

Ján Jakubík

Maximal Dedekind completion of an abelian lattice-ordered group

*Czechoslovak Mathematical Journal*, Vol. 28 (1978), No. 4, 611–631

Persistent URL: <http://dml.cz/dmlcz/101563>

## Terms of use:

© Institute of Mathematics AS CR, 1978

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

MAXIMAL DEDEKIND COMPLETION OF AN ABELIAN LATTICE  
ORDERED GROUP

JÁN JAKUBÍK, Košice

(Received October 5, 1976)

For an archimedean lattice ordered group  $H$  we denote by  $D(H)$  the Dedekind completion of  $H$  (cf. BIRKHOFF [1], Chap. XIII, § 13). Let  $K$  be an abelian lattice ordered group. EVERETT [5] defined an extension  $M(K)$  of  $K$  that was constructed by means of Dedekind cuts of the lattice  $(K; \leq)$ ; we shall call  $M(K)$  the maximal Dedekind completion of  $K$ . (In [5],  $M(K)$  was said to be the Dedekind completion of  $K$ .) The generalized Dedekind completion  $D_1(G)$  of a lattice ordered group  $G$  has been defined in [8]; cf. also [9]. If  $G$  is archimedean, then both  $M(G)$  and  $D_1(G)$  coincide with the Dedekind completion  $D(G)$ .

A lattice ordered group  $K$  will be called  $M$ -complete if it is abelian and  $M(K) = K$ . In § 1 it will be shown that each lattice ordered group  $G$  possesses a largest convex  $M$ -complete  $l$ -subgroup  $m(G)$ . From this it follows that the class of all  $M$ -complete lattice ordered groups is a radical class [7].

In § 2 it will be proved that if an abelian lattice ordered group is a direct product of its  $l$ -subgroups  $B_i$  ( $i \in I$ ), then  $M(G)$  is a direct product of its  $l$ -subgroups  $M(B_i)$  ( $i \in I$ ). An analogous assertion is valid for direct sums.

A natural question arises what are the relations between  $M(G)$  and  $D_1(G)$  for an abelian lattice ordered group  $G$  and, in particular, when do the both completions  $M(G)$  and  $D_1(G)$  coincide. It will be shown in § 3 that  $D_1(G)$  is always an  $l$ -subgroup of  $M(G)$ . In each lattice ordered group  $G$  there exists a largest archimedean convex  $l$ -subgroup  $A(G)$  of  $G$ ; this is said to be the archimedean kernel of  $G$  (cf. [8]). If  $G$  is an abelian lattice ordered group such that (i)  $G$  is linearly ordered, and (ii)  $A(G) \neq \{0\}$ , then  $M(G) = D_1(G)$ . If either (i) or (ii) fails to hold, then  $M(G)$  need not coincide with  $D_1(G)$ . In Proposition 3.11 a necessary and sufficient condition is given for  $M(G) = D_1(G)$  to be valid.

Some results on the relationship between  $G$  and  $M(G)$  are established in § 4. E.g., it is shown that if  $G$  fulfils the condition

(F) each upper bounded disjoint subset of  $G$  is finite,  
then  $M(G)$  fulfils the condition (F) as well. CONRAD [3] has proved that if  $G$  satisfies

(F), then there exists a system  $S = \{A_i\}$  ( $i \in I$ ) of convex linearly ordered subgroups of  $G$  such that  $G$  can be constructed from  $S$  by means of the operations of direct sums and lexicographic extensions; it will be shown that then  $M(G)$  can be constructed in an analogous way from the system  $S' = \{M(A_i)\}$  ( $i \in I$ ).

We shall use the standard notation for lattices and lattice ordered groups (cf. BIRKHOFF [1], CONRAD [2], FUCHS [6]).

## 1. MAXIMAL DEDEKIND COMPLETION

Let  $L$  be a lattice. For  $X \subseteq L$  we denote by  $X^u$  or  $X^l$ , respectively, the set of all upper and the set of all lower bounds of the set  $X$  in  $L$ . Let  $d(L)$  be the system of all sets  $(X^u)^l$ , where  $X$  runs over the system of all nonempty upper bounded subsets of  $L$ . The system  $d(L)$  is partially ordered by the set theoretical inclusion. Then  $d(L)$  is a conditionally complete lattice. The lattice operations in  $d(L)$  will be denoted by  $\wedge$  and  $\vee$ . If  $(X_i^u)^l$  ( $i \in I$ ) are subsets of  $L$  such that the system  $\{(X_i^u)^l\}_{i \in I}$  is lower bounded in  $L$ , then

$$\bigwedge_{i \in I} (X_i^u)^l = \bigcap_{i \in I} (X_i^u)^l;$$

if the system  $S$  is upper bounded in  $L$ , then

$$\bigvee_{i \in I} (X_i^u)^l = ((\bigcup_{i \in I} (X_i^u)^u)^u)^l.$$

The mapping  $\varphi : L \rightarrow d(L)$  defined by

$$\varphi(x) = (\{x\}^u)^l \quad \text{for each } x \in L$$

is an isomorphism of  $L$  into  $d(L)$ . We shall identify  $x$  with  $\varphi(x)$  for each  $x \in L$ . Then  $L$  is a sublattice of  $d(L)$ . If  $X_1$  is a subset of  $L$  and  $x_1$  is the supremum of  $X_1$  in  $L$ , then  $x_1$  is also the supremum of  $X_1$  in  $d(L)$ ; the analogous dual assertion is also valid.

Let  $(G; \leq; +)$  be an abelian lattice ordered group. Consider the lattice  $(G; \leq)$ . We define a binary operation  $+$  in  $d(G)$  as follows: for  $Y_1, Y_2 \in d(G)$  we set

$$Y_1 + Y_2 = (\{y_1 + y_2 : y_1 \in Y_1, y_2 \in Y_2\}^u)^l.$$

The following results 1.1–1.3 have been proved in [5].

