Maximal cluster sets of L-analytic functions along arbitrary curves

L. Bernal-González, A. Bonilla, M.C. Calderón-Moreno and J. A. Prado-Bassas*

Abstract

Let Ω be a domain in the N-dimensional real space, L be an elliptic differential operator, and (T_n) be a sequence whose members belong to a certain class of operators defined on the space of L-analytic functions on Ω. It is proved in this paper the existence of a dense linear manifold of L-analytic functions all of whose nonzero members have maximal cluster sets under the action of every T_n along any curve ending at the boundary of Ω such that its ω-limit does not contain any component of the boundary. The above class contains all partial differentiation operators ∂^α, hence the statement extends earlier results due to Boivin, Gauthier and Paramonov, and to the first, third and fourth authors.

2000 Mathematics Subject Classification: Primary 30D40. Secondary 30E10, 31B35, 41A30, 47F05.

Key words and phrases: maximal cluster set, L-analytic function, dense linear manifold, admissible path, elliptic operator, internally controlled operator.

*The first, third and fourth authors have been partially supported by the Plan Andaluz de Investigación de la Junta de Andalucía FQM-127 and by MCyT Grants BFM2003-03893-C02-01 and MTM2004-21420-E. The second author has been partially supported by MCYT-FEDER Project no. BFM2002-02098.

1 Introduction

The behavior of the complex-valued functions near the boundary of the domain where they are defined has attracted the attention of many mathematicians. Such a behavior can be considered either globally or restricted to certain subsets (mainly, adequate curves) near the boundary, and is defined by the so-called cluster sets, see below. For background about classical results on cluster sets, we refer to [5] and [9]. Next, we fix some standard notation that will be used throughout this paper.

The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{N}_0$ will stand for the real line, the complex plane, the set of positive integers, and the set $\mathbb{N} \cup \{0\}$, respectively. If $N \in \mathbb{N}$ then $\mathbb{R}^N$ is the N-dimensional real space; specially, $\mathbb{C} = \mathbb{R}^2$. If $A \subset \mathbb{R}^N$ then A^0 (∂A, resp.) represents its interior (its boundary, resp.) in $\mathbb{R}^N$. In addition, we set $A^c := \mathbb{R}^N \setminus A$, as usual. The open ball with center $a \in \mathbb{R}^N$ and radius $r > 0$ –with respect to the euclidean distance d on $\mathbb{R}^N$– is $B(a, r)$. By Ω we denote a domain in $\mathbb{R}^N$, that is, a nonempty connected open subset of $\mathbb{R}^N$. Moreover, Ω^* will denote the one-point compactification of Ω. A Jordan domain is a domain $\Omega \subset \mathbb{C}$ whose boundary in $\mathbb{C}^*$ is a topological image of the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. If $A \subset \mathbb{R}^N$ and f is a complex-valued function defined on A then $\|f\|_A := \sup_A |f|$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$, we let $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ for $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $\partial^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_N})^{\alpha_N}$.

Let $r, N \in \mathbb{N}$ with $N \geq 2$ and let $L(\xi) = \sum_{|\alpha|=r} a_\alpha \xi^\alpha$ ($\xi \in \mathbb{R}^N$) be a fixed homogeneous polynomial of degree r with complex constant coefficients and which satisfies the ellipticity condition $L(\xi) \neq 0$ ($\xi \in \mathbb{R}^N \setminus \{0\}$). We associate to L the homogeneous elliptic operator of order r given by

$$L = L(\partial) = \sum_{|\alpha|=r} a_\alpha \partial^\alpha.$$

The symbol $\mathcal{L}_r^N$ stands for the class of all homogeneous elliptic operators of order r in $\mathbb{R}^N$ with constant complex coefficients.

If $\Omega \subset \mathbb{R}^N$ is a domain and $f : \Omega \to \mathbb{C}$ is a C^∞-function, then f is called *L-analytic* (or *L-holomorphic*) on Ω if it satisfies the equation

$$L(\partial) f = 0 \quad \text{on} \quad \Omega.$$

We denote by $L(\Omega)$ the linear space of L-holomorphic functions on Ω endowed with the compact-open topology. For instance, if $N = 2$ then the space of

$\overline{\partial}$-holomorphic functions is the space $H(\Omega)$ of holomorphic functions in the usual sense, and for arbitrary N, Δ-holomorphic is the same as harmonic in $\mathbb{R}^N$ if Δ denotes the Laplace operator. In general, $L(\Omega)$ turns out to be a Fréchet space (see [12]), which is separable because it is a subspace of $C(\Omega)$, the space of complex continuous functions on Ω, also endowed with the compact-open topology. By an operator on $L(\Omega)$ we mean a continuous linear selfmapping $T : L(\Omega) \to L(\Omega)$.

Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, $\Omega \neq \mathbb{R}^N$ and $\gamma : [0,1) \to \Omega$ be a curve in Ω tending to $\partial\Omega$, that is, γ is continuous and, given a compact set $K \subset \Omega$, there is $t_0 = t_0(K) \in (0,1)$ for which $K \cap \gamma([t_0,1)) = \emptyset$. Then the *$\omega$-limit* (or *oscillation set*) of γ is the set $Osc\,(\gamma)$ of points in $\partial\Omega$ which are in the closure of $\gamma([0,1))$. Let $b \in \partial\Omega$. Following [3, Section 6], we shall say that a continuous path $\gamma : [0,1] \to \mathbb{R}^N$ is *admissible* for Ω with end point b if $\gamma([0,1)) \subset \Omega$ and $\gamma(1) = b$. Note that, in particular, if $\partial\Omega$ has no connected component consisting of a single point then for any admissible path γ we have that $Osc\,(\gamma)$ contains no component of $\partial\Omega$. By abuse of language, we sometimes identify $\gamma = \gamma([0,1))$.

Given a continuous function $f : \Omega \to \mathbb{C}$ and a curve γ in Ω tending to its boundary, we denote by $\mathcal{C}_\gamma(f)$ the *cluster set* of f along γ, that is, $\mathcal{C}_\gamma(f) = \{w \in \mathbb{C}^* :$ there exists a sequence $\{t_n\} \subset [0,1)$ such that $t_n \to 1$ and $f(\gamma(t_n)) \to w$ as $n \to \infty\}$.

In [3, Theorem 5], Boivin, Gauthier and Paramonov proved that if $L \in \mathcal{L}_r^N$ and $\Omega \subset \mathbb{R}^N$ is a domain with $\Omega \neq \mathbb{R}^N$ such that $\partial\Omega$ has no component consisting of a single point, then there is at least *one* function $g \in L(\Omega)$ with the property that for each $b \in \partial\Omega$, each admissible path γ ending at b and each $\alpha \in \mathbb{N}_0^N$, one has $C_\gamma(\partial^\alpha g) = \mathbb{C}^*$, that is, each such cluster set is *maximal.* On the other hand, in [1, Theorem 2.1] it is shown that if Ω is a Jordan domain in $\mathbb{C}$ then there is a *dense linear manifold* $\mathcal{D} \subset H(\Omega)$ all of whose nonzero members f satisfy that the cluster set of f along γ is maximal for every curve γ in Ω tending to the boundary with $Osc\,(\gamma) \neq \partial\Omega$. In this paper, we state a theorem (see Section 4) that unifies and extends largely both results above; our theorem will be valid for a large class of operators on $L(\Omega)$ which will be introduced in Section 2. Section 3 is devoted to present some additional terminology together with a preparatory result about approximation.

2 A new class of operators

We collect in the following definition the adequate class of operators to be used later.

Definition 2.1. Let $r, N \in \mathbb{N}$ with $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a domain. Assume that $L \in \mathcal{L}_r^N$ and that T is an operator on $L(\Omega)$. We say that T is *internally controlled* if given $\varepsilon > 0$ and a pair of compact sets F, $G \subset \Omega$ with $F \subset G^0$, there exists $\delta = \delta(\varepsilon, F, G) > 0$ such that

$$[f \in L(\Omega) \ \text{ and } \ \|f\|_G < \delta] \ \text{ implies } \ \|Tf\|_F < \varepsilon.$$

Note that the notion of "internal control" is stronger than the mere continuity. In fact, the former requires in particular the continuity of *every* mapping $T_{1,2} : X_1 \to X_2$, where X_i $(i = 1, 2)$ is the normed space consisting of all functions of $L(\Omega)$ under the respective norms $\| \cdot \|_F$, $\| \cdot \|_G$ and (F, G) is any pair of compact subsets of Ω with $F \subset G^0$, while the continuity of T only requires to fix a compact subset F and find *some* compact set G in Ω such that the corresponding mapping $T_{1,2}$ is continuous.

