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MAXIMAL CANCELLATIVE SUBSEMIGROUPS AND CANCELLATIVE CONGRUENCES

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ABSTRACT. A subsemigroup T of a commutative semigroup S is called a mild ideal if for any $a \in S$, $aT \cap T \neq \emptyset$. It is shown here that any maximal cancellative subsemigroup T of a commutative, idempotent-free, archimedean semigroup S must be a mild ideal of S. Maximal cancellative subsemigroups exist in abundance due to Zorn's lemma. It is also shown that if T is mild ideal of a commutative semigroup S, then every cancellative congruence of T has a unique extension to a cancellative congruence of S.

1. **Maximal cancellative subsemigroups.** Let S be a commutative archimedean semigroup with no idempotents. Let A be a cancellative subsemigroup of S. By the Hausdorff maximal principle (Zorn's lemma), there will exist a maximal[1] cancellative subsemigroup T such that $A \subseteq T$. In particular if $a \in S$, then the cyclic semigroup $\langle a \rangle$ is cancellative, and hence there exists a maximal cancellative subsemigroup of S containing a. In what follows, Z^+ denotes the set of positive integers.

We start with

Lemma 1.1. *Let S be a commutative, archimedean, idempotent-free semigroup and let T be a maximal cancellative subsemigroup of S. Then for any $a \in S\setminus T$, there exists $i \in Z^+$ and $t_1, t_2 \in T^1$, $u \in T$, such that $a^i t_1 u = t_2 u$ but $a^i t_1 \neq t_2$.*

Proof. We use, without further comment, a result of Tamura (see [2] or [3]) that for any $a, b \in S$, $ab \neq b$. Now let $a \in S\setminus T$. By maximality of T, the semigroup generated by a and T is not cancellative. So there exist nonnegative integers j, k and $t_1, t_2 \in T^1$, $x \in S$, such that $a^j t_1 \neq a^k t_2$; $a^j t_1 x = a^k t_2 x$. If $j = k$, then $t_1 a^j x = t_2 a^j x$. Since S is archimedean,

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[1] Maximal as a cancellative subsemigroup, not as a subsemigroup.

$t_1y = t_2y$ for some $y \in T$ whence $t_1 = t_2$. So $a^jt_1 = a^kt_2$, a contradiction. So $j \neq k$. Let us say $j > k$. Then $i = j - k \in Z^+$. Now $a^it_1 \neq t_2$ lest $a^jt_1 = a^kt_2$. Now $a^it_1(a^kx) = t_2(a^kx)$. Since S is archimedean, $a^it_1u = t_2u$ for some $u \in T$. This proves the lemma.

Definition. Let S be a commutative semigroup and T a subsemigroup of S. Then T is a mild ideal of S if for every $a \in S$, $aT \cap T \neq \emptyset$.

Theorem 1.2. *Let S be a commutative, archimedean, idempotent-free semigroup and T a maximal cancellative subsemigroup of S. Then T is a mild ideal of S.*

Proof. Let $a \in S \setminus T$. We must show that $aT \cap T \neq \emptyset$. By Lemma 1.1, there exists $i \in Z^+$ such that $a^iT \cap T \neq \emptyset$. Choose i minimal. We assume $i > 1$ and obtain a contradiction. Now $a^it \in T$ for some $t \in T$. Let $b = at$. Then $b \notin T$ but $b^i \in T$. Again by Lemma 1.1 there exist $j \in Z^+$ and $t_1, t_2 \in T^1$, $u \in T$, such that $b^jt_1u = t_2u$ but $b^jt_1 \neq t_2$. So $b^j \notin T$. Thus $i \nmid j$. So there exist $k \in Z^+$ and an integer l $(l \geq 0)$ such that $j = il + k$, $k < i$. Let $c = b^{il}$. Then $c \in T^1$. Also $b^k(ct_1u) = b^jt_1u = t_2u \in T$. Therefore $b^kT \cap T \neq \emptyset$. Since $b \in aT$, $a^kT \cap T \neq \emptyset$. But this contradicts the minimality of i. So it must be that T is a mild ideal of S.

Remark. In general there is no hope of T being an ideal of S. In fact Professor Takayuki Tamura has shown that there exist commutative archimedean semigroups with no cancellative ideals. He has also obtained necessary and sufficient conditions for the existence of a cancellative ideal.

A commutative, cancellative, idempotent-free archimedean semigroup is known as an $\mathfrak{N}$-semigroup. They have been well studied by Tamura and others. Let T be a commutative, cancellative idempotent-free semigroup. Suppose there is a partial order $\leq$ defined on T such that $(T, \leq)$ is a partially ordered semigroup, $\leq$ is positive ($ab \geq a, b$ for all $a, b \in T$), and T is $\leq$-strongly archimedean, i.e. $a < b$ implies that for any $c \in T$ there exists $i \in Z^+$ such that $a^ic < b^i$. Then according to the author [1], one can in a very natural way embed T in an $\mathfrak{N}$-semigroup $N(T, \leq)$ called the quotient $\mathfrak{N}$-semigroup of T. In particular, if S is a commutative, archimedean, idempotent-free semigroup and T is a cancellative subsemigroup, then we can consider $\leq$ on T induced by division in S. Then $(T, \leq)$ has all the properties discussed above. So we can construct the quotient $\mathfrak{N}$-semigroup $N(T, \leq)$ containing T. Precisely, if G is the quotient group of T, then $N(T, \leq) = \{x | x \in G, x = ab^{-1}$ for some $a, b \in T$ and $b < a\}$.

