Maximal decidable fragments of Halpern and Shoham's modal logic of intervals

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Abstract. In this paper, we focus our attention on the fragment of Halpern and Shoham's modal logic of intervals (HS) that features four modal operators corresponding to the relations "meets", "met by", "begun by", and "begins" of Allen's interval algebra ($\mathsf{A\bar{A}B\bar{B}}$ logic). $\mathsf{A\bar{A}B\bar{B}}$ properly extends interesting interval temporal logics recently investigated in the literature, such as the logic $\mathsf{B\bar{B}}$ of Allen's "begun by/begins" relations and propositional neighborhood logic $\mathsf{A\bar{A}}$, in its many variants (including metric ones). We prove that the satisfiability problem for $\mathsf{A\bar{A}B\bar{B}}$, interpreted over finite linear orders, is decidable, but not primitive recursive (as a matter of fact, $\mathsf{A\bar{A}B\bar{B}}$ turns out to be maximal with respect to decidability). Then, we show that it becomes undecidable when $\mathsf{A\bar{A}B\bar{B}}$ is interpreted over classes of linear orders that contains at least one linear order with an infinitely ascending sequence, thus including the natural time flows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

1 Introduction

For a long time, the role of interval temporal logics in computer science has been controversial. On the one hand, it is commonly recognized that they provide a natural framework for representing and reasoning about temporal properties in many computer science areas (quoting Kamp and Reyle [11], "truth, as it pertains to language in the way we use it, relates sentences not to instants but to temporal intervals"), including specification and design of hardware components, concurrent real-time processes, event modeling, temporal aggregation in databases, temporal knowledge representation, systems for temporal planning and maintenance, qualitative reasoning, and natural language semantics [9]. On the other hand, the computational complexity of most interval temporal logics proposed in the literature has been a barrier to their systematic investigation and their extensive use in practical applications. This is the case with the modal logic of time intervals HS introduced by Halpern and Shoham in [10]. HS makes it possible to express all basic binary relations that may hold between any pair of intervals (the so-called Allen's relations [1]) by means of four unary modalities, namely, $\langle \mathsf{B} \rangle$, $\langle \mathsf{E} \rangle$ and their transposes $\langle \bar{\mathsf{B}} \rangle$, $\langle \bar{\mathsf{E}} \rangle$, corresponding to Allen's relations "begun by", "ended by" and their inverses "begins", "ends", provided that singleton intervals are included in the temporal structure [18]. HS turns out to be highly undecidable under very weak assumptions on the class of linear orders over which its formulas are interpreted [10]. In particular, undecidability holds for any class of linear orders that contains at least one linear order with an infinitely ascending or descending sequence, thus including the natural time flows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. In fact, undecidability occurs even without infinitely ascending/descending sequences: undecidability also holds for any class of linear orders with unboundedly ascending sequences, that is, for any class such that for every n, there is a structure in the class with an ascending sequence of length at least n, e.g., for the class of all finite linear orders. In [12], Lodaya sharpens the undecidability of HS showing that the two modalities $\langle \mathsf{B} \rangle, \langle \mathsf{E} \rangle$ suffice for undecidability over dense linear orders (in fact, the result applies to the class of all linear orders [9]).

The recent identification of expressive decidable fragments of HS, whose decidability does not depend on simplifying semantic assumptions such as locality and homogeneity [9], shows that such a trade-off between expressiveness and decidability of interval temporal logics can actually be overcome. The most significant ones are the logic $\mathsf{B\bar{B}}$ (resp., $\mathsf{E\bar{E}}$) of Allen's "begun by/begins" (resp., "ended by/ends") relations [9], the logic $\mathsf{A\bar{A}}$ of temporal neighborhood, whose modalities correspond to Allen's "meets/met by" relations (it can be easily shown that Allen's "before/after" relations can be expressed in $\mathsf{A\bar{A}}$) [8], and the logic $\mathsf{D\bar{D}}$ of the subinterval/superinterval relations, whose modalities correspond to Allen's "contains/during" relations [14]. In this paper, we focus our attention on the logic $\mathsf{A\bar{A}B\bar{B}}$ that joins $\mathsf{B\bar{B}}$ and $\mathsf{A\bar{A}}$ (the case of $\mathsf{A\bar{A}E\bar{E}}$ is fully symmetric). The decidability of $\mathsf{B\bar{B}}$ (resp., $\mathsf{E\bar{E}}$) can be proved by translating it into the point-based propositional temporal logic of linear time with temporal modalities F (sometime in the future) and P (sometime in the past), which has the finite (pseudo-)model property and is decidable [9]. Unfortunately, such a reduction to point-based temporal logics does not work for most interval temporal logics as their propositional variables are evaluated over pairs of points and translate into binary relations. This is the case with $\mathsf{A\bar{A}}$. Unlike the case of $\mathsf{B\bar{B}}$ (resp., $\mathsf{E\bar{E}}$), when dealing with $\mathsf{A\bar{A}}$ one cannot abstract away from the left (resp., right) endpoint of intervals, as contradictory formulas may hold over intervals with the same right (resp., left) endpoint and a different left (resp., right) one. The decidability of $\mathsf{A\bar{A}}$, over various classes of linear orders, has been proved by Bresolin et al. [3] by reducing its satisfiability problem to that of the two-variable fragment of first-order logic over the same classes of linear orders [16]. An optimal (NEXPTIME) tableau-based decision procedure for $\mathsf{A\bar{A}}$ over the integers has been given in [5] and later extended to the classes of all (resp., dense, discrete) linear orders [6], while a decidable metric extension of the future fragment of $\mathsf{A\bar{A}}$ over the natural numbers has been proposed in [7] and later extended to the full logic [4]. Finally, a number of undecidable extensions of $\mathsf{A\bar{A}}$ have been given in [2, 3].

In [15], Montanari et al. consider the effects of adding the modality $\langle \mathsf{A} \rangle$ to $\mathsf{B\bar{B}}$, interpreted over the natural numbers. They show that $\mathsf{AB\bar{B}}$ retains the simplicity of its constituents, but it improves a lot on their expressive power. In particular, besides making it possible to easily encode the until operator of point-based temporal logic (this is possible neither with $\mathsf{B\bar{B}}$ nor with A), $\mathsf{AB\bar{B}}$ allows one to express accomplishment conditions as well as metric constraints. Such an increase in expressiveness is achieved at the cost of an increase in complexity: the satisfiability problem for $\mathsf{AB\bar{B}}$ is EXPSPACE-complete (that for A is NEXPTIME-complete). In this paper, we show that the addition of the modality $\langle\bar{\mathsf{A}}\rangle$ to $\mathsf{AB\bar{B}}$ drastically changes the characteristics of the logic. First, decidability is preserved (only) if $\mathsf{A\bar{A}B\bar{B}}$ is interpreted over finite linear orders, but there is a non-elementary blow up in complexity: the satisfiability problem is not primitive recursive anymore. Moreover, we show that the addition of any modality in the set $\{\langle\mathsf{D}\rangle, \langle\bar{\mathsf{D}}\rangle, \langle\mathsf{E}\rangle, \langle\bar{\mathsf{E}}\rangle, \langle\mathsf{O}\rangle, \langle\bar{\mathsf{O}}\rangle\}$ (modalities $\langle\mathsf{O}\rangle, \langle\bar{\mathsf{O}}\rangle$ correspond to Allen's "overlaps/overlapped by" relations) to $\mathsf{A\bar{A}B\bar{B}}$ leads to undecidability. This allows us to conclude that $\mathsf{A\bar{A}B\bar{B}}$, interpreted over finite linear orders, is maximal with respect to decidability. Next, we prove that the satisfiability problem for $\mathsf{A\bar{A}B\bar{B}}$ becomes undecidable when it is interpreted over any class of linear orders that contains at least one linear order with an infinitely ascending sequence, thus including the natural time flows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. As matter of fact, we prove that the addition of B to $\mathsf{A\bar{A}}$ suffices to yield undecidability (the proof can be easily adapted to the case of $\bar{\mathsf{B}}$). Paired with undecidability results in [2, 3], this shows the maximality of $\mathsf{A\bar{A}}$ with respect to decidability when interpreted over these classes of linear orders.

2 The interval temporal logic $\mathsf{A\bar{A}B\bar{B}}$

In this section, we first give syntax and semantics of the logic $\mathsf{A\bar{A}B\bar{B}}$. Then, we introduce the basic notions of atom, type, and dependency. We conclude the section by providing an alternative interpretation of $\mathsf{A\bar{A}B\bar{B}}$ over labeled grid-like structures (such an interpretation is quite common in the interval temporal logic setting).

2.1 Syntax and semantics

Given a set $\mathcal{P}rop$ of propositional variables, formulas of $\mathsf{A\bar{A}B\bar{B}}$ are built up from $\mathcal{P}rop$ using the boolean connectives $\neg$ and $\vee$ and the unary modal operators $\langle\mathsf{A}\rangle$, $\langle\bar{\mathsf{A}}\rangle$, $\langle\mathsf{B}\rangle$, $\langle\bar{\mathsf{B}}\rangle$. As usual, we shall take advantage of shorthands like $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$, $[\mathsf{A}]\varphi = \neg\langle\mathsf{A}\rangle\neg\varphi$, $[\mathsf{B}]\varphi = \neg\langle\mathsf{B}\rangle\neg\varphi$, $\top = p \vee \neg p$, and $\bot = p \wedge \neg p$, with $p \in \mathcal{P}rop$. Hereafter, we denote by $|\varphi|$ the size of φ.

We interpret formulas of $\mathsf{A\bar{A}B\bar{B}}$ in interval temporal structures over finite linear orders with the relations "meets", "met by", "begins", and "begun by". Precisely, given $N \in \mathbb{N}$, we define $\mathbb{I}_N$ as the set of all (non-singleton) closed intervals $[x, y]$, with $0 \le x < y \le N$. For any pair of intervals $[x, y], [x', y'] \in \mathbb{I}_N$, Allen's relations "meets" A, "met by" $\bar{\mathsf{A}}$, "begun by" B, and "begins" $\bar{\mathsf{B}}$ are defined as follows:

- **"meets":** $[x, y] \ \mathsf{A} \ [x', y']$ iff $y = x'$;

- **"met by":** $[x,y]\ \bar{A}\ [x',y']$ iff $x = y'$;
- **"begun by":** $[x,y]\ B\ [x',y']$ iff $x = x'$ and $y' < y$;
- **"begins":** $[x,y]\ \bar{B}\ [x',y']$ iff $x = x'$ and $y < y'$.

Given an *interval structure* $\mathcal{S} = (\mathbb{I}_N, A, \bar{A}, B, \bar{B}, \sigma)$, where $\sigma : \mathbb{I}_N \to \mathscr{P}(\mathcal{P}rop)$ is a labeling function that maps intervals in $\mathbb{I}_N$ to sets of propositional variables, and an initial interval I, we define the semantics of an $A\bar{A}B\bar{B}$ formula as follows:

- $\mathcal{S}, I \models a$ iff $a \in \sigma(I)$, for any $a \in \mathcal{P}rop$;
- $\mathcal{S}, I \models \neg\varphi$ iff $\mathcal{S}, I \not\models \varphi$;
- $\mathcal{S}, I \models \varphi_1 \vee \varphi_2$ iff $\mathcal{S}, I \models \varphi_1$ or $\mathcal{S}, I \models \varphi_2$;
- for every relation $R \in \{A, \bar{A}, B, \bar{B}\}$, $\mathcal{S}, I \models \langle R\rangle\varphi$ iff there is an interval $J \in \mathbb{I}_N$ such that $I\ R\ J$ and $\mathcal{S}, J \models \varphi$.

