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Matter Representations and Gauge Symmetry Breaking via Compactified Space

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We study dynamical gauge symmetry breaking via compactified space in the framework of $SU(N)$ gauge theory in $M^{d-1} \times S^1$ $(d = 4, 5, 6)$ space-time. In particular, we study in detail the gauge symmetry breaking in $SU(2)$ and $SU(3)$ gauge theories when the models contain both fundamental and adjoint matter. As a result, we find that any pattern of gauge symmetry breaking can be realized by selecting an appropriate set of numbers $(N_{\rm f}, N_{\rm ad})$ in these cases. This is achieved without tuning boundary conditions of the matter fields. As a by-product, in some cases we obtain an effective potential which has no curvature at the minimum, thus leading to massless Higgs scalars, irrespective of the size of the compactified space.

§1. Introduction

1.1. *The gauge hierarchy problem and higher-dimensional gauge theories*

We have proposed a possibility of solving the gauge hierarchy problem[1)] by invoking higher-dimensional gauge symmetry.

Our main idea in this scenario is the following. Suppose we consider a gauge theory on the d-dimensional manifold $M^{\rm Minkow} \times M^{\rm extra}$, where $M^{\rm Minkow}$ is an ordinary Minkowskian 4-dimensional space-time, and $M^{\rm extra}$ is a compactified spatial $(d-4)$-dimensional manifold. The gauge fields A_M on $M^{\rm Minkow} \times M^{\rm extra}$ have $4 + (d - 4)$ components:

$$A_\mu \ (\mu = 0, \cdots, 3), \qquad A_m \ (m = 4, \cdots, d - 1). \tag{1}$$

If we regard zero-modes of $d - 4$ components of gauge fields as "Higgs scalars", we can avoid quadratically divergent quantum corrections of Higgs boson masses thanks to the higher-dimensional gauge symmetry.

In a previous work,[1)] we considered a QED model on $M^{d-1} \times S^1$ space-time and took a zero-mode of d-th the component of the gauge field as a "Higgs scalar". We calculated also the quantum mass correction of the d-th component of the gauge field. As a result we confirmed the disappearance of the quadratic divergence. We, however, have found that a finite mass correction to the "Higgs" remains, which is generally proportional to the inverse of the S^1 radius.

Although the gauge hierarchy problem in its original sense has been solved, as the quadratic divergence has disappeared, several issues have to be settled for the scenario to be realistic. Here we address the following two important issues. (i) In our toy model on $M^4 \times S^1$, the Higgs mass $M_{\rm H}$ experiences a finite but generally

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large quantum correction provided that the radius of S^1 is as small as the GUT or Planck length. [1] Then the issue is how we can make the finite correction to $M_{\rm H}$ small, approximately comparable to the weak scale. (ii) The toy model on $M^4 \times S^1$ is higher dimensional QED. The $U(1)$ gauge symmetry, therefore, is not spontaneously broken, even though the Higgs d-th component of the photon field has non-vanishing vacuum expectation value (VEV). This is simply because the Higgs field is electrically neutral, and there appears the possibility to break the gauge symmetry, once the theory is extended to the non-Abelian gauge (Yang-Mills) theory, where the Higgs behaves as an adjoint representation of gauge group.

With regard to the first issue above, we have already discussed a few possibilities to settle the problem in a previous work. [1] The first rather trivial possibility is to assume a "large" compactification scale such as TeV^{-1}. [2),3)] It is interesting to note that such a large scale compactification has attracted revived interest, partially inspired by the progress in superstring/M theories. [4] The second possibility is to utilize a heavy electron where mass m satisfies $mL \gg 1$ (L: circumference of S^1). Then the correction to $M_{\rm H}$ is suppressed roughly as $\exp(-mL)/L$. (Here $\exp(-mL)$ corresponds to the Boltzmann factor in finite temperature field theory.) It has also been pointed out that the finite mass $M_{\rm H}$ can be regarded as coming from a sort of Aharonov-Bohm effect on S^1 space and may disappear if the compactified space is a simply-connected manifold. We have confirmed this expectation in a toy model on S^2, showing that $M_{\rm H}$ identically vanishes.

The main purpose of this paper is to settle the remaining second issue mentioned above. In theories with S^1 as the extra space, the constant background and VEV of gauge fields have physical meaning as a Wilson loop, and they can be dynamically fixed by the minimization of effective potential as functions of the background fields. Thus in Yang-Mills theories the VEV behave just as the VEV of adjoint Higgs, and they generally break the gauge symmetry spontaneously: the so-called "Hosotani mechanism". [5]

To obtain realistic GUT-type theories, starting from higher-dimensional Yang-Mills theories, it is quite important to study how the gauge symmetry is dynamically broken by the Hosotani mechanism. In particular, we are interested in the question of how the breaking pattern of gauge symmetry depends on the choice of the representation (fundamental or adjoint or $\cdots$) of matter fields. We will extensively study this point by considering two representative gauge groups, $SU(2)$ and $SU(3)$. As a bonus we also find an interesting property of the theory, concerning the first issue mentioned above. Namely, we realize in some cases that the curvature of the curve of the potential function identically vanishes, leading to $M_{\rm H} = 0$. It should be noted that this vanishing $M_{\rm H}$ is obtained for arbitrary compactification scale and without the introduction of a simply-connected space like S^2. [1]

1.2. *Gauge symmetry breaking and matter representation in Hosotani mechanism*

We now briefly review the history of the investigation of the gauge symmetry breaking via the Hosotani mechanism, and in particular the case of $M^{d-1} \times S^1$ space-time configuration.

