶ the Higgsing chain for $SU(6)$, $Sp(3)$, and $SU(3) \times SU(3)$, illustrated in figure 2, involves two distinct deformations. $SU(6)$ is Higgsed to $SU(3) \times SU(3)$ by removing the central node in the A_5 Dynkin diagram. Meanwhile, the $Sp(3)$ model is produced by introducing monodromy effects in the $SU(6)$ model that cause a \mathbb{Z}_2 folding of the A_5 Dynkin diagram. The $SU(3)$ symmetry is produced by applying both deformations to $SU(6)$. One can think of the $SU(3)$ singularity as consisting of two A_2 singularities that are mapped onto each other through the monodromy-induced \mathbb{Z}_2 folding. Thus, the two $SU(3)$ algebras in the $SU(3) \times SU(3)$ product model reduce to a single $SU(3)$ algebra.

For an explicit construction of $SU(3)$, consider an F-theory compactification with base \mathbb{F}_m and an $SU(6)$ singularity tuned on a curve $\sigma = 0$ of divisor class \tilde{S} . Using a strategy similar to that of section 3.4.3, the $SU(6)$ symmetry can then be Higgsed to $SU(3) \times SU(3)$, with the two $SU(3)$ singularities tuned on the curves

$$\sigma \pm \frac{\alpha\epsilon_1}{2} = 0.$$

This situation could be thought of as a single A_2 singularity tuned on the reducible curve

$$\sigma^2 - \frac{\alpha^2\epsilon_1^2}{4} = 0;$$

⁶Here, we essentially describe the F-theory realization of the level-two $SU(3)$ (or $SU(3)_2$) discussed in [14]. However, our notation differs slightly. We take n to refer to $b \cdot b$ or the self-intersection number of the curve with the $SU(3)$ singularity; in [14], n refers to the base \mathbb{F}_n . r is also smaller by 2 in our conventions.

since this curve can be reduced into the product of two components, it actually represents two distinct $SU(3)$ algebras. The reducible curve can then be smoothed into a non-factorizable quadratic polynomial in σ , thereby reducing the product group to a single $SU(3)$. Specifically, α always appears in even powers in the $SU(3) \times SU(3)$ Weierstrass model, so α^2 can be consistently replaced with a parameter h . Note that this deformation is the same as that for the $SU(6) \rightarrow Sp(3)$ Higgsing process described earlier. Assuming that neither h nor ϵ_1 is constant, the A_2 singularity is now tuned on the non-factorizable curve

$$\sigma^2 - \frac{h\epsilon_1^2}{4} = 0, \tag{6.23}$$

so the resulting gauge algebra is a single $SU(3)$ tuned on the divisor class $2\tilde{S}$. The corresponding Weierstrass model has

$$f = -\frac{1}{48} \left(h\beta^2 + \epsilon_1^2\nu^2 + 4\beta\nu\sigma \right)^2 - \frac{1}{6} \left(9\epsilon_1^2\lambda\nu + h\beta\phi_2 + (18\beta\lambda + 2\nu\phi_2)\sigma \right) \left(\sigma^2 - \frac{h\epsilon_1^2}{4} \right) + (f_4 + f_5\sigma) \left(\sigma^2 - \frac{h\epsilon_1^2}{4} \right)^2 + \mathcal{O} \left(\sigma^2 - \frac{h\epsilon_1^2}{4} \right)^3 \tag{6.24}$$

and

$$g = \frac{1}{864} \left(h\beta^2 + \epsilon_1^2\nu^2 + 4\beta\nu\sigma \right)^3 + \frac{1}{72} \left(h\beta^2 + \epsilon_1^2\nu^2 + 4\beta\nu\sigma \right) \left(9\epsilon_1^2\lambda\nu + h\beta\phi_2 + (18\beta\lambda + 2\nu\phi_2)\sigma \right) \left(\sigma^2 - \frac{h\epsilon_1^2}{4} \right) + \frac{1}{36} \left(81\epsilon_1^2\lambda^2 + h\phi_2^2 + 36\lambda\phi_2\sigma - 3(f_4 + f_5\sigma)(h\beta^2 + \epsilon_1^2\nu^2 + 4\beta\nu\sigma) \right) \left(\sigma^2 - \frac{h\epsilon_1^2}{4} \right)^2 + \mathcal{O} \left(\sigma^2 - \frac{h\epsilon_1^2}{4} \right)^3 \tag{6.25}$$

Taking the $\epsilon_1 \rightarrow 0$ limit recovers the $Sp(3)$ model, even though the $Sp(3)$ model was never directly used to find the $SU(3)$ Weierstrass tuning. This fact confirms that our $SU(3)$ tuning agrees with the Higgsing chain given in figure 2, as we could reach the $Sp(3)$ model indirectly via $SU(3) \times SU(3)$ and $SU(3)$. Moreover, we can directly see that the monodromy “inherited” from the $Sp(3)$ model is crucial in the $SU(3)$ tuning. The $SU(3)$ singularity could alternatively be thought of as two A_2 singularities tuned on the two curves

$$\sigma \pm \frac{h^{1/2}\epsilon_1}{2} = 0.$$

However, the curves would interchange under the transformation $h \rightarrow e^{2\pi i}h$, the same transformation involved in the $Sp(3)$ monodromy, and the two A_2 subdiagrams should be identified with one another. We are therefore left with a single $SU(3)$ algebra, with the $h \rightarrow e^{2\pi i}h$ transformation providing the \mathbb{Z}_2 folding depicted in figure 2.

Turning to the matter content, a curve with divisor class $2\tilde{S}$ has a genus g given by

$$g = 1 + \frac{1}{2} \left(2\tilde{S} \right) \cdot \left(K_B + 2\tilde{S} \right) = m - 1. \tag{6.26}$$

The $SU(3)$ model therefore has a total of $m - 1$ charged multiplets in either the adjoint or symmetric representation. Distinguishing between the two representations requires examining the two possible sources of double point singularities in the $2\tilde{S}$ curve: double point singularities that can be deformed away contribute adjoint ($\mathbf{8}$) matter, whereas non-deformable double point singularities gives one $\square\square$ ($\mathbf{6}$) multiplet and one fundamental ($\mathbf{3}$) multiplet. The situation is the Higgsed version of a similar feature in the $SU(3) \times SU(3)$ model, where there are subtle differences between the $(\mathbf{3}, \bar{\mathbf{3}})$ and $(\bar{\mathbf{3}}, \mathbf{3})$ bifundamental representations [5]. There, the distinction between the two bifundamental representations does not have significant physical implications. But upon Higgsing to $SU(3)$, the two bifundamentals branch to dramatically different representations, turning into either $\mathbf{8} + \mathbf{1}$ or $\mathbf{6} + \mathbf{3}$. From an F-theory perspective, monodromy identifies the nodes of the two A_2 subdiagrams in a particular way. With this identification, the redefinitions of the gauge algebras that made the bifundamental representations essentially equivalent are no longer valid.

We can now find the specific double-point singularities that contribute symmetric matter. h can have perfect square factors that lead to double point singularities. But since h is not required to have any perfect square factors, such double points can be deformed away by modifying the form of h . These double point singularities therefore contribute localized adjoints. The zeroes of ϵ_1 lead to double point singularities as well, but these double points cannot be removed by simply deforming one of the free parameters. Letting r be the order of ϵ_1 , ϵ_1 contributes r multiplets of $\square\square$ ($\mathbf{6}$) matter and r fundamental multiplets. To find the additional fundamentals provided by the discriminant loci, it is easiest to expand the discriminant around $\sigma \pm \frac{1}{2}h^{1/2}\epsilon_1$. When expanded around $\sigma + \frac{1}{2}h^{1/2}\epsilon_1$, there are $3m + 18$ codimension two loci where the discriminant vanishes to order 4; there are the same number of loci when the discriminant is expanded around $\sigma - \frac{1}{2}h^{1/2}\epsilon_1$. A total of $6m + 36$ fundamentals therefore come from the discriminant. Considering all of these contributions, the $SU(3)$ models have r $\mathbf{6}$ multiplets, $m - 1 - r$ $\mathbf{8}$ multiplets, and $6m + 36 + r$ $\mathbf{3}$ multiplets. Noting that $2\tilde{S}$ has self-intersection number $n = 4m$ and genus $g = m - 1$, this is in agreement with (6.22).

It is tempting to transfer factors from h to ϵ_1 in order to systematically introduce non-deformable double-point singularities. If this were possible, we could transform adjoint matter into symmetric matter. However, there are terms in the Weierstrass model of equations (6.24) and (6.25) that depend on ϵ_1 and not on h , or vice versa. The only consistent way to transfer factors from h to ϵ_1 is to use a matter transition similar to that of $SU(6)$, which requires moving through a superconformal point. Suppose we wish to convert an adjoint $\mathbf{8}$ multiplet to a $\mathbf{6}$ multiplet (along with other fundamentals and singlets). Just as in the $SU(3) \times SU(3)$ transition, h, ν , and λ develop common factors:

$$h \rightarrow a^2 h', \tag{6.27}$$

$$\nu \rightarrow a \nu', \tag{6.28}$$

$$\lambda \rightarrow a \lambda'. \tag{6.29}$$

Note that there is now superconformal point at the locus $a = \sigma = 0$. ϵ_1, β , and ϕ_2 then

absorb the common factor:

$$a\epsilon_1 \rightarrow \epsilon'_1, \tag{6.30}$$

$$a\beta \rightarrow \beta', \tag{6.31}$$

$$a\phi_2 \rightarrow \phi'_2. \tag{6.32}$$

We have thus introduced a non-deformable double point singularity through the matter transition. Overall, the transition is summarized as

$$\mathbf{Adj} + 6 \times \square + 3 \times \mathbf{1} \rightarrow \mathbf{Superconformal Matter} \rightarrow \square\square + 7 \times \square + 2 \times \mathbf{1}, \tag{6.33}$$

giving a net matter change of

$$\mathbf{Adj} + \mathbf{1} \rightarrow \square\square + \square. \tag{6.34}$$

Note the usual appearance of 29 matter fields in the spectrum on each side of the full transition (6.33).

Finally, we note that the Weierstrass model of equations (6.24) and (6.25) has a “non-Tate” structure. The SU(3) tuning cannot be written in the generic SU(3) form given in [5]. Instead, the curve with the A_2 singularity has a specific structure that depends on variables used in the tuned Weierstrass coefficients. The coefficients then conspire to ensure all terms in the discriminant are proportional $(\sigma^2 - \frac{h\epsilon_1^2}{4})^3$. But for the special case with no symmetric matter, the SU(3) tuning can be written in the standard form. If $r = 0$, ϵ_1 is a non-zero constant, and we can set ϵ_1 to 1 without loss of generality. With this simplification, f and g can be rewritten as

$$f = \frac{-1}{48}\Phi_0^4 + \frac{1}{2}\Phi_0\Psi_0\left(\sigma^2 - \frac{h}{4}\right) + F_2\left(\sigma^2 - \frac{h}{4}\right)^2 + \mathcal{O}\left(\sigma^2 - \frac{h}{4}\right)^3, \tag{6.35}$$

$$g = \frac{1}{864}\Phi_0^6 - \frac{1}{24}\Phi_0^3\Psi_1\left(\sigma^2 - \frac{h}{4}\right) + \left(\frac{\Psi_1^2}{4} - \frac{1}{12}F_2\Phi_0^2\right)\left(\sigma^2 - \frac{h}{4}\right)^2 + \mathcal{O}\left(\sigma^2 - \frac{h}{4}\right)^3, \tag{6.36}$$

where

$$\Phi_0 = \nu + 2\beta\sigma, \tag{6.37}$$

$$\Psi_1 = -3\lambda - \frac{2}{3}\phi_2\sigma + \frac{1}{3}\beta^2\Phi_0, \tag{6.38}$$

$$F_2 = f_4 + f_5\sigma - \frac{1}{3}\beta^4 + \frac{2}{3}\beta\phi_2. \tag{6.39}$$

In fact, equations (6.35) and (6.36) are in the standard SU(3) forms given in [5].

This behavior parallels that observed in the SU(3) models derived in [80]. There, all of the higher-genus SU(3) models with symmetric matter had non-Tate structures. The construction presented here further supports the idea that non-Tate structures are necessary for symmetric matter; indeed, our models can only be expressed in standard forms for exactly those cases without symmetric matter. However, our SU(3) tuning seems to be different from the tuning given in [80]. The connection between these classes of models is left as a question for future investigations.