Given an interval structure $\mathcal{S}$ and a formula φ, we say that $\mathcal{S}$ *satisfies* φ if there is an interval I in $\mathcal{S}$ such that $\mathcal{S}, I \models \varphi$. We say that φ is *satisfiable* if there exists an interval structure that satisfies it. We define the *satisfiability problem* for $A\bar{A}B\bar{B}$ as the problem of establishing whether a given $A\bar{A}B\bar{B}$-formula φ is satisfiable.

2.2 Atoms, types, and dependencies

Let $\mathcal{S} = (\mathbb{I}_N, A, \bar{A}, B, \bar{B}, \sigma)$ be an interval structure and φ be a formula of $A\bar{A}B\bar{B}$. In the sequel, we shall compare intervals in $\mathcal{S}$ with respect to the set of subformulas of φ they satisfy. To do that, we introduce the key notions of φ-atom and φ-type.

First of all, we define the *closure* $\mathcal{C}l(\varphi)$ of φ as the set of all subformulas of φ and of their negations (we identify $\neg\neg\alpha$ with α, $\neg\langle A\rangle\alpha$ with $[A]\neg\alpha$, etc.). For technical reasons, we also introduce the *extended closure* $\mathcal{C}l^+(\varphi)$, which is defined as the set of all formulas in $\mathcal{C}l(\varphi)$ plus all formulas of the forms $\langle R\rangle\alpha$ and $\neg\langle R\rangle\alpha$, with $R \in \{A, \bar{A}, B, \bar{B}\}$ and $\alpha \in \mathcal{C}l(\varphi)$.

A φ-*atom* is any non-empty set $F \subseteq \mathcal{C}l^+(\varphi)$ such that (i) for every $\alpha \in \mathcal{C}l^+(\varphi)$, we have $\alpha \in F$ iff $\neg\alpha \notin F$ and (ii) for every $\gamma = \alpha \vee \beta \in \mathcal{C}l^+(\varphi)$, we have $\gamma \in F$ iff $\alpha \in F$ or $\beta \in F$ (intuitively, a φ-*atom* is a maximal *locally consistent* set of formulas chosen from $\mathcal{C}l^+(\varphi)$). Note that the cardinalities of both sets $\mathcal{C}l(\varphi)$ and $\mathcal{C}l^+(\varphi)$ are *linear* in the number $|\varphi|$ of subformulas of φ, while the number of φ-atoms is *at most exponential* in $|\varphi|$ (precisely, we have $|\mathcal{C}l(\varphi)| = 2|\varphi|$, $|\mathcal{C}l^+(\varphi)| = 18|\varphi|$, and there are at most $2^{9|\varphi|}$ distinct atoms).

We also associate with each interval $I \in \mathcal{S}$ the set of all formulas $\alpha \in \mathcal{C}l^+(\varphi)$ such that $\mathcal{S}, I \models \alpha$. Such a set is called φ-*type* of I and it is denoted by $\mathcal{T}ype_{\mathcal{S}}(I)$. We have that every φ-type is a φ-atom, but not vice versa. Hereafter, we shall omit the argument φ, thus calling a φ-atom (resp., a φ-type) simply an atom (resp., a type).

Given an atom F, we denote by $\mathcal{O}bs(F)$ the set of all *observables* of F, namely, the formulas $\alpha \in \mathcal{C}l(\varphi)$ such that $\alpha \in F$. Similarly, given an atom F and a relation $R \in \{A, \bar{A}, B, \bar{B}\}$, we denote by $\mathcal{R}eq_R(F)$ the set of all R-*requests* of F, namely,

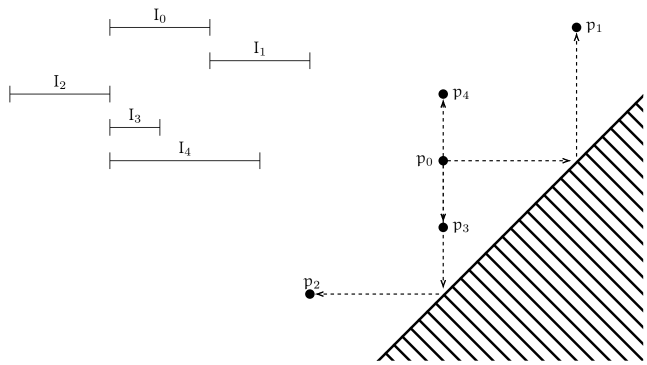

Fig. 1. A compass structure.

the formulas $\alpha \in \mathcal{C}l(\varphi)$ such that $\langle \mathsf{R} \rangle \alpha \in \mathsf{F}$. Note that, for every pair of intervals $I = (x, y)$ and $J = (x', y')$ in $\mathcal{S}$, if $y = y'$ (resp., $x = x'$) holds, then $\mathcal{R}eq_{\mathsf{A}}(\mathcal{T}ype_{\mathcal{S}}(I)) = \mathcal{R}eq_{\mathsf{A}}(\mathcal{T}ype_{\mathcal{S}}(J))$ (resp., $\mathcal{R}eq_{\bar{\mathsf{A}}}(\mathcal{T}ype_{\mathcal{S}}(I)) = \mathcal{R}eq_{\bar{\mathsf{A}}}(\mathcal{T}ype_{\mathcal{S}}(J)))$ follows. Taking advantage of the above sets, we can define the following relations between atoms F and G:

$$\mathsf{F} \xmapsto{\mathsf{A}} \mathsf{G} \quad \text{iff} \quad \begin{cases} \mathcal{R}eq_{\mathsf{A}}(\mathsf{F}) = \mathcal{O}bs(\mathsf{G}) \cup \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{G}) \\ \mathcal{R}eq_{\mathsf{B}}(\mathsf{G}) = \varnothing \\ \mathcal{O}bs(\mathsf{F}) \subseteq \mathcal{R}eq_{\bar{\mathsf{A}}}(\mathsf{G}) \end{cases}$$

$$\mathsf{F} \xmapsto{\mathsf{B}} \mathsf{G} \quad \text{iff} \quad \begin{cases} \mathcal{R}eq_{\mathsf{B}}(\mathsf{F}) = \mathcal{O}bs(\mathsf{G}) \cup \mathcal{R}eq_{\mathsf{B}}(\mathsf{G}) \\ \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{G}) = \mathcal{O}bs(\mathsf{F}) \cup \mathcal{R}eq_{\bar{\mathsf{B}}}(\mathsf{F}). \end{cases}$$

Note that the above relations satisfy a *view-to-type dependency*, namely, for every pair of intervals $I = [x, y]$ and $I' = [x', y']$, we have

$$\begin{array}{lll} x' = y \ \wedge y' = y + 1 & \text{implies} & \mathcal{T}ype_{\mathcal{S}}(I) \xmapsto{\mathsf{A}} \mathcal{T}ype_{\mathcal{S}}(I') \\ x' = x \ \wedge y' = y - 1 & \text{implies} & \mathcal{T}ype_{\mathcal{S}}(I) \xmapsto{\mathsf{B}} \mathcal{T}ype_{\mathcal{S}}(I'). \end{array}$$

2.3 Compass structures

The logic $\mathsf{A}\bar{\mathsf{A}}\mathsf{B}\bar{\mathsf{B}}$ can be equivalently interpreted over the so-called compass structures [18], namely, over grid-like structures. Such an alternative interpretation exploits the existence of a natural bijection between the intervals $I = [x, y]$ and the points $p = (x, y)$ of an $\mathbb{N} \times \mathbb{N}$ grid such that $x < y$. As an example, Figure 1 depicts five intervals $I_0, ..., I_4$ such that $I_0\ \mathsf{A}\ I_1$, $I_0\ \bar{\mathsf{A}}\ I_2$, $I_0\ \mathsf{B}\ I_3$, and $I_0\ \bar{\mathsf{B}}\ I_4$, together with the corresponding points $p_0, ..., p_4$ of a discrete grid (note that the four Allen's relations $\mathsf{A}, \bar{\mathsf{A}}, \mathsf{B}, \bar{\mathsf{B}}$ between intervals are mapped to corresponding spatial relations between points; for the sake of readability, we name the latter ones as the former ones).

Definition 1. *Given an* $\mathsf{A\bar{A}B\bar{B}}$ *formula* φ*, a (finite, consistent, and fulfilling)* compass *(*φ*-)*structure *of length* $N \in \mathbb{N}$ *is a pair* $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$*, where* $\mathbb{P}_N$ *is the set of points* $p = (x, y)$*, with* $0 \le x < y \le N$*, and* $\mathcal{L}$ *is function that maps any point* $p \in \mathbb{P}_N$ *to a (*φ*-)atom* $\mathcal{L}(p)$ *in such a way that*

- *for every relation* $R \in \{A, \bar{A}, B, \bar{B}\}$ *and every pair of points* $p, q \in \mathbb{P}_N$ *such that* $p \; R \; q$*, we have* $\mathcal{O}bs(\mathcal{L}(q)) \subseteq \mathcal{R}eq_R(\mathcal{L}(p))$ *(***consistency***);*
- *for every relation* $R \in \{A, \bar{A}, B, \bar{B}\}$*, every point* $p \in \mathbb{P}_N$*, and every formula* $\alpha \in \mathcal{R}eq_R\big(\mathcal{L}(p)\big)$*, there is a point* $q \in \mathbb{P}_N$ *such that* $p \; R \; q$ *and* $\alpha \in \mathcal{O}bs\big(\mathcal{L}(q)\big)$ *(***fulfillment***).*

It is easy to see that the (finite, consistent, and fulfilling) compass structures are exactly those structures $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$, with $N \in \mathbb{N}$, that satisfy the following conditions for all pair of points p, q in $\mathcal{G}$:

i) if $p = (x, y)$ and $q = (y, y+1)$, then $\mathcal{L}(p) \xmapsto{A} \mathcal{L}(q)$;

ii) if $p = (x, y)$ and $q = (x, y+1)$, then $\mathcal{L}(q) \xmapsto{B} \mathcal{L}(p)$;

iii) if $p = (y-1, y)$, then $\mathcal{R}eq_{\bar{A}}(\mathcal{L}(p)) = \bigcup_{0 \le x < y-1} \mathcal{O}bs(\mathcal{L}(x, y-1))$;

iv) if $p = (x, N)$, then $\mathcal{R}eq_A(\mathcal{L}(p)) = \emptyset$ and $\mathcal{R}eq_{\bar{B}}(\mathcal{L}(p)) = \emptyset$.

We say that a compass structure $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ *features* a formula α if there is a point $p \in \mathbb{P}_N$ such that $\alpha \in \mathcal{L}(p)$. We conclude the section with the following basic result (the proof is straightforward and thus omitted).