In 1983, Hosotani found that a gauge symmetry may break down in a system of

$SU(N)$ gauge theories on $M^3 \times S^1$ space-time. But in his model in order to break the gauge symmetry, it was necessary to have both many (more than 16 flavors) complex scalar fields as the matter field and a non-trivial boundary conditions for the matter fields, $\delta = \pi/2$.[5)] Later it became obvious that for the case in which fermions belong to the fundamental representation of the $SU(N)$ gauge group, gauge symmetry is never broken. In 1989, It was found that $SU(2)$ gauge symmetry breaking is possible when fermions are in an adjoint representation,[6),7)] and it was shown that the pattern of gauge symmetry breaking depends on the *congruency class representation of the gauge group.*[7),8)]

Many works have been done along this direction, giving hope for using the Hosotani mechanism as a possible mechanism of GUT symmetry breaking. This scenario has been investigated to some extent.[9),10)] However, the case in which the theory contains both fundamental and adjoint representations of fermions has never been investigated, except the E_6 model in Ref. 7) and supersymmetric models.[12),13)] I was partially motivated by these works in supersymmetric models and attempt to study the non-supersymmetric version of gauge symmetry breaking via the Hosotani mechanism in theories with both fundamental and adjoint matter.

§2. Dynamical gauge symmetry breaking through extra dimension

2.1. *Effective potential*

We consider $SU(N)$ gauge theories in $M^{d-1} \times S^1$ space-time with both fundamental and adjoint fermionic fields.

The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}F^{MN}F_{MN} + \sum_{j=1}^{N_{\mathrm{f}}} \bar{\psi}_{\mathrm{f},j} i\gamma^M D_M \psi_{\mathrm{f},j} + \sum_{k=1}^{N_{\mathrm{ad}}} \bar{\psi}_{\mathrm{ad},k} i\gamma^M D_M \psi_{\mathrm{ad},k}, \tag{2}$$

where

$$F_{MN} = \partial_M A_N - \partial_N A_M + ig[A_M, A_N], \qquad A_M = A_M^a T^a, \tag{3}$$
$$D_M \psi_{\mathrm{f}} = \partial_M \psi_{\mathrm{f}} + igT^a A_M^a \psi_{\mathrm{f}}, \tag{4}$$
$$D_M \psi_{\mathrm{ad}} = \partial_M \psi_{\mathrm{ad}} + ig[T^a A_M^a, \psi_{\mathrm{ad}}], \tag{5}$$

ψ_{f} (ψ_{ad}) are fermions in the fundamental (adjoint) representation, and N_{f} (N_{ad}) are the flavor numbers of the Dirac fermion in the fundamental (adjoint) representation. The indices M and N run from 0 to $d-1$ (whereas μ and ν run from 0 to $d-2$) and j and k are flavor indices. The color index has been omitted here. We use x and y as the coordinates on M^{d-1} and S^1, respectively.

The boundary conditions on S^1 are taken as

$$A_\mu(x, y+L) = A_\mu(x, y), \tag{6}$$
$$\psi_{\mathrm{f}}(x, y+L) = e^{i\delta_{\mathrm{f}}} \psi_{\mathrm{f}}(x, y), \tag{7}$$
$$\psi_{\mathrm{ad}}(x, y+L) = e^{i\delta_{\mathrm{ad}}} \psi_{\mathrm{ad}}(x, y), \tag{8}$$

where L is the compactification scale of the S^1 subspace and is related to the radius of S^1 as $L = 2\pi R$.

The effective potential is separated into three parts:[9),10)]

$$V_{\rm eff} = V_{\rm eff}^{\rm g+gh} + V_{\rm eff}^{\rm f} + V_{\rm eff}^{\rm ad}, \tag{9}$$

$$V_{\rm eff}^{\rm g+gh} = -(d-2)\frac{\Gamma(d/2)}{\pi^{d/2}L^d}\sum_{i,j=1}^{N}\sum_{n=1}^{\infty}\frac{\cos[n(\theta_i-\theta_j)]}{n^d}, \tag{10}$$

$$V_{\rm eff}^{\rm f} = N_{\rm f}\cdot 2^{[d/2]}\frac{\Gamma(d/2)}{\pi^{d/2}L^d}\sum_{i=1}^{N}\sum_{n=1}^{\infty}\frac{\cos[n(\theta_i-\delta_{\rm f})]}{n^d}, \tag{11}$$

$$V_{\rm eff}^{\rm ad} = N_{\rm ad}\cdot 2^{[d/2]}\frac{\Gamma(d/2)}{\pi^{d/2}L^d}\sum_{i,j=1}^{N}\sum_{n=1}^{\infty}\frac{\cos[n(\theta_i-\theta_j-\delta_{\rm ad})]}{n^d}. \tag{12}$$