Representation	Cohomology	Multiplicity
1	$H^1(\text{End}_0(\mathcal{V}_2)) \oplus H^1(\text{End}_0(\mathcal{V}_3))$	$(4c_2(\mathcal{V}_2) - 28) + (6c_2(\mathcal{V}_3) - 16)$
3	$H^1(\mathcal{V}_2 \otimes \mathcal{V}_3^\vee)$	$(3c_2(\mathcal{V}_2) + 7c_2(\mathcal{V}_3) - 42)$
$\bar{\mathbf{3}}$	$H^1(\mathcal{V}_2 \otimes \mathcal{V}_3)$	$(3c_2(\mathcal{V}_2) + 7c_2(\mathcal{V}_3) - 42)$
6	$H^1(\mathcal{V}_3)$	$c_2(\mathcal{V}_3) - 6$
$\bar{\mathbf{6}}$	$H^1(\mathcal{V}_3^\vee)$	$c_2(\mathcal{V}_3) - 6$
8	$H^1(\mathcal{V}_2)$	$c_2(\mathcal{V}_2) - 14$

Table 22. The cohomology associated to each representation of the low-energy gauge group SU(3).

Matter transitions in SU(3) models with symmetric matter can also be realized within the heterotic context. The relevant group theory in this instance is as follows.

$$E_8 \supset G_2 \times \text{SU}(3) \times \text{SU}(3) \tag{6.40}$$

$$\mathbf{248} = (\mathbf{14}, \mathbf{1}, \mathbf{1}) + (\mathbf{7}, \mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{8}) + (\mathbf{7}, \bar{\mathbf{3}}, \mathbf{3}) + (\mathbf{1}, \mathbf{6}, \mathbf{3}) + (\mathbf{7}, \mathbf{3}, \bar{\mathbf{3}}) + (\mathbf{1}, \bar{\mathbf{6}}, \bar{\mathbf{3}})$$

We denote the G_2 gauge bundle by \mathcal{V}_2 and the SU(3) gauge bundle by \mathcal{V}_3 . It should be noted that we choose the second of the two SU(3)'s to be associated with the bundle structure group. We then have the multiplicities of representations in the six-dimensional theory given in table 22.

The anomaly cancelation condition in this case gives the relations

$$n_H + 29n_T - n_V = 273 \tag{6.41}$$

$$\Rightarrow \frac{1}{2}c_2(\mathcal{V}_2) + c_2(\mathcal{V}_3) + \frac{1}{60}c_2(\text{End}_0(\mathcal{V}_{E_8})) = 24. \tag{6.42}$$

Equation (6.42) is of course simply the 10D anomaly cancelation condition as one would expect.

Given equation (6.42) and the matter multiplicities given in table 22 we arrive at the following matter transition induced by small instanton transitions in ten dimensions.

$$\mathbf{1} + \mathbf{8} \leftrightarrow \mathbf{3} + \mathbf{6} \tag{6.43}$$

This is exactly of the form given in equation (6.21) in the six-dimensional field theory discussion.

7 Conclusions

7.1 Summary of results

Novel matter transitions. We have identified a new class of field theory transitions through which matter fields in one set of representations transform into matter fields in another set of representations without changing the gauge group. We have explicitly described these transitions in 6D models where the field theory is coupled to gravity, from

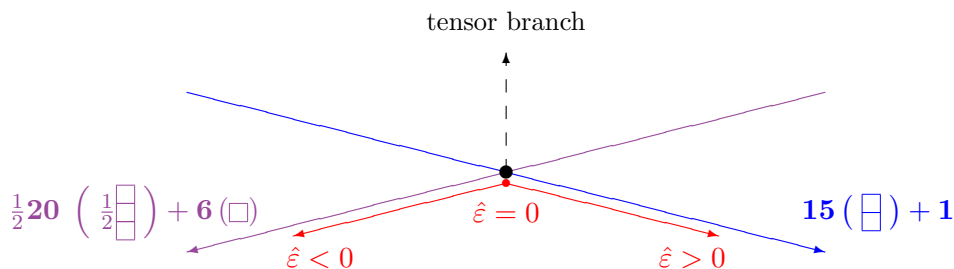


Figure 3. The matter transitions studied in this paper can be seen as arising along one-parameter families of theories, with the transition point at $\hat{\epsilon} = 0$, and field theories with the same gauge group but distinct matter representation content at $\hat{\epsilon} > 0$ and $\hat{\epsilon} < 0$. For these matter transitions in 6D theories, the transition point at $\hat{\epsilon} = 0$ is a superconformal field theory, from which an additional tensor branch generally extends, as well as separate branches for each of the field theories with distinct matter contents, though the tensor branch is incidental to the matter transition. The matter representations shown are for the simplest $SU(6)$ matter transition.

both the heterotic and F-theory perspectives. These pictures match, and agree with constraints from gauge and gravitational anomaly conditions in 6D. In both pictures these transitions involve passing through a superconformal fixed point (figure 3), at which an infinite family of light fields appear and the simple perturbative low-energy field theory picture breaks down. Similar transitions should be possible in 4D theories, though in some cases may be obstructed by a superpotential (see section 7.2). The simplest example we have studied is the transition between a matter field in the three-index antisymmetric (Λ^3) representation of $SU(6)$, along with a fundamental ($\frac{1}{2}\mathbf{20} + \mathbf{6}$) to a two-index antisymmetric representation and a singlet ($\mathbf{15} + \mathbf{1}$). Similar transitions occur for other groups such as $SU(7)$, $Sp(3)$, $SO(12)$, and $SU(3)$.

Heterotic picture. In the heterotic picture, these transitions occur by moving an instanton between different simple factors in the structure group of a single E_8 bundle, such as $SU(3) \times SU(2)$ in the $SU(6)$ Λ^3 case. Such transitions had not been previously explored systematically in heterotic theories. Several novel features arise in these heterotic models. For the $SU(7)$ and $SU(8)$ models with Λ^3 matter, the necessary bundles are only possible for a fixed subset of the moduli space of K3 compact spaces, giving a nontrivial coupling between bundle and complex structure moduli of the K3 surface. We find a new form of the stable degeneration limit in these cases. The $SU(7)$ construction also leads to an increased Mordell-Weil rank on one side of the stable degeneration limit, associated with a massive $U(1)$ field in the low-energy theory.

F-theory picture. In the F-theory picture, we have an explicit description of these transitions in terms of Weierstrass models. At the transition points between models with different representations, codimension two singular loci L develop in the Weierstrass model where f and g vanish to orders 4 and 6, signaling SCFTs in the low-energy theory. Matter transitions occur when the Weierstrass model moves from one branch to another without changing the geometry of the compactification base B . Tensionless string transitions, on

the other hand, are associated with a blowup of the locus L and a corresponding change in the topology of the base B that in 6D changes the number of tensor multiplets. The Weierstrass models for all theories with non-generic matter representations (e.g., for $SU(N)$ anything but the fundamental, Λ^2 and adjoint representations) are realized in a way that is not captured by the simple general Tate form for the associated gauge group.

Heterotic/F-theory duality. In our analysis we have carried out a careful matching of the degrees of freedom in the spectral cover construction of the heterotic bundles with the parameters in the F-theory Weierstrass model. Matching these parameters gives a clear picture of the duality between these classes of models and illuminates many features such as the appearance of SCFTs at the transition points and the matching of constraints between the two pictures. We view this work as important first step into extending heterotic/F-theory duality to include more complex and phenomenologically relevant Calabi-Yau geometries and vector bundles (including, for example the geometries in [85–88] which involve bundles with reducible structure groups and Green-Schwarz massive $U(1)$ symmetries).

Higgsing deformations. In the process of constructing models with transitions, we have discussed the F-theory and heterotic manifestations of a variety of Higgsing processes. On the F-theory side, this analysis involved identifying explicit Weierstrass deformations for particular Higgsing processes. For instance, our analysis has described the F-theory deformations that correspond to giving VEVs to antisymmetric representations of $SU(N)$. Examining the Higgsing connections between different models clarified how the transitions are related by Higgsing and allowed us to investigate transitions in product group models. Moreover, the explicit F-theory deformations illuminate how the F-theory degrees of freedom correspond to those of the low-energy theory, at least in six dimensions. Further investigations into Higgsing deformations could help to develop a more explicit dictionary between F-theory, the heterotic effective theory, and their low-energy limits.

Higher genus matter. An interesting subject of study in recent work is the appearance of “higher genus” matter in 6D supergravity theories. Each representation R of a group G can be associated with a genus contribution g , which in F-theory should have the interpretation of an arithmetic genus contribution from a singularity in the divisor supporting the gauge factor G . We have found an explicit example of such a realization in models with the symmetric $(\mathbf{6})$ representation of $SU(3)$. These models are connected through a matter transition to other models with an adjoint representation, but are realized, as in [80], by non-Tate Weierstrass models that exhibit a highly nontrivial cancellation in the vanishing of the discriminant at higher orders. The role of SCFTs in matter transitions that we have uncovered here suggests a resolution of a question regarding under what circumstances a Weierstrass model can exhibit such an exotic higher genus matter representation other than the adjoint: it seems in particular that a model formed by starting with a Tate model for a group G on a smooth divisor D and then deforming the divisor to a singular geometry will not develop an exotic matter representation in the singular limit without moving through a superconformal fixed point to a different branch of the set of Weierstrass models where

the algebraic structure of the model is changed, modifying the cancellation mechanism in the discriminant.

7.2 Further directions

4D realizations. While we have focused here on 6D models, over which we have the greatest level of control, similar transitions should be possible in 4D $\mathcal{N} = 1$ field theories, particularly in the context of global models coupled to supergravity. From the heterotic and F-theory points of view, the 4D constructions are almost identical to the 6D constructions. In the heterotic picture, bundles with structure groups within E_8 that are products like $SU(3) \times SU(2)$ can arise on Calabi-Yau threefolds as well as on K3. And in the F-theory picture, the algebraic structure of Weierstrass models giving these matter transitions is formally identical in 4D or in 6D. One issue that may arise, however, is that while in 6D the geometric moduli space precisely matches the moduli space of flat directions in the low-energy theory, in four dimensions there is a superpotential that lifts some of the flat directions. Such a superpotential could in principle either obstruct the passage between branches with different matter content, or render unstable one of the branches. In 4D there are fewer constraints from anomaly cancellation in the $\mathcal{N} = 1$ low-energy theory, so it is less clear why the specific combinations of matter representations that admit matter transitions in the 6D theories should be singled out particularly. Recent work [20] has given a description of a very broad class of F-theory/heterotic dual pairs for 4D F-theory compactifications. Further study of 4D versions of the matter transitions explored here in some of these dual geometries would provide an interesting direction for further investigations.

Geometry of transitions. The matter transitions described here should correspond geometrically to geometric transitions between distinct elliptically fibered Calabi-Yau manifolds with the same $h^{1,1}(X)$ but different intersection structure. Such transitions would be interesting to study further from the purely geometric perspective, or in the context of other types of string compactification.

Other groups and representations. We have focused here on a set of gauge groups and matter transitions for the classical groups $SU(N)$, $SO(N)$ and $Sp(N)$ that are related to the basic $SU(6)$ Λ^3 example by simple Higgsing transitions. It would be interesting to explore further other possibilities, including exceptional groups and higher representations, such as tri-fundamental representations of $SU(2) \times SU(2) \times SU(2)$, etc.

Global models and string universality. The models with unusual matter representations that we have considered here provide an interesting test of the 6D string universality conjecture [89], which suggests that all consistent low-energy theories of matter fields, gauge fields and tensor fields coupled to supergravity in six dimensions should have a UV description in string theory or F-theory. The low-energy models with $SU(6)$ and $SU(7)$ gauge groups and Λ^3 representations that are acceptable from low-energy anomaly cancellation conditions we have identified here in both heterotic and F-theory constructions as models with good UV completions in string theory. The $SU(8)$ theory with a Λ^4 matter representation, however, seems acceptable from the low-energy point of view but does not

seem to have a clear realization in either the heterotic or F-theory pictures. This raises the question of whether this theory suffers from some as-yet-undiscovered inconsistency, or can be realized in some new way in F-theory or another string construction, or whether it actually represents a counterexample to the string universality conjecture. Similar questions can be asked of the $SU(9)$ $n_T = 1$ model with Λ^3 representations. An interesting feature of these models that seem acceptable from the low-energy point of view but for which we cannot identify a consistent F-theory or heterotic construction is that on the F-theory side they would involve in principle an embedding of a Dynkin diagram of the gauge group into \hat{E}_8 and not E_8 at the singular point. Resolving whether there is some low-energy problem with such configurations or some novel F-theory mechanism for realizing such constructions is an interesting open problem. It seems promising that models with larger N and/or higher-index antisymmetric representations, such as $SU(10)$ with Λ^3 or $SU(9)$ with Λ^4 seem to violate anomaly cancellation so there is a close connection between what is allowed in the low-energy theory and what can be realized through F-theory, with the difference between these conditions giving only a small intermediate zone of uncertainty. One possibly surprising feature here is that the constraints from the low-energy theory seem to depend on the gravitational anomaly cancellation condition, while the F-theory constraints on gauge groups that admit a Λ^3 representation seem to come from local considerations. This should be understood better.