Another property to be imposed in Section 3 on an operator T on $L(\Omega)$ will be that $\operatorname{ran} T$ $(:= T(L(\Omega)))$ contains the constant functions. Of course, this is equivalent to $1 \in \operatorname{ran} T$.

Let us provide some examples of "classical" operators that satisfy some of the above conditions.

Examples 2.2. 1. If $L \in \mathcal{L}_r^N$ and Ω is a domain in $\mathbb{R}^N$ then from known estimations about the norms of the solutions of a homogeneous elliptic partial differential equation (see for instance [10, pages 188–189, Lemma 1]) it is deduced that each operator ∂^α is internally controlled on $L(\Omega)$. On the other hand, in [3, Lemma 3] it is proved that $1 \in \operatorname{ran} L$.

2. In particular, by Example 1 –or by using Cauchy's integral formula for derivatives– we obtain for $N = 2$ and $L = \overline{\partial}$ that each derivative operator D^n $(n \in \mathbb{N}_0)$ is internally controlled on $H(\Omega)$. Here $D^0 f = f$ and $D^{n+1} f = (D^n f)'$ for every $n \in \mathbb{N}_0$ and every $f \in H(\Omega)$. More, if $\Phi(z) = \sum_{n=0}^\infty a_n z^n$ is an entire function of subexponential type (that is, given $\varepsilon > 0$ there is a constant $A \in (0, +\infty)$ such that $|\Phi(z)| \leq A \exp(\varepsilon |z|)$ for all $z \in \mathbb{C}$), then its associated (in general, infinite order) differential operator $\Phi(D) = \sum_{n=0}^\infty a_n D^n$ is well defined and internally controlled on $H(\Omega)$. This is easy to see just by taking into account that Φ is of subexponential type if and only if

$\lim_{n\to\infty}(n|a_n|^{1/n}) = 0$ (see for instance [2]). Moreover, $1 \in \operatorname{ran}\Phi(D)$ as soon as $\Phi \not\equiv 0$. Indeed, let $m = \min\{n \in \mathbb{N}_0 : \Phi^{(n)}(0) \neq 0\}$; then $\Phi(D)(h) = 1$ if $h(z) := z^m/\Phi^{(m)}(0)$.

3. Again in the classical holomorphic case, if $\alpha \in H(\Omega)$ then its associated multiplication operator $M_\alpha : f \mapsto \alpha f$ is always internally controlled on $H(\Omega)$. And $1 \in \operatorname{ran} M_\alpha$ if and only if α has no zeros in Ω. On the contrary, if $\varphi : \Omega \to \Omega$ is a holomorphic selfmapping, then the composition operator $C_\varphi : f \in H(\Omega) \mapsto f \circ \varphi \in H(\Omega)$ is internally controlled only if φ is the identity. We have that $1 \in \operatorname{ran} C_\varphi$ for any holomorphic selfmapping φ.

4. We know that internal control implies continuity. In fact, the former property is strictly stronger. For instance, the Volterra operator V on $H(\mathbb{C})$ given as $(Vf)(z) = \int_0^z f(t)\,dt$ is easily seen to be not internally controlled. Observe that, in addition, $1 \notin \operatorname{ran} V$ because $(Vf)(0) = 0$ for every $f \in H(\mathbb{C})$.

5. The family of internally controlled operators is a vector algebra in the space of all operators on $L(\Omega)$, that is, if α, β are scalars and T, S are internally controlled operators on $L(\Omega)$, then the operators $\alpha T + \beta S$ and $T \circ S$ are internally controlled too. Indeed, this is evident for $\alpha T + \beta S$. As for the composition $T \circ S$, fix a pair $(F, G) = (F_1, F_2)$ of compact sets as well as a number $\varepsilon > 0$ as in Definition 2.1. Then choose a compact set F_3 with $F_1 \subset F_3^0 \subset F_3 \subset F_2^0$, for instance, $F_3 := \overline{\bigcup_{x\in F_1} B(x, \alpha/2)}$, where $\alpha = d(F_1, F_2^c)$. Now apply that T is internally controlled to the pair (F_3, F_2) and, finally, apply that S is internally controlled to (F_1, F_3).