On the other hand, if σ is the finest cancellative congruence on S (i.e.

$a \sigma b$ iff $ac = bc$ for some $c \in S$), then $S' = S/\sigma$ is an $\mathfrak{N}$-semigroup [3]. The relationship between these two ways of associating $\mathfrak{N}$-semigroups with S lies in the following result.

Theorem 1.3. *Let S be a commutative, archimedean, idempotent-free semigroup and let T be a maximal cancellative subsemigroup of S. If S' is the greatest cancellative image of S, then $S' \cong N(T, \leq)$ where $\leq$ is the partial order on T induced by division in S.*

Proof. Let $\phi: S \to S'$ be the natural homomorphism. Let $a, b \in T$ and $\phi(a) = \phi(b)$. Then $ac = bc$ for some $c \in S$. Since S is archimedean, $au = bu$ for some $u \in T$ and consequently $a = b$. So $T \cong \phi(T) = T'$. Let G be the quotient group of S' and H the quotient group of T'. Then $N(T, \leq) \cong N(T', \leq') \subseteq H \subseteq G$. Here $\leq'$ is induced by $\leq$ on T. First let a, $b \in T$, $b < a$. Then $bx = a$ for some $x \in S$. Hence $\phi(a)\phi(b)^{-1} = \phi(x) \in S'$. So $N(T', \leq') \subseteq S'$. Conversely let $x \in S$. By Theorem 1.2, T is a mild ideal of S. Therefore there exist $a, b \in T$ such that $bx = a$. Then $b < a$ and $ab^{-1} \in N(T, \leq)$. Now $\phi(x) = \phi(a)\phi(b)^{-1} \in N(T', \leq')$. So $N(T', \leq') = S'$, proving the theorem.

Remark. $T = N(T, \leq)$ iff $\leq$ is equal to division in T.

Problem. Let S be a commutative, archimedean, idempotent-free semigroup and T a maximal cancellative subsemigroup of S. Is T necessarily an $\mathfrak{N}$-semigroup?

2. **Cancellative congruences.** Mild ideals are also nice when dealing with cancellative congruences (i.e. congruences σ on S such that S/σ is cancellative).

Theorem 2.1. *Let S be a commutative semigroup and T a mild ideal of S. Then every cancellative congruence on T extends uniquely to a cancellative congruence on S. Thus there is a one-to-one correspondence between cancellative congruences of S and those of T.*

Proof. Let σ be a cancellative congruence on T. Define $\hat{\sigma}$ on S as follows: for $a, b \in S$, $a \hat{\sigma} b$ iff $at \sigma bt$ for some $t \in T$ such that $at, bt \in T$. Evidently $\hat{\sigma}$ is symmetric. It is reflexive since T is a mild ideal. Let a, $b, c \in S$ such that $a \hat{\sigma} b \hat{\sigma} c$. Then for some $t_1, t_2 \in T$ and at_1, bt_1, bt_2, $ct_2 \in T$, $at_1 \sigma bt_1$ and $bt_2 \sigma ct_2$. With $t = t_1 t_2$, $at \sigma ct$ and $at, ct \in T$. So $\hat{\sigma}$ is an equivalence relation on S. Since σ is cancellative, $\hat{\sigma}|T = \sigma$. Next let $a, b \in S$, $a \hat{\sigma} b$. Then $at_1 \sigma bt_1$ for some $t_1 \in T$ such that at_1, $bt_1 \in T$. Let $c \in S$. Then $ct_2 \in T$ for some $t_2 \in T$. Thus $at_1ct_2 \sigma bt_1ct_2$, and so $ac(t_1t_2)\ \sigma\ bc(t_1t_2)$ showing $ac\ \hat{\sigma}\ bc$. So $\hat{\sigma}$ is a congruence on S. Finally let $a, b, c \in S$, $ac\ \hat{\sigma}\ bc$. So $act_1\ \sigma\ bct_1$ for some $t_1 \in T$ such that act_1 and $bct_1 \in T$. Now $ct_2 \in T$ for some $t_2 \in T$. Thus $a(ct_2t_1)\ \sigma\ b(ct_2t_1)$ whence $a\ \hat{\sigma}\ b$. So $\hat{\sigma}$ is a cancellative congruence on S.

Next let σ_1, σ_2 be two cancellative congruences on S such that $\sigma_i|T = \sigma$ $(i = 1, 2)$. Let $a, b \in S$, $a\ \sigma_1\ b$. Then for some $t_1, t_2 \in T$, $at_1, bt_2 \in T$. So $at, bt \in T$ with $t = t_1t_2$. Moreover, $at\ \sigma_1\ bt$, whence $at\ \sigma\ bt$. Now it must be that $at\ \sigma_2\ bt$ whereupon $a\ \sigma_2\ b$. So $\sigma_1 \subseteq \sigma_2$. Similarly $\sigma_2 \subseteq \sigma_1$ whence $\sigma_1 = \sigma_2$. Conversely, any cancellative congruence on S has a restriction to a cancellative congruence on T.

Remark. Cancellative congruences of $\mathfrak{N}$-semigroups have been determined in different ways by Tamura [4]. Of course the cancellative congruences of S are derived from those of S'. But Theorem 2.1 tells how they can also be derived from those of any maximal cancellative subsemigroup T of S. Note that T is a mild ideal of $N(T, \leq)$. Therefore the cancellative congruences of T are the restrictions of the cancellative congruences of the $\mathfrak{N}$-semigroup $N(T, \leq)$.

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