Higher genus matter. The explicit F-theory construction found here of matter in the symmetric representation of $SU(3)$ complements another family of Weierstrass models constructed recently [80] that realize the same kind of matter fields. These constructions, however, seem to give slightly different classes of models. It would be desirable to have a more general understanding of how such Weierstrass models are constructed and a general framework that would encompass both of these classes of models. It would also be interesting to construct more general types of matter, such as a 3-symmetric representation of $SU(N)$, using the kinds of analysis developed here.

Transitions between conjugate representations. An interesting question, which we have not explored here, is the extent to which matter transitions can occur for smaller groups like $SU(5)$. While in 6D supergravity theories, the Λ^3 representation of $SU(5)$ is the conjugate of the Λ^2 representation and therefore lies in the same hypermultiplets, a Higgsing of the $SU(6)$ Λ^3 matter transition appears to give a class of $SU(5)$ Weierstrass models where there is a transition between Λ^3 and Λ^2 representations through an SCFT. For 6D theories, both branches of the theory seem to represent special cases of the general $SU(5)$ construction, so that there is no obstacle to deforming the theory from one branch to the other without passing through the SCFT. In the F-theory picture this follows by taking a general form of the parameter ϕ_0 from (3.7), (3.8) without any factorization as in (3.10)–(3.12). In the heterotic picture the transition can be realized by building an $SU(5)$ bundle with a specialized structure group $S(U(2) \times U(3))$, where there can be small instanton transitions between the factors, but for compactifications on K3 there is no obstruction to deformation to a general $SU(5)$ bundle (so long as the required matter is present for higgsing). For 4D theories, on the other hand, there may be obstructions to

moving between these branches, which would stem physically from the fact that in 4D $\mathcal{N} = 1$ theories there is a distinction between chiral multiplets in the Λ^2 and conjugate representations. This would be interesting to investigate further.

Massive $U(1)$'s in heterotic/F-theory duality. We have found that some models such as the $SU(7)$ and $SU(8)$ models with exotic matter seem to always give rise to massive $U(1)$'s in the heterotic description, which correspond to an enhanced Mordell-Weil group on one side of the stable degeneration limit in the corresponding F-theory picture, but not both. It would be interesting to further explore the physical consequences of these massive abelian symmetries in the low-energy theory away from the stable degeneration limit. Particularly in 4D compactifications, it is expected that the discrete remnants of such massive symmetries can significantly affect the structure of the theory — including Yukawa couplings, Kähler potentials and the vacuum structure of the theory [50, 90–93].

Superconformal points. We have found that the transitions between different matter fields occur at points in the heterotic moduli space where instantons have shrunk to a point, corresponding to points in the F-theory Weierstrass moduli space where $(4, 6)$ singularities have arisen at codimension two loci in the base. These are the same transition points that give superconformal fixed points in the low-energy theory, and which lead to tensionless string transitions to models with more tensor multiplets in the 6D framework. A large class of 6D SCFT's were constructed from the F-theory point of view in [39] and couplings of these theories to supergravity were explored in [42]. It would be interesting to investigate further how the SCFT's that play the role of mediating the matter transitions considered here fit into this picture. In particular, it would be nice to find some clear way of following the transition from one field theory to another through the superconformal point strictly in the language of the low-energy theory.

Acknowledgments

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A Addition of sections

In this section we review briefly the addition of points on the elliptic fiber (see [70, 94] for reviews). Given a Weierstrass model in $\mathbb{P}[1, 2, 3]$

$$y^2 = x^3 + fxz^4 + gz^6 \tag{A.1}$$

the zero section can be defined to be $(x, y, z) = (1, 1, 0)$. On the affine patch where $z = 1$, the addition (denoted $[+]$) of two points (a, b) and (A, B) on the elliptic fiber can be defined as follows. On this patch the Weierstrass equation can be written as

$$y^2 = x^3 + fx + g = (x - a)(x^2 + ax + c) + b^2 \tag{A.2}$$

$$= (x - A)(x^2 + Ax + C) + B^2 \tag{A.3}$$

Then, $\mathcal{P} = p[+]P = (\mathcal{A}, \mathcal{B})$ can be defined by demanding that $(\mathcal{A}, -\mathcal{B})$ is the third intersection point of the line that joins the points p and P . Then, in the notation of [70],

$$\mathcal{P} = \left(\left(\frac{B - b}{A - a} \right)^2 - (a + A), - \left(\frac{B - b}{A - a} \right)^3 + (2a + A) \left(\frac{B - b}{A - a} \right) - b \right) \tag{A.4}$$

Likewise, the point $2P = P[+]P$ is defined via

$$2P = \left(\left(\frac{C + 2A^2}{2B} \right)^2 - 2A, - \left(\frac{C + 2A^2}{2B} \right)^3 + (3A) \left(\frac{C + 2A^2}{2B} \right) - B \right) \tag{A.5}$$

B Three stable degeneration limits for SU(8)

As described in section 5.6, there are three possible paths that lead to stable degeneration limits for the SU(8) F-theory geometry. Of these, only one leads to a smooth $K3$ manifold in the heterotic dual theory. In order to take the stable degeneration limit given by equations (5.2) and (5.4), it is necessary to decide on a limit which takes $f_i, g_j \rightarrow 0$ for $i > 4$ and $j > 6$. A Groebner basis calculation [79] demonstrates that the most general path to such a solution can be described by the following equations (the primary decomposition of the stable degeneration locus):

$$4\zeta_3\psi_5^2\phi_4 - 24\lambda_2\phi_4^3 - \tau^2\psi_5^3 - 4\psi_5\omega_1\phi_4^2 = 0 \tag{B.1}$$

$$\delta^2\psi_5 + 2\zeta_4\phi_4 = 0 \tag{B.2}$$

$$12\delta^2\lambda_2\phi_4^2 + 4\zeta_3\zeta_4\psi_5\phi_4 - \zeta_4\tau^2\psi_5^2 - 4\zeta_4\omega_1\phi_4^2 = 0 \tag{B.3}$$

$$6\delta^4\lambda_2\phi_4 - 2\delta^2\zeta_4\omega_1\phi_4 - 4\zeta_3\zeta_4^2\phi_4 + \zeta_4^2\tau^2\psi_5 = 0 \tag{B.4}$$

$$3\delta^6\lambda_2 - \delta^4\zeta_4\omega_1 - 2\delta^2\zeta_3\zeta_4^2 - \zeta_4^3\tau^2 = 0 \tag{B.5}$$

Note that if either ζ_4 or τ share factors with δ , (4, 6) singularities are unavoidable on the shared factor.

We will first consider (B.2). This equation would imply that $\zeta_4\phi_4$ is proportional to δ^2 , and every zero of δ must be in either ζ_4 or ϕ_4 . But δ and ζ_4 cannot share any zeroes if we wish to avoid (4,6) singularities. As a result, ϕ_4 must be proportional to δ^2 , or

$$\phi_4 = \phi'_4\delta^2. \tag{B.6}$$

This in turn implies that

$$\psi_5 = -2\zeta_4\phi'_4 \tag{B.7}$$

If the expression for ψ_5 is substituted into (B.1), it leads to

$$4\phi_4'^3 (2\zeta_4^3 \tau^2 + \delta^2 (\zeta_3 \zeta_4^2 + 2\delta^2 \zeta_4 \omega_1 - 6\delta^4 \lambda_2)) = 0 \tag{B.8}$$

Unless $\phi_4' = 0$, we must have that $\zeta_4^3 \tau^2$ is proportional to δ^2 . However, this possibility would introduce (4, 6) singularities and will therefore not be a valid geometry. Thus, the only option is to take $\phi_4' = 0$, and both ϕ_4 and ψ_5 must go to zero. In fact, once (B.2) is solved as above, all of the other equations similarly lead to singular geometries. As a result the $\phi_4 = \psi_5 = 0$ branch is the unique smooth solution.

C Stable degeneration limits and U(1) symmetries

In this section, we briefly explore the compatibility of the stable degeneration limit and the existence of non-zero rank Mordell-Weil group in the Calabi-Yau geometry. The existence of a log semi-stable degeneration limit [57, 58] depends globally on being able to consistently define dP_9 -fibered n -folds $Y^{(1)}$ and $Y^{(2)}$ and their ability to share a common divisor $D = X_n$ a CY manifold of one dimension lower. Moreover, the limit requires that the fibrations of $Y^{(i)}$ and X_n be compatible — that is, the elliptic fiber of X_n should be the same form as the elliptic fiber of $Y^{(i)}$, etc. In what follows, we will demonstrate that the process of stable degeneration and presentation of a manifold in Weierstrass form do not necessarily commute. When this paper was in the final stages of preparation, [78] appeared which comprehensively studies the above questions using a different approach. For concreteness, here we will illustrate the relevant ambiguities in stable degeneration with elliptically fibered threefold, Y , with base \mathbb{F}_n .

To begin, we briefly review the “standard” stable degeneration limit in the case that the elliptic fibration of Y admits a single section (i.e. the rank of $MW(Y)$ is vanishing). That is, Y is a generic Weierstrass model over \mathbb{F}_n . We can realize this torically as a hypersurface with $P_{123}[6]$ fiber. That is, the charge matrix

$$Y = \begin{array}{c|ccccccc} & y & x & z & x_0 & x_1 & y_0 & y_1 \\ \hline 6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2-n & 1 & 1 & n & 0 \end{array} \tag{C.1}$$

where the first column denotes a degree (6, 0, 0) hypersurface in the toric ambient space. For this choice of global description of the manifold, the stable degeneration limit corresponds to choosing

$$Y^{(1)} = \begin{array}{c|ccccccc} & y & x & z & x_0 & x_1 & y_0 & y_1 \\ \hline 6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2-n & 1 & 1 & n & 0 \end{array} \tag{C.2}$$

(with $Y^{(2)}$ similar). The defining equation of $Y^{(1)}$ is of the same form as that of Y (i.e. of Weierstrass form: $y^2 = x^3 + fxz^4 + gz^6$) but with f, g truncated at degree $\leq 4, 6$ in

the coordinate ($\sigma = y_0$). This global description of $Y^{(1)}$ is equivalent to the scaling limit defined in (5.4). Briefly, for the Weierstrass coefficients of Y above, we choose

$$f \sim \sum_{i=0}^8 f_i \sigma^i, \quad g \sim \sum_{j=0}^{12} g_j \sigma^j \tag{C.3}$$

where $\sigma = 0$ and $\sigma = \infty$ define the poles of the \mathbb{P}^1 fiber of \mathbb{F}_n . To take the stable degeneration limit, a scaling is chosen in which

$$\begin{aligned} f_i & \text{ scales as } \epsilon^{(i-4)} \\ g_j & \text{ scales as } \epsilon^{(j-6)} \end{aligned} \tag{C.4}$$

which in the limit that $\epsilon \rightarrow 0$ “separates” the dP_9 -fibered halves of the Y . Note that in this scaling, the divisor $D = K3$ along which $Y^{(1)}$ and $Y^{(2)}$ are glued is defined by f_4, g_6 (the coefficients of weight zero in ϵ). These “middle” coefficients define the moduli of an elliptically fibered $K3$ surface with a compatible (i.e. $P_{123}[6]$) fibration.

Now, in contrast, when the F-theory geometry has a higher rank Mordell-Weil group then the stable degeneration limit may differ with the global description of Y . The following example provides a simple example of a geometry where the process of presenting a manifold in Weierstrass form and the stable degeneration limit do not commute.