Proposition 1. *An* $\mathsf{A\bar{A}B\bar{B}}$*-formula* φ *is satisfied by some finite interval structure iff it is featured by some finite* φ*-compass structure.*

3 Decidability and complexity of the satisfiability problem for $\mathsf{A\bar{A}B\bar{B}}$ over finite linear orders

In this section, we prove that the satisfiability problem for $\mathsf{A\bar{A}B\bar{B}}$ interpreted over *finite* linear orders is decidable, but not primitive recursive. In order to do that, we use a technique similar to [15], namely, we fix a formula φ and a finite compass structure $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ satisfying φ and we show that, under suitable conditions, $\mathcal{G}$ can be reduced in length while preserving the existence of atoms featuring φ. For the sake of brevity, we call *contraction* the operation that reduces the length of a given compass structure $\mathcal{G}$ while preserving the existence of atoms featuring φ. Such an operation has been introduced in its simple variant in [15] and it precisely consists of removing the portion of the compass structure $\mathcal{G}$ included between two distinguished rows y_0 and y_1 and selecting a subset of atoms from the upper row y_1 that match with the atoms of the lower row y_0. Hereafter, we refer the reader to Figure 2 for an intuitive account of the contraction operation (the colored nodes represent the atoms associated with the points of $\mathcal{G}$). According to the definition given in [15], the contraction operation is applicable whenever the set of atoms of the lower row y_0 is included in the set of atoms of the upper row y_1 (the arrows in Figure 2 represent a matching function f between the atoms of the lower row y_0 and the

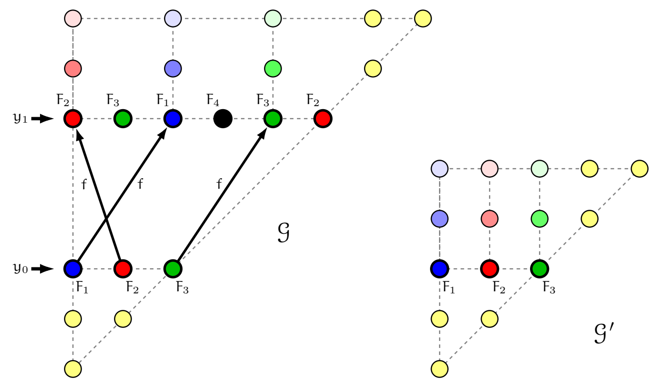

Fig. 2. Contraction of a compass structure.

atoms of the upper row y_1). Such a condition on the set of atoms associated with the rows y_0 and y_1 guarantees the correctness of the contraction operation with respect to the definition of consistent and fulfilling compass structure, provided that the use of the modal operator $\langle \bar{A} \rangle$ is avoided. However, in the presence of the modal operator $\langle \bar{A} \rangle$, things get more involved, since some points $p = (x, y_1)$ from the upper row y_1 (e.g., the one labeled by F_4 in Figure 2) might be necessary in order to fulfill the $\bar{A}$-requests enforced by other points $p' = (x', y')$, with $x' = y_1$ and $y' > y_1$. In the following, we describe a suitable variant of the contraction operation which is applicable to models of $A\bar{A}B\bar{B}$ formulas.

Let us fix an $A\bar{A}B\bar{B}$ formula φ that is featured by a finite compass structure $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$. Without loss of generality, we can assume that φ is of the form $(\phi \wedge [B]\bot) \vee (\langle \bar{B} \rangle \phi) \vee (\langle \bar{B} \rangle \langle A \rangle \phi)$ and, furthermore, it belongs to the atom associated with the point $p = (0, 1)$ at the bottom of the structure $\mathcal{G}$. Before turning to our main result, we need to introduce some preliminary notation and terminology.

For every $1 \leq y \leq N$, we denote by $\mathcal{R}ow_{\mathcal{G}}(y)$ the row y of $\mathcal{G}$, namely, the set of all points $p = (x, y)$ of $\mathcal{G}$. We associate with each row y the set $\mathcal{S}hading_{\mathcal{G}}(y) = \mathcal{L}(\mathcal{R}ow_{\mathcal{G}}(y))$, which consists of the atoms associated with the points in $\mathcal{R}ow_{\mathcal{G}}(y)$. Clearly, for every pair of atoms F, G in $\mathcal{S}hading_{\mathcal{G}}(y)$, we have $\mathcal{R}eq_A(F) = \mathcal{R}eq_A(G)$. We also associate with the row y the function $\mathcal{C}ount_{\mathcal{G}}(y)$, which maps an atom F to the number $\mathcal{C}ount_{\mathcal{G}}(y)(F)$ of F-labeled points in $\mathcal{R}ow_{\mathcal{G}}(y)$.

In order to deal with $\bar{A}$-requests, we need to introduce the notion of cover of a compass structure. Intuitively, this is a selection of points that fulfills all $\bar{A}$-requests coming from other points (hence the points in a cover should not disappear during the operation of contraction). Formally, a *cover* of a compass structure $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ is a subset C of $\mathbb{P}_N$ that satisfies the following two conditions:

- if $(x, y) \in C$ and $x < y - 1$, then $(x, y - 1) \in C$ as well;

- for every point $q = (y-1, y) \in \mathbb{P}_N$, the set $\mathcal{R}eq_{\bar{A}}(\mathcal{L}(q))$ coincides with the union of the sets $\mathcal{O}bs(\mathcal{L}(p))$ for all $p = (x, y-1) \in C$.

Given a cover C of $\mathcal{G}$, we extend the notations $\mathcal{R}ow_{\mathcal{G}}(y)$, $\mathcal{S}hading_{\mathcal{G}}(y)$, and $\mathcal{C}ount_{\mathcal{G}}(y)$ respectively to $\mathcal{R}ow_{\mathcal{G}|C}(y)$, $\mathcal{S}hading_{\mathcal{G}|C}(y)$, and $\mathcal{C}ount_{\mathcal{G}|C}(y)$, having the obvious meaning (e.g., $\mathcal{R}ow_{\mathcal{G}|C}(y)$ is the set of all points of $\mathcal{G}$ along the row y that also belong to C). Moreover, we say that a cover is *minimal* if it does not include properly any other cover. We can easily verify that every *minimal* cover C of $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ satisfies

$$\begin{aligned} &\mathcal{R}ow_{\mathcal{G}|C}(N) = \varnothing \\ &\left|\mathcal{R}ow_{\mathcal{G}|C}(y)\right| - 1 \le \left|\mathcal{R}ow_{\mathcal{G}|C}(y-1)\right| \le \left|\mathcal{R}ow_{\mathcal{G}|C}(y)\right| + |\varphi|. \end{aligned} \tag{1}$$

The following proposition shows that, under suitable conditions, a given compass structure $\mathcal{G}$ can be reduced in length while preserving the existence of atoms featuring φ. Note that such a result can be thought of as a strenghtening of the original "contraction lemma" for structures over the signature $A, B, \bar{B}$ (indeed, if the logic does not allow the modal operator $\langle\bar{A}\rangle$, then the empty set is the unique minimal cover of any compass structure $\mathcal{G}$ and hence the proposition below becomes equivalent to Lemma 3.2 in [15]). For the sake of brevity, hereafter we use $\le$ to denote the componentwise partial order between functions that map atoms to natural numbers, i.e., $f \le g$ iff $f(F) \le g(F)$ holds for all atoms F.

Proposition 2. *Let $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ be a compass structure that features a formula φ in its bottom row. If there exist a cover C of $\mathcal{G}$ and two rows y_0 and y_1 in $\mathcal{G}$, with $1 < y_0 < y_1 \le N$, such that*

i) $\mathcal{S}hading_{\mathcal{G}}(y_0) \subseteq \mathcal{S}hading_{\mathcal{G}}(y_1)$,

ii) $\mathcal{C}ount_{\mathcal{G}}(y_0) \ge \mathcal{C}ount_{\mathcal{G}|C}(y_1)$,

then there exists a compass structure $\mathcal{G}'$ of length $N' < N$ that features φ.

On the grounds of Proposition 2, it makes sense to restrict ourselves to the *minimal* models of φ and, in particular, to those compass structures $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ that feature φ ($= (\phi \wedge [B]\bot) \vee (\langle\bar{B}\rangle\phi) \vee (\langle\bar{B}\rangle\langle A\rangle\phi)$) in the bottom row and that cannot be contracted. The above argument leads to a non-deterministic procedure that decides whether a given formula ϕ is satisfied by a (contraction-free) interval structure $\mathcal{S}$. The pseudo-code of such an algorithm is given in Figure 3: the variable Δ represents the value $N - y + 1$, where N is the length of the model $\mathcal{G}$ to be guessed and y is the current row (note that we cannot use y in place of Δ since there is no a priori bound on the length N of the model), the variable F_Δ represents the atom associated with the rightmost point $p = (y-1, y)$ along the current row y, the variable S_Δ represents an *over-approximation* of the set $\mathcal{S}hading_{\mathcal{G}}(y)$, and the variable C_Δ represents the function $\mathcal{C}ount_{\mathcal{G}|C}(y)$ for a suitable cover C of $\mathcal{G}$ (note that the content of such a variable can be guessed because the sum of its values is bounded in virtue of Equation 1).

The decidability of the satisfiability problem for $A\bar{A}B\bar{B}$ interpreted over finite linear orders is thus reduced to a proof of termination, soundness, and completeness for the algorithm given in Figure 3 as formally stated by Theorem 1 (its

let ϕ **be an input formula**
let φ **be** $(\phi \wedge [B]\bot) \vee (\langle\bar{B}\rangle\phi) \vee (\langle\bar{B}\rangle\langle A\rangle\phi)$

proc CheckRows $\begin{pmatrix} F_\Delta, S_\Delta, C_\Delta, \\ F_{\Delta+1}, S_{\Delta+1}, C_{\Delta+1} \end{pmatrix}$

- $S^+_{\Delta+1} \leftarrow S_{\Delta+1} \cup \{F_{\Delta+1}\}$
- **if there is** $F \in S^+_{\Delta+1}$ **such that** $F \overset{A}{\not\mapsto} F_\Delta$ **then return false**
- $S^{cov}_{\Delta+1} \leftarrow \{F : C_{\Delta+1}(F) > 0\}$
- **if** $\mathcal{R}eq_{\bar{A}}(F_\Delta) \neq \bigcup \mathcal{O}bs(S^{cov}_{\Delta+1})$ **then return false**
- $f \leftarrow$ **any function from** $S^+_{\Delta+1}$ **to** S_Δ
- **if there is** $F \in S^+_{\Delta+1}$ **such that** $f(F) \overset{B}{\not\mapsto} F$ **then return false**
- $M_\Delta \leftarrow \{(F, i) : F \in S_\Delta,\ 1 \leq i \leq C_\Delta(F)\}$
- $M^+_{\Delta+1} \leftarrow \{(F, i) : F \in S^+_{\Delta+1},\ 1 \leq i \leq C_{\Delta+1}(F)\}$
- $g \leftarrow$ **any injective function from** M_Δ **to** $M^+_{\Delta+1}$
- **if there is** $(F, i) \in M_\Delta$ **and** $(F', i') = g(F, i)$ **such that** $F \overset{B}{\not\mapsto} F'$ **then return false**
- **return true**

proc CheckContraction $\begin{pmatrix} F_1, S_1, C_1, \ldots, \\ F_\Delta, S_\Delta, C_\Delta \end{pmatrix}$