Here $V_{\rm eff}^{\rm g+gh}$, $V_{\rm eff}^{\rm f}$ and $V_{\rm eff}^{\rm ad}$ are the contributions from the loops of gauge and ghost fields, fundamental fermions and adjoint fermions, respectively. The θ_i $(i = 1\cdots N)$ denote the components of the diagonalized constant background gauge field (or VEV of the gauge field):

$$\langle A_y\rangle = \frac{1}{gL}\begin{pmatrix}\theta_1 & & \\ & \ddots & \\ & & \theta_N\end{pmatrix}, \qquad \sum_{i=1}^{N}\theta_i = 0. \tag{13}$$

The θ_i are called *the non-integrable phases of the gauge field*[9)] and cannot be constrained by the pure-gauge condition $F_{MN} \equiv 0$. They should be determined dynamically at the quantum level as a result of the minimization of $V_{\rm eff}$.

When non-zero θ_i are induced dynamically, only generators of the gauge group, which commute with the Wilson line surrounding S^1,

$$\langle W(x)\rangle \equiv \mathcal{P}\exp\left[ig\int_{x,y}^{x,y+L}\langle A_y\rangle dy\right] = \begin{pmatrix}\exp i\theta_1 & & \\ & \ddots & \\ & & \exp i\theta_N\end{pmatrix}, \tag{14}$$

form a remaining subgroup of $SU(N)$, and other symmetries are broken. Thus gauge symmetry breaking is induced through dynamics in the extra dimension.

The phases $\delta^{\rm f}$ and $\delta^{\rm ad}$, relevant for the fermions' boundary conditions, remain as non-dynamical parameters of the model and must be set by hand. In the $N_{\rm ad} \neq 0$ and $N_{\rm f} = 0$ cases,[9),10)] the gauge symmetry breaking is controlled by changing the boundary condition $\delta_{\rm ad}$s. In this paper, however, we show that we can also realize various patterns of gauge symmetry breaking by varying the appropriate choice of $N_{\rm f}$ and $N_{\rm ad}$, even if we simply take the periodic boundary condition of fermions,

$$\delta_{\rm f} = \delta_{\rm ad} = 0. \tag{15}$$

When the periodic boundary condition is taken for fermions, $V_{\rm eff}$ is simplified as

$$V_{\rm eff} = \frac{\Gamma(d/2)}{\pi^{d/2}L^d}\left\{\left[-(d-2)+2^{[d/2]}N_{\rm ad}\right]\sum_{i,j=1}^{N}F_d(\theta_i-\theta_j) + 2^{[d/2]}N_{\rm f}\sum_{i=1}^{N}F_d(\theta_i)\right\}, \tag{16}$$

where the function $F_d(\theta)$ is defined as

$$F_d(\theta) \equiv \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^d}. \tag{17}$$

It can also be written in terms of the *polylogarithmic functions* $\mathrm{Li}_d(x) \equiv \sum_{n=1}^{\infty} \frac{x^n}{n^d}$:

$$F_d(\theta) = \mathrm{Re} \sum_{n=1}^{\infty} \frac{\exp(in\theta)}{n^d} = \mathrm{Re}\, \mathrm{Li}_d(\exp(i\theta)). \tag{18}$$

Thus we can calculate $F_d(\theta)$ analytically for some special values, or even numerically with arbitrary precision.

It is easy to show how each sector of the effective potential plays a role in the gauge symmetry breaking. The combination of gauge and fundamental fermion sectors has global minima in a symmetric configuration of θ, where $\langle W \rangle$ is $SU(N)$ gauge symmetric. On the other hand, the adjoint fermion sector has global minima at another θ, where $\langle W \rangle$ allows only $U(1)^N$ symmetry. Thus we can expect that by changing N_f and N_ad we get a variety of gauge symmetry breaking.

Some comments are now in order.

2.1.1. Effects of scalar field

It is obvious that if there are also some scalars in the theory, the contribution of the $N_{\mathrm{f,ad}}^{\mathrm{scalar}}$ flavor complex scalar fields to the effective potential is given by replacing $2^{[d/2]} N_{\mathrm{f,ad}}$ with $-2N_{\mathrm{f,ad}}^{\mathrm{scalar}}$. Thus, N_f and N_ad in Eqs. (11) and (12) should be understood as

$$N_\mathrm{R} = N_\mathrm{R}^{\mathrm{fermion}} - \frac{2}{2^{[d/2]}} N_\mathrm{R}^{\mathrm{scalar}}, \tag{19}$$

where R stands for the representation and $N_\mathrm{R}^{\mathrm{fermion}}$ and $N_\mathrm{R}^{\mathrm{scalar}}$ are the number of flavors of fermions and scalars in the representation R, respectively. Therefore N_f and N_ad can be negative and even take rational number values.