Following [70], let us realize the elliptic fiber of a generic $rk(MW) = 1$ F-theory geometry, Y , as a toric blow up of $P_{112}[4]$. For concreteness, consider the global geometry

	u	v	w	t	s	x_0	x_1	y_0	y_1	
1	1	0	1	-1	0	0	0	0	0	
4	1	1	2	0	0	0	0	0	0	
2	0	0	1	0	1	0	0	0	0	
0	0	-2	0	0	0	0	0	1	1	
0	0	-n-2	0	0	0	1	1	n	0	

(C.5)

This leads to generic defining equation of the form

$$w^2 s + b_0 u^2 w s^2 t + b_1 u v w s t + b_2 v^2 w t = c_0 u^4 t^2 s^3 + c_1 u^3 v t^2 s^2 + c_2 u^2 v^2 t^2 s + c_3 u v^3 t^2 \tag{C.6}$$

where b_i, c_j are functions of the base coordinates (x, y) . By shifting w by a multiple of u it is possible to set $b_0 = b_1 = 0$. Labeling b_2 simply as b , and letting subscripts denote degree in x_i , we have explicitly

$$\begin{aligned} b &= y_0^4 b_{4-2n}(x) + y_0^3 y_1 b_{4-n}(x) + y_0^2 y_1^2 b_4(x) + y_0 y_1^3 b_{n+1}(x) + y_1^4 b_{2n+4}(x) \\ c_1 &= y_0^2 (c_1)_{2-n}(x) + y_0 y_1 (c_1)_2(x) + y_1^2 (c_1)_{n+2}(x) \\ c_2 &= y_0^2 (c_2)_{4-2n}(x) + y_0^3 y_1 (c_2)_{4-n}(x) + y_0^2 y_1^2 (c_2)_4(x) + y_0 y_1^3 (c_2)_{n+4}(x) + y_1^4 (c_2)_{2n+4}(x) \\ c_3 &= y_0^6 (c_3)_{6-3n}(x) + y_0^5 y_1 (c_3)_{6-2n}(x) + y_0^4 y_1^2 (c_3)_{6-n}(x) + y_0^3 y_1^3 (c_3)_6(x) \\ &\quad + y_0^2 y_1^4 (c_3)_{n+6}(x) + y_0 y_1^5 (c_3)_{2n+6}(x) + y_1^6 (c_3)_{3n+6}(x) \end{aligned} \tag{C.7}$$

Moreover, Y in (C.5) has a compatible $K3$ fibration given by

$$K3 = \begin{array}{c|ccccccc} & u & v & w & t & s & y_0 & y_1 \\ \hline 1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{array} \quad (C.8)$$

Here once again we have the defining relation (C.6) where the degrees of the coefficients in the $K3$ case in terms of the base variables (y) are given by

$$\deg(c_0) = 0, \quad \deg(c_1) = 2, \quad \deg(b) = \deg(c_2) = 4, \quad \deg(c_3) = 6 \quad (C.9)$$

Now to take the stable degeneration limit of the threefold given in (C.5) we need to identify the “middle” $K3$ along which $Y^{(i)}$ will be glued. Here the “middle” $K3$ coefficients inside of X are given by

$$b \sim y_0^2 y_1^2 b_4 \quad (C.10)$$

$$c_1 \sim y_0 y_1 (c_1)_2 \quad (C.11)$$

$$c_2 \sim y_0^2 y_1^2 (c_2)_4 \quad (C.12)$$

$$c_3 \sim y_0^3 y_1^3 (c_3)_6 \quad (C.13)$$

More precisely, as in the standard case above, the stable generation limit can be defined via a scaling of the coefficients of b, c_i . To take the stable degeneration limit $Y \rightarrow Y^{(1)} \cup_D Y^{(2)}$ with Y defined by (C.5) and D defined via (C.8) it is possible to choose

$$\begin{aligned} b_i & \text{ scales as } \epsilon^{(i-4)} \\ (c_1)_j & \text{ scales as } \epsilon^{(j-2)} \\ (c_2)_k & \text{ scales as } \epsilon^{(k-4)} \\ (c_3)_l & \text{ scales as } \epsilon^{(l-6)} \end{aligned} \quad (C.14)$$

In the limit that $\epsilon \rightarrow 0$ this separates Y into two dP_9 fibered (non-CY) 3-folds. Now we come to the central observation to be made from the above geometry: in this stable degeneration limit, it is straightforward to verify that *the presence of a non-trivial Mordell-Weil group is fully preserved* not only in each of $Y^{(1)}, Y^{(2)}$ but also in the “middle” $K3$ surface (C.8) (defined by the ϵ weight zero terms above).

Having come thus far, it should now be recalled [70] that it is always possible via coordinate redefinitions (i.e. the Jacobian procedure) to put a two-section model such as Y in (C.5) explicitly into Weierstrass form. The dictionary to Weierstrass form defines the coefficients

$$f = c_1 c_3 - b^2 c_0 - \frac{1}{3} c_2^2 \quad (C.15)$$

$$g = c_0 c_3^2 - \frac{1}{3} c_1 c_2 c_3 + \frac{2}{27} c_2^3 - \frac{2}{3} b^2 c_0 c_2 + \frac{1}{4} b^2 c_1^2 \quad (C.16)$$

With this explicit defining relation we are once again considering a manifold with $\mathbb{P}_{123}[6]$ fiber type of the form given in (C.1). For such a geometry, the usual stable degeneration limit (i.e. the splitting given in (C.2)) and the scalings defined in (C.4) can be employed.

To begin, the “middle” $K3$ coefficients can be readily identified as those of ϵ weight zero. For example combining (C.7) with (C.15) the coefficients of $y_0^4 y_1^4$ in f include the following terms from the product $c_1 c_3$

$$c_1 c_3 \sim (y_0^4 y_1^4) \left((c_1)_2 (c_3)_6 + (c_1)_{n+2} (c_3)_{-n+6} + (c_1)_{-n+2} (c_3)_{n+6} \right) \quad (\text{C.17})$$

(with many further terms of this order arising from $b^2 c_0$ and c_2^2). These coefficients (and the coefficients of $y_0^6 y_1^6$ in g) determine a “middle” $K3$ surface along which the standard stable degeneration limit would glue the new $\mathbb{P}_{123}[6]$ -fibered manifolds $Y^{(i)}$. Note that this same result could be obtained simply by performing the scaling rule given in (C.14) since according to that rule, the epsilon factors cancel in terms like $(c_1)_{n+2} (c_3)_{-n+6}$ above.

It is clear by inspection of the Weierstrass coefficients of the $K3$ surface (including the terms in (C.17)) that it is not of the form required for a higher rank Mordell-Weil group. Moreover it *does not* define the same $K3$ surface as we obtained in (C.8) and (C.10). Although the Weierstrass form for Y given by (C.15) initially has a non-trivial Mordell-Weil group, that structure is *not in general preserved in either $Y^{(1)}, Y^{(2)}$ or the gluing divisor $D = K3$* . In this case, unlike in the stable degeneration limit of (C.5), the rank of Mordell-Weil is reduced in stable degeneration. Thus finally, we have reached our central observation: the procedures of stable degeneration of a global elliptically fibered geometry and putting that fibration into explicit Weierstrass form, do not in general commute. That is, varying the order of these operations leads to different weakly coupled limits.

D Details of heterotic Higgsing analysis

In this appendix, we provide some of the details of the computations which underly the heterotic description of the Higgsing processes as described in tables 12, 15 and 18. The cases of Higgsing $SU(6)$ on fundamental, $SU(7)$ on two-index antisymmetric and $SU(8)$ on three-index antisymmetric matter are described in the main text.

D.1 Higgsing on fundamental matter

D.1.1 $SU(7)$

Let us consider what happens to bundle topology as we Higgs from $SU(7)$ to $SU(6)$. The relevant group theory is

$$\begin{aligned} SU(7) &\rightarrow SU(6) \times U(1) & (\text{D.1}) \\ \mathbf{7} &= \mathbf{1}_{-6} + \mathbf{6}_1 \\ \mathbf{21} &= \mathbf{6}_{-5} + \mathbf{15}_2 \\ \mathbf{35} &= \mathbf{15}_{-4} + \mathbf{20}_3 \end{aligned}$$

Clearly, we wish to turn on the singlet of $SU(6)$ inside $SU(7)$ to achieve the Higgsing. However, we see from table 8 that there are two types of $\mathbf{7}$ (and indeed $\bar{\mathbf{7}}$) one associated

to $H^1(\mathcal{V}_2^\vee \otimes \mathcal{L})$ and another to $H^1(\mathcal{L}^{\vee 2})$. It turns out that, at an arbitrary point in the moduli space of the base K3, we can not simply choose which of these to give a VEV too. A combination of both must be turned on simultaneously.

We can see this structure by examining the change in bundle geometry in going from an SU(7) to SU(6) visible symmetry. Turning on the VEV associated to $H^1(\mathcal{V}_2^\vee \otimes \mathcal{L})$ corresponds to forming the following extension.

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{V}_{\text{SU}(3)} \rightarrow \mathcal{V}_2 \rightarrow 0 \tag{D.2}$$

Turning on the field associated to $H^1(\mathcal{L}^{\vee 2})$ corresponds to forming this extension.

$$0 \rightarrow \mathcal{L}^\vee \rightarrow \mathcal{V}_{\text{SU}(2)} \rightarrow \mathcal{L} \rightarrow 0 \tag{D.3}$$

The conjugate representations that must also be given VEVs correspond to the dual to the above two sequences and the actual SU(3) and SU(2) bundles are formed from the combination of these mutually dual extensions in the usual manner [54].

We see now that the need to turn on both types of **7** corresponds to the need to form a bundle with structure group $\text{SU}(3) \times \text{SU}(2)$, which is the relevant case to arrive at SU(6). At special loci in the moduli space of K3 one can leave one of the two extensions (D.2) and (D.3) split and still maintain bundle poly-stability (this is actually the same locus on which the original $\text{S}(\text{U}(2) \times \text{U}(1))$ bundle is poly-stable). Splitting one of these two bundles in this fashion would induce an extra U(1) factor in the commutant of the bundle structure group inside E_8 . This additional abelian factor will be Green-Schwarz anomalous however [47–52].

One can check that the matter one achieves in the SU(6) case obtained by Higgsing SU(7) in this fashion agrees with what one would get by plugging in the topology of the above bundles into a direct computation of the spectrum of an SU(6) model, as given in table 7, a calculation to which we now turn.

Equation (D.1) describes how the SU(7) representations that we have in our initial theory branch to SU(6) representations under the breaking. We start with a number of each of these representations, determined by the topology of the bundle $\mathcal{V} = \mathcal{V}_2 \oplus \mathcal{L}$, as detailed in table 8. This information is enough to determine the spectrum after breaking.

Alternatively, from equations (D.2) and (D.3) we can determine the topology of the bundle $\mathcal{V}_{\text{SU}(3)} \oplus \mathcal{V}_{\text{SU}(2)}$ that we transition to and from there, using table 7, the matter content after the Higgsing. The second Chern classes of the two bundles can easily be determined to be,

$$\begin{aligned} c_2(\mathcal{V}_{\text{SU}(3)}) &= c_2(\mathcal{V}_2) - c_1(\mathcal{L})^2 \\ c_2(\mathcal{V}_{\text{SU}(2)}) &= -c_1(\mathcal{L})^2, \end{aligned} \tag{D.4}$$

which can be then used to determine the matter content of the resulting SU(6) theory.

In table 23 we present the matter content of an SU(6) bundle with the topology just described, together with the matter content that would be naively expected under a transition from the SU(7) theory to SU(6) using table 8 and the branching rules (D.1).

We can now observe that this result matches the field theory analysis of the Higgs mechanism given in section 2.3.1.

SU(6) Representation	# from SU(7) multiplet decomposition	# found after transition
1	$6c_2(\mathcal{V}_2) - 10c_1(\mathcal{L})^2 - 18$	$6c_2(\mathcal{V}_2) - 10c_1(\mathcal{L})^2 - 22$
6	$2c_2(\mathcal{V}_2) - 10c_1(\mathcal{L})^2 - 10$	$2c_2(\mathcal{V}_2) - 5c_1(\mathcal{L})^2 - 12$
$\bar{\mathbf{6}}$	$2c_2(\mathcal{V}_2) - 10c_1(\mathcal{L})^2 - 10$	$2c_2(\mathcal{V}_2) - 5c_1(\mathcal{L})^2 - 12$
15	$c_2(\mathcal{V}_2) - c_1(\mathcal{L})^2 - 6$	$c_2(\mathcal{V}_2) - c_1(\mathcal{L})^2 - 6$
$\bar{\mathbf{15}}$	$c_2(\mathcal{V}_2) - c_1(\mathcal{L})^2 - 6$	$c_2(\mathcal{V}_2) - c_1(\mathcal{L})^2 - 6$
20	$-c_1(\mathcal{L})^2 - 4$	$-c_1(\mathcal{L})^2 - 4$

Table 23. Matter content after Higgsing an SU(7) to an SU(6) theory, both via a naive decomposition of the initial SU(7) multiplets and via a direct computation from the resulting SU(6) bundle.