**1.1. Lemma.**  *$(d(G); +)$  is a semigroup. The element  $0 \in G$  is a neutral element in  $(d(G); +)$ . If  $a, b, c \in d(G)$ ,  $a \leq b$ , then  $c + a \leq c + b$ . The set  $G$  is a subsemigroup of  $(d(G); +)$ .*

**1.2. Theorem.** *Let  $M(G)$  be the set of all elements of  $d(G)$  having an inverse in the semigroup  $d(G)$ . Then*

- (a)  $M(G)$  is a group with respect to the operation  $+$ ;
- (b)  $M(G)$  is a sublattice of the lattice  $d(G)$ .

From 1.1 and 1.2 it follows that  $M(G)$  is a lattice ordered group. From the definition of  $M(G)$  we obtain immediately that  $M(G)$  is a maximal subsemigroup of  $d(G)$  with respect to the property of being a group.  $G$  is obviously an  $l$ -subgroup of  $M(G)$ . We shall call  $M(G)$  the maximal Dedekind completion of  $G$ . If  $M(G) = G$ , then  $G$  is said to be  $M$ -complete.

For  $a \in d(G)$  we denote

$$u(a) = \{g \in G : a \leq g\}, \quad l(a) = \{g \in G : g \leq a\},$$

$$v(a) = \{g_1 - g_2 : g_1 \in u(a), g_2 \in l(a)\}.$$

**1.3. Theorem.** *Let  $a \in d(G)$ . Then  $a$  belongs to  $M(G)$  if and only if  $\inf v(a) = 0$  holds in  $G$ .*

Let  $x, y \in G, x \leq y$ . The set  $[x, y] = \{z \in G : x \leq z \leq y\}$  is said to be an interval of  $G$ . The interval  $[x, y]$  is nontrivial, if  $x < y$ . Let us consider the following condition for a nontrivial interval  $[x, y]$  of  $G$ :

(m) If  $X \neq \emptyset \neq Y$  are subsets of  $[x, y]$  such that  $x_1 \leq y_1$  for each  $x_1 \in X$  and each  $y_1 \in Y$  and

$$(1) \quad \inf \{y_1 - x_1 : x_1 \in X, y_1 \in Y\} = 0,$$

then  $\sup X$  exists in  $[x, y]$ .

**1.4. Lemma.**  *$G$  is  $M$ -complete if and only if each nontrivial interval of  $G$  fulfils the condition (m). If each nontrivial interval of  $G^+$  fulfils (m), then  $G$  is  $M$ -complete.*

*Proof.* Let  $G$  be  $M$ -complete. Suppose that  $[x, y]$  is a nontrivial interval of  $G$  and let  $X, Y$  be as in (m). Then  $X$  is upper bounded in  $G$  and hence  $z = (X^u)^l$  belongs to  $d(G)$ . Clearly

$$X \subseteq l(z), \quad Y \subseteq u(z)$$

and hence

$$\{y_1 - x_1 : x_1 \in X, y_1 \in Y\} \subseteq v(z).$$

From this and from (1) we infer that  $\inf v(z) = 0$  holds in  $G$ . Thus according to Theorem 1.3,  $z$  belongs to  $M(G)$ . Because  $G$  is  $M$ -complete, we have  $z \in G$ . In  $M(G)$  the relation  $\sup X = z$  is valid; thus  $\sup X = z$  holds in  $G$ . Obviously  $z \in [x, y]$ . Hence  $z = \sup X$  in  $[x, y]$ . Therefore  $[x, y]$  fulfils (m).

Conversely, assume that each nontrivial interval of  $G$  fulfils the condition (m). If  $G = \{0\}$ , then clearly  $M(G) = \{0\}$ . Suppose that  $G \neq \{0\}$  and let  $a \in M(G)$ . There are elements  $x, y \in G$  with  $x < a < y$ . Put

$$X = l(a) \cap [x, y], \quad Y = u(a) \cap [x, y].$$

Then  $x_1 \leq y_1$  for each  $x_1 \in X$  and each  $y_1 \in Y$ . If  $g_1 \in u(a), g_2 \in l(a)$ , then  $g_1 \wedge y \in$

$\in Y, g_2 \vee x \in X$  and

$$g_1 \wedge y - g_2 \vee x \leq g_1 - g_2.$$

From this and from  $\inf v(a) = 0$  we obtain that (1) is valid. Hence by (m),  $\sup X = x_1$  exists in  $G$ . Thus  $\sup X = x_1$  holds in  $d(G)$ . On the other hand, according to the construction of  $X$  we have  $\sup X = a$  in  $d(G)$ . Hence  $a = x_1 \in G$  and thus  $G = M(G)$ .

Suppose that each nontrivial interval of  $G^+$  fulfils (m) and let  $I_1 = [a, b]$  be a nontrivial interval of  $G$ . Then  $I_2 = [0, b - a]$  is a nontrivial interval of  $G^+$  isomorphic with  $I_1$ ; hence  $I_1$  fulfils (m) and thus  $G$  is  $M$ -complete.