Our investigation can be applied to any region of $(N_\mathrm{f}, N_\mathrm{ad})$ plane. The $N_\mathrm{ad} < (d-2)/2^{[d/2]}$ region is investigated without loss of generality by investigating the $N_\mathrm{ad} = 0$ case and has been already investigated by many authors. The region defined by $N_\mathrm{ad} > (d-2)/2^{[d/2]}$ and $N_\mathrm{f} = 0$ has also been investigated,[6)-10)] and it has been claimed that the gauge symmetry breaking is mainly handled by the boundary conditions of fermion fields.[9),10)] But the region defined by $N_\mathrm{ad} > (d-2)/2^{[d/2]}$ and $N_\mathrm{f} \neq 0$ has not yet been investigated. Hence we study what happens if both adjoint and fundamental matter are included in the Hosotani mechanism.

2.1.2. Supersymmetry

If Majorana, Weyl or Majorana-Weyl fermions exist in d-dimensional space-time and if we take such a type of fermion as the matter field, instead of the Dirac fermion, N_R should be replaced by $\frac{1}{2}N_\mathrm{R}^{\mathrm{Weyl}}$, $\frac{1}{2}N_\mathrm{R}^{\mathrm{Majorana}}$ or $\frac{1}{4}N_\mathrm{R}^{\mathrm{Majorana\text{-}Weyl}}$, respectively. It also must be pointed out that if the specific relations $2^{[d/2]}N_\mathrm{ad} - (d-2) = 0$ and $N_\mathrm{f} = 0$ hold, and all N_ad matter fields are fermionic, the fermionic fields can be regarded as "gaugino", and the system may be regarded as a super Yang-Mills theory. For supersymmetric pure Yang-Mills theories, however, the effective potentials calculated in this article all vanish and will not be investigated further. When the boundary conditions of bosons and fermions are different, this is not the case, and symmetry breaking results even if $N_{\rm ad} = (d-2)/2^{[d/2]}$ and $N_{\rm f} = 0$. This is related to the Scherk-Schwarz mechanism of supersymmetry breaking,[11)-13)] and we will present the results of an investigation along this line in the near future.

2.2. *SU*(2) *case*

In the $SU(2)$ case, the parameterization of background gauge fields is given as

$$\langle A_y \rangle gL = {\rm diag}(\theta, -\theta). \quad (20)$$

The $SU(2)$ symmetric gauge configuration corresponds to $\langle A_y \rangle gL = {\rm diag}(\pi, -\pi)$ or ${\rm diag}(0, 0)$. Hence, when $V_{\rm eff}(\theta)$ has a minimum at $\theta = \pi$ or $\theta = 0$, the $SU(2)$ symmetry remains dynamically.

We search for global minima of $V(\theta)$ for various $N_{\rm f}$ and $N_{\rm ad}$ with periodic boundary conditions ($\delta_{\rm f} = \delta_{\rm ad} = 0$). To see the contribution of each sector, $V_{\rm eff}^{\rm g+gh}, V_{\rm eff}^{\rm f}$ and $V_{\rm eff}^{\rm ad}$ are plotted in Figs. 1(a), (b) and (c), respectively. In these figures, we find that the points $\theta = 0$ and π always become global minima of $V_{\rm eff}^{\rm g+gh} + V_{\rm eff}^{\rm f}$. On the other hand, $V_{\rm eff}^{\rm ad}$ has global minima at $\theta = \pi/2$ and $3\pi/2$, at which the $SU(2)$ symmetry is broken. Thus to judge whether there appears another local minimum to break $SU(2)$ that is lower than those at $\theta = 0$ and π, it is useful to calculate the second derivative of $V_{\rm eff}$ at $\theta = 0$ and π. If $SU(2)$ starts to break at some number of $N_{\rm f}$ and $N_{\rm ad}$, the second derivative at $\theta = 0$ or π is expected to vanish for such a pair of numbers. For the specific case of $d = 4$, the calculated second derivatives are found to behave as

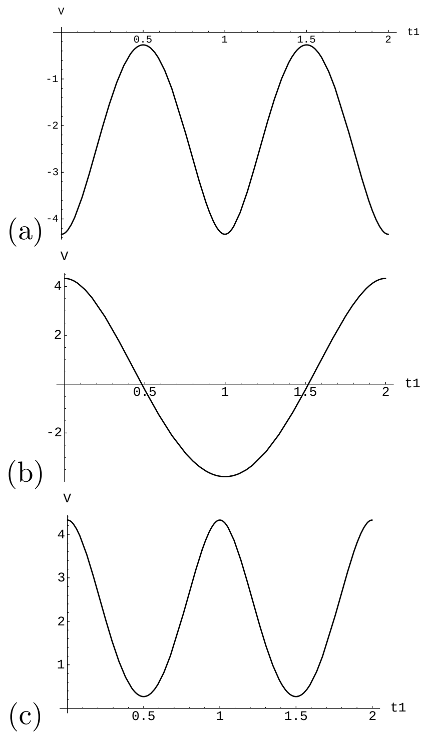


Fig. 1. Plots of $V_{\rm eff}$ as a function of θ_1 (indicated as t1). (a) $V_{\rm eff}^{\rm g+gh}$ (for $N_{\rm ad} = N_{\rm f} = 0$), (b) $V_{\rm eff}^{\rm f}$ (for $N_{\rm ad} = 1/2$, $N_{\rm f} = 1$), and (c) $V_{\rm eff}^{\rm ad}$ (for $N_{\rm ad} = 1$, $N_{\rm f} = 0$).