3 Some auxiliary statements

Following [4], a relatively closed subset F in Ω will be called a Roth-Keldysh-Lavrentiev set, or more simply an *Ω-RKL set*, if $\Omega^* \setminus F$ is connected and locally connected. The following lemma (see [3] and [6]) will reveal useful in the proof of our main result. Note that Arakelian's theorem (see [7]) covers the assertion for $N = 2$ in the holomorphic case.

Lemma 3.1. *Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$ and F be a Ω-RKL set. Assume that f be L-analytic in some neighbourhood of F in Ω. Then, for each $\varepsilon > 0$, there exists $g \in L(\Omega)$ such that*

$$\|f - g\|_F < \varepsilon.$$

In order to simplify the notation, we denote by $\mathcal{O}$ the family of domains $\Omega \subset \mathbb{R}^N$ such that $\Omega \neq \mathbb{R}^N$ and the boundary of Ω in $(\mathbb{R}^N)^*$ has no connected components that consist of a single point. Moreover, for each domain Ω as before, we denote by $\Gamma(\Omega)$ the family of curves $\gamma \subset \Omega$ tending to the boundary whose ω-limit does not contain any component of the boundary. Then, we establish the following technical lemma.

Lemma 3.2. *Assume that* $\Omega \in \mathcal{O}$. *Then there are three sequences* (K_j), (G_j), (F_j) *of compact subsets of* Ω *satisfying the following properties:*

(i) $\Omega = \bigcup_{j=1}^{\infty} K_j$ *and* $K_j \subset K_{j+1}^0$ $(j \in \mathbb{N})$.

(ii) *The* G_j*'s are pairwise disjoint and* $G_j \subset K_j$ *for all* $j \in \mathbb{N}$.

(iii) *For every* $j \in \mathbb{N}$ *one has that* $K_j \cap G_{j+1} = \emptyset$ *and* $K_j \cup G_{j+1}$ *is an RKL-set.*

(iv) $F_j \subset G_j^0$ $(j \in \mathbb{N})$ *and every* $\gamma \in \Gamma(\Omega)$ *intersects all sets* F_j*'s except possibly finitely many of them.*

The content of the last lemma is essentially proved in [3, Proof of Theorem 5], so we omit a detailed proof. Suffice it to say that (i), (iii) and the first part of (ii) and of (iv) are obtained in the cited proof, in which each G_j is constructed as a thin (closed) neighborhood of F_j, which in turn is a subset of K_j. In order to obtain the contention $G_j \subset K_j$ of (ii), it suffices to replace each K_j by a set $\widetilde{K}_j$ slightly larger. The sets K_j, G_j, F_j in [3] are closed, but in addition they are constructed by using the level sets of an adequate real analytic function Ψ that satisfies an inequality [3, inequality (6) on page 960], which produces that the above closed sets are bounded; hence they are compact. Finally, the last part of (iv) is shown in [3] for admissible curves, but the same holds for every $\gamma \in \Gamma(\Omega)$ just by using the same argument of [3, page 962].

4 The main result

Once the basic terminology and background have been established, we are ready to present our main assertion.

Theorem 4.1. *Let* $\Omega \in \mathcal{O}$. *Assume that* $T_n : L(\Omega) \to L(\Omega)$ $(n \in \mathbb{N})$ *is a sequence of operators satisfying the following conditions:*

(a) *Each* T_n *is internally controlled.*

(b) *For each* $n \in \mathbb{N}$, $\operatorname{ran} T_n$ *contains the constants.*

Then there exists a dense linear manifold $\mathcal{D}$ *in* $L(\Omega)$ *such that for every* $g \in \mathcal{D} \setminus \{0\}$, *every* $n \in \mathbb{N}$, *and every curve* $\gamma \in \Gamma(\Omega)$, *we have*

$$\mathcal{C}_\gamma(T_n\, g) = \mathbb{C}^*.$$

Proof. From Lemma 3.2, we can choose sequences (K_j), (F_j), (G_j) satisfying properties (i) to (iv) in such lemma. For every pair $(k,j) \in \mathbb{N}^2$, we set the disjoint union $M_{k,j} := K_{k+j-1} \cup G_{k+j}$, which is an RKL-set by (iii). Moreover, we can fix sequences $\{q_j : j \in \mathbb{N}\}$, $\{Q_k : k \in \mathbb{N}\}$ which are dense in $\mathbb{C}$ and $L(\Omega)$, respectively.