**1.5. Lemma.** *Let  $x, y, z \in G, x < y < z$ . Suppose that both intervals  $[x, y]$  and  $[y, z]$  fulfil the condition (m). Then  $[x, z]$  fulfils (m).*

*Proof.* Let  $X, Y$  be nonempty subsets of the interval  $[x, z]$  such that  $x_1 \leq y_1$  for each  $x_1 \in X$  and each  $y_1 \in Y$ . Suppose that (1) is valid. Put

$$X' = \{x_1 \wedge y : x_1 \in X\}, \quad X'' = \{x_1 \vee y : x_1 \in X\}$$

and let  $Y', Y''$  be defined analogously.  $X'$  and  $Y'$  are nonempty subsets of the interval  $[x, y]$  and  $x'_1 \leq y'_1$  for each  $x'_1 \in X'$  and each  $y'_1 \in Y'$ . For  $x_1 \in X$  and  $y_1 \in Y$  we have  $y_1 \wedge y - x_1 \wedge y \leq y_1 - x_1$ ; hence from (1) we obtain

$$(2) \quad \inf \{y'_1 - x'_1 : y'_1 \in Y', x'_1 \in X'\} = 0.$$

Because  $[x, y]$  fulfils (m), there exists  $\sup X' = p$  in  $[x, y]$ . Similarly, by considering the subsets  $X''$  and  $Y''$  in  $[y, z]$  we conclude that  $q = \sup X''$  exists in  $[y, z]$ . Put  $r = p + q - y$ . For each  $x_1 \in X$  and each  $y_1 \in Y$  denote

$$\begin{aligned} x'_1 &= x_1 \wedge y, & x''_1 &= x_1 \vee y, \\ y'_1 &= y_1 \wedge y, & y''_1 &= y_1 \vee y. \end{aligned}$$

Then  $x_1 = x'_1 + x''_1 - y \leq r$  and  $y_1 = y'_1 + y''_1 - y \geq r$  holds for each  $x_1 \in X$  and each  $y_1 \in Y$ . Suppose that  $r \neq \sup X$  in  $[x, z]$ ; then there is  $r_1 \in [x, z]$  such that  $r_1 < r$  and  $r_1 \leq x_1$  for each  $x_1 \in X$ . Hence  $0 < r - r_1 \leq y_1 - x_1$  for each  $x_1 \in X$  and each  $y_1 \in Y$ , which contradicts (1). Thus  $[x, z]$  fulfils (m).

**1.6. Corollary.** *Let  $x, y \in G, 0 < x, 0 < y$ . If both  $[0, x]$  and  $[0, y]$  fulfil (m), then  $[0, x + y]$  fulfils (m).*

Let  $m(G)$  be the set of all elements  $g \in G$  such that either  $g = 0$  or  $g \neq 0$  and the interval  $[0, |g|]$  fulfils (m).

**1.7. Theorem.** *For each abelian lattice ordered group  $G$ ,  $m(G)$  is a convex  $l$ -subgroup of  $G$ . Moreover,  $m(G)$  is  $M$ -complete and  $A \subseteq m(G)$  whenever  $A$  is an  $M$ -complete convex  $l$ -subgroup of  $G$ . For each automorphism  $\varphi$  of the lattice ordered group  $G$  we have  $\varphi(m(G)) = m(G)$ .*

Proof. If  $g \in m(G)$ , then clearly  $-g \in m(G)$ . Let  $g_1, g_2 \in m(G)$ ,  $g_1 \neq 0 \neq g_2$ . Put  $|g_1| + |g_2| = g$ . According to 1.6, the interval  $[0, g]$  fulfils (m). We have

$$0 < |g_1 + g_2| \leq g,$$

hence the interval  $[0, |g_1 + g_2|]$  fulfils (m). Thus  $g_1 + g_2 \in m(G)$ . Hence  $m(G)$  is a subgroup of  $G$ . It is obvious that  $m(G)$  is upper-directed and convex, hence it is a convex  $l$ -subgroup of  $G$ . If  $x, y \in m(G)$ ,  $x < y$ , then  $0 < y - x \in m(G)$  and the intervals  $[x, y]$ ,  $[0, y - x]$  are isomorphic; hence  $[x, y]$  fulfils the condition (m). Thus according to 1.4,  $m(G)$  is  $M$ -complete. Let  $A$  be a convex  $l$ -subgroup of  $G$  and suppose that  $A$  is  $M$ -complete. Let  $0 < a \in A$ . By 1.4, the interval  $[0, a]$  of  $G$  fulfils (m) and thus  $a \in m(G)$ . Therefore  $A \subseteq m(G)$ . Let  $\varphi$  be an automorphism of the lattice ordered group  $G$ . Then  $\varphi(m(G))$  is isomorphic with  $m(G)$ , hence  $\varphi(m(G))$  is  $M$ -complete and hence  $\varphi(m(G)) \subseteq m(G)$ . Similarly  $\varphi^{-1}(m(G)) \subseteq m(G)$  and thus  $\varphi(m(G)) = m(G)$ .

**1.7.1. Theorem.** *Let  $H$  be a lattice ordered group. There exists a convex  $l$ -subgroup  $m_1(H)$  of  $H$  with the following properties:*

- (i)  $m_1(H)$  is  $M$ -complete;
- (ii) if  $A$  is a convex  $l$ -subgroup of  $H$  such that  $A$  is  $M$ -complete, then  $A \subseteq m_1(H)$ .

Proof. There exists a largest convex abelian  $l$ -subgroup  $a(H)$  of  $H$  (cf. [8], Lemma 1.1). Put  $a(H) = G$ ,  $m_1(H) = m(G)$ . Then  $m_1(H)$  is abelian and by 1.7,  $m_1(H)$  is  $M$ -complete. Let  $A$  be a convex  $l$ -subgroup of  $H$  such that  $A$  is  $M$ -complete. Thus  $A \subseteq a(H)$  and hence according to 1.7 we have  $A \subseteq m_1(H)$ .

Let  $\mathcal{K}$  be a class of lattice ordered groups. Suppose that  $\mathcal{K}$  fulfils the following conditions:

- (a)  $\mathcal{K}$  is closed with respect to isomorphisms.
- (b) If  $A \in \mathcal{K}$  and  $B$  is a convex  $l$ -subgroup of  $A$ , then  $B \in \mathcal{K}$ .
- (c) If  $\{A_i\}$  is a system of convex  $l$ -subgroups of a lattice ordered group  $H$  such that each  $A_i$  belongs to  $\mathcal{K}$ , then  $\bigvee A_i$  belongs to  $\mathcal{K}$ .

Under these assumptions  $\mathcal{K}$  is said to be a radical class [7].