D.1.2 SU(8)

The relevant group theory in this case is as follows.

$$\text{SU}(8) \rightarrow \text{SU}(7) \times \text{U}(1) \tag{D.5}$$

$$\mathbf{8} = \mathbf{7}_1 + \mathbf{1}_{-7} \tag{D.6}$$

$$\mathbf{28} = \mathbf{7}_{-6} + \mathbf{21}_2 \tag{D.7}$$

$$\mathbf{56} = \mathbf{21}_{-5} + \mathbf{35}_3 \tag{D.8}$$

Given this, giving a VEV to the singlet of SU(7) inside the $\bar{\mathbf{8}}$ of SU(8) will Higgs SU(8) to SU(7). Looking at table 9, we see that the $\bar{\mathbf{8}}$'s correspond to elements of $H^1(\mathcal{L}^{\vee 3})$. Giving such a field a VEV corresponds to forming the following extension (and its dual via the associated $\mathbf{8}$ VEV).

$$0 \rightarrow \mathcal{L}^{\vee 2} \rightarrow \mathcal{V}_2 \rightarrow \mathcal{L} \rightarrow 0 \tag{D.9}$$

This is a U(2) bundle which can form part of an $\text{S}(\text{U}(2) \times \text{U}(1))$ object, in order to break to SU(7) (with, in addition, a Green-Schwarz anomalous U(1)), as follows.

$$\mathcal{V} = \mathcal{V}_2 \oplus \mathcal{L} \tag{D.10}$$

As in previous cases, we now compare the SU(7) spectrum that is achieved by a decomposition of the parent SU(8) theory to the spectrum associated to the SU(7) theory defined by (D.9) and (D.10).

In table 24 we have used the fact that, for the bundle given in (D.9),

$$c_2(\mathcal{V}_2) = -2c_1(\mathcal{L})^2. \tag{D.11}$$

As in previous cases, the differences between the second and third columns in table 24 precisely match what we would expect from an analysis of the Higgs mechanism in such a situation. This Higgsing is precisely of the form described in a field theory context in section 2.3.2.

It should also be mentioned that, during this transition, the ray in the moduli space of K3 where the $\text{S}(\text{U}(1) \times \text{U}(1))$ bundle and $\text{S}(\text{U}(2) \times \text{U}(1))$ bundle are slope poly-stable is the same. Therefore the K3 moduli expectation values need not change during this process.

SU(7) Representation	# from SU(8) multiplet decomposition	# found after transition
1	$-9c_1(\mathcal{L})^2 - 4$	$-9c_1(\mathcal{L})^2 - 6$
7	$-\frac{13}{2}c_1(\mathcal{L})^2 - 4$	$-\frac{13}{2}c_1(\mathcal{L})^2 - 6$
$\bar{\mathbf{7}}$	$-\frac{13}{2}c_1(\mathcal{L})^2 - 4$	$-\frac{13}{2}c_1(\mathcal{L})^2 - 6$
21	$-\frac{5}{2}c_1(\mathcal{L})^2 - 4$	$-\frac{5}{2}c_1(\mathcal{L})^2 - 4$
$\bar{\mathbf{21}}$	$-\frac{5}{2}c_1(\mathcal{L})^2 - 4$	$-\frac{5}{2}c_1(\mathcal{L})^2 - 4$
35	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$
$\bar{\mathbf{35}}$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$

Table 24. Matter content after Higgsing an SU(8) to an SU(7) theory, both via a naive decomposition of the initial SU(8) multiplets and via a direct computation from the resulting SU(7) bundle.

D.2 Higgsing on two-index antisymmetric matter

D.2.1 SU(6)

The relevant group theory in this case is as follows.

$$\text{SU}(6) \rightarrow \text{SU}(4) \times \text{SU}(2) \times \text{U}(1) \tag{D.12}$$

$$\mathbf{6} = (\mathbf{1}, \mathbf{2})_{-2} + (\mathbf{4}, \mathbf{1})_1 \tag{D.13}$$

$$\mathbf{15} = (\mathbf{1}, \mathbf{1})_{-4} + (\mathbf{4}, \mathbf{2})_{-1} + (\mathbf{6}, \mathbf{1})_2 \tag{D.14}$$

$$\mathbf{20} = (\mathbf{4}, \mathbf{1})_{-3} + (\bar{\mathbf{4}}, \mathbf{1})_3 + (\mathbf{6}, \mathbf{2})_0 \tag{D.15}$$

The breaking pattern in (D.12) corresponds to giving a VEV to a $\mathbf{15}$, $\bar{\mathbf{15}}$ pair. The $\mathbf{15}$'s, according to table 7, lie in the cohomology $H^1(\mathcal{V}_3^\vee)$. In terms of bundle topology, giving an expectation value to such a field corresponds to forming the following bundle.

$$\mathcal{V} = \mathcal{V}_2 \oplus \mathcal{V}_4 \tag{D.16}$$

$$\text{where } 0 \rightarrow \mathcal{V}_3 \rightarrow \mathcal{V}_4 \rightarrow \mathcal{O} \rightarrow 0$$

The conjugate representation which must also be given a VEV corresponds to the dual to the above sequence and the actual SU(3) bundle is formed from the combination of these mutually dual extensions in the usual manner [54].

The group theory for a $\text{SU}(4) \times \text{SU}(2)$ compactification of heterotic is as follows.

$$E_8 \supset \text{SU}(4) \times \text{SU}(2) \times \text{SU}(4) \times \text{SU}(2) \tag{D.17}$$

$$\begin{aligned} \mathbf{248} = & (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{6}, \mathbf{2}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{2}) + (\mathbf{4}, \mathbf{1}, \bar{\mathbf{4}}, \mathbf{2}) + (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{4}, \mathbf{2}) \\ & + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{4}, \mathbf{1}) + (\mathbf{4}, \mathbf{2}, \bar{\mathbf{4}}, \mathbf{1}) + (\mathbf{6}, \mathbf{1}, \mathbf{6}, \mathbf{1}) \\ & + (\mathbf{15}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{15}, \mathbf{1}) \end{aligned}$$

This leads to the low-energy spectrum given in table 25. The number of massless hy-

Representation	Cohomology	Multiplicity
(1, 1)	$H^1(\text{End}_0(\mathcal{V}_2)) \oplus H^1(\text{End}_0(\mathcal{V}_4))$	$4c_2(\mathcal{V}_2) + 8c_2(\mathcal{V}_4) - 36$
(6, 2)	$H^1(\mathcal{V}_2)$	$c_2(\mathcal{V}_2) - 4$
(1, 2)	$H^1(\mathcal{V}_2 \otimes \wedge^2 \mathcal{V}_4)$	$6c_2(\mathcal{V}_2) + 4c_2(\mathcal{V}_4) - 24$
(4, 1)	$H^1(\mathcal{V}_4' \otimes \mathcal{V}_2)$	$4c_2(\mathcal{V}_2) + 2c_2(\mathcal{V}_4) - 16$
(4, 2)	$H^1(\mathcal{V}_4')$	$c_2(\mathcal{V}_4) - 8$
(6, 1)	$H^1(\wedge^2 \mathcal{V}_4)$	$2c_2(\mathcal{V}_4) - 12$

Table 25. The cohomology associated to each representation of the low-energy gauge group $SU(4) \times SU(2)$.

$SU(4) \times SU(2)$ Representation	# from $SU(6)$ multiplet decomposition	# found after transition
(1, 1)	$4c_2(\mathcal{V}_2) + 8c_2(\mathcal{V}_3) - 34$	$4c_2(\mathcal{V}_2) + 8c_2(\mathcal{V}_3) - 36$
(6, 2)	$c_2(\mathcal{V}_2) - 4$	$c_2(\mathcal{V}_2) - 4$
(1, 2)	$6c_2(\mathcal{V}_2) + 4c_2(\mathcal{V}_3) - 24$	$6c_2(\mathcal{V}_2) + 4c_2(\mathcal{V}_3) - 24$
(4, 1)	$4c_2(\mathcal{V}_2) + 2c_2(\mathcal{V}_3) - 16$	$4c_2(\mathcal{V}_2) + 2c_2(\mathcal{V}_3) - 16$
(4, 2)	$c_2(\mathcal{V}_3) - 6$	$c_2(\mathcal{V}_3) - 8$
(6, 1)	$2c_2(\mathcal{V}_3) - 12$	$2c_2(\mathcal{V}_3) - 12$

Table 26. Matter content after Higgsing an $SU(6)$ to an $SU(4) \times SU(2)$ theory, both via a naive decomposition of the initial $SU(6)$ multiplets and via a direct computation from the resulting $SU(4) \times SU(2)$ bundle.

permultiplets can be read off from table 25 and leads to the following anomaly cancellation condition.

$$\begin{aligned}
 n_H + 29n_T - n_V &= 273 \\
 \Rightarrow c_2(\mathcal{V}_2) + c_2(\mathcal{V}_4) + \frac{1}{60}c_2(\text{End}_0(\mathcal{V}_{E_8})) &= 24
 \end{aligned}
 \tag{D.18}$$

We can now specialize this result to the particular $SU(4) \times SU(2)$ bundle that we found in (D.16). In this instance we have,

$$c_2(\mathcal{V}_4) = c_2(\mathcal{V}_3) \tag{D.19}$$

We can now construct the equivalent table to those that have been formed in the proceeding cases. We compare the spectrum which is obtained by decomposing the initial $SU(6)$ multiples with that obtained by direct computation from the bundle after transition. The results of this comparison are in table 26.

As in all of the cases in this section, the result is fully consistent with the field theory analysis given in section 2.3.2.

Representation	Cohomology	Multiplicity
(1, 1)	$H^1(\text{End}_0(\mathcal{V}_2))$	$4c_2(\mathcal{V}_2) - 6$
(1, 2)	$H^1(\mathcal{L}^3) \oplus H^1(\mathcal{L}^{\vee 3})$	$2(-\frac{9}{2}c_1(\mathcal{L})^2 - 2)$
(15, 1)	$H^1(\mathcal{L}^2)$	$-2c_1(\mathcal{L})^2 - 2$
(15, 2)	$H^1(\mathcal{L}^\vee)$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$
(6, 1)	$H^1(\mathcal{V}_2 \otimes \mathcal{L}^{\vee 2})$	$c_2(\mathcal{V}_2) - 4c_1(\mathcal{L})^2 - 4$
(6, 2)	$H^1(\mathcal{V}_2 \otimes \mathcal{L})$	$c_2(\mathcal{V}_2) - c_1(\mathcal{L})^2 - 4$
(20, 1)	$H^1(\mathcal{V}_2)$	$c_2(\mathcal{V}_2) - 4$

Table 27. The cohomology associated to each representation of the low-energy gauge group $SU(6) \times SU(2)$.

D.2.2 SU(8)

The relevant group theory in this case is the following.

$$\begin{aligned}
 SU(8) &\supset SU(6) \times SU(2) \times U(1) & (D.20) \\
 \mathbf{28} &= (\mathbf{1}, \mathbf{1})_{-6} + (\mathbf{6}, \mathbf{2})_{-2} + (\mathbf{15}, \mathbf{1})_2 \\
 \mathbf{8} &= (\mathbf{1}, \mathbf{2})_{-3} + (\mathbf{6}, \mathbf{1})_1 \\
 \mathbf{56} &= (\mathbf{6}, \mathbf{1})_{-5} + (\mathbf{20}, \mathbf{1})_3 + (\mathbf{15}, \mathbf{2})_{-1}
 \end{aligned}$$

This breaking pattern of interest in (D.20) corresponds to giving a VEV to a $\mathbf{28}, \overline{\mathbf{28}}$ pair. The $\mathbf{28}$'s, according to table 9, lie in the cohomology $H^1(\mathcal{L}^{\vee 2})$. In terms of bundle topology, giving an expectation value to such a field corresponds to forming the following bundle.

$$\mathcal{V} = \mathcal{V}_2 \oplus \mathcal{L} \oplus \mathcal{L}^\vee \tag{D.21}$$

$$\text{where } 0 \rightarrow \mathcal{L}^\vee \rightarrow \mathcal{V}_2 \rightarrow \mathcal{L} \rightarrow 0 \tag{D.22}$$

As in previous cases one should really think of \mathcal{V}_2 as being a deformation of this extension and its dual [54]. Here \mathcal{V} is an $SU(2) \times S(U(1) \times U(1))$ bundle. The correct embedding of $SU(2) \times U(1)$ does indeed break E_8 to $SU(6) \times SU(2)$, with an additional Green-Schwarz massive $U(1)$ being present.