$$\left.\frac{\partial^2 V_{\rm eff}^{d=4}}{\partial \theta^2}\right|_{\theta=\pi} \propto N_{\rm f} - 8N_{\rm ad} + 4, \quad (21)$$

$$\left.\frac{\partial^2 V_{\rm eff}^{d=4}}{\partial \theta^2}\right|_{\theta=0} \propto 2 - N_{\rm f} - 4N_{\rm ad}, \quad (22)$$

where we have used $F_4''(0) = -\pi^2/6$ and $F_4''(\pi) = \pi^2/12$. Thus the conditions

$N_{\rm f}-8N_{\rm ad}+4=0$ and $2-N_{\rm f}-4N_{\rm ad}=0$ give critical numbers for $N_{\rm f}$ and $N_{\rm ad}$. In this way, the breaking pattern of $SU(2)$ for given $N_{\rm f}$ and $N_{\rm ad}$ is known to be

$$SU(2)\to\left\{\begin{array}{lllcll} SU(2) & \text{for} & & & N_{\rm f} & \le & 2-4N_{\rm ad}, \\ U(1) & \text{for} & 2-4N_{\rm ad} & < & N_{\rm f} & < & 8N_{\rm ad}-4, \\ SU(2) & \text{for} & 8N_{\rm ad}-4 & \le & N_{\rm f}. & & \end{array}\right. \tag{23}$$

It is quite interesting to note the fact that when equality is satisfied in (23), not only is $SU(2)$ symmetry restored, but also the second derivative of gauge potential is exactly 0, which means that the Higgs scalar, as an extra gauge boson, has a vanishing mass for arbitrary L, even in the model on S^1.

2.2.1. $d=5$, $d=6$ case

Using $F_5''(0)=F_5''(2\pi n)=-\zeta(3)$, $F_5''(\pi)=3\zeta(3)/4$, $F_6''(0)=F_6''(2\pi n)=-\zeta(4)$ $=-\pi^4/90$, $F_6''(\pi)=7\pi^4/720$, where $\zeta(\alpha)$ is Riemann's zeta function, we can also determine the symmetry breaking pattern for the $d=5$ and $d=6$ cases. For $d=5$,

$$SU(2)\to\left\{\begin{array}{lllcll} SU(2) & \text{for} & & & N_{\rm f} & \le & 3-4N_{\rm ad}, \\ U(1) & \text{for} & 3-4N_{\rm ad} & < & N_{\rm f} & < & (16N_{\rm ad}-12)/3, \\ SU(2) & \text{for} & (16N_{\rm ad}-12)/3 & \le & N_{\rm f}, & & \end{array}\right. \tag{24}$$

and for $d=6$,

$$SU(2)\to\left\{\begin{array}{lllcll} SU(2) & \text{for} & & & N_{\rm f} & \le & 2-4N_{\rm ad}, \\ U(1) & \text{for} & 2-4N_{\rm ad} & < & N_{\rm f} & < & (32N_{\rm ad}-16)/7, \\ SU(2) & \text{for} & (32N_{\rm ad}-16)/7 & \le & N_{\rm f}. & & \end{array}\right. \tag{25}$$

We thus have found that in both cases, gauge symmetry breaking and its restoration occur, though the critical values of $(N_{\rm f},N_{\rm ad})$ are different.

2.3. $SU(3)$ *case*

We take the parameterization of the background $SU(3)$ gauge field as*)

$$\langle A_y\rangle gL=\text{diag}(\theta_1,\theta_2,-(\theta_1+\theta_2)). \tag{26}$$

The case in which $SU(3)$ is exact and not broken is realized for

$$\langle A_y\rangle gL=\text{diag}(0,0,0), \tag{27}$$

$$\langle A_y\rangle gL=\text{diag}(2\pi/3,2\pi/3,-4\pi/3)\ \text{ and its permutations.} \tag{28}$$