Let  $\mathcal{M}$  be the class of all lattice ordered groups that are  $M$ -complete.  $\mathcal{M}$  obviously fulfils (a). From 1.4 it follows that (b) is valid for  $\mathcal{M}$ . Let  $\{A_i\}$  be a system of convex  $l$ -subgroups of a lattice ordered group  $H$  such that each  $A_i$  is  $M$ -complete. According to 1.7.1 we have  $A_i \subseteq m_1(H)$  for each  $A_i$  and hence  $\bigvee A_i \subseteq m_1(H)$ . Thus by (b),  $\bigvee A_i$  belongs to  $\mathcal{M}$  and hence  $\mathcal{M}$  fulfils (c). Therefore  $\mathcal{M}$  is a radical class.

**1.8. Lemma.** *Let  $K$  be a lattice ordered group,  $k \in K$ ,  $X \subseteq K$ ,  $Y \subseteq K$ ,  $\sup X = k = \inf Y$ ,  $Z = \{y - x : x \in X, y \in Y\}$ . Then  $\inf Z = 0$ .*

Proof. Denote  $X = \{x_i\}_{i \in I}$ ,  $Y = \{y_j\}_{j \in J}$ . Suppose that  $\inf Z \neq 0$ . Then there is  $0 < t \in K$  such that  $y_j - x_i \geq t$  for each  $i \in I$  and each  $j \in J$ . Hence  $y_j \geq t + x_i$

and thus

$$k < t + k = t + \bigvee_{i \in I} x_i = \bigvee_{i \in I} (t + x_i) \leq \bigwedge_{j \in J} y_j = k,$$

which is a contradiction.

**1.9. Lemma.**  *$M(G)$  is  $M$ -complete.*

*Proof.* Denote  $M(G) = H$  and let  $h_0 \in M(H)$ . There are sets  $X, Y \subseteq H$  such that  $\sup X = h_0 = \inf Y$  holds in  $M(H)$ . Put  $X = \{x_i\}$ ,  $Y = \{y_j\}$ . For each  $x_i$  there is a subset  $X_i$  of  $G$  such that  $x_i = \sup X_i$  is valid in  $H$ . Similarly, for each  $y_j$  there is a subset  $Y_j$  of  $G$  such that  $y_j = \inf Y_j$  holds in  $H$ . Denote  $X' = \bigcup X_i$ ,  $Y' = \bigcup Y_j$ . Then

$$\sup X' = h_0 = \inf Y'$$

is valid in  $M(H)$ . Put  $Z' = \{y' - x' : y' \in Y', x' \in X'\}$ . According to 1.8 we have  $\inf Z' = 0$  in  $M(H)$ , and hence  $\inf Z' = 0$  in  $G$ . Hence in view of 1.3 there is  $h_1 \in H$  such that

$$(*) \quad \sup X' = h_1 = \inf Y'$$

is valid in  $H$ ; thus  $(*)$  holds in  $M(H)$  as well and therefore  $h_1 = h_0$ . Hence  $h_0 \in H$  and so  $M(H) = H$ .

**1.10. Theorem.** *Let  $G$  be an abelian lattice ordered group. Let  $H$  be a lattice ordered group fulfilling the conditions*

- (a)  *$H$  is  $M$ -complete;*
- (b)  *$G$  is an  $l$ -subgroup of  $H$ ;*
- (c) *for each  $h \in H$  there are sets  $X \subseteq G$ ,  $Y \subseteq G$  such that  $\sup X = h = \inf Y$ .*

*Then there exists an isomorphism  $\varphi$  of  $M(G)$  onto  $H$  such that  $\varphi(g) = g$  for each  $g \in G$ .*

*Proof.* Let  $a \in M(G)$ . There exists a subset  $X_1 \neq \emptyset$  of  $G$  such that  $\sup X_1 = a$  holds in  $M(G)$ . The set  $X_1$  is upper bounded in  $G$ . Put  $Y = X_1^u$ ,  $X = (X_1^l)^l$  in  $G$ . Then we have in  $M(G)$  the relations

$$\sup X = a = \inf Y.$$

Put  $Z = \{y - x : x \in X, y \in Y\}$ . According to Lemma 1.8,

$$(2) \quad \inf Z = 0$$

holds in  $M(G)$ . Hence (2) is valid in  $G$ .

Denote  $Y_2 = X^u$ ,  $X_2 = Y_2^l$ , where the symbols  $u$  and  $l$  are taken with respect to  $H$ . Put  $Z_2 = \{y_2 - x_2 : x_2 \in X_2, y_2 \in Y_2\}$ . From (2) we obtain

$$(3) \quad \inf Z_2 = 0.$$

Hence according to 1.3 there exists  $h \in M(H)$  such that

$$\sup X_2 = h = \inf Y_2 .$$

Because  $H$  is  $M$ -complete, we have  $h \in H$ . We put  $\varphi(a) = h$ . Clearly  $\varphi(g) = g$  for each  $g \in G$ . The element  $h$  is correctly defined (i.e., it is uniquely defined by the element  $a$ ); namely, if  $X'_1$  is another subset of  $G$  with  $\sup X'_1 = a$  in  $M(G)$ , then  $(X'_1)^u = X'_1$  in  $G$ . From the definition of the operations  $+$ ,  $\wedge$  and  $\vee$  in  $M(G)$  it follows that  $\varphi$  is a homomorphism of the lattice ordered group  $M(G)$  into  $H$ .

Suppose that  $\varphi(a) = 0$  for some  $a \in M(G)$ ,  $a \neq 0$ . Then  $\varphi(|a|) = 0$ . There is  $0 < g \in G$  with  $g \leq |a|$ ; thus  $\varphi(g) = 0$ . This is a contradiction, since  $\varphi(g) = g$ . Hence  $\varphi$  is an isomorphism of  $M(G)$  into  $H$ .