The group theory for a $SU(6) \times SU(2)$ compactification of heterotic is as follows.

$$\begin{aligned}
 E_8 &\supset SU(6) \times SU(2) \times SU(2) \times U(1) & (D.23) \\
 \mathbf{248} &= (\mathbf{1}, \mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{2}, \mathbf{1})_3 + (\mathbf{1}, \mathbf{2}, \mathbf{1})_{-3} + (\mathbf{1}, \mathbf{3}, \mathbf{1})_0 \\
 &+ (\mathbf{35}, \mathbf{1}, \mathbf{1})_0 + (\overline{\mathbf{15}}, \mathbf{1}, \mathbf{1})_{-2} + (\overline{\mathbf{15}}, \mathbf{2}, \mathbf{1})_1 + (\mathbf{15}, \mathbf{1}, \mathbf{1})_2 + (\mathbf{15}, \mathbf{2}, \mathbf{1})_{-1} \\
 &+ (\mathbf{6}, \mathbf{1}, \mathbf{2})_{-2} + (\mathbf{6}, \mathbf{2}, \mathbf{2})_1 + (\overline{\mathbf{6}}, \mathbf{1}, \mathbf{2})_2 + (\overline{\mathbf{6}}, \mathbf{2}, \mathbf{2})_{-1} + (\mathbf{20}, \mathbf{1}, \mathbf{2})_0
 \end{aligned}$$

This leads to the low-energy spectrum given in table 27. The number of massless hy-

SU(6) × SU(2) Representation	# from SU(8) multiplet decomposition	# found after transition
(1, 1)	$-4c_1(\mathcal{L})^2 - 4$	$-4c_1(\mathcal{L})^2 - 6$
(1, 2)	$-9c_1(\mathcal{L})^2 - 4$	$-9c_1(\mathcal{L})^2 - 4$
(15, 1)	$-2c_1(\mathcal{L})^2 - 2$	$-2c_1(\mathcal{L})^2 - 2$
(15, 2)	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$
(6, 1)	$-5c_1(\mathcal{L})^2 - 4$	$-5c_1(\mathcal{L})^2 - 4$
(6, 2)	$-2c_1(\mathcal{L})^2 - 2$	$-2c_1(\mathcal{L})^2 - 4$
(20, 1)	$-c_1(\mathcal{L})^2 - 4$	$-c_1(\mathcal{L})^2 - 4$

Table 28. Matter content after Higgsing an SU(8) to an SU(6) × SU(2) theory, both via a naive decomposition of the initial SU(8) multiplets and via a direct computation from the resulting SU(2) × S(U(1) × U(1)) bundle.

permultiplets can be read off from this table leading to the following anomaly cancellation condition.

$$\begin{aligned}
 n_H + 29n_T - n_V &= 273 & (D.24) \\
 \Rightarrow c_2(\mathcal{V}_2) - 3c_1(\mathcal{L})^2 + \frac{1}{60}c_2(\text{End}_0(\mathcal{V}_{E_8})) &= 24
 \end{aligned}$$

We can now specialize this result to the particular SU(2) × S(U(1) × U(1)) bundle that we obtained in (D.21). In this instance we have,

$$c_2(\mathcal{V}_2) = -c_1(\mathcal{L})^2. \tag{D.25}$$

We are now in a position to construct the equivalent table to those that have been formed in the other cases. We compare the spectrum which is obtained by a decomposition of the initial SU(8) multiplets with that obtained by direct computation from the bundle after transition. The results of this comparison are in table 28.

As in all of the cases in this section, the result is fully consistent with the field theory analysis given in section 2.3.2.

D.3 Higgsing on three-index antisymmetric matter

D.3.1 SU(6)

The group theory relevant to this case is as follows.

$$\text{SU}(6) \rightarrow \text{SU}(3) \times \text{SU}(3) \times \text{U}(1) \tag{D.26}$$

$$\mathbf{6} = (\mathbf{3}, \mathbf{1})_1 + (\mathbf{1}, \mathbf{3})_{-1} \tag{D.27}$$

$$\mathbf{15} = (\bar{\mathbf{3}}, \mathbf{1})_2 + (\mathbf{1}, \bar{\mathbf{3}})_{-2} + (\mathbf{3}, \mathbf{3})_0 \tag{D.28}$$

$$\mathbf{20} = (\mathbf{1}, \mathbf{1})_3 + (\mathbf{1}, \mathbf{1})_{-3} + (\mathbf{3}, \bar{\mathbf{3}})_{-1} + (\bar{\mathbf{3}}, \mathbf{3})_1 \tag{D.29}$$

The breaking pattern we are interested in thus corresponds to giving a VEV to matter in the **20** representation. The **20**, according to table 7, is associated with the cohomology $H^1(\mathcal{V}_2)$.

Representation	Cohomology	Multiplicity
(1, 1)	$H^1(\text{End}_0(\mathcal{V}_3)) \oplus H^1(\text{End}_0(\tilde{\mathcal{V}}_3))$	$6c_2(\mathcal{V}_3) + 6c_2(\tilde{\mathcal{V}}_3) - 32$
(3, 1)	$H^1(\mathcal{V}_3 \otimes \tilde{\mathcal{V}}_3)$	$3c_2(\mathcal{V}_3) + 3c_2(\tilde{\mathcal{V}}_3) - 18$
(1, 3)	$H^1(\mathcal{V}_3 \otimes \tilde{\mathcal{V}}_3^\vee)$	$3c_2(\mathcal{V}_3) + 3c_2(\tilde{\mathcal{V}}_3) - 18$
(3, 3)	$H^1(\mathcal{V}_3^\vee)$	$c_2(\mathcal{V}_3) - 6$
(3, $\bar{3}$)	$H^1(\tilde{\mathcal{V}}_3^\vee)$	$c_2(\tilde{\mathcal{V}}_3) - 6$

Table 29. The cohomology associated to each representation of the low-energy gauge group $\text{SU}(3) \times \text{SU}(3)$.

In terms of bundle geometry, therefore, giving a VEV to matter in this representation corresponds to forming the following bundle.

$$\mathcal{V} = \mathcal{V}_3 + \tilde{\mathcal{V}}_3 \tag{D.30}$$

where $0 \rightarrow \mathcal{V}_2 \rightarrow \tilde{\mathcal{V}}_3 \rightarrow \mathcal{O} \rightarrow 0$

As in the other cases in this appendix, one should really think of \mathcal{V}_3 as being a deformation of this extension and its dual [54].

The group theory associated to a $\text{SU}(3) \times \text{SU}(3)$ compactification of heterotic string theory is as follows.

$$E_8 \supset \text{SU}(3) \times \text{SU}(3) \times \text{SU}(3) \times \text{SU}(3) \tag{D.31}$$

$$\begin{aligned} \mathbf{248} = & (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{8}) + (\mathbf{3}, \mathbf{1}, \mathbf{3}, \mathbf{3}) + (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}, \mathbf{3}) + (\bar{\mathbf{3}}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}) + (\bar{\mathbf{3}}, \mathbf{1}, \bar{\mathbf{3}}, \bar{\mathbf{3}}) \\ & + (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{3}, \bar{\mathbf{3}}) + (\mathbf{3}, \mathbf{3}, \mathbf{1}, \bar{\mathbf{3}}) + (\bar{\mathbf{3}}, \mathbf{3}, \mathbf{3}, \mathbf{1}) + (\mathbf{3}, \bar{\mathbf{3}}, \bar{\mathbf{3}}, \mathbf{1}) \\ & + (\mathbf{8}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{8}, \mathbf{1}) \end{aligned} \tag{D.32}$$

This leads to the matter content given in table 29.

The matter content in table 29 leads to the following anomaly cancellation condition.

$$n_H + 29n_T - n_V = 273 \tag{D.33}$$

$$\Rightarrow c_2(\mathcal{V}_3) + c_2(\tilde{\mathcal{V}}_3) + \frac{1}{60}c_2(\text{End}_0(\mathcal{V}_{E_8})) = 24 \tag{D.34}$$

As in previous cases, we can now specialize this result to the particular $\text{SU}(3) \times \text{SU}(3)$ bundle which we obtain after transition, as given in equation (D.30). We have that,

$$c_2(\tilde{\mathcal{V}}_3) = c_2(\mathcal{V}_2) . \tag{D.35}$$

Using this, we can compare the spectrum which is obtained by a decomposition of the initial $\text{SU}(6)$ multiplets to that obtained by a direct computation from the bundle after transition. The results of this comparison are in table 30.

As in all of the cases we look at, the result is fully consistent with the field theory analysis given in section 2.3.2

SU(3) × SU(3) Representation	# from SU(6) multiplet decomposition	# found after transition
(1, 1)	$6c_2(\mathcal{V}_3) + 6c_2(\mathcal{V}_2) - 30$	$6c_2(\mathcal{V}_3) + 6c_2(\mathcal{V}_2) - 32$
(3, 1)	$3c_2(\mathcal{V}_3) + 3c_2(\mathcal{V}_2) - 18$	$3c_2(\mathcal{V}_3) + 3c_2(\mathcal{V}_2) - 18$
(1, 3)	$3c_2(\mathcal{V}_3) + 3c_2(\mathcal{V}_2) - 18$	$3c_2(\mathcal{V}_3) + 3c_2(\mathcal{V}_2) - 18$
(3, 3)	$c_2(\mathcal{V}_3) - 6$	$c_2(\mathcal{V}_3) - 6$
(3, $\bar{3}$)	$c_2(\mathcal{V}_2) - 4$	$c_2(\mathcal{V}_2) - 6$

Table 30. Matter content after Higgsing an SU(6) to an SU(3) × SU(3) theory, both via a naive decomposition of the initial SU(6) multiplets and via a direct computation from the resulting SU(3) × SU(3) bundle.

D.3.2 SU(7)

The analysis for Higgsing SU(7) on triple antisymmetrics is extremely similar. We have,

$$\begin{aligned}
 \text{SU}(7) &\supset \text{SU}(4) \times \text{SU}(3) \times \text{U}(1) & \text{(D.36)} \\
 \mathbf{35} &= (\mathbf{1}, \mathbf{1})_{-12} + (\bar{\mathbf{4}}, \mathbf{1})_9 + (\mathbf{4}, \bar{\mathbf{3}})_{-5} + (\mathbf{6}, \mathbf{3})_2 \\
 \mathbf{7} &= (\mathbf{1}, \mathbf{3})_{-4} + (\mathbf{4}, \mathbf{1})_3 \\
 \mathbf{21} &= (\mathbf{1}, \bar{\mathbf{3}})_{-8} + (\mathbf{4}, \mathbf{3})_{-1} + (\mathbf{6}, \mathbf{1})_6 .
 \end{aligned}$$

The breaking pattern we are interested in thus corresponds to giving a VEV to a $\mathbf{35}, \bar{\mathbf{35}}$ pair. The $\mathbf{35}$'s, according to table 8, lie in the cohomology $H^1(\mathcal{L})$. In terms of bundle topology, giving an expectation value to such a field corresponds to forming the following bundle.

$$\begin{aligned}
 \mathcal{V} &= \mathcal{V}_2 \oplus \tilde{\mathcal{V}}_2 & \text{(D.37)} \\
 \text{where } 0 &\rightarrow \mathcal{L} \rightarrow \tilde{\mathcal{V}}_2 \rightarrow \mathcal{O} \rightarrow 0
 \end{aligned}$$

As in previous cases one should really think of $\tilde{\mathcal{V}}_2$ as being a deformation of this extension and its dual [54]. Here $\tilde{\mathcal{V}}_2$ is an U(2) bundle and the bundle \mathcal{V}_2 is unaffected by the transition. The overall structure group is $S(U(2) \times U(2))$ which does indeed break E_8 to $SU(4) \times SU(3) \times U(1)$ (where the last, U(1), factor is Green-Schwarz anomalous).

The group theory for a $SU(4) \times SU(3) \times U(1)$ heterotic compactification is as follows,

$$E_8 \supset \text{SU}(4) \times \text{SU}(3) \times \text{U}(1) \times \text{SU}(2) \times \text{SU}(2) \tag{D.38}$$

$$\begin{aligned}
 \mathbf{248} &= (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{2})_{-2} + (\mathbf{4}, \mathbf{3}, \mathbf{1}, \mathbf{2})_1 + (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{2}, \mathbf{2})_2 + (\bar{\mathbf{4}}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{2})_{-1} & \text{(D.39)} \\
 &+ (\mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{2})_{-3} + (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{2})_3 + (\mathbf{6}, \mathbf{1}, \mathbf{2}, \mathbf{2})_0 + (\mathbf{1}, \mathbf{8}, \mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{1})_{-4} \\
 &+ (\mathbf{4}, \bar{\mathbf{3}}, \mathbf{2}, \mathbf{1})_{-1} + (\mathbf{6}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{1})_2 + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1})_4 + (\bar{\mathbf{4}}, \mathbf{3}, \mathbf{2}, \mathbf{1})_1 + (\bar{\mathbf{6}}, \mathbf{3}, \mathbf{1}, \mathbf{1})_{-2} + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_0 \\
 &+ (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1})_0 + (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{1})_{-3} + (\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{1})_3 + (\mathbf{15}, \mathbf{1}, \mathbf{1}, \mathbf{1})_0 ,
 \end{aligned}$$

which leads to the matter content given in table 31.