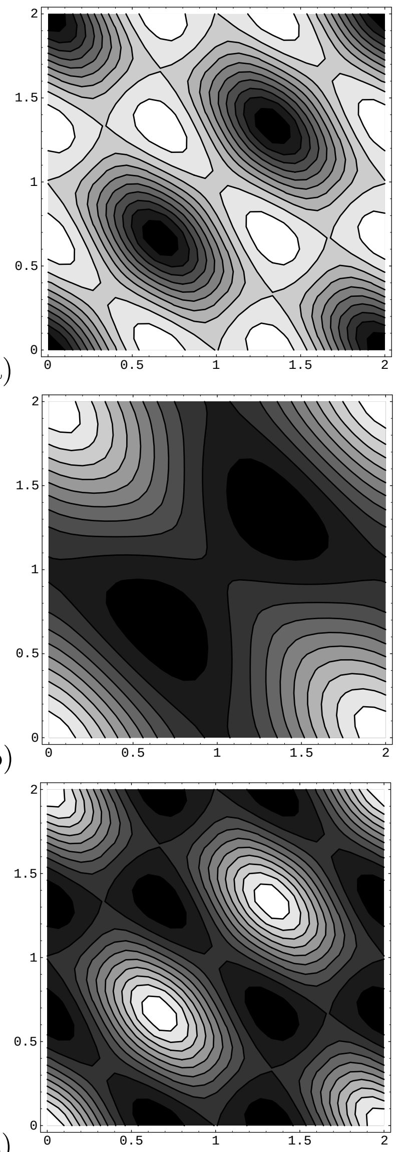
The symmetry breaking of $SU(3)$ into $SU(2)\times U(1)$ is realized provided

$$\langle A_y\rangle gL=\text{diag}(0,\pi,-\pi)\ \text{ and its permutations,} \tag{29}$$

or generally,

$$\langle A_y\rangle gL=\text{diag}(\theta,\theta,-2\theta)\ (\theta\neq-2\theta \bmod 2\pi)\ \text{ and its permutations.} \tag{30}$$

*) We can also use another parameterization such as $\langle A\rangle gL=\theta_1\lambda_3+\theta_2\lambda_8$, where the λ_i are Gell-Mann's λ matrices. But the following discussion does not depend on the choice of parameterization.



(a) (b) (c)

Fig. 2. Plots of the $V_{\rm eff}$. (a) $V_{\rm eff}^{\rm g+gh}(\theta_1,\theta_2)$ (for $N_{\rm ad}=N_{\rm f}=0$), (b) $V_{\rm eff}^{\rm f}(\theta_1,\theta_2)$ (for $N_{\rm ad}=1/2$, $N_{\rm ad}=1$), and (c) $V_{\rm eff}^{\rm ad}(\theta_1,\theta_2)$ (for $N_{\rm ad}=1$, $N_{\rm f}=0$).

The contribution of each sector to $V_{\rm eff}$ is shown in Fig. 2. We can see that $V_{\rm eff}^{\rm g+gh}+V_{\rm eff}^{\rm f}$ does not have a minima at $\theta_1=\theta_2=0$ for sufficiently large $N_{\rm f}$, and $V_{\rm eff}$ has global minima at points such as $(\theta_1,\theta_2)=(2\pi/3,2\pi/3)$, as pointed out by Hosotani. [5] The points $(\theta_1,\theta_2)=(0,0)$ and $(2\pi/3,2\pi/3)$ are all $SU(3)$ symmetric. On the other hand, $V_{\rm eff}^{\rm ad}$ has minima at other points, for example at $(\theta_1,\theta_2)=(0,2\pi/3)$, where $SU(3)$ symmetry is broken.

Our main interest here is whether the breaking $SU(3)\to SU(2)\times U(1)$ is possible. Such a study will be helpful when we apply the method to more realistic GUT theories, where partial breaking like $SU(5)\to SU(3)\times SU(2)\times U(1)$ is favored. When $N_{\rm ad}=1$ and $N_{\rm f}=3$, $V_{\rm eff}$ seems to have a minimum at $(\theta_1,\theta_2,-(\theta_1+\theta_2))=(0,\pi,-\pi)$ and its permutations (Fig. 3).

To determine the position of the global minimum, evaluating the Hessian of $V_{\rm eff}$,

$$H_{V_{\rm eff}}(\theta_1,\theta_2;N_{\rm ad},N_{\rm f})_{i,j} \equiv \left.\frac{\partial^2 V_{\rm eff}(\theta_1,\theta_2;N_{\rm ad},N_{\rm f})}{\partial\theta_i\partial\theta_j}\right|_{\theta_1,\theta_2}, \tag{31}$$

and its eigenvalues and eigenvectors is useful. By utilizing such a method, we study how the gauge symmetry breaks by changing $N_{\rm f}$ with $N_{\rm ad}=1$ for brevity.

(1) $0\le N_{\rm f}<3$

For $N_{\rm f}=0$, the minima of the potential coincides with those of $V_{\rm eff}^{\rm ad}$. As $N_{\rm f}$ increases, such minima move to the points where $SU(2)\times U(1)$ symmetry is realized.