Let  $h \in H$ . Put  $X' = \{g \in G : g \leq h\}$ ,  $Y' = \{g \in G : h \leq g\}$ ,  $Z' = \{y' - x' : x' \in X', y' \in Y'\}$ . From (c) we obtain

$$\sup X' = h = \inf Y' .$$

Hence according to 1.8 we have

$$(4) \quad \inf Z' = 0$$

in  $H$ . From this and from (b) it follows that (4) holds in  $G$  as well. Thus by 1.3, there exists  $a \in M(G)$  with  $\sup X' = a$  and then  $\varphi(a) = h$ . Hence  $\varphi$  is onto. This completes the proof.

According to 1.2 and 1.9 the lattice ordered group  $H = M(G)$  fulfils the conditions (a) and (b). From the construction of  $M(G)$  it follows that it satisfies also the condition (c). Hence these conditions may serve as an intrinsic definition of the maximal Dedekind completion of  $G$ .

From the definition of  $M(G)$  and from the definition of the Dedekind closure of an archimedean lattice ordered group it follows immediately that if  $G$  is archimedean, then  $M(G)$  coincides with the Dedekind closure  $D(G)$  of  $G$ .

The lattice ordered group  $M(G)$  is said to be the  $M$ -closure of  $G$ .

**1.11. Lemma.** *Let  $G_1$  be a convex  $l$ -subgroup of  $G$ . Let  $H$  be as in 1.10 and let*

$$H_1 = \{h \in H : \text{there are } g, g' \in G_1 \text{ with } g \leq h \leq g'\} .$$

*Then*

- (i)  $H_1$  is a convex  $l$ -subgroup of  $H$ ;
- (ii)  $H_1$  is the maximal Dedekind closure of  $G_1$ .

*Proof.* Let  $h_i \in H_1$  ( $i = 1, 2$ ). There are  $g_i, g'_i \in G$  with  $g_i \leq h_i \leq g'_i$  ( $i = 1, 2$ ). Then  $-g'_i \leq -h_i \leq -g_i$  ( $i = 1, 2$ ),  $g_1 + g_2 \leq h_1 + h_2 \leq g'_1 + g'_2$ , hence  $H_1$  is a subgroup of  $H$ . Since  $g_1 \wedge g_2 \leq x \leq g'_1 \vee g'_2$  holds for  $x = h_1 \wedge h_2$  and for  $x = h_1 \vee h_2$ ,  $H_1$  is a sublattice of  $H$ . It is obvious that  $H_1$  is a convex subset of  $H$ . Thus (i) is valid.



In order to prove (ii) we have to verify that the conditions (a), (b) and (c) from 1.10 are fulfilled with  $H$  and  $G$  replaced by  $H_1$  and  $G_1$ . Since  $H_1$  is a convex  $l$ -subgroup of  $H$  by (i), it follows from 1.4 that  $H_1$  is  $M$ -complete, thus (a) holds. The condition (b) is obviously valid. Let  $h \in H_1$ . Since  $h \in H$ , there are  $X, Y \subseteq G$  such that (c) is valid. Put  $X_1 = X \cap G_1, Y_1 = Y \cap G_1$ . From (c) and from the definition of  $H_1$  it follows that  $\sup X_1 = h = \inf Y_1$ . Hence (ii) holds.

## 2. DIRECT DECOMPOSITIONS

For the notions and notation concerning direct decompositions of a lattice ordered group cf., e.g., [9], § 2. The notion of a completely subdirect product of lattice ordered groups has been introduced by ŠIK [10]. Let  $G$  be an abelian lattice ordered group and let  $H$  be the  $M$ -completion of  $G$ .

**2.1. Lemma.** *Let  $B$  be a direct factor of  $G$ . If  $G$  is  $M$ -complete, then  $B$  is  $M$ -complete as well.*

*Proof.* From Lemma 1.4 it follows that each convex  $l$ -subgroup of an  $M$ -complete lattice ordered group is  $M$ -complete. Since  $B$  is a convex  $l$ -subgroup of  $G$ , the  $M$ -completeness of  $G$  implies that  $B$  is  $M$ -complete.

**2.2. Proposition.** *Let  $G = \prod_{i \in I}^0 B_i$ . Then  $G$  is  $M$ -complete if and only if all  $B_i$  are  $M$ -complete.*

*Proof.* The assertion "only if" follows from 2.1. Suppose that all  $B_i$  are  $M$ -complete and let  $[a, b]$  be a nontrivial interval of  $G$ . Let  $X, Y$  be nonempty subsets of  $[a, b]$  such that  $x \leq y$  for each  $x \in X$  and each  $y \in Y$ . Assume that (1) is valid. For each  $i \in I$  let  $a_i = a(B_i), X_i = X(B_i)$  and let  $b_i, Y_i$  have the analogous meanings. For each  $x \in X, y \in Y, i \in I$  we have

$$0 \leq y(B_i) - x(B_i) = (y - x)(B_i) \leq y - x.$$

Hence from (1) we obtain that

$$(1') \quad \inf \{y_1 - x_1 : y_1 \in Y_i, x_1 \in X_i\} = 0$$

is valid for each  $i \in I$ . Because  $B_i$  is  $M$ -complete,  $z_i = \sup X_i$  exists  $B_i$  for each  $i \in I$ . Let  $z \in G$  be such that  $z(B_i) = z_i$  for each  $i \in I$ . It is easy to verify that  $z = \sup X$  is valid in  $G$ . Hence in view of 1.4,  $G$  is  $M$ -complete.

For  $X \subseteq G$  we denote

$$c(X) = \{y \in H : x_1 \leq y \leq x_2 \text{ for some } x_1, x_2 \in X\}.$$

If  $X$  is a convex  $l$ -subgroup of  $G$ , then  $c(X)$  is a convex  $l$ -subgroup of  $H$ .