This matter content leads to the following anomaly cancellation condition.

$$\begin{aligned}
 n_H + 29n_T - n_V &= 273 & \text{(D.40)} \\
 \Rightarrow c_2(\tilde{\mathcal{V}}_2) + c_2(\mathcal{V}_1) - 2c_1(\tilde{\mathcal{V}}_2)^2 + \frac{1}{60}c_2(\text{End}_0(\mathcal{V}_{E_8})) &= 24
 \end{aligned}$$

Representation	Cohomology	Multiplicity
(1, 1)	$H^1(\text{End}_0(\mathcal{V}_2)) \oplus H^1(\text{End}_0(\tilde{\mathcal{V}}_2))$	$4c_2(\mathcal{V}_2) + 4c_2(\tilde{\mathcal{V}}_2) - 2c_1(\tilde{\mathcal{V}}_2)^2 - 12$
(1, 3)	$H^1(\mathcal{V}_2 \otimes \tilde{\mathcal{V}}_2^\vee) \oplus H^1(\wedge^2 \mathcal{V}_2^\vee \otimes \wedge^2 \tilde{\mathcal{V}}_2)$	$2c_2(\tilde{\mathcal{V}}_2) + 2c_2(\mathcal{V}_2) - 5c_1(\tilde{\mathcal{V}}_2)^2 - 10$
(4, 3)	$H^1(\mathcal{V}_2^\vee)$	$c_2(\mathcal{V}_2) - \frac{1}{2}c_1(\tilde{\mathcal{V}}_2)^2 - 4$
(4, 1)	$H^1(\tilde{\mathcal{V}}_2 \otimes \wedge^2 \mathcal{V}_2^\vee) \oplus H^1(\mathcal{V}_2 \otimes \wedge^2 \tilde{\mathcal{V}}_2^\vee)$	$c_2(\tilde{\mathcal{V}}_2) + c_2(\mathcal{V}_2) - 5c_1(\tilde{\mathcal{V}}_2)^2 - 8$
(6, 1)	$H^1(\tilde{\mathcal{V}}_2 \otimes \mathcal{V}_2)$	$2c_2(\mathcal{V}_2) + 2c_2(\tilde{\mathcal{V}}_2) - c_1(\tilde{\mathcal{V}}_2)^2 - 8$
(6, $\bar{3}$)	$H^1(\wedge^2 \mathcal{V}_2^\vee)$	$-\frac{1}{2}c_1(\tilde{\mathcal{V}}_2)^2 - 2$
($\bar{4}$, 3)	$H^1(\tilde{\mathcal{V}}_2)$	$c_2(\tilde{\mathcal{V}}_2) - \frac{1}{2}c_1(\tilde{\mathcal{V}}_2)^2 - 4$

Table 31. The cohomology associated to each representation of the low-energy gauge group $\text{SU}(4) \times \text{SU}(3) \times \text{U}(1)$.

$\text{SU}(4) \times \text{SU}(3)$ Representation	# from $\text{SU}(7)$ multiplet decomposition	# found after transition
(1, 1)	$4c_2(\mathcal{V}_2) - 2c_1(\mathcal{L})^2 - 10$	$4c_2(\mathcal{V}_2) - 2c_1(\mathcal{L})^2 - 12$
(1, 3)	$2c_2(\mathcal{V}_2) - 5c_1(\mathcal{L})^2 - 10$	$2c_2(\mathcal{V}_2) - 5c_1(\mathcal{L})^2 - 10$
(4, 3)	$c_2(\mathcal{V}_2) - \frac{1}{2}c_1(\mathcal{L})^2 - 4$	$c_2(\mathcal{V}_2) - \frac{1}{2}c_1(\mathcal{L})^2 - 4$
(4, 1)	$c_2(\mathcal{V}_2) - 5c_1(\mathcal{L})^2 - 8$	$c_2(\mathcal{V}_2) - 5c_1(\mathcal{L})^2 - 8$
(6, 1)	$2c_2(\mathcal{V}_2) - c_1(\mathcal{L})^2 - 8$	$2c_2(\mathcal{V}_2) - c_1(\mathcal{L})^2 - 8$
(6, $\bar{3}$)	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$
($\bar{4}$, 3)	$-\frac{1}{2}c_1(\mathcal{L})^2 - 2$	$-\frac{1}{2}c_1(\mathcal{L})^2 - 4$

Table 32. Matter content after Higgsing an $\text{SU}(7)$ to an $\text{SU}(4) \times \text{SU}(3)$ theory, both via a naive decomposition of the initial $\text{SU}(7)$ multiplets and via a direct computation from the resulting $\text{S}(\text{U}(2) \times \text{U}(2))$ bundle.

For the case of a $\text{S}(\text{U}(2) \times \text{U}(2))$ bundle obtained by a transition of the form (D.37) we have the following topology.

$$c_2(\tilde{\mathcal{V}}_2) = 0 \tag{D.41}$$

$$c_1(\tilde{\mathcal{V}}_2) = c_1(\mathcal{L}) \tag{D.42}$$

Given this, and noting that the **6** of $\text{SU}(4)$ is real, we obtain table 32, which gives the number of $\text{SU}(4) \times \text{SU}(3)$ multiplets obtained by decomposing the $\text{SU}(7)$ matter content, compared to a direct computation of the spectrum of the four-dimensional theory given the bundle topology after transition. Once more these results are consistent with the usual understanding of such a Higgsing process and with the field theory analysis in section 2.3.2.

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References

- [1] N. Seiberg and E. Witten, *Comments on string dynamics in six-dimensions*, *Nucl. Phys. B* **471** (1996) 121 [[hep-th/9603003](#)] [[INSPIRE](#)].
- [2] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 2.*, *Nucl. Phys. B* **476** (1996) 437 [[hep-th/9603161](#)] [[INSPIRE](#)].
- [3] S. Kachru and E. Silverstein, *Chirality changing phase transitions in 4 – D string vacua*, *Nucl. Phys. B* **504** (1997) 272 [[hep-th/9704185](#)] [[INSPIRE](#)].
- [4] B.A. Ovrut, T. Pantev and J. Park, *Small instanton transitions in heterotic M-theory*, *JHEP* **05** (2000) 045 [[hep-th/0001133](#)] [[INSPIRE](#)].
- [5] D.R. Morrison and W. Taylor, *Matter and singularities*, *JHEP* **01** (2012) 022 [[arXiv:1106.3563](#)] [[INSPIRE](#)].
- [6] A. Grassi and D.R. Morrison, *Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds*, *Commun. Num. Theor. Phys.* **6** (2012) 51 [[arXiv:1109.0042](#)] [[INSPIRE](#)].
- [7] M.B. Green, J.H. Schwarz and P.C. West, *Anomaly Free Chiral Theories in Six-Dimensions*, *Nucl. Phys. B* **254** (1985) 327 [[INSPIRE](#)].
- [8] A. Sagnotti, *A Note on the Green-Schwarz mechanism in open string theories*, *Phys. Lett. B* **294** (1992) 196 [[hep-th/9210127](#)] [[INSPIRE](#)].
- [9] V. Kumar, D.R. Morrison and W. Taylor, *Global aspects of the space of 6D $\mathcal{N} = 1$ supergravities*, *JHEP* **11** (2010) 118 [[arXiv:1008.1062](#)] [[INSPIRE](#)].
- [10] R. Donagi, B.A. Ovrut and D. Waldram, *Moduli spaces of five-branes on elliptic Calabi-Yau threefolds*, *JHEP* **11** (1999) 030 [[hep-th/9904054](#)] [[INSPIRE](#)].
- [11] E. Buchbinder, R. Donagi and B.A. Ovrut, *Vector bundle moduli and small instanton transitions*, *JHEP* **06** (2002) 054 [[hep-th/0202084](#)] [[INSPIRE](#)].
- [12] J. Erler, *Anomaly cancellation in six-dimensions*, *J. Math. Phys.* **35** (1994) 1819 [[hep-th/9304104](#)] [[INSPIRE](#)].
- [13] V. Kumar, D.S. Park and W. Taylor, *6D supergravity without tensor multiplets*, *JHEP* **04** (2011) 080 [[arXiv:1011.0726](#)] [[INSPIRE](#)].
- [14] M. Bershadsky, K.A. Intriligator, S. Kachru, D.R. Morrison, V. Sadov and C. Vafa, *Geometric singularities and enhanced gauge symmetries*, *Nucl. Phys. B* **481** (1996) 215 [[hep-th/9605200](#)] [[INSPIRE](#)].
- [15] D.R. Morrison, *TASI lectures on compactification and duality*, [hep-th/0411120](#) [[INSPIRE](#)].
- [16] F. Denef, *Les Houches Lectures on Constructing String Vacua*, [arXiv:0803.1194](#) [[INSPIRE](#)].
- [17] W. Taylor, *TASI Lectures on Supergravity and String Vacua in Various Dimensions*, [arXiv:1104.2051](#) [[INSPIRE](#)].
- [18] C. Vafa, *Evidence for F-theory*, *Nucl. Phys. B* **469** (1996) 403 [[hep-th/9602022](#)] [[INSPIRE](#)].
- [19] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 1.*, *Nucl. Phys. B* **473** (1996) 74 [[hep-th/9602114](#)] [[INSPIRE](#)].
- [20] L.B. Anderson and W. Taylor, *Geometric constraints in dual F-theory and heterotic string compactifications*, *JHEP* **08** (2014) 025 [[arXiv:1405.2074](#)] [[INSPIRE](#)].

- [21] D.R. Morrison and W. Taylor, *Non-Higgsable clusters for 4D F-theory models*, *JHEP* **05** (2015) 080 [[arXiv:1412.6112](#)] [[INSPIRE](#)].
- [22] K. Kodaira, *On compact analytic surfaces. II*, *Ann. Math.* **77** (1963) 563.
- [23] K. Kodaira, K. Kodaira, *On compact analytic surfaces. III*, *Ann. Math.* **78** (1963) 1.
- [24] S. Katz, D.R. Morrison, S. Schäfer-Nameki and J. Sully, *Tate's algorithm and F-theory*, *JHEP* **08** (2011) 094 [[arXiv:1106.3854](#)] [[INSPIRE](#)].
- [25] D.R. Morrison and W. Taylor, *Classifying bases for 6D F-theory models*, *Central Eur. J. Phys.* **10** (2012) 1072 [[arXiv:1201.1943](#)] [[INSPIRE](#)].
- [26] S.H. Katz and C. Vafa, *Matter from geometry*, *Nucl. Phys. B* **497** (1997) 146 [[hep-th/9606086](#)] [[INSPIRE](#)].
- [27] M. Esole and S.-T. Yau, *Small resolutions of SU(5)-models in F-theory*, *Adv. Theor. Math. Phys.* **17** (2013) 1195 [[arXiv:1107.0733](#)] [[INSPIRE](#)].
- [28] C. Lawrie and S. Schäfer-Nameki, *The Tate Form on Steroids: Resolution and Higher Codimension Fibers*, *JHEP* **04** (2013) 061 [[arXiv:1212.2949](#)] [[INSPIRE](#)].
- [29] H. Hayashi, C. Lawrie and S. Schäfer-Nameki, *Phases, Flops and F-theory: SU(5) Gauge Theories*, *JHEP* **10** (2013) 046 [[arXiv:1304.1678](#)] [[INSPIRE](#)].
- [30] H. Hayashi, C. Lawrie, D.R. Morrison and S. Schäfer-Nameki, *Box Graphs and Singular Fibers*, *JHEP* **05** (2014) 048 [[arXiv:1402.2653](#)] [[INSPIRE](#)].
- [31] M. Esole, S.-H. Shao and S.-T. Yau, *Singularities and Gauge Theory Phases*, [arXiv:1402.6331](#) [[INSPIRE](#)].
- [32] A.P. Braun and S. Schäfer-Nameki, *Box Graphs and Resolutions I*, *Nucl. Phys. B* **905** (2016) 447 [[arXiv:1407.3520](#)] [[INSPIRE](#)].
- [33] A. Grassi, J. Halverson and J.L. Shaneson, *Matter From Geometry Without Resolution*, *JHEP* **10** (2013) 205 [[arXiv:1306.1832](#)] [[INSPIRE](#)].
- [34] G. Martini and W. Taylor, *6D F-theory models and elliptically fibered Calabi-Yau threefolds over semi-toric base surfaces*, *JHEP* **06** (2015) 061 [[arXiv:1404.6300](#)] [[INSPIRE](#)].
- [35] E. Witten, *Phase transitions in M-theory and F-theory*, *Nucl. Phys. B* **471** (1996) 195 [[hep-th/9603150](#)] [[INSPIRE](#)].
- [36] M. Bershadsky and A. Johansen, *Colliding singularities in F-theory and phase transitions*, *Nucl. Phys. B* **489** (1997) 122 [[hep-th/9610111](#)] [[INSPIRE](#)].
- [37] O.J. Ganor and A. Hanany, *Small E_8 instantons and tensionless noncritical strings*, *Nucl. Phys. B* **474** (1996) 122 [[hep-th/9602120](#)] [[INSPIRE](#)].
- [38] N. Seiberg, *Nontrivial fixed points of the renormalization group in six-dimensions*, *Phys. Lett. B* **390** (1997) 169 [[hep-th/9609161](#)] [[INSPIRE](#)].
- [39] J.J. Heckman, D.R. Morrison and C. Vafa, *On the Classification of 6D SCFTs and Generalized ADE Orbifolds*, *JHEP* **05** (2014) 028 [*Erratum ibid.* **1506** (2015) 017] [[arXiv:1312.5746](#)] [[INSPIRE](#)].
- [40] M. Del Zotto, J.J. Heckman, A. Tomasiello and C. Vafa, *6d Conformal Matter*, *JHEP* **02** (2015) 054 [[arXiv:1407.6359](#)] [[INSPIRE](#)].
- [41] B. Haghighat, A. Klemm, G. Lockhart and C. Vafa, *Strings of Minimal 6d SCFTs*, *Fortsch. Phys.* **63** (2015) 294 [[arXiv:1412.3152](#)] [[INSPIRE](#)].