(2) $3\le N_{\rm f}<9$

When $N_{\rm f}=3$, global minima are at the points $(0,\pi,-\pi)$ (and its permutations). It also should be noted that the Hessian of $V_{\rm eff}$ exactly vanishes at such points:

$$\det H_{V_{\rm eff}}(0,\pi;N_{\rm ad}=1,N_{\rm f}=3)\propto -(6N_{\rm ad}-3-N_{\rm f})^2|_{N_{\rm ad}=1,N_{\rm f}=3}=0. \tag{32}$$

This means that the mass of the adjoint Higgs as a curvature of the effective potential vanishes exactly. Then as $N_{\rm f}$ increases, global minima of $V_{\rm eff}$ leave from $SU(2)\times U(1)$ symmetric points $((0,0),\ (0,\pi),\ (\pi,0))$ and move to the $SU(3)$ symmetric points $(2\pi/3,\ 2\pi/3)$, keeping $SU(2)\times U(1)$ symmetry.

(3) $9 \le N_{\rm f}$

When $N_{\rm f} \geq 9$, global minima are at $SU(3)$ symmetric (non-trivial) points, such as $(2\pi/3, 2\pi/3)$.

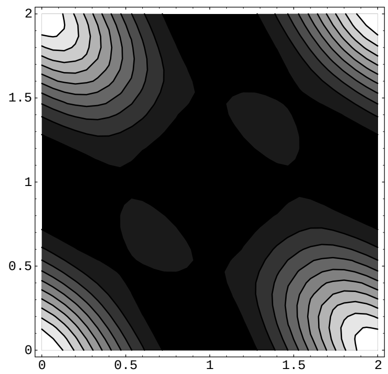


Fig. 3. $V_{\rm eff}$ plot ($N_{\rm ad}=1$, $N_{\rm f}=3$).

(4) Negative $N_{\rm f}$

As $N_{\rm f}$ decreases, the global minima move to the trivial VEV, such as $(0,0)$. When $N_{\rm f} \le -3$, $SU(3)$ symmetry is recovered.

2.3.1. Summary

The results obtained above can be summarized as follows:

$$SU(3) \to \begin{cases} SU(3) & \text{for} & & & N_{\rm f} & \leq & 3-6N_{\rm ad}, \\ U(1)^2 & \text{for} & 3-6N_{\rm ad} & < & N_{\rm f} & < & 6N_{\rm ad}-3, \\ SU(2)\times U(1) & \text{for} & 6N_{\rm ad}-3 & \leq & N_{\rm f} & < & 18N_{\rm ad}-9, \\ SU(3) & \text{for} & 18N_{\rm ad}-9 & \leq & N_{\rm f}. & & \end{cases} \tag{33}$$

Hence all possible symmetry breaking patterns turn out to be realized.

It is interesting to note that in the case $N_{\rm f} = 6N_{\rm ad}-3$, not only does the Wilson line allow $SU(2)\times U(1)$ symmetry, but also all components of the Hessian at $SU(2)\times U(1)$ symmetric points vanish. This means that by the Hosotani mechanism, $SU(3)$ symmetry breaks into $SU(2)\times U(1)$, and at this minimum, the effective potential has a vanishing second derivative, which again suggests that the bosons remain massless.

We must remark that we get such a massless state without any fine-tuning of the parameters because the equality $N_{\rm f}-6N_{\rm ad}+3=0$ is exactly satisfied without any fine-tuning of parameters, as both $N_{\rm ad}$ and $N_{\rm f}$ are discrete numbers.

2.3.2. $d=5$, $d=6$ cases

As pointed out for the $SU(2)$ model, the situation of symmetry breaking depends also on the dimensionality of space-time.

Dependences on the matter content is more complicated for $d=5$ and $d=6$ case. For example, in the $d=5$ case, $\det H_{V_{\rm eff}}(2\pi/3, 2\pi/3) > 0$ suggests that $(2\pi/3, 2\pi/3)$ is at least a local minimum of $V_{\rm eff}$. However, for $N_{\rm f} = (108N_{\rm ad}-81)/8$, as shown in Fig. 4, such a point is not the global minimum. $SU(3)$ symmetry is recovered for sufficiently large $N_{\rm f}$, though the critical value at which $SU(3)$ symmetry is recovered is more ambiguous than in the $d=4$ case. We see that $SU(3)$ is recovered at least when $N_{\rm f} > 4$ for $N_{\rm ad}=1$ case.

For $d = 6$, we have

$$SU(3) \to \begin{cases} SU(3) & \text{for} & & & N_{\rm f} & \leq & 3 - 6N_{\rm ad}, \\ U(1)^2 & \text{for} & 3 - 6N_{\rm ad} & < & N_{\rm f} & < & (18N_{\rm ad} - 9)/7. \end{cases} \tag{34}$$

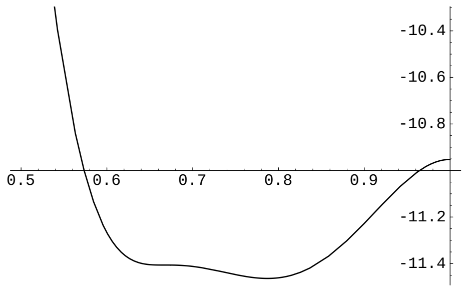


Fig. 4. Plot of $V_{\rm eff}(\theta, \theta)$ for $\theta = \pi/2$ to π with $N_{\rm ad} = 1$ and $N_{\rm f} = 27/8$.

The critical value at which $SU(2) \times U(1)$ is enlarged to $SU(3)$ is ambiguous, as in the case of $d = 5$. The transition $U(1)^2 \to SU(2) \times U(1)$ occurs at $(18N_{\rm ad} - 9)/7 = N_{\rm f}$, and $SU(3)$ symmetry is recovered at least at $N_{\rm f} = 8$.

It also must be pointed out that in the $d = 5$ and 6 cases, the curvatures at $SU(2) \times U(1)$ symmetric points never vanish, unlike in the $d = 4$ case.

2.4. $F_d(\theta) \to \cos\theta$ *(F2C) approximation*

We found that gauge symmetry breaking depends not only on the matter content of the theory but also on the dimensionalities of the space-time. This is partially because the factors $2^{[2/d]}$ and $-(d-2)$ depend on the dimensionality d. The other reason is that the form of the function $F_d(\theta)$ depends on d. Generally, $F_d(\theta)$ becomes close to $\cos\theta$ as d becomes large. Thus we can approximate $F_d(\theta)$ by $\cos\theta$ for sufficiently large d.

By taking $F_d(\theta) \to \cos\theta$ (F2C, for short) as an approximation, we can more easily calculate and evaluate the $V_{\rm eff}$ for the case of $SU(3)$ with $d = 10$.

To summarize the results,

$$SU(3) \to \begin{cases} SU(3) & \text{for} & & & N_{\rm f} & \leq & (3 - 12N_{\rm ad})/2, \\ U(1)^2 & \text{for} & (3 - 12N_{\rm ad})/2 & < & N_{\rm f} & < & (4N_{\rm ad} - 1)/2, \\ SU(2) \times U(1) & \text{for} & (4N_{\rm ad} - 1)/2 & \leq & N_{\rm f} & < & 16N_{\rm ad} - 4, \\ SU(3) & \text{for} & 16N_{\rm ad} - 4 & \leq & N_{\rm f}, & & \end{cases} \tag{35}$$

where the critical value for the $SU(2) \times U(1) \to SU(3)$ transition is obtained by simply comparing the values of $V_{\rm eff}(2\pi/3, 2\pi/3)$ and $V_{\rm eff}(\pi, \pi)$, because minima of $V_{\rm eff}$ cannot be realized except at these points for $(4N_{\rm ad} - 1)/2 < N_{\rm f} < 16N_{\rm ad} - 4$ in this approximation.

§3. Discussion

In this paper, it has been shown that in the Hosotani mechanism, the breaking pattern of gauge symmetry depends on the choice of the representation of matter fields. The manner in which the matter in the fundamental and adjoint representations contribute to the gauge symmetry breaking has been clearly shown in the $SU(2)$ case.

By explicit calculation we have also shown for the $SU(3)$ model that we can realize not only breaking of $SU(3)$ gauge symmetry but also breaking into $SU(2) \times U(1)$, i.e., partial gauge symmetry breaking, by suitably selecting the combination of $N_{\rm f}$ and $N_{\rm ad}$. This feature may be very desirable in constructing more realistic GUT models where a breaking into a subsymmetry with the same rank, such as $SU(5) \to SU(3) \times SU(2) \times U(1)$, is needed. Ho and Hosotani's model [10] contains only an adjoint matter field, and the $SU(N)$ gauge group breaks only into $U(1)^{N-1}$ with periodic boundary conditions, and nontrivial boundary conditions were required for $SU(5)$ to break into $SU(3) \times SU(2) \times U(1)$.

Including the desirable feature mentioned just above, our model has the following advantages, when considered as a prototype model of GUT.

1. Our model contains fermions with the fundamental representation in addition to the adjoint representation, while the models discussed by Hosotani et al. have only fermions with the adjoint representation. The situation we discussed may be understood as more realistic, since matter with the fundamental representation is needed in GUT.
2. It should be stressed that in our model, once the representation of the matter field is fixed, there is no arbitrariness in $V_{\rm eff}$ and in the breaking pattern of gauge symmetry, whereas in ordinary GUT they depend crucially on the values of many parameters of self-coupling of the Higgs, such as the **24** representation of the Higgs in $SU(5)$. We have shown that in our model the VEVs of Higgs (as extra space components of the gauge boson) are dynamically (and uniquely) chosen as the bottom of $V_{\rm eff}$. As a bonus we have also shown that our 4-dimensional gauge theory must live at a "flat bottom" of $V_{\rm eff}$, namely that the Higgs does not suffer from the large finite mass correction encountered in our previous work, in which an Abelian gauge theory is discussed. [1]

To construct a realistic model, we must investigate models with larger gauge groups, such as $SU(5)$, and with other representations, such as **10** of $SU(5)$. Moreover, the compactified manifold may be a larger and more complicated one, such as $T^n (n \geq 2)$. The analysis may become somewhat more complicated. The F2C approximation mentioned in this paper should help to simplify the analysis.

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