Let us pick countably many pairwise disjoint infinite sets $J(k)$ ($k \in \mathbb{N}$) such that $J(k) \subset \{j \in \mathbb{N} : j > k\}$ for all $k \in \mathbb{N}$ and, in addition, the sets $J(k) - k := \{j - k : j \in J(k)\}$ ($k \in \mathbb{N}$) are also mutually disjoint. This can be made in many ways; for instance, we may choose $J(k) = \{3^\nu - k^2 : \nu > k\}$. We set $\mathcal{N} := \bigcup_{k \in \mathbb{N}} (J(k) - k)$. In turn, we divide each $J(k)$ into infinitely many strictly increasing sequences $I(k,n) = \{p(k,n,l) : l \in \mathbb{N}\}$ ($n \in \mathbb{N}$). Then we have the disjoint union $\mathcal{N} := \bigcup_{k,n \in \mathbb{N}} (I(k,n) - k)$. For every pair $(k,n) \in \mathbb{N}^2$, the mapping $\varphi(k,n,\cdot) : j \mapsto l$ –where l is the unique positive integer with $k+j = p(k,n,l)$– is a strictly increasing bijection from $I(k,n) - k$ onto $\mathbb{N}$. Therefore, trivially, the set $\{q_{\varphi(k,n,j)} : j \in I(k,n) - k\}$ is dense in $\mathbb{C}$.

Given $j \in \mathcal{N}$, there exist a unique $k = k(j) \in \mathbb{N}$ and also a unique $n = n(j) \in \mathbb{N}$ such that $j \in I(k,n) - k \subset J(k) - k$. From (a), there is $\delta_j > 0$ such that

$$\|T_n\, h\|_{F_{k+j}} < \frac{1}{2^j} \quad \text{for all} \quad h \in L(\Omega) \quad \text{with} \quad \|h\|_{G_{k+j}} < \delta_j. \tag{1}$$

With no loss of generality, we may assume that the sequence (δ_j) is strictly decreasing and tends to zero. Then we can define $\tau_j := \delta_j - \delta_{j+1} > 0$ ($j \ge 1$). On the other hand, thanks to (b), there exists a sequence $(\Phi_n) \subset L(\Omega)$ satisfying

$$T_n\, \Phi_n = 1 \quad \text{for all} \quad n \in \mathbb{N}. \tag{2}$$

Fix now a number $k \in \mathbb{N}$. Inspired by the approach of [11] (see also [8]), we define $g_{k,0} := Q_k$ and $f_{k,1} : M_{k,1} \to \mathbb{C}$ as

$$f_{k,1}(x) = \begin{cases} g_{k,0}(x) & \text{for every } x \in K_k \\ q_l \Phi_n(x) & \text{for every } x \in G_{k+1} \ \text{ if } \ k+1 = p(k,n,l) \in J(k) \\ 0 & \text{for every } x \in G_{k+1} \ \text{ if } \ 1 \notin J(k) - k. \end{cases}$$

Notice that $f_{k,1}$ is well defined because if $1 \in J(k) - k$ then there is a unique $n \in \mathbb{N}$ with $1 \in I(k,n) - k$; so $k+1 = p(k,n,l)$ for a unique $l \in \mathbb{N}$, namely, $l = \varphi(k,n,1)$. Observe also that $f_{k,1}$ is L-analytic in some neighborhood of $M_{k,1}$. By Lemma 3.1, there is a function $g_{k,1} \in L(\Omega)$ such that

$$\|g_{k,1} - f_{k,1}\|_{M_{k,1}} < \frac{\tau_1}{2^k}.$$

From this step, we proceed by induction. Define for $j \geq 2$ the function $f_{k,j} : M_{k,j} \to \mathbb{C}$ by

$$f_{k,j}(x) = \begin{cases} g_{k,j-1}(x) & \text{for every } x \in K_{k+j-1} \\ q_l \Phi_n(x) & \text{for every } x \in G_{k+j} \text{ if } k+j = p(k,n,l) \in J(k) \\ 0 & \text{for every } x \in G_{k+j} \text{ if } j \notin J(k) - k. \end{cases}$$