**2.3. Lemma.** Let  $B$  be a direct factor of  $G$ . Then  $c(B)$  is a direct factor of  $H$ . For each  $g \in G$ ,  $g(B) = g(c(B))$ .

*Proof.* Let  $0 < h \in H$ . There are subsets  $X, Y \subseteq G^+$  such that  $\sup X = h = \inf Y$  holds in  $H$ . Hence (1) is valid in  $G$  (cf. Lemma 1.8). Put  $X_1 = X(B)$ ,  $Y_1 = Y(B)$ . From (1) it follows that

$$(1'') \quad \inf \{y_1 - x_1 : x_1 \in X_1, y_1 \in Y_1\} = 0$$

is valid in  $G$ . Hence there is  $h_1 \in H$  such that  $\sup X_1 = h_1 = \inf Y_1$ . Clearly  $h_1 \in c(B)$  and  $0 \leq h_1 \leq h$ . Assume that there is  $t \in c(B)$  such that  $h_1 < t \leq h$ . We have  $0 < -h_1 + t \in c(B)$ ,  $-h_1 + t \in H$  and hence there is  $0 < v \in B$  such that  $v \leq -h_1 + t$ . Hence  $h_1 + v \leq h$  and therefore  $x_1 + v \leq y_1$  for each  $x_1 \in X_1$  and each  $y_1 \in Y_1$ , which contradicts (1''). Thus  $h_1$  is the greatest element of the set  $\{z \in c(B) : 0 \leq z \leq h\}$ . From this it follows that  $c(B)$  is a direct factor of  $H$  and that  $h_1 = h(c(B))$ .

Now suppose that the element  $0 < h$  belongs to  $G$ . Then we may suppose that  $h \in X$  and hence  $h(B)$  is the greatest element of  $X_1$ . This implies that  $h_1 = \sup X_1 = h(B)$ , hence  $h(c(B)) = h(B)$ . If  $g$  is an arbitrary element of  $G$ , then there are  $g_1, g_2 \in G^+$  with  $g = g_1 - g_2$ ; then  $g(B) = g_1(B) - g_2(B) = g_1(c(B)) - g_2(c(B)) = g(c(B))$ .

**2.4. Lemma.** Let  $B$  be a convex  $l$ -subgroup of  $G$ . Then  $c(B)$  is the  $M$ -closure of  $B$ .

*Proof.* We have to verify that the conditions (a), (b) and (c) from 1.10 are valid if we replace  $G$  and  $H$  by  $B$  and  $c(B)$ . The conditions (a) and (b) are obviously fulfilled. Let  $h \in c(B)$ . There are elements  $b_1, b_2 \in B$  with  $b_1 \leq h \leq b_2$ . Further, there are subsets  $X, Y$  of  $G$  such that  $\sup X = h = \inf Y$ . Put

$$X_1 = \{x \vee b_1 : x \in X\}, \quad Y_1 = \{y \wedge b_2 : y \in Y\}.$$

Then  $X_1, Y_1 \subseteq B$  and  $\sup X_1 = h = \inf Y_1$  holds in  $c(B)$ . Hence (c) is valid.

**2.5. Theorem.** Let  $G$  be a completely subdirect product of lattice ordered groups  $B_i$  ( $i \in I$ ). Then  $M(G)$  is a completely subdirect product of lattice ordered groups  $M(B_i)$  ( $i \in I$ ).

*Proof.* As above, we put  $M(G) = H$ . According to 2.4 we have  $M(B_i) = c(B_i)$  for each  $i \in I$ . By 2.3, each  $c(B_i)$  is a direct factor in  $H$ . If  $i, j \in I$ ,  $i \neq j$ , then  $B_i \cap B_j = \{0\}$ ; this implies immediately that  $c(B_i) \cap c(B_j) = \{0\}$ . Thus it suffices to verify that for each  $h \in H$  we have

$$h = \bigvee_{i \in I} h_i,$$

where  $h_i = h(c(B_i))$ . Clearly  $h_i \leq h$  for each  $i \in I$ . Assume that there is  $f \in H$  with  $f < h$  such that  $h_i \leq f$  for each  $i \in I$ . There is  $0 < g \in G$  with  $g \leq -f + h$ . We have

$g_i = g(B_i) > 0$  for some  $i \in I$ . Hence

$$h_i < h_i + g_i \leq f + g < h,$$

which is a contradiction, because  $h_i + g_i \in c(B_i)$  and  $h_i$  is the greatest element of the set  $\{z \in c(B_i) : 0 \leq z \leq h\}$ .

**2.5.1. Corollary.** (Cf. ČERNÁK [4].) *Let  $G$  be an archimedean lattice ordered group. Suppose that  $G$  is a completely subdirect product of lattice ordered groups  $B_i$  ( $i \in I$ ). Then  $D(G)$  is a completely subdirect product of lattice ordered groups  $D(G_i)$  ( $i \in I$ ).*

**2.6. Lemma.** *Let  $G = \prod^0 G_i$  ( $i \in I$ ) and let  $X, Y$  be nonempty subsets of  $G$  such that  $x \leq y$  for each  $x \in X$  and each  $y \in Y$ . For  $i \in I$  let  $X_i$  and  $Y_i$  be as in 2.2. Suppose that (1') holds for each  $i \in I$ . Then (1) is valid.*

*Proof.* Assume that (1) fails to hold. Then there is  $0 < g \in G$  such that  $y - x \geq g$  for each  $x \in X$  and each  $y \in Y$ . There is  $i \in I$  with  $g_i = g(B_i) > 0$ . Then  $y_i - x_i \geq g_i$  for each  $y_i \in Y_i$  and each  $x_i \in X_i$ , which is a contradiction.

**2.7. Theorem.** *Let  $G = \prod^0 G_i$  ( $i \in I$ ). Then  $M(G) = \prod^0 M(G_i)$  ( $i \in I$ ).*

*Proof.* We shall use the same notation as in 2.6. In view of 2.5 it suffices to prove that if  $0 \leq h^i \in c(B_i)$  for each  $i \in I$ , then  $\bigvee_{i \in I} h^i$  exists in  $H$ .