- **if** $S_\Delta \neq \emptyset$ **and there is** $1 \leq \Delta' < \Delta$ **such that** $S_\Delta \cup \{F_\Delta\} \subseteq S_{\Delta'} \cup \{F_{\Delta'}\}$ **and** $C_\Delta \geq C_{\Delta'}$ **then return true**
- **return false**

main

- $\Delta \leftarrow 1$
- $F_1 \leftarrow$ **any** φ**-atom** F **such that** $\mathcal{R}eq_A(F) = \mathcal{R}eq_{\bar{B}}(F) = \mathcal{R}eq_B(F) = \emptyset$
- $S_1 \leftarrow$ **any set of** φ**-atoms** F **such that** $\mathcal{R}eq_A(F) = \mathcal{R}eq_{\bar{B}}(F) = \emptyset$, $\mathcal{R}eq_B(F) \neq \emptyset$
- $C_1 \leftarrow$ **the function** $C : S_1 \cup \{F_1\} \to \mathbb{N}$ **such that** $C(F) = 0$ **for all** $F \in S_1 \cup \{F_1\}$
- **while** $S_\Delta \neq \emptyset$ **or** $\varphi \notin F_\Delta$ **do**
  - $F_{\Delta+1} \leftarrow$ **any** φ**-atom** F **such that** $\mathcal{R}eq_B(F) = \emptyset$
  - $S_{\Delta+1} \leftarrow$ **any set of** φ**-atoms** F **such that** $\mathcal{R}eq_B(F) \neq \emptyset$
  - $C_{\Delta+1} \leftarrow$ **any** $C : S_{\Delta+1} \cup \{F_{\Delta+1}\} \to \mathbb{N}$ **such that** $\begin{cases} 0 \leq C(F_{\Delta+1}) \leq 1 \\ \sum_F C(F) \geq \sum_{F \neq F_\Delta} C_\Delta(F) \\ \sum_F C(F) \leq \sum_F C_\Delta(F) + |\varphi| \end{cases}$
  - **if not CheckRows** $\begin{pmatrix} F_\Delta, S_\Delta, C_\Delta, \\ F_{\Delta+1}, S_{\Delta+1}, C_{\Delta+1} \end{pmatrix}$ **then return false**
  - **if CheckContraction** $\begin{pmatrix} F_1, S_1, C_1, \ldots, \\ F_{\Delta+1}, S_{\Delta+1}, C_{\Delta+1} \end{pmatrix}$ **then return false**
  - $\Delta \leftarrow \Delta + 1$
- **return true**

Fig. 3. Decision algorithm for the satisfiability problem over finite linear orders.

proof is reported in the Appendix). As a matter of fact, termination relies on the following crucial lemma, which is often attributed to Dickson.

Lemma 1 (Dickson's Lemma). *Let* $(\mathbb{N}^k, \leq)$ *be the* k*-dimensional vector space over* $\mathbb{N}$ *equipped with the componentwise partial order* $\leq$*. Then,* $(\mathbb{N}^k, \leq)$ *admits no infinite anti-chains, namely, every subset of* $\mathbb{N}^d$ *that consists of pairwise* $\leq$*-incomparable vectors must be finite.*

Theorem 1. *The satisfiability problem for* $A\bar{A}B\bar{B}$*, interpreted over finite linear orders, is decidable.*

We conclude the section by analyzing the complexity of the satisfiability problem for $A\bar{A}B\bar{B}$. In [15], Montanari et al. show that the satisfiability problem for $AB\bar{B}$ is EXPSPACE-complete. Here we prove that, quite surprisingly, the satisfiability problem for $A\bar{A}B\bar{B}$ (in fact, also that for the fragment $A\bar{A}B$) has much higher complexity, precisely, it is not primitive recursive.

Theorem 2. *The satisfiability problem for* $A\bar{A}B$*, and hence that for* $A\bar{A}B\bar{B}$*, interpreted over finite linear orders, is not primitive recursive.*

The proof of Theorem 2 is given in the Appendix and it is based on a reduction from the reachability problem for lossy counter machines, which is known to have strictly non-primitive recursive complexity [17], to the satisfiability problem for $\mathsf{A\bar{A}B}$. In particular, it shows that there is an $\mathsf{A\bar{A}B}$ formula that defines a set of encodings of all possible computations of a given lossy counter machine. The key ingredients of the proof are as follows. First, we represent the value $c(t)$ of each counter c, at each instant t of a computation, by means of a sequence consisting of exactly $c(t)$ unit-length intervals labeled by c. Then, we guarantee that suitable disequalities of the form $c(t+1) \leq c(t) + h$, with $h \in \{-1, 0, 1\}$, hold between the values of the counter c at consecutive time instants. This can be done by enforcing the existence of a surjective partial function g from the set of c-labeled unit-length intervals corresponding to the time instant t to the set of c-labeled unit-length intervals corresponding to the next time instant $t+1$. Finally, we exploit the fact that surjective partial functions between sets of unit-length intervals can be specified in the logic $\mathsf{A\bar{A}B}$.

4 Undecidabiliy is the rule, decidability the exception

We conclude the paper by proving that $\mathsf{A\bar{A}B\bar{B}}$, interpreted over finite linear orders, is maximal with respect to decidability. The addition of a modality for any one of the remaining Allen's relations, that is, of any modality in the set $\{\langle D\rangle, \langle \bar{D}\rangle, \langle E\rangle, \langle \bar{E}\rangle, \langle O\rangle, \langle \bar{O}\rangle\}$, indeed leads to undecidability. The proof of the following theorem is given in the Appendix.

Theorem 3. *The satisfiability problem for the logic* $\mathsf{A\bar{A}B\bar{B}D}$ *(resp.,* $\mathsf{A\bar{A}B\bar{B}\bar{D}}$*,* $\mathsf{A\bar{A}B\bar{B}E}$*,* $\mathsf{A\bar{A}B\bar{B}\bar{E}}$*,* $\mathsf{A\bar{A}B\bar{B}O}$*,* $\mathsf{A\bar{A}B\bar{B}\bar{O}}$*), interpreted over finite linear orders, is undecidable.*

It is possible to show that the satisfiability problem for $\mathsf{A\bar{A}B\bar{B}}$ (in fact, this holds for its proper fragment $\mathsf{A\bar{A}B}$) becomes undecidable if we interpret it over any class of linear orders that contains at least one linear order with an infinitely ascending sequence. It follows that, in particular, it is undecidable when $\mathsf{A\bar{A}B\bar{B}}$ is interpreted over natural time flows like $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

We first consider the satisfiability problem for $\mathsf{A\bar{A}B}$ interpreted over $\mathbb{N}$. By definition, φ is satisfiable over $\mathbb{N}$ if there exists an interval structure of the form $\mathcal{S} = (\mathbb{I}_\omega, A, \bar{A}, B, \sigma)$, with $\mathbb{I}_\omega = \{[x, y] : 0 \leq x < y < \omega\}$ and $\sigma : \mathbb{I}_\omega \rightarrow \mathscr{P}(\mathcal{P}rop)$, that satisfies it. A straightforward adaptation of the proof of Theorem 2 (see the proof of Theorem 4 in the Appendix) shows that an undecidable variant of the universal reachability problem for lossy counter machines, called "structural termination" [13], is reducible to the satisfiability problem for $\mathsf{A\bar{A}B}$ interpreted over interval structures of the form $\mathcal{S} = (\mathbb{I}_\omega, A, \bar{A}, B, \sigma)$. It immediately follows that the latter problem is undecidable as well. Such a negative result can be easily transferred to any class of linear orders that contains at least one linear order with an infinitely ascending sequence.

Theorem 4. *The satisfiability problem for the logic* $\mathsf{A\bar{A}B}$*, and hence that for the logic* $\mathsf{A\bar{A}B\bar{B}}$*, interpreted over over any class of linear orders that contains at least one linear order with an infinitely ascending sequence is undecidable.*

5 Conclusions

In this paper, we proved that the satisfiability problem for $\mathsf{A\bar{A}B\bar{B}}$, interpreted over finite linear orders, is decidable, but not primitive recursive. We also showed that all proper extensions of $\mathsf{A\bar{A}B\bar{B}}$ with a modality corresponding to one of the remaining Allen's relations yields undecidability, thus proving maximality of $\mathsf{A\bar{A}B\bar{B}}$ with respect to finite linear orders. Moreover, we proved that the satisfiability problem for $\mathsf{A\bar{A}B}$ (in fact, the proof for $\mathsf{A\bar{A}B}$ can be adapted to the case of $\mathsf{A\bar{A}\bar{B}}$), interpreted over any class of linear orders that contains at least one linear order with an infinitely ascending sequence, is undecidable. The same results obviously hold for $\mathsf{A\bar{A}E}$ and $\mathsf{A\bar{A}\bar{E}}$, provided that we replace the infinitely ascending sequence by an infinitely descending one. As Bresolin et al. already proved that the extension of $\mathsf{A\bar{A}}$ with the operator $\langle \mathsf{D} \rangle$ (resp., $\langle \mathsf{\bar{D}} \rangle$, $\langle \mathsf{O} \rangle$, $\langle \mathsf{\bar{O}} \rangle$) is undecidable [2, 3], maximality of $\mathsf{A\bar{A}}$, interpreted over any class of linear orders that contains at least one linear order with an infinitely ascending/descending sequence, immediately follows. As a matter of fact, this is the first case in the interval temporal logic setting where the decidability/undecidability of a logic depends on the class of linear orders over which it is interpreted.

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A Appendix

In this appendix, we report the proofs that have been omitted in the previous sections.

A.1 Proof of Proposition 2

Proposition 2. *Let $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ be a compass structure that features a formula φ in its bottom row. If there exist a cover C of $\mathcal{G}$ and two rows y_0 and y_1 in $\mathcal{G}$, with $1 < y_0 < y_1 \le N$, such that*

i) $\mathcal{S}hading_{\mathcal{G}}(y_0) \subseteq \mathcal{S}hading_{\mathcal{G}}(y_1)$,

ii) $\mathcal{C}ount_{\mathcal{G}}(y_0) \ge \mathcal{C}ount_{\mathcal{G}|C}(y_1)$,

then there exists a compass structure $\mathcal{G}'$ of length $N' < N$ that features φ.

Proof. Suppose that C is a cover of $\mathcal{G}$ and that $1 < y_0 < y_1 \le N$ are two rows satisfying the hypothesis of the proposition. Then we know that there is a function $f : \{0, ..., y_0 - 1\} \to \{0, ..., y_1 - 1\}$ such that

i) for every point $p = (x, y_0)$ along the row y_0, the corresponding point $q = (f(x), y_1)$ along the row y_1 satisfies $\mathcal{L}(q) = \mathcal{L}(p)$;

ii) for every point $q = (x', y_1)$ along the row y_1 that also belongs to the cover C, there is a point $p = (x, y_0)$ along the row y_0 such that $f(x) = x'$ (and hence, from the previous property, $\mathcal{L}(q) = \mathcal{L}(p)$).

Let $k = y_1 - y_0$, $N' = N - k$ $(< N)$, and $\mathbb{P}_{N'}$ be the portion of the grid that consists of all points $p = (x, y)$, with $0 \le x < y \le N'$. We extend the above function f to a function that maps points in $\mathbb{P}_{N'}$ to points in $\mathbb{P}_N$ as follows:

- if $p = (x, y)$, with $0 \le x < y < y_0$, then we simply let $f(p) = p$;
- if $p = (x, y)$, with $0 \le x < y_0 \le y$, then we let $f(p) = (f(x), y + k)$;
- if $p = (x, y)$, with $y_0 \le x < y$, then we let $f(p) = (x + k, y + k)$.

We denote by $\mathcal{L}'$ the labeling of $\mathbb{P}_{N'}$ such that, for every point $p \in \mathbb{P}_{N'}$, $\mathcal{L}'(p) = \mathcal{L}(f(p))$ and we denote by $\mathcal{G}'$ the resulting structure $(\mathbb{P}_{N'}, \mathcal{L}')$ (see Figure 2). We have to prove that $\mathcal{G}'$ is a *consistent* and *fulfilling* compass structure that features φ (see Definition 1). As a preliminary remark, we recall that, by hypothesis, the bottom row of $\mathcal{G}$, and hence the bottom row of $\mathcal{G}'$, features the formula φ. Moreover, since the above definition of matching function f is a specialization of the definition given in [15], the proof that $\mathcal{G}'$ is consistent and fulfilling with respect to the relations A, B, and $\bar{B}$ is the exactly same as the proof of Lemma 3.2 of [15]. In that proof, it is also implicitly shown that $\mathcal{G}'$ is consistent with respect to the relation $\bar{A}$. Thus, in order to conclude the proof, it is sufficient to show that $\mathcal{G}'$ is fulfilling with respect to the relation $\bar{A}$.

FULFILLMENT OF $\bar{A}$-REQUESTS. Let $p = (x, y)$ be a point in $\mathcal{G}'$ and let α be a subformula in $\mathcal{R}eq_{\bar{A}}(\mathcal{L}'(p))$. The following cases arise:

1. $x < y_0$ and $y \le y_0$. In such a case, we have $f(p) = p$ and, since $\mathcal{G}$ is a (fulfilling) compass structure, there exists a point $p' = (x', y')$ such that $p \ \bar{A} \ p'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(p'))$. Moreover, since $x' < x < y_0$, we have $f(p') = p'$ and hence $f(p) = p \ \bar{A} \ p' = f(p')$ and $\alpha \in \mathcal{O}bs(\mathcal{L}'(p'))$.
2. $x < y_0$ and $y > y_0$. In such a case, we define $p' = (x', y')$, where $x' = f(x)$ and $y' = y + k$, in such a way that $\mathcal{L}'(p) = \mathcal{L}(f(p)) = \mathcal{L}(p')$. By construction, we have $\mathcal{L}(x', y_1) = \mathcal{L}(f(x), y_1)) = \mathcal{L}(x, y_0)$ and hence, from basic properties of types, $\mathcal{R}eq_{\bar{A}}(\mathcal{L}'(p)) = \mathcal{R}eq_{\bar{A}}(\mathcal{L}(p')) = \mathcal{R}eq_{\bar{A}}(\mathcal{L}(x', y_1)) = \mathcal{R}eq_{\bar{A}}(\mathcal{L}(x, y_0))$. Now, since $\alpha \in \mathcal{R}eq_{\bar{A}}(\mathcal{L}(x, y_0))$ and $\mathcal{G}$ is a (fulfilled) compass structure, we know that there is a point $p'' = (x'', y'')$ in $\mathcal{G}$ such that $(x, y_0) \ \bar{A} \ p''$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(p''))$. Moreover, since $y'' = x < y_0$, we have $f(p'') = p''$, from which we obtain $p'' \ \bar{A} \ p$ and $\alpha \in \mathcal{O}bs(\mathcal{L}'(p''))$.
3. $x \ge y_0$ (and hence $y > y_0$). In such a case, we have $f(p) = (x + k, y + k)$ and, since $\mathcal{G}$ is a (fulfilling) compass structure, there is a point $p' = (x', y')$ such that $f(p) \ \bar{A} \ p'$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(p'))$. Note that $y' = x + k \ge y_1$. We further distinguish between three subcases. If $x' \ge y_1$, then we simply define $p'' = (x' - k, y' - k)$ in such a way that $p' = f(p'')$ and hence $p \ \bar{A} \ p''$ and $\alpha \in \mathcal{O}bs(\mathcal{L}'(p''))$. Otherwise, if $x' < y_1$ and $(x', y_1) \in C$ (namely, (x', y_1) is a point inside the cover C of $\mathcal{G}$), then, from the properties satisfied by the function f, we know that there is $x'' < y_0$ such that $f(x'') = x'$. We thus define $p''' = (x'', y' - k)$ in such a way that $f(p''') = p'$, from which we obtain $p \ \bar{A} \ p'''$ and $\alpha \in \mathcal{O}bs(\mathcal{L}(p')) = \mathcal{O}bs(\mathcal{L}'(p'''))$. Finally, if $x' < y_1$ but $(x', y_1) \notin C$, then, by definition of cover, we know that there exists another point $(\widetilde{x}', y_1)$ along the same row such that $\alpha \in \mathcal{O}bs(\mathcal{L}(\widetilde{x}', y_1))$. We can then use an argument similar to the previous case to devise the fulfillment of the $\bar{A}$-request α in $\mathcal{G}'$. □

A.2 Proof of Theorem 1

Theorem 1. *The satisfiability problem for* $\mathsf{A\bar{A}B\bar{B}}$, *interpreted over finite linear orders, is decidable.*

Proof. We prove that the non-deterministic algorithm described in Figure 3 terminates on every input formula ϕ and it returns **true** iff ϕ is satisfied by some finite interval structure. It is convenient to divide the proof into three parts: first, we prove termination (i.e., every computation of the algorithm terminates), then soundness (i.e., if there is a computation of the algorithm that returns **true** on ϕ, then ϕ is satisfiable), and finally completeness (i.e., if ϕ is satisfiable, then there is a computation on ϕ that returns **true**).

Termination. Let ϕ be an input formula and suppose, by way of contradiction, that there is a non-terminating computation of the algorithm. In particular, this means that the function **CheckContraction** returns **false** on all sequences of arguments $F_1, S_1, C_1, \ldots, F_\Delta, S_\Delta, C_\Delta$. Therefore, for all pair of positive natural numbers $\Delta' < \Delta$, one of the following conditions must hold:

1. $S_\Delta \cup \{F_\Delta\} \nsubseteq S_{\Delta'} \cup \{F_{\Delta'}\}$,
2. $C_\Delta \ngeq C_{\Delta'}$.

We now recall that there only exist finitely many distinct φ-atoms and hence finitely many distinct sets S_Δ. This implies that there is an infinite sequence of indices $\Delta_1 < \Delta_2 < ...$ such that, for all $i > i'$, $S_{\Delta_i} = S_{\Delta_{i'}}$ and hence, by previous assumptions, $C_{\Delta_i} \ngeq C_{\Delta_{i'}}$. Similarly, since every function C_{Δ_i} dominates (with respect to the componentwise partial order $\leq$) only finitely many functions $C_{\Delta_{i'}}$, with $i' < i$, we can find an infinite subsequence $i_1 < i_2 < ...$ of indices for which the functions $C_{\Delta_{i_1}}, C_{\Delta_{i_2}}, ...$ (thought of as vectors in the k-dimensional space $\mathbb{N}^k$) turn out to be pairwise $\leq$-incomparable. This is in contradiction with Lemma 1 and therefore the algorithm must terminate.

Soundness. We consider a successful computation of the algorithm on a formula ϕ and we show that there is a finite compass structure $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ that features φ, where the length N coincides with the value of the variable Δ at the end of the computation. For every $1 \leq \Delta \leq N$, we denote by F_Δ, S_Δ, and C_Δ the content of the omonimous variables which are guessed during the computation. Moreover, we use y (resp., $y-1$) as a shorthand for the value $N - \Delta + 1$ (resp., $N - (\Delta+1) - 1$). Below, we specify the atom $\mathcal{L}(x, y)$ associated with each point $p = (x, y)$ of the compass structure $\mathcal{G} = (\mathbb{P}_N, \mathcal{L})$ by exploting an induction on $y = N - \Delta + 1$ (that is, starting from the lower rows and going upward). While doing this, we also build a cover C of $\mathcal{G}$ in such a way that the two conditions $\mathit{Shading}_{\mathcal{G}}(y) \subseteq S_\Delta \cup \{F_\Delta\}$ and $\mathit{Count}_{\mathcal{G}|C}(y)(F) = C_\Delta(F)$ are guaranteed for every row $y = N - \Delta + 1$ and every atom F. Let us consider a point $p = (x, y)$, with $0 \leq x < y \leq N$ and $y = N - \Delta + 1$:

- If $x = y - 1$, then we let $\mathcal{L}(p) = F_\Delta$. Moreover, we let p belong to the set C iff $C_\Delta(F_\Delta) = 1$ (we can shortly write $|C \cap \{p\}| = C_\Delta(F_\Delta)$). Note that, when $y = 1$, we have $\mathit{Shading}_{\mathcal{G}}(y) = \{F_\Delta\} \subseteq S_\Delta \cup \{F_\Delta\}$, and $\mathit{Count}_{\mathcal{G}|C}(y)(\mathcal{L}(p)) = C_\Delta(F_\Delta) \leq 1 = \mathit{Count}_{\mathcal{G}}(y)(\mathcal{L}(p))$.
- If $x < y - 1$, then, by exploiting the inductive hypothesis, we assume that both $\mathcal{L}(q)$ and $C \cap \{q\}$ are specified for all points $q = (x', y-1)$ along the row $y - 1$ and we accordingly define $\mathcal{L}(p)$ and $C \cap \{p\}$, as follows. First, we denote by $f : S^+_{\Delta+1} \to S_\Delta$ and $g : M_\Delta \to M^+_{\Delta+1}$ the two functions that have been guessed during the execution of the procedure **CheckRows** on arguments $(F_\Delta, S_\Delta, C_\Delta, F_{\Delta+1}, S_{\Delta+1}, C_{\Delta+1})$ (the sets $S^+_{\Delta+1}$, M_Δ, and $M^+_{\Delta+1}$ are defined as in the body of the procedure). Then, given an atom F, we shortly denote by C^F_{y-1} the set of all F-labeled points that lie along the row $y - 1$ and belong to the cover C. From the inductive hypothesis, we know that $|C^F_{y-1}| = \mathit{Count}_{\mathcal{G}|C}(y-1)(F) = C_{\Delta+1}(F)$ and hence, by construction, there is a bijection h^F_{y-1} from the set C^F_{y-1} to the set of all pairs (F, i) in $M^+_{\Delta+1}$, with $1 \leq i \leq C_{\Delta+1}(F)$ (we fix a unique bijection h^F_{y-1} for each row $y - 1$ and for each atom F). We now let $q = (x, y - 1)$ (i.e., the point just below p) and we distinguish between two cases, depending on whether $g^{-1}(h^F_{y-1}(q))$ is defined or not (recall that the inverse g^{-1} of the injective function g is a partial surjective function from $M^+_{\Delta+1}$ to M_Δ). If $g^{-1}(h^F_{y-1}(q))$ is defined

and equal to the pair $(F', i') \in M_\Delta$, then we let $\mathcal{L}(p) = F'$ and $p \in C$. Otherwise, if $g^{-1}(h^F_{y-1}(q))$ is not defined, then we let $\mathcal{L}(p) = f(\mathcal{L}(q))$ and $p \notin C$. Note that, if we apply the above definitions of $\mathcal{L}(p)$ and $C \cap \{p\}$ for all points p along the same row y, we then obtain $\mathcal{S}hading_{\mathcal{G}}(y) \subseteq S_\Delta \cup \{F_\Delta\}$ and $\mathcal{C}ount_{\mathcal{G}|C}(y)(F) = C_\Delta(F)$ for all atoms F.

By exploiting the fact that every call to the procedure **CheckConsistency** is successful, we can easily verify that, for every pair of points p, q in $\mathcal{G}$, the following conditions hold:

i) if $p = (x, y)$ and $q = (y, y+1)$, then $\mathcal{L}(p) \xmapsto{A} \mathcal{L}(q)$;

ii) if $p = (x, y)$ and $q = (x, y+1)$, then $\mathcal{L}(q) \xmapsto{B} \mathcal{L}(p)$;

iii) if $p = (y-1, y)$, then $\mathcal{R}eq_{\bar{A}}(\mathcal{L}(p)) = \bigcup_{0 \le x < y-1} \mathcal{O}bs(\mathcal{L}(x, y-1))$;

iv) if $p = (x, N)$, then $\mathcal{R}eq_A(\mathcal{L}(p)) = \emptyset$ and $\mathcal{R}eq_{\bar{B}}(\mathcal{L}(p)) = \emptyset$.

This shows that $\mathcal{G}$ is a (consistent and fulfilling) compass structure that features φ in its bottom row. Therefore, by Proposition 1, we can conclude that the input formula ϕ is satisfied over a finite interval structure.

Completeness. As for completeness, we consider a finite labeled interval structure $\mathcal{S} = (\mathbb{I}_N, A, \bar{A}, B, \bar{B}, \sigma)$ that satisfies ϕ. By Proposition 1, we know that there is a (consistent and fulfilling) compass structure $\mathcal{G} = (\mathbb{I}_N, \mathcal{L})$ that features the formula $\varphi = (\phi \wedge [B]\bot) \vee (\langle\bar{B}\rangle\phi) \vee (\langle\bar{B}\rangle\langle A\rangle\phi)$ in its bottom row. Let us also fix a minimal cover C of $\mathcal{G}$. We can exploit the existence of $\mathcal{G}$ and C to devise the existence of a successful computation of the algorithm. Precisely, we let the guessed contents for the variables F_Δ, S_Δ, and C_Δ be, respectively, the atom $\mathcal{L}(p)$ associated with the rightmost point $p = (y-1, y)$ along the row $y = N - \Delta + 1$, the set of atoms associated with the non-rightmost points $p = (x, y)$, with $x < y - 1$, along the same row $y = N - \Delta + 1$, and the function $\mathcal{C}ount_{\mathcal{G}|C}(y)$ that maps every atom F to the number of F-labeled points along the row y that also belong to the cover C. On the grounds of Equation 1, it is clear that the above defined values can be correctly guessed at each iteration of the main loop. Moreover, for each call to the procedure **CheckRows** with arguments $(F_\Delta, S_\Delta, C_\Delta, F_{\Delta+1}, S_{\Delta+1}, C_{\Delta+1})$, we assume that the variables f and g are guessed as follows:

- f is any function between atoms such that, for every $F \in S_{\Delta+1} \cup \{F_{\Delta+1}\}$, there exist two points $p = (x, y-1)$ and $q = (x, y)$, with $0 \le x < y - 1$, satisfying $\mathcal{L}(p) = F$ and $\mathcal{L}(q) = f(F)$ (note that such a function f exists since, by construction, $F \in S_{\Delta+1} \cup \{F_{\Delta-1}\} = \mathcal{S}hading_{\mathcal{G}}(y-1)$ and $f(F) \in S_\Delta \cup \{F_\Delta\} = \mathcal{S}hading_{\mathcal{G}}(y)$, where $y = N - \Delta + 1$);
- g is any injective function from $M_\Delta = \{(F, i) : F \in S_\Delta, 1 \le i \le C_\Delta(F)\}$ to $M^+_{\Delta+1} = \{(F, i) : F \in S^+_{\Delta+1}, 1 \le i \le C_{\Delta+1}(F)\}$ such that, for every pair $(F, i) \in M_\Delta$, the cover C contains two points $p = (x, y)$ and $q = (x, y-1)$ satisfying $\mathcal{L}(p) = F$ and $\mathcal{L}(q) = F'$, with $g(F, i) = (F', i')$ (note that such an injective function g exists since, by construction, $C_\Delta(F) = \mathcal{C}ount_{\mathcal{G}|C}(y)(F)$ and $C_{\Delta+1}(F') = \mathcal{C}ount_{\mathcal{G}|C}(y-1)(F')$).

The above definitions guarantee that every call to the procedure **CheckRows** terminates by returning **true**. As for the calls to the procedure **CheckContraction**, we can assume, without loss of generality, that $\mathcal{G}$ has minimal length. In particular, by Proposition 2, this means that, for every pair of rows $y = N-\Delta+1$ and $y' = N-\Delta'+1$, with $1 \le \Delta' < \Delta < N$ (hence $S_\Delta \neq \emptyset$), at least one of the following two conditions holds:

1. $S_\Delta \cup \{F_\Delta\} = \mathcal{S}hading_{\mathcal{G}}(y) \notin \mathcal{S}hading_{\mathcal{G}}(y') = S_{\Delta'} \cup \{F_{\Delta'}\}$,
2. $C_\Delta = \mathcal{C}ount_{\mathcal{G}|C}(y) \not\geq \mathcal{C}ount_{\mathcal{G}|C}(y') = C_{\Delta'}$.

This immediately implies that every call to the procedure **CheckContraction** terminates by returning **true**. Finally, since the algorithm terminates, the formula φ must belong to the atom F_N associated with the point $p = (0,1)$ of $\mathcal{G}$. We have just shown that there is a successful computation of the algorithm. □

A.3 Proof of Theorem 2

Theorem 2. *The satisfiability problem for* $A\bar{A}B$*, and hence that for* $A\bar{A}B\bar{B}$*, interpreted over finite linear orders, is not primitive recursive.*

Proof. We first give a precise notion of *lossy (Minsky) counter machine*. This is a triple of the form $\mathcal{A} = (Q, k, \delta)$, where Q is a finite set of control states, k is the number of counters (whose values range over $\mathbb{N}$), and δ is a function that maps each state $q \in Q$ to a transition rule having one of the following forms:

1. $i \leftarrow i+1;$ `goto` q', for some $1 \le i \le k$ and $q' \in Q$, meaning that, whenever $\mathcal{A}$ is at state q, it increases the counter i and it moves to state q';
2. `if` $i = 0$ `then goto` q' `else` $i \leftarrow i-1;$ `goto` q'', for some $1 \le i \le k$ and $q', q'' \in Q$, meaning that, whenever $\mathcal{A}$ is at state q and the value of the counter i is 0 (resp., greater than 0), it moves to state q' (resp., it decrements the counter i and it moves to state q'').

In addition, from each configuration $(q, \bar{z}) \in Q \times \mathbb{N}^k$, a lossy counter machine $\mathcal{A}$ can non-deterministically activate an internal (lossy) transition and move to a configuration $(q, \bar{z}')$, with $\bar{z}' \le \bar{z}$ (the relation $\le$ is defined componentwise on the values of the counters, as in Lemma 1). A computation of $\mathcal{A}$ is any sequence of configurations that respects the obvious semantics of the transition relation. The *rechability problem* for a lossy counter machine $\mathcal{A} = (Q, k, \delta)$ consists of deciding, given two configurations $(q_{source}, \bar{z}_{source})$ and $(q_{target}, \bar{z}_{target})$, whether or not there is a computation that takes $\mathcal{A}$ from $(q_{source}, \bar{z}_{source})$ to $(q_{target}, \bar{z}_{target})$. Below, we show how to reduce the reachability problem for lossy counter machines to the satisfiability problem for the logic $A\bar{A}B$.

Let us fix a lossy counter machine $\mathcal{A} = (Q, k, \delta)$ together with a source configuration $(q_{source}, \bar{z}_{source})$ and a target configuration $(q_{target}, \bar{z}_{target})$. Without loss of generality, we can assume that $\bar{z}_{source} = \bar{z}_{target} = \bar{0} = (0, ..., 0)$ (indeed, if this were not the case, we can modify $\mathcal{A}$ by introducing some fresh control states $p_0, p_1, ..$ and $p'_0, p'_1, ...$, some increment-transitions that take the machine

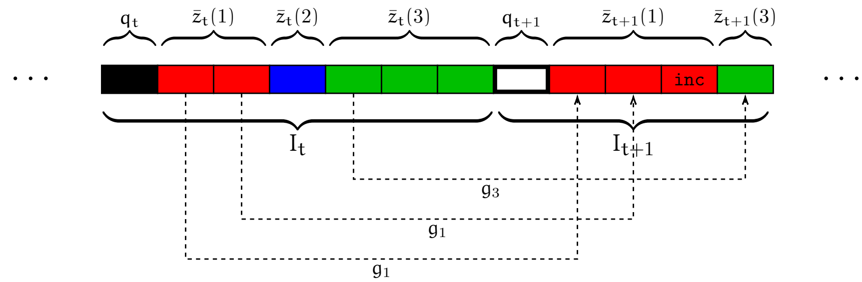

Fig. 4. Encoding of part a computation of a lossy counter machine.

from $(p_0, \bar{0})$ to $(q_{source}, \bar{z}_{source})$, and some decrement-transitions that take the machine from $(q_{target}, \bar{z}_{target})$ to $(p'_0, \bar{0})$). Moreover, we can assume that q_{target} is a sink state, namely, the only state accessible from q_{target} is q_{target} itself.

We first show how to encode a generic computation $(q_1, \bar{z}_1)$ ... $(q_n, \bar{z}_n)$ of $\mathcal{A}$ into an interval structure $\mathcal{S} = (\mathbb{I}_N, A, \bar{A}, B, \bar{B}, \sigma)$. To do that, we first introduce $|Q| + k$ propositional variables that label *unit-length* intervals (i.e., intervals of the form $[x, x+1]$): the first $|Q|$ propositional variables will be identified with the control states of $\mathcal{A}$, while the last k propositional variables, denoted $c_1, .., c_k$, will be identified with the k counters of $\mathcal{A}$. We then divide the underlying domain $\{0, ..., N\}$ of the interval structure $\mathcal{S}$ into exactly $n+2$ intervals $I_0 = [0, x_1]$, $I_1 = [x_1, x_2]$, ..., $I_n = [x_n, x_{n+1}]$, $I_{n+1} = [x_{n+1}, N]$, with $1 = x_1 < ... < x_{n+1} = N - 1$ and $x_{t+1} - x_t = 1 + \sum_{1 \le i \le k} \bar{z}_t(i)$ for all $1 \le t \le n$ (hence the length N of the interval structure $\mathcal{S}$ is exactly $2 + n + \sum_{1 \le t \le n} \sum_{1 \le i \le k} \bar{z}_t(i)$). The intervals I_1, ..., I_n will be used to encode, respective,y the configurations $(q_1, \bar{z}_1)$, ..., $(q_n, \bar{z}_n)$ of the computation of $\mathcal{A}$, while the two additional intervals I_0 and I_{n+1} will be used as to correctly move between the various intervals via the modal operators $\langle A \rangle$ and $\langle \bar{A} \rangle$. Finally, we let the labeling function σ associate a unique propositional variable in $Q \cup \{c_1, ..., c_k\}$ with each unit-length subinterval of I_t, for all $1 \le t \le n$, as follows:

i) the subinterval $[x_t, x_t + 1]$ is labeled by the control state q_t;

ii) for every $1 \le i \le k$, the number of c_i-labeled intervals of the form $[x, x+1]$, with $x_t < x < x_{t+1}$, coincides with the value $\bar{z}_t(i)$ of the counter i

(note that there may exist different encodings of the same computation of $\mathcal{A}$). As an example, Figure 4 represents part of an encoding of a computation for a lossy counter machine $\mathcal{A}$ with two control states, whose occurrences are represented by black-colored and white-colored intervals, and three counters, whose values are represented by the numbers of occurrences of intervals colored, respectively, by red, blue, and green (the meaning of the dashed arrows is explained below).

The next ingredient of the reduction is the specification of all encodings of all computations of $\mathcal{A}$ by means of a suitable $A\bar{A}B$ formula. In particular, we are interested into enforcing disequalities between counters of the form $\bar{z}_{t+1}(i) \le$

$\bar{z}_t(i) + h$, with $h \in \{-1, 0, 1\}$. We first explain how this is done in the case $h = 0$. By definition, enforcing a constraint of the form $\bar{z}_{t+1}(i) \le \bar{z}_t(i)$ is equivalent to enforcing the existence of a *surjective partial function* g_i from the set of c_i-labeled subintervals of I_t to the set of c_i-labeled subintervals of I_{t+1}. As an example, the dashed arrow labeled by g_3 in Figure 4 represents one instance of a surjective partial function representing a constraint of the form $\bar{z}_{t+1}(3) \le \bar{z}_t(3)$. In its turn, each partial function g_i can encoded by a set of g_i-labeled intervals of the form $[x, g(x)]$, where g_i is viewed as a fresh propositional variable, $x_t < x < x_{t+1} < g(x) < x_{t+2}$, and $\sigma([x, x+1]) = \sigma([g(x), g(x)+1]) = c_i$. The relevant properties of these g_i-labeled intervals are then translated into a suitable formula $\varphi_i^{\le}$ evaluated on the interval I_t. Precisely, we let

$$\begin{aligned}\varphi_i^{\le} \;=\; & [B][A]\big(g_i \;\rightarrow\; \varphi_Q^{\exists!} \wedge [B]\neg g_i \wedge \langle B\rangle c_i \wedge \langle A\rangle c_i\big) \wedge \\ & [B][A]\big(\varphi_Q^{\exists!} \wedge \langle A\rangle c_i \;\rightarrow\; \langle A\rangle\langle \bar{A}\rangle g_i\big)\end{aligned}$$

where $\varphi_A^{\exists!} = \langle B\rangle\langle A\rangle \bigvee_{a\in A} a \;\wedge\; [B]\big([A]\bigvee_{a\in A} a \rightarrow [B][A]\bigwedge_{a\in A} \neg a\big)$ for any given set A (e.g., $A = Q$) of propositional variables (note that $\varphi_A^{\exists!}$ holds at a interval J iff J contains exactly one unit-length subinterval labeled by some propositional variable $a \in A$). Intuitively, the first line of the formula $\varphi_i^{\le}$ enforces the condition that the set of g_i-labeled intervals that start inside I_t represent a partial function from the c_i-labeled subintervals of I_t to the c_i-labeled subintervals of I_{t+1}, while the second line gurantees that such a partial function is surjective.

In a similar way, one can specify the constraints of the form $\bar{z}_{t+1}(i) \le \bar{z}_t(i) - 1$ (resp., $\bar{z}_{t+1}(i) \le \bar{z}_t(i) + 1$) by means of a formula $\varphi_i^{\le,-1}$ (resp., $\varphi_i^{\le,+1}$): this is done by excluding from the domain (resp., from the range) of the surjective partial function g_i exactly one c_i-labeled subinterval of I_t (resp., I_{t+1}), which is then distinguished by using an additional propositional variable `dec` (resp., `inc`). Precisely, we let

$$\begin{aligned}\varphi_i^{\le,-1} \;=\; & [B][A]\big(g_i \;\rightarrow\; \varphi_Q^{\exists!} \wedge [B]\neg g_i \wedge \langle B\rangle(c_i \wedge \neg\mathtt{dec}) \wedge \langle A\rangle c_i\big) \wedge \\ & [B][A]\big(\varphi_Q^{\exists!} \wedge \langle A\rangle c_i \;\rightarrow\; \langle A\rangle\langle \bar{A}\rangle g_i\big) \wedge \\ & \varphi_{\{\mathtt{dec}\}}^{\exists!} \\ \varphi_i^{\le,+1} \;=\; & [B][A]\big(g_i \;\rightarrow\; \varphi_Q^{\exists!} \wedge [B]\neg g_i \wedge \langle B\rangle c_i \wedge \langle A\rangle(c_i \wedge \neg\mathtt{inc})\big) \wedge \\ & [B][A]\big(\varphi_Q^{\exists!} \wedge \langle A\rangle(c_i \wedge \neg\mathtt{inc}) \;\rightarrow\; \langle A\rangle\langle \bar{A}\rangle g_i\big) \wedge \\ & \langle A\rangle\big(\varphi_Q^{\exists!} \wedge [A]\varphi_Q^{\exists!} \wedge \varphi_{\{\mathtt{inc}\}}^{\exists!}\big).\end{aligned}$$

Now, we rewrite each transition rule $\delta(q)$ of $\mathcal{A}$ into a formula φ_q^{δ}, which is defined by case analysis as follows:

1. if $\delta(q)$ is a rule of the form $\mathtt{i \leftarrow i{+}1;\ goto\ q'}$, then we let φ_q^{δ} be the formula $\langle A\rangle q' \;\wedge\; \varphi_i^{\le,+1} \;\wedge\; \bigwedge_{j\ne i} \varphi_j^{\le}$;
2. if $\delta(q)$ is a rule of the form $\mathtt{if\ i = 0\ then\ goto\ q'\ else\ i \leftarrow i{-}1;\ goto\ q''}$, then we let φ_q^{δ} be $\big([B][A]\neg c_i \;\rightarrow\; \varphi_{q,0}^{\delta}\big) \;\wedge\; \big(\langle B\rangle\langle A\rangle c_i \;\rightarrow\; \varphi_{q,1}^{\delta}\big)$, where $\varphi_{q,0}^{\delta} = \langle A\rangle q' \;\wedge\; \bigwedge_{1\le i\le k} \varphi_i^{\le}$ and $\varphi_{q,1}^{\delta} = \langle A\rangle q'' \;\wedge\; \varphi_i^{\le,-1} \;\wedge\; \bigwedge_{j\ne i} \varphi_j^{\le}\big)$.

We can specify the set of all encodings of all computations of $\mathcal{A}$ by means of the following formula (here we shortly denote by C the set of k propositional variables $c_1, ..., c_k$):

$$\begin{aligned}
\varphi^{\mathcal{A}} \;=\; & [U]\Big(\langle A\rangle\top \wedge \langle\bar{A}\rangle\top \wedge [B]\bot \;\rightarrow\; \bigvee_{a\in Q\cup C} a \;\wedge \bigwedge_{a\neq b\in Q\cup C}\neg(a \wedge b)\Big) \wedge \\
& [U]\Big([A]\bot \vee [\bar{A}]\bot \vee \langle B\rangle\top \;\rightarrow \bigwedge_{a\in Q\cup C\cup\{\mathtt{inc},\mathtt{dec}\}} \neg a\Big) \wedge \\
& [U]\bigwedge_{q\in Q\smallsetminus\{q_{target}\}}\Big(\langle B\rangle q \;\rightarrow\; \langle\bar{A}\rangle\langle A\rangle\varphi^{\delta}_{q}\Big) \wedge \\
& \langle U\rangle\Big(\langle B\rangle\big(q_{source} \wedge [A]\bigwedge_{c\in C}\neg c\big) \;\wedge\; \langle A\rangle\big(q_{target} \wedge [A]\bigwedge_{c\in C}\neg c\big)\Big).
\end{aligned}$$

where $[U]\alpha$ is a shorthand for ... and $\langle U\rangle$ is its dual. Intuitively, the first two lines of the formula $\varphi^{\mathcal{A}}$ guarantees that all unit-length intervals, but the first and the last ones, are associated with exactly one propositional variable in $Q \cup C$ and, possibly, with the variables `inc` and `dec`; the third line of the formula $\varphi^{\mathcal{A}}$ defines a valid computation $(q_1, \bar{z}_1)$, ..., $(q_n, \bar{z}_n)$ of $\mathcal{A}$; finally, the fourth line requires that the first configuration $(q_1, \bar{z}_1)$ coincides with the source configuration $(q_{source}, \bar{0})$ and the last configuration $(q_n, \bar{z}_m)$ coincides with the target configuration $(q_{target}, \bar{0})$. Therefore, we can conclude that $\varphi^{\mathcal{A}}$ is satisfiable iff $\mathcal{A}$ admits a computation from $(q_{source}, \bar{0})$ to $(q_{target}, \bar{0})$. This shows that the satisfiability problem for the logic $A\bar{A}B$ is not primitive recursive. □

A.4 Proof of Theorem 3

Theorem 3. *The satisfiability problem for the logic* $A\bar{A}B\bar{B}D$ *(resp.,* $A\bar{A}B\bar{B}\bar{D}$*,* $A\bar{A}B\bar{B}E$*,* $A\bar{A}B\bar{B}\bar{E}$*,* $A\bar{A}B\bar{B}O$*,* $A\bar{A}B\bar{B}\bar{O}$*), interpreted over finite linear orders, is undecidable.*

Proof. First of all, we recall the definitions for the Allen's relations "contains" D, "during" $\bar{D}$, "ended by" E, "ends" $\bar{E}$, "overlaps" O, and "overlapped by" $\bar{O}$:

- **"contains":** $[x,y]\ D\ [x',y']$ iff $x < x' < y' < y$;
- **"during":** $[x,y]\ \bar{D}\ [x',y']$ iff $x' < x < y < y'$;
- **"ended by ":** $[x,y]\ E\ [x',y']$ iff $x < x'$ and $y' = y$;
- **"ends":** $[x,y]\ \bar{E}\ [x',y']$ iff $x' < x$ and $y' = y$;
- **"overlaps":** $[x,y]\ O\ [x',y']$ iff $x < x' < y$ and $y < y'$;
- **"overlapped by":** $[x,y]\ \bar{O}\ [x',y']$ iff $x' < x$ and $x < y' < y$.

The semantics of the corresponding formulas $\langle D\rangle\alpha$, $\langle\bar{D}\rangle\alpha$, $\langle E\rangle\alpha$, $\langle\bar{E}\rangle\alpha$, $\langle O\rangle\alpha$, and $\langle\bar{O}\rangle\alpha$ is defined, as usual, for a given interval structure $\mathcal{S}$ and a given interval I as follows: for any relation $R \in \{D, \bar{D}, E, \bar{E}, O, \bar{O}\}$, we write $\mathcal{S}, I \models \langle R\rangle\alpha$ iff there is an interval $J \in \mathbb{I}_N$ such that $I\ R\ J$ and $\mathcal{S}, J \models \alpha$.

Since Allen's "contains" relation D (resp., Allen's "during" relation $\bar{D}$) is definable in terms of Allen's "begun by" and "ended by" relations B and E (resp., in terms of Allen's "begins" and "ends" relations $\bar{\mathsf{B}}$ and $\bar{\mathsf{E}}$), to prove the theorem it is sufficient to show that the extension of $\mathsf{A\bar{A}B\bar{B}}$ with any modal operator among $\langle\mathsf{D}\rangle$, $\langle\bar{\mathsf{D}}\rangle$, $\langle\mathsf{O}\rangle$, and $\langle\bar{\mathsf{O}}\rangle$ has an undecidable satisfiability problem over finite linear orders. To do that, we will reduce the (undecidable) reachability problem for *non-lossy* (Minsky) counter machines to the satisfiability problem for each of the relevant extensions of $\mathsf{A\bar{A}B\bar{B}}$. One can think of these reductions as slight modifications of the proof of Theorem 2, where inequalities between counter values of the form $\bar{z}_{t+1}(i) \le \bar{z}_t(i) + h$ are replaced by more restrictive constraints of the form $\bar{z}_{t+1}(i) = \bar{z}_t(i) + h$. Thus, from now on, we use the same notation as in the proof of Theorem 2. Replacing inequalities of the form $\bar{z}_{t+1}(i) \le \bar{z}_t(i) + h$ by corresponding equalities $\bar{z}_{t+1}(i) = \bar{z}_t(i) + h$ amounts at enforcing all partial surjective functions g_i that match c_i-labeled subintervals of I_t with c_i-labeled subintervals of I_{t+1} to be in fact *bijections*. Therefore, given a counter machine $\mathcal{A}$, the set of possible encodings of the (unique, non-lossy) computation of $\mathcal{A}$ is specified in terms of a new formula $\varphi^{\mathcal{A}}_{non-lossy}$, which is obtained from $\varphi^{\mathcal{A}}$ (i.e., the formula introduced at the end of the proof of Theorem 2) by adding new conjuncts of the form $\varphi^{=}_i$ for all indices $1 \le i \le k$. Each of these conjuncts $\varphi^{=}_i$ precisely requires the partial surjective function g_i matching c_i-labeled subintervals of I_t with c_i-labeled subintervals of I_{t+1} to be, in addition, total and injective. In the sequel, we define the conjuncts $\varphi^{=}_i$ within the various logics $\mathsf{A\bar{A}B\bar{B}D}$, $\mathsf{A\bar{A}B\bar{B}\bar{D}}$, $\mathsf{A\bar{A}B\bar{B}O}$, and $\mathsf{A\bar{A}B\bar{B}\bar{O}}$, and we briefly discuss their semantics.

Logic $\mathsf{A\bar{A}B\bar{B}D}$. For every $1 \le i \le k$, we define

$$\begin{aligned}\varphi^{=}_i \;=\; & [\mathsf{U}]\big(\langle\mathsf{A}\rangle(c_i \wedge \neg\mathtt{dec}) \;\rightarrow\; \langle\mathsf{A}\rangle g_i\big) \;\wedge \\ & [\mathsf{U}]\big(\langle\mathsf{B}\rangle g_i \wedge [\mathsf{B}][\mathsf{B}]\neg g_i \;\rightarrow\; [\mathsf{D}]\neg g_i\big).\end{aligned}$$

Intuitively, the first line of the formula $\varphi^{=}_i$ requires that every subinterval of I_t which is labeled with c_i, but not with $\mathtt{dec}$, is matched with a c_i-labeled subinterval of I_{t+1}, that is, the function g_i is total. The second line of the formula tries to avoid the existence of pairs of g_i-labeled intervals that end in the same time point, that is, the function g_i is injective; in fact, it enforces a *stronger* condition, namely, that there exist no intervals I, J, K such that (i) both I and J are labeled by g_i, (ii) I is the maximal interval that begins K, and (iii) J is contained in, but does not begin or end, K. Even though the latter condition discards some valid encodings of the non-lossy computation of $\mathcal{A}$ (precisely, those that feature g_i-labeled intervals contained one into each other), we can easily see that there exist equivalent encodings that guarantee that all g_i-labeled intervals are pairwise overlapping or non-intersecting. Under such an assumption, the second line of the formula $\varphi^{=}_i$ turns out to be equivalent to the condition that all g_i-labeled intervals end in pairwise distinct time points.

Logic $\mathsf{A\bar{A}B\bar{B}\bar{D}}$. The encoding of the equality constraints in the logic $\mathsf{A\bar{A}B\bar{B}\bar{D}}$ is analogous to that for the logic $\mathsf{A\bar{A}B\bar{B}D}$. Precisely, for every $1 \le i \le k$, we define

$$\begin{aligned}\varphi^{=}_i \;=\; & [\mathsf{U}]\big(\langle\mathsf{A}\rangle(c_i \wedge \neg\mathtt{dec}) \;\rightarrow\; \langle\mathsf{A}\rangle g_i\big) \;\wedge \\ & [\mathsf{U}]\big(\langle\bar{\mathsf{B}}\rangle g_i \wedge [\bar{\mathsf{B}}][\bar{\mathsf{B}}]\neg g_i \;\rightarrow\; [\bar{\mathsf{D}}]\neg g_i\big).\end{aligned}$$

As a matter of fact, the above formula defines exactly the same models as those of the $A\bar{A}B\bar{B}D$ formula $\varphi_i^=$ introduced before.

Logic $A\bar{A}B\bar{B}O$. For every $1 \le i \le k$, we define

$$\begin{aligned}\varphi_i^= \ = \ & [U]\big(\langle A\rangle(c_i \wedge \neg\mathtt{dec}) \ \to \ \langle A\rangle g_i\big) \ \wedge \\ & [U]\big(\langle \bar{B}\rangle g_i \wedge [\bar{B}][\bar{B}]\neg g_i \ \to \ [O]\neg g_i\big).\end{aligned}$$

The semantics of the above formula is similar to the previous one. The only difference now is that the second line avoids the existence of g_i-labeled intervals that are overlapping (rather than contained one into each other). By using arguments analogous to the previous cases, one can show that such an assumption is not too restrictive, since it still captures some valid encodings of the non-lossy computation of $\mathcal{A}$.

Logic $A\bar{A}B\bar{B}\bar{O}$. The definitions and the arguments for the encoding in the logic $A\bar{A}B\bar{B}\bar{O}$ are symmetric to those for the logic $A\bar{A}B\bar{B}O$:

$$\begin{aligned}\varphi_i^= \ = \ & [U]\big(\langle A\rangle(c_i \wedge \neg\mathtt{dec}) \ \to \ \langle A\rangle g_i\big) \ \wedge \\ & [U]\big(\langle B\rangle g_i \wedge [B][B]\neg g_i \ \to \ [\bar{O}]\neg g_i\big).\end{aligned}$$

□

A.5 Proof of Theorem 4

Theorem 4. *The satisfiability problem for the logic $A\bar{A}B$, and hence that for the logic $A\bar{A}B\bar{B}$, interpreted over over any class of linear orders that contains at least one linear order with an infinitely ascending sequence is undecidable.*

Proof. We first reduce an undecidable variant of the universal reachability problem for lossy counter machines to the satisfiability problem for the logic $A\bar{A}B$ interpreted over structures of the form $\mathcal{S} = (\mathbb{I}_\omega, A, \bar{A}, B, \sigma)$. The variant of the universal reachability problem we focus on is called structural termination and it consists of deciding, given a lossy counter machine $\mathcal{A} = (Q, k, \delta)$ and a pair of control states q_{source} and q_{target}, whether *every* computation of $\mathcal{A}$ that stars at state q_{source}, with any abitrary assignment for the counters, eventually reaches the state q_{target}, again with some arbitrary assignment for the counters. Given the results in [13], it is clear that the problem of structural termination is undecidable.

In order to reduce the above problem to a satisfiability problem for the logic $A\bar{A}B$, we use a technique similar to that of Theorem 2 and we encode an infinite computation $(q_1, \bar{z}_1)\ (q_2, \bar{z}_2)\ \ldots$ of $\mathcal{A}$, with $q_1 = q_{source}$ and $q_t \neq q_{target}$ for all $t \geq 1$, into a suitable interval structure over the domain $(\mathbb{N}, <)$. Precisely, we divide $(\mathbb{N}, <)$ into an infinite sequence of intervals $I_0 = [0, x_1]$, $I_1 = [x_1, x_2]$, ..., where $1 = x_1 < x_2 \ldots$ and $x_{t+1} - x_t = 1 + \sum_{1 \le i \le k} \bar{z}_t(i)$ for all $t \ge 1$. Then, we proceed exactly like in the proof of Theorem 2, by introducing suitable propositional variables and the auxiliary formulas $\varphi_A^{\exists!}$, $\varphi_i^{\le}$, etc. Finally, we represent the set of all infinite computations of $\mathcal{A}$ that start in q_{source} and avoid q_{target} by means of the following formula:

$$\begin{aligned}
\varphi^{\mathcal{A}} \;=\; & [U]\Big(\langle A\rangle\top \wedge \langle \bar{A}\rangle\top \wedge [B]\bot \;\rightarrow\; \bigvee_{a\in Q\cup C} a \;\wedge \bigwedge_{a\neq b\in Q\cup C} \neg(a \wedge b)\Big) \;\wedge \\
& [U]\Big([A]\bot \vee [\bar{A}]\bot \vee \langle B\rangle\top \;\rightarrow \bigwedge_{a\in Q\cup C\cup\{\mathtt{inc},\mathtt{dec}\}} \neg a\Big) \;\wedge \\
& [U]\bigwedge_{q\in Q}\Big(\langle B\rangle q \;\rightarrow\; \langle\bar{A}\rangle\langle A\rangle\varphi^{\delta}_{q}\Big) \;\wedge\; \langle U\rangle q_{source} \;\wedge\; [U]\neg q_{target}.
\end{aligned}$$

Clearly, $\varphi^{\mathcal{A}}$ is satisfiabiable over a right-infinite interval structure of the form $\mathcal{S} = (\mathbb{I}_{\omega}, A, \bar{A}, B, \sigma)$ iff there is an initial configuration of the form $(q_{source}, \bar{z})$, with $\bar{z} \in \mathbb{N}^{k}$, for which the lossy counter machine $\mathcal{A}$ never halts.

We conclude the proof by showing how the above undecidability result can be transferred to all *right-infinite* interval structures, that is, to all structures of the form $\mathcal{S} = (\mathbb{I}, A, \bar{A}, B, \sigma)$, where $\mathbb{I}$ contains intervals over a fixed linear order $(L, <)$ that embeds $(\mathbb{N}, <)$. To do that, we introduce a fresh propositional variable $\#$ and a suitable formula $\varphi_{\#}$ that enforces the following property: the set of all *left-endpoints* of $\#$-labeled intervals of $\mathbb{I}$ is an *infinite* and *discrete* subordering of $(L, <)$ (hence, it embeds $(\mathbb{N}, <)$). Precisely, we let

$$\varphi_{\#} \;=\; \langle U\rangle\# \wedge [U]\big(\# \rightarrow [\bar{A}][A]\#\big) \wedge \langle U\rangle\big(\# \rightarrow \langle\bar{A}\rangle\langle A\rangle(\langle A\rangle\# \wedge [B][A]\neg\#)\big)$$

(intutively, the above formula enforces that (i) there is at least one interval labeled by $\#$, (ii) for every pair of intervals I, J that start at the same point, either both I and J are labeled by $\#$ or neither I nor J are labeled by $\#$, and (iii) the set of left-endpoints of $\#$-labeled intervals, equipped with the underlying order, is right-infinite and nowhere dense, namely, discrete). Finally, in order to correctly encode the set of relevant computations of $\mathcal{A}$ inside the interval structure $\mathcal{S} = (\mathbb{I}, A, \bar{A}, B, \sigma)$, it is sufficient to restrict the quantifications in the above formula $\varphi^{\mathcal{A}}$ to range only over those intervals that satisfy $\# \wedge \langle A\rangle\#$. $\square$