Again, $f_{k,j}$ is L-analytic in some neighborhood of the RKL-set $M_{k,j}$. Hence Lemma 3.1 guarantees the existence of a function $g_{k,j} \in L(\Omega)$ such that

$$\|g_{k,j} - f_{k,j}\|_{M_{k,j}} < \frac{\tau_j}{2^k}. \tag{3}$$

Let us consider the series

$$g_{k,0} + \sum_{j=1}^{\infty} (g_{k,j} - g_{k,j-1}). \tag{4}$$

For a prescribed compact subset $K \subset \Omega$, there is $m \in \mathbb{N}$ such that $K \subset K_{k+m-1}$. Then for $j \geq m$ one has that

$$\|g_{k,j} - g_{k,j-1}\|_K \leq \|g_{k,j} - f_{k,j}\|_{K_{k+j-1}} \leq \|g_{k,j} - f_{k,j}\|_{M_{k,j}} < \tau_j,$$

due to (3). But $\sum_{j \geq 1} \tau_j = \delta_1 < \infty$, therefore (4) converges uniformly on compacta in Ω. Thus, its sum defines a function g_k belonging to $L(\Omega)$. Notice that, from the shape of (4), we have that

$$g_k = \lim_{\nu \to \infty} g_{k,\nu} = g_{k,j} + \sum_{\nu=j+1}^{\infty} (g_{k,\nu} - g_{k,\nu-1}) \quad \text{for all } j \in \mathbb{N}. \tag{5}$$

Finally, we define the desired linear submanifold $\mathcal{D}$ in $L(\Omega)$ as the linear span

$$\mathcal{D} = \operatorname{span}\{g_k : k \in \mathbb{N}\}$$

Let us fix a function $h \in L(\Omega)$, a compact set $K \subset \Omega$ and a number $\varepsilon > 0$. Then by the exhaustion property (i) we get an $m \in \mathbb{N}$ with $K \subset K_m$; so $K_{k+j-1} \supset K$ for all $k \geq m$ and all $j \in \mathbb{N}$. Therefore, due to (3), (4) and the definition of $f_{k,j}$, we obtain for $k \geq m$ that

$$\|g_k - Q_k\|_K = \|g_k - g_{k,0}\|_K \leq \sum_{j=1}^{\infty} \|g_{k,j} - g_{k,j-1}\|_{K_{k+j-1}}$$

$$\leq \sum_{j=1}^{\infty} \|g_{k,j} - f_{k,j}\|_{M_{k,j}} < \sum_{j=1}^{\infty} \frac{\tau_j}{2^k} = \frac{\delta_1}{2^k}.$$

Since $\{Q_k : k \in \mathbb{N}\}$ is dense in $L(\Omega)$ there exists $k \geq m$ with $2^k > 2\delta_1/\varepsilon$ and $\|Q_k - h\|_K < \varepsilon/2$. Hence, by the triangle inequality, $\|g_k - h\|_K < \varepsilon$. Thus, the set $\{g_k : k \in \mathbb{N}\}$ is dense in $L(\Omega)$. But $\mathcal{D} \supset \{g_k : k \in \mathbb{N}\}$, so $\mathcal{D}$ is also dense.

It remains to show that if a number $n \in \mathbb{N}$, a function $g \in \mathcal{D} \setminus \{0\}$ and a curve $\gamma \in \Gamma(\Omega)$ are prescribed, then $C_\gamma(T_n\, g) = \mathbb{C}^*$. For this, fix n, g, γ as before. We can write $g = c_1 g_1 + \cdots + c_k g_k$ for certain $k \in \mathbb{N}$ and $c_1, \ldots, c_k \in \mathbb{C}$ with $c_k \neq 0$. Moreover, by the property (iv) we can select a number $j_0 \in \mathbb{N}$ and points $x_j \in \gamma \cap F_{k+j}$ $(j \geq j_0)$. Observe that the construction of the set $I(k,n) - k$ guarantees that for any $j \in I(k,n) - k$ the corresponding $k(j)$, $n(j)$ for which (1) is satified are exactly our prescribed indexes k, n. Consequently, from (2), (5) and the linearity of T_n, we obtain for every $j \in I(k,n) - k$ with $j \geq j_0$ that

$$\begin{aligned}
&|(T_n\, g)(x_j) - c_k q_{\varphi(k,n,j)}| \\
&= \left| T_n \left(c_k g_k + \sum_{i=1}^{k-1} c_i g_i \right)(x_j) - c_k q_{\varphi(k,n,j)} T_n \Phi_n(x_j) \right| \\
&= \Bigg| c_k T_n(g_{k,j} - q_{\varphi(k,n,j)}\Phi_n)(x_j) + c_k T_n \left(\sum_{\nu=j+1}^{\infty} (g_{k,\nu} - g_{k,\nu-1}) \right)(x_j) \\
&\qquad + \sum_{i=1}^{k-1} c_i T_n(g_i)(x_j) \Bigg| \\
&\leq \left\| c_k T_n(g_{k,j} - q_{\varphi(k,n,j)}\Phi_n) + c_k T_n \left(\sum_{\nu=j+1}^{\infty} (g_{k,\nu} - g_{k,\nu-1}) \right) + \sum_{i=1}^{k-1} c_i T_n g_i \right\|_{F_{k+j}}
\end{aligned}$$