- [42] M. Del Zotto, J.J. Heckman, D.R. Morrison and D.S. Park, *6D SCFTs and Gravity*, *JHEP* **06** (2015) 158 [[arXiv:1412.6526](#)] [[INSPIRE](#)].
- [43] J.J. Heckman, D.R. Morrison, T. Rudelius and C. Vafa, *Geometry of 6D RG Flows*, *JHEP* **09** (2015) 052 [[arXiv:1505.00009](#)] [[INSPIRE](#)].
- [44] A. Grassi, J. Halverson and J.L. Shaneson, *Non-Abelian Gauge Symmetry and the Higgs Mechanism in F-theory*, *Commun. Math. Phys.* **336** (2015) 1231 [[arXiv:1402.5962](#)] [[INSPIRE](#)].
- [45] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology*, *Cambridge Monographs On Mathematical Physics*, Cambridge University Press, Cambridge U.K. (1987).
- [46] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer, Heidelberg Germany (1995).
- [47] M. Dine, N. Seiberg and E. Witten, *Fayet-Iliopoulos Terms in String Theory*, *Nucl. Phys. B* **289** (1987) 589 [[INSPIRE](#)].
- [48] E.R. Sharpe, *Kähler cone substructure*, *Adv. Theor. Math. Phys.* **2** (1999) 1441 [[hep-th/9810064](#)] [[INSPIRE](#)].
- [49] A. Lukas and K.S. Stelle, *Heterotic anomaly cancellation in five-dimensions*, *JHEP* **01** (2000) 010 [[hep-th/9911156](#)] [[INSPIRE](#)].
- [50] R. Blumenhagen, G. Honecker and T. Weigand, *Loop-corrected compactifications of the heterotic string with line bundles*, *JHEP* **06** (2005) 020 [[hep-th/0504232](#)] [[INSPIRE](#)].
- [51] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, *The Edge Of Supersymmetry: Stability Walls in Heterotic Theory*, *Phys. Lett. B* **677** (2009) 190 [[arXiv:0903.5088](#)] [[INSPIRE](#)].
- [52] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, *Stability Walls in Heterotic Theories*, *JHEP* **09** (2009) 026 [[arXiv:0905.1748](#)] [[INSPIRE](#)].
- [53] P.S. Aspinwall and D.R. Morrison, *Nonsimply connected gauge groups and rational points on elliptic curves*, *JHEP* **07** (1998) 012 [[hep-th/9805206](#)] [[INSPIRE](#)].
- [54] J. Li and S.-T. Yau, *The Existence of supersymmetric string theory with torsion*, *J. Diff. Geom.* **70** (2005) 143 [[hep-th/0411136](#)] [[INSPIRE](#)].
- [55] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Commun. Math. Phys.* **187** (1997) 679 [[hep-th/9701162](#)] [[INSPIRE](#)].
- [56] G. Curio and R.Y. Donagi, *Moduli in $N = 1$ heterotic/F theory duality*, *Nucl. Phys. B* **518** (1998) 603 [[hep-th/9801057](#)] [[INSPIRE](#)].
- [57] P.S. Aspinwall, *Aspects of the hypermultiplet moduli space in string duality*, *JHEP* **04** (1998) 019 [[hep-th/9802194](#)] [[INSPIRE](#)].
- [58] R. Donagi, S. Katz and M. Wijnholt, *Weak Coupling, Degeneration and Log Calabi-Yau Spaces*, [arXiv:1212.0553](#) [[INSPIRE](#)].
- [59] R. Friedman and J. Morgan, *Smooth Four-Manifolds and Complex Surfaces*, Springer, Heidelberg Germany (1994).
- [60] R. Friedman, J.W. Morgan and E. Witten, *Vector bundles over elliptic fibrations*, [alg-geom/9709029](#) [[INSPIRE](#)].

- [61] R.Y. Donagi, *Principal bundles on elliptic fibrations*, *Asian J. Math.* **1** (1997) 214 [[alg-geom/9702002](#)] [[INSPIRE](#)].
- [62] R. Donagi, *Spectral Covers*, in *Mathematical Sciences Research Institute Publications. Book 28: Current topics in complex algebraic geometry*, Cambridge University Press, Cambridge U.K. (1992), pg. 65 [[alg-geom/9505009](#)].
- [63] R. Donagi, *Heterotic/F theory duality: ICMP lecture*, [hep-th/9802093](#) [[INSPIRE](#)].
- [64] B.A. Ovrut, T. Pantev and J. Park, *Small instanton transitions in heterotic M-theory*, *JHEP* **05** (2000) 045 [[hep-th/0001133](#)] [[INSPIRE](#)].
- [65] G. Curio, *Chiral matter and transitions in heterotic string models*, *Phys. Lett. B* **435** (1998) 39 [[hep-th/9803224](#)] [[INSPIRE](#)].
- [66] E. Witten, *Small instantons in string theory*, *Nucl. Phys. B* **460** (1996) 541 [[hep-th/9511030](#)] [[INSPIRE](#)].
- [67] C. Lazaroiu, *On degree zero semistable bundles over an elliptic curve*, [physics/9712054](#).
- [68] K.-S. Choi and H. Hayashi, *U(n) Spectral Covers from Decomposition*, *JHEP* **06** (2012) 009 [[arXiv:1203.3812](#)] [[INSPIRE](#)].
- [69] L.B. Anderson, J.J. Heckman and S. Katz, *T-Branes and Geometry*, *JHEP* **05** (2014) 080 [[arXiv:1310.1931](#)] [[INSPIRE](#)].
- [70] D.R. Morrison and D.S. Park, *F-Theory and the Mordell-Weil Group of Elliptically-Fibered Calabi-Yau Threefolds*, *JHEP* **10** (2012) 128 [[arXiv:1208.2695](#)] [[INSPIRE](#)].
- [71] H. Jockers and J. Louis, *D-terms and F-terms from D7-brane fluxes*, *Nucl. Phys. B* **718** (2005) 203 [[hep-th/0502059](#)] [[INSPIRE](#)].
- [72] M. Buican, D. Malyshev, D.R. Morrison, H. Verlinde and M. Wijnholt, *D-branes at Singularities, Compactification and Hypercharge*, *JHEP* **01** (2007) 107 [[hep-th/0610007](#)] [[INSPIRE](#)].
- [73] T.W. Grimm and T. Weigand, *On Abelian Gauge Symmetries and Proton Decay in Global F-theory GUTs*, *Phys. Rev. D* **82** (2010) 086009 [[arXiv:1006.0226](#)] [[INSPIRE](#)].
- [74] T.W. Grimm, M. Kerstan, E. Palti and T. Weigand, *Massive Abelian Gauge Symmetries and Fluxes in F-theory*, *JHEP* **12** (2011) 004 [[arXiv:1107.3842](#)] [[INSPIRE](#)].
- [75] A.P. Braun, A. Collinucci and R. Valandro, *The fate of U(1)'s at strong coupling in F-theory*, *JHEP* **07** (2014) 028 [[arXiv:1402.4054](#)] [[INSPIRE](#)].
- [76] L.B. Anderson, I. García-Etxebarria, T.W. Grimm and J. Keitel, *Physics of F-theory compactifications without section*, *JHEP* **12** (2014) 156 [[arXiv:1406.5180](#)] [[INSPIRE](#)].
- [77] I. García-Etxebarria, T.W. Grimm and J. Keitel, *Yukawas and discrete symmetries in F-theory compactifications without section*, *JHEP* **11** (2014) 125 [[arXiv:1408.6448](#)] [[INSPIRE](#)].
- [78] M. Cvetič, A. Grassi, D. Klevers, M. Poretschkin and P. Song, *Origin of Abelian Gauge Symmetries in Heterotic/F-theory Duality*, *JHEP* **04** (2016) 041 [[arXiv:1511.08208](#)] [[INSPIRE](#)].
- [79] J. Gray, Y.-H. He, A. Ilderton and A. Lukas, *STRINGVACUA: A Mathematica Package for Studying Vacuum Configurations in String Phenomenology*, *Comput. Phys. Commun.* **180** (2009) 107 [[arXiv:0801.1508](#)] [[INSPIRE](#)].

- [80] M. Cvetič, D. Klevers, H. Piragua and W. Taylor, *General $U(1) \times U(1)$ F -theory compactifications and beyond: geometry of unHiggsings and novel matter structure*, *JHEP* **11** (2015) 204 [[arXiv:1507.05954](#)] [[INSPIRE](#)].
- [81] M.R. Douglas and C.-g. Zhou, *Chirality change in string theory*, *JHEP* **06** (2004) 014 [[hep-th/0403018](#)] [[INSPIRE](#)].
- [82] M. Cicoli, S. Krippendorff, C. Mayrhofer, F. Quevedo and R. Valandro, *The Web of D -branes at Singularities in Compact Calabi-Yau Manifolds*, *JHEP* **05** (2013) 114 [[arXiv:1304.2771](#)] [[INSPIRE](#)].
- [83] P.S. Aspinwall and M. Gross, *The $SO(32)$ heterotic string on a $K3$ surface*, *Phys. Lett. B* **387** (1996) 735 [[hep-th/9605131](#)] [[INSPIRE](#)].
- [84] V. Sadov, *Generalized Green-Schwarz mechanism in F -theory*, *Phys. Lett. B* **388** (1996) 45 [[hep-th/9606008](#)] [[INSPIRE](#)].
- [85] L.B. Anderson, A. Constantin, S.-J. Lee and A. Lukas, *Hypercharge Flux in Heterotic Compactifications*, *Phys. Rev. D* **91** (2015) 046008 [[arXiv:1411.0034](#)] [[INSPIRE](#)].
- [86] L.B. Anderson, A. Constantin, J. Gray, A. Lukas and E. Palti, *A Comprehensive Scan for Heterotic $SU(5)$ GUT models*, *JHEP* **01** (2014) 047 [[arXiv:1307.4787](#)] [[INSPIRE](#)].
- [87] L.B. Anderson, J. Gray, A. Lukas and E. Palti, *Heterotic Line Bundle Standard Models*, *JHEP* **06** (2012) 113 [[arXiv:1202.1757](#)] [[INSPIRE](#)].
- [88] L.B. Anderson, J. Gray, A. Lukas and E. Palti, *Two Hundred Heterotic Standard Models on Smooth Calabi-Yau Threefolds*, *Phys. Rev. D* **84** (2011) 106005 [[arXiv:1106.4804](#)] [[INSPIRE](#)].
- [89] V. Kumar and W. Taylor, *String Universality in Six Dimensions*, *Adv. Theor. Math. Phys.* **15** (2011) 325 [[arXiv:0906.0987](#)] [[INSPIRE](#)].
- [90] R. Blumenhagen, S. Moster and T. Weigand, *Heterotic GUT and standard model vacua from simply connected Calabi-Yau manifolds*, *Nucl. Phys. B* **751** (2006) 186 [[hep-th/0603015](#)] [[INSPIRE](#)].
- [91] M. Kuriyama, H. Nakaajima and T. Watari, *Theoretical Framework for R -parity Violation*, *Phys. Rev. D* **79** (2009) 075002 [[arXiv:0802.2584](#)] [[INSPIRE](#)].
- [92] L.B. Anderson, J. Gray and B. Ovrut, *Yukawa Textures From Heterotic Stability Walls*, *JHEP* **05** (2010) 086 [[arXiv:1001.2317](#)] [[INSPIRE](#)].
- [93] L.B. Anderson, J. Gray and B.A. Ovrut, *Transitions in the Web of Heterotic Vacua*, *Fortsch. Phys.* **59** (2011) 327 [[arXiv:1012.3179](#)] [[INSPIRE](#)].
- [94] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, Heidelberg Germany (1977).