Let  $0 \leq h^i \in c(B_i)$  for each  $i \in I$ . Then for each  $i \in I$  there are nonempty subsets  $X^i, Y^i \subseteq B_i$  such that  $\sup X^i = h^i = \inf Y^i$  is valid in  $c(B_i)$ . Let  $X$  be the set of all elements  $x \in G$  such that  $x(B_i) \in X^i$  holds for each  $i \in I$  and let  $Y$  be defined analogously. Then  $X \neq \emptyset \neq Y$ . According to 2.6, the condition (1) holds for  $X, Y$  and hence there is  $h \in H$  such that  $\sup X = h = \inf Y$ . Put  $h_i = h(c(B_i))$  for each  $i \in I$ . Let  $i$  be a fixed element of  $I$  and choose  $x_i \in X^i$ . There exists  $x \in X$  such that  $x_i = x(B_i)$ , hence  $x_i \leq x \leq h$ . From this we obtain  $x_i = x_i(c(B_i)) \leq h(c(B_i)) = h_i$ . Thus  $h^i = \sup X^i \leq h_i$ . Analogously, by considering the elements  $y_i \in Y^i$ , we get  $h_i \leq h^i$ ; thus  $h^i = h_i$ . By the same method as in 2.5 we can now prove that

$$\bigvee h^i = h.$$

**2.8. Proposition.** *Let  $G = \sum^0 B_i$  ( $i \in I$ ). Then  $H = \sum^0 M(B_i)$  ( $i \in I$ ).*

*Proof.* According to 2.3 and 2.4, each  $M(B_i)$  is a direct factor of  $H$ . If  $i, j \in I$ ,  $i \neq j$ , then  $B_i \cap B_j = \{0\}$  and from this we infer that  $M(B_i) \cap M(B_j) = \{0\}$ . Now in view of 2.5 we have only to verify that for each  $x_0 \in H$ , the set

$$I(x_0) = \{i \in I : x_0(M(B_i)) \neq 0\}$$

is finite. Since each element of  $H$  is a difference of positive elements, it suffices to

consider the case  $x_0 > 0$ . There is  $y \in G$  with  $x_0 < y$ . According to 2.3 we have  $y(M(B_i)) = y(B_i)$ , hence

$$I(y) = \{i \in I : y(B_i) \neq 0\};$$

thus  $I(y)$  is finite. For each  $i \in I$ ,

$$0 \leq x_0(M(B_i)) \leq y(M(B_i));$$

therefore the set  $I(x_0)$  is finite.

### 3. THE ARCHIMEDEAN KERNEL

Let  $G$  be a lattice ordered group. An element  $0 < g \in G$  is called archimedean in  $G$  if for each  $0 < x \in G$  there is a positive integer  $n$  such that  $nx \prec g$ . A lattice ordered group  $G$  is archimedean if and only if all its strictly positive elements are archimedean in  $G$ . Let  $A(G)$  be the set of all elements  $g \in G$  such that either  $g = 0$  or  $|g|$  is archimedean in  $G$ . It has been proved in [8] that  $A(G)$  is a closed  $l$ -ideal of  $G$ ; it is said to be the archimedean kernel of  $G$ .

Assume that  $G$  is abelian and let  $H$  be the maximal Dedekind completion of  $G$ .

**3.1. Proposition.** *The archimedean kernel of  $H$  is the set of all elements  $h \in H$  with the property that  $|h| = \sup Z$  for a subset  $Z \subseteq A(G)$ .*

*Proof.* Let  $h \in H$ ,  $|h| = \sup Z$  for some  $Z \subseteq A(G)$ ,  $h \neq 0$ . Without loss of generality we can suppose that  $0 < z$  for each  $z \in Z$ . Assume that there is  $z \in Z$  that fails to be archimedean in  $H$ . Hence there is  $0 < x \in H$  such that  $nx \leq z$  for each positive integer  $n$ . There exists  $0 < g \in G$  with  $g \leq x$ ; hence  $ng \leq z$  for each positive integer  $n$ , which is a contradiction, because  $z \in A(G)$ . Thus  $Z \subseteq A(H)$ ; since  $A(H)$  is a closed  $l$ -ideal in  $H$ , we infer that  $|h| \in A(H)$ . This implies  $h \in A(H)$ .

Conversely, let  $h \in A(H)$ ,  $h \neq 0$ . Then  $0 < |h| \in A(H)$ . There exists a subset  $X \subseteq G^+$  with  $\sup X = |h|$ . If some  $0 \neq x \in X$  is not archimedean in  $G$ , then  $|h|$  fails to be archimedean in  $H$ , which is a contradiction. Thus  $X \subseteq A(G)$ .

**3.1.1. Corollary.**  $c(A(G)) \subseteq A(H)$ .

**3.2. Proposition.** *Let  $G$  be a linearly ordered group. Then  $c(A(G)) = A(H)$ .*

*Proof.* If  $A(G) = \{0\}$ , then it follows from 3.1 that  $A(H) = \{0\}$ . Suppose that  $A(G) \neq \{0\}$ . Hence according to 3.1 we have  $A(H) \neq \{0\}$ . Let  $0 < h \in A(H)$  and assume that  $h$  does not belong to  $c(A(G))$ . Hence  $a < h$  for each  $a \in A(G)$ . Let  $h_1 \in A(H)$ ,  $h_1 \geq 0$ . According to 3.1 there is  $Z \subseteq A(G)$  with  $\sup Z = h_1$ . Hence  $h_1 \leq h$  and therefore  $h$  is the greatest element of  $A(H)$ . But no lattice ordered group distinct from  $\{0\}$  can have a greatest element and thus we arrive at a contradiction.

Now we can ask whether  $c(A(G)) = A(H)$  is valid for each abelian lattice ordered group  $G$ . The following example shows that the answer is negative.

**3.3. Example.** Let  $N$  be the set of all positive integers and let  $A$  be the additive group of all integers with the natural linear order. For each  $n \in N$  let

$$G_n = B_n \circ C_n$$

(the symbol  $\circ$  denoting the lexicographic product, cf. [6]) with  $B_n = C_n = A$  for each  $n \in N$ . Put

$$G_0 = \prod_{n \in N} G_n.$$

The elements  $g \in G_0$  will be written as pairs  $g = (g_1, g_2)$ , where

$$g(n) = (g_1(n), g_2(n)), \quad g_1(n) \in B_n, \quad g_2(n) \in C_n.$$

Denote

$$s_2(g) = \{n \in N : g_2(n) \neq 0\}.$$

Let  $G$  be the set of all elements  $g \in G_0$  fulfilling the following conditions:

- (i) the set  $s_2(g)$  is finite;
- (ii) there exists  $n_0 \in N$  such that  $g_1(n_1) = g_1(n_2)$  for each pair  $n_1, n_2 \in N$  with  $n_0 \leq n_1 \leq n_2$ .

Then  $G$  is an  $l$ -subgroup of the lattice ordered group  $G_0$ . Clearly

$$A(G) = \{g \in G : g_1(n) = 0 \text{ for each } n \in N\}.$$

We denote by  $g^0$  the element of  $G_0$  satisfying  $g_1^0(n) = 0$  and  $g_2^0(n) = 1$  for each  $n \in N$ . Put

$$X = \{g \in G : g < g^0\}, \quad Y = \{g \in G : g > g^0\},$$

$$Z = \{y - x : x \in X, y \in Y\}.$$

We have  $\emptyset \neq Z \subset G^+$ . Assume that there is  $0 < u \in G$  such that  $u \leq z$  for each  $z \in Z$ . Then there is  $n_0 \in N$  with  $u(n_0) > 0$ .

Define  $x, y \in G$  as follows:

$$x_1(n) = 0 \text{ for each } n \in N,$$

$$x_2(n_0) = 1 \text{ and } x_2(n) = 0 \text{ for each } n \in N, n \neq n_0;$$

$$y_1(n_0) = 0 \text{ and } y_1(n) = 1 \text{ for each } n \in N, n \neq n_0;$$

$$y_2(n_0) = 1 \text{ and } y_2(n) = 0 \text{ for each } n \in N, n \neq n_0.$$

Then  $x \in X, y \in Y$  and hence  $u \leq y - x$ . Thus

$$u(n_0) \leq (y - x)(n_0) = 0,$$

which is a contradiction. Therefore

$$\inf Z = 0.$$

Hence according to 1.3, the element

$$h = \sup X = \inf Y$$

exists in  $M(G) = H$ . Since  $X \subset A(G)$ , it follows from 3.1 that  $h$  belongs to  $A(H)$ .

$A(G)$  is a complete lattice ordered group. This implies that  $c(A(G)) = A(G)$ .

A subset  $\emptyset \neq S$  of  $G$  such that  $0 < s$  for each  $s \in S$  and  $s_1 \wedge s_2 = 0$  for any two distinct elements of  $S$  is said to be disjoint. Each subset of  $A(G)$  that is upper bounded in  $A(G)$  is finite. There exists an infinite disjoint subset of  $X$ . Since  $h$  is an upper bound of  $X$ , the element  $h$  cannot belong to  $A(G)$  and thus  $h \notin c(A(G))$ . Therefore  $c(A(G)) \neq A(H)$ .

Let us recall the notion of the generalized Dedekind completion of a lattice ordered group  $G$  that was introduced in [8] (without assuming the commutativity of  $G$ ).

Let  $D_1(G)$  be a lattice ordered group fulfilling the following conditions:

- (i)  $G$  is an  $l$ -subgroup of  $D_1(G)$ .
- (ii)  $D(A(G))$  is an  $l$ -ideal in  $D_1(G)$ .
- (iii) If  $x \in G$  and  $X$  is a nonempty subset of  $x + A(G)$  such that  $X$  is upper bounded in  $x + A(G)$ , then there is  $x_0 \in D_1(G)$  with  $\sup X = x_0$ .
- (iv) For each  $x_0 \in D_1(G)$  there exist  $x \in G$  and  $X \subseteq x + A(G)$  such that  $X$  is upper bounded in  $x + A(G)$  and  $x_0 = \sup X$ .

Under these conditions  $D_1(G)$  is said to be a generalized Dedekind completion of  $G$ . The following results have been obtained in [8]:

- (a) Each lattice ordered group possesses a generalized Dedekind completion.
- (b) If  $D_1(G)$  and  $D_2(G)$  are generalized Dedekind completions of  $G$ , then there is an isomorphism  $\varphi$  of  $D_1(G)$  onto  $D_2(G)$  such that  $\varphi(g) = g$  for each  $g \in G$ .

Again, let  $G$  be abelian and let  $H$  be as above. Denote  $G_1 = \{h \in H : \text{there is } x \in G \text{ and there are } g_1, g_2 \in x + A(G) \text{ with } g_1 \leq h \leq g_2\}$ .

**3.4. Lemma.**  $G_1$  is an  $l$ -subgroup of  $H$ .

*Proof.* Let  $h_1, h_2 \in G_1$ . There are elements  $x, y, g_i \in G$  ( $i = 1, 2, 3, 4$ ) such that  $g_1, g_2 \in x + A(G)$ ,  $g_3, g_4 \in y + A(G)$ ,  $g_1 \leq h_1 \leq g_2$ ,  $g_3 \leq h_2 \leq g_4$ . Then  $-g_2 \leq \leq -h_1 \leq -g_1$  and  $-g_1, -g_2$  belong to  $-x + A(G)$ . Let  $f \in \{\wedge, \vee, +\}$