$$\leq |c_k| \|T_n(g_{k,j} - q_{\varphi(k,n,j)}\Phi_n)\|_{F_{k+j}} + |c_k| \left\| T_n \left(\sum_{\nu=j+1}^{\infty} (g_{k,\nu} - g_{k,\nu-1}) \right) \right\|_{F_{k+j}}$$

$$+ \sum_{i=1}^{k-1} |c_i| \|T_n g_i\|_{F_{k+j}}$$

First, we get from (3) that $\|g_{k,j} - q_{\varphi(k,n,j)}\Phi_n\|_{G_{k+j}} < \tau_j$ $(< \delta_j)$. Hence by (1) it is derived that

$$\|T_n(g_{k,j} - q_{\varphi(k,n,j)}\Phi_n)\|_{F_{k+j}} < \frac{1}{2^j}. \tag{6}$$

Second, since $G_{k+j} \subset K_{k+j} \subset K_{k+\nu-1}$ for all $\nu \geq j+1$ and $g_{k,\nu-1} = f_{k,\nu}$ on $K_{k+\nu-1}$, one deduces via (3) that $\|g_{k,\nu} - g_{k,\nu-1}\|_{G_{k+j}} < \tau_\nu$. Hence, by the triangle inequality,

$$\left\| \sum_{\nu=j+1}^{\infty} (g_{k,\nu} - g_{k,\nu-1}) \right\|_{F_{k+j}} < \sum_{\nu=j+1}^{\infty} \tau_\nu = \delta_{j+1} < \delta_j.$$

Thus, again from (1), we obtain

$$\left\| T_n \left(\sum_{\nu=j+1}^{\infty} (g_{k,\nu} - g_{k,\nu-1}) \right) \right\|_{F_{k+j}} < \frac{1}{2^j}. \tag{7}$$

Next, each quantity $\|T_n g_i\|_{F_{k+j}}$ $(i = 1, \ldots, k-1)$ should be estimated. This is a bit more involved. According to (5), we can expand g_i as

$$g_i = g_{i,k-i+j} + \sum_{\nu=k-i+j+1}^{\infty} (g_{i,\nu} - g_{i,\nu-1})$$

Analogously to the second estimation, observe that $G_{k+j} \subset K_{k+j} \subset K_{i+\nu-1}$ for all $\nu \geq k+j-i+1$, and $g_{i,\nu-1} = f_{i,\nu}$ on $K_{i+\nu-1}$. Therefore, due to (3), $\|g_{i,\nu} - g_{i,\nu-1}\|_{G_{k+j}} < \tau_\nu$. Furthermore,

$$\|g_{i,k-i+j}\|_{G_{k+j}} = \|g_{i,k-i+j} - f_{i,k-i+j}\|_{G_{i+k-i+j}} < \|g_{i,k-i+j} - f_{i,k-i+j}\|_{M_{i,k-i+j}} < \tau_{k-i+j}.$$

This holds because of (3) and of the fact that $f_{i,k-i+j} = 0$ on G_{k+j}, which in turn is true because $k+j \in J(k)$, so $i + (k-i+j) = k+j \notin J(i)$ (recall that the sets $J(\nu)$'s are pairwise disjoint). The triangle inequality drives us to

$$\|g_i\|_{G_{k+j}} < \tau_{k-i+j} + \sum_{\nu=k-i+j+1}^{\infty} \tau_\nu = \tau_{k-i+j} + \delta_{k-i+j+1} = \delta_{k-i+j} < \delta_j.$$

Once more, (3) tells us that

$$\|T_n g_i\|_{F_{k+j}} < \frac{1}{2^j} \quad (i = 1, \ldots, k-1). \tag{8}$$

Putting (6), (7) and (8) together and setting $M := 2\sum_{i=1}^k |c_i| < \infty$, we finally get

$$|(T_n\, g)(x_j) - c_k q_{\varphi(k,n,j)}| < \frac{M}{2^j} \to 0 \;\; (j \to \infty). \tag{9}$$

As a final step, let us fix $c \in \mathbb{C}$ and choose an increasing sequence $\{j(1) < j(2) < \cdots\} \subset I(k,n) - k$ such that $q_{\varphi(k,n,j(s))} \to c/c_k$ as $s \to \infty$. Observe that $x_{j(s)}$ $(\in \gamma)$ tends to $\partial\Omega$. By using (9) we get $\lim_{s\to\infty}(T_n\, g)(x_{j(s)}) = c$. Consequently, $c \in C_\gamma(T_n\, g)$ for all $c \in \mathbb{C}$. Thus, $C_\gamma(T_n\, g) = \mathbb{C}^\star$, as desired. $\square$

Note that thanks to the Examples 2.2 our Theorem 4.1 contains and strengthens both results [3, Theorem 5] and [1, Theorem 2.1] mentioned in the Introduction. Indeed, for the first case take as (T_n) a sequence whose members are all partial derivations ∂^α, and for the second case take $N = 2$, $\Omega =$ a Jordan domain, $L = \overline{\partial}$, and $T_n =$ the identity $(n \in \mathbb{N})$.

As a final remark, we point out that, at least for $L = \Delta$ in $\mathbb{R}^N$ and $L = \overline{\partial}$ in $\mathbb{R}^2$, our theorem is close to being sharp. Indeed, it is shown in [3, Proposition 4] that if Ω is a domain in $\mathbb{R}^N$ such that its $(\mathbb{R}^N)^*$-boundary has an isolated point b, then for each function f harmonic in Ω or (if $N = 2$) for each function f holomorphic in Ω, there exists an admissible path $\gamma \subset \Omega$ ending at b such that $\mathcal{C}_\gamma(f)$ is a *single point* in $\mathbb{C}^*$.

References

[1] L. Bernal-González, M.C. Calderón-Moreno and J.A. Prado-Bassas, Maximal cluster sets along arbitrary curves, J. Approx. Theory **129** (2004), 207–216.

[2] R.P. Boas, Entire functions, Academic Press, New York, 1954.

[3] A. Boivin, P.M. Gauthier and P.V. Paramonov, Approximation on closed sets by analytic or meromorphic solutions of elliptic equations and applications, Canadian J. Math. **54** (2002), 945–969.

[4] A. Boivin, and P.V. Paramonov, Approximation by meromorphic and entire solutions of elliptic equations in Banach spaces of distributions, Math. Sb. (4) **189** (1998), 481–502.

[5] E.F. Collingwood and A.J. Lohwater, The theory of cluster sets, Cambridge University Press, Londres, 1966.

[6] A. Dufresnoy, P.M. Gauthier and W.H. Ow, Uniform approximation on closed sets by solutions of elliptic partial differential equations, Complex Variables Theory Appl. **6** (1986), 235–247.

[7] D. Gaier, Lectures on Complex Approximation, London Math. Society Lecture Notes **221**, Cambridge Univ. Press, 1995.

[8] P.M. Gauthier, Tangential approximation and universality, J. Contemp. Math. Anal. **36** (2001), 2–13.

[9] K. Noshiro, Cluster sets, Springer-Verlag, Berlin, 1960.

[10] G.E. Shilov, Generalized functions and partial differential equations, Gordon and Breach, New York, 1968.

[11] A. Sinclair, A general solution for a class of approximation problems, Pacific J. Math. **8** (1958), 857–866.

[12] F. Trèves, Linear partial differential equations with constant coefficients, Gordon and Breach, New York, 1966.

L. Bernal-González, M.C. Calderón-Moreno
Departamento de Análisis Matemático
Facultad de Matemáticas, Apdo. 1160
Avda. Reina Mercedes, 41080 Sevilla, Spain
E-mails: lbernal@us.es, mccm@us.es

Antonio Bonilla
Departamento de Análisis Matemático
Universidad de La Laguna
C/Astrofísico Fco. Sánchez, s/n
38271 La Laguna, Tenerife, Spain
E-mail: abonilla@ull.es

J.A. Prado-Bassas
Departamento de Matemáticas
Universidad Autónoma de Madrid
Ctra. Colmenar Viejo Km. 15
28049, Madrid, Spain
E-mail: joseantonio.prado@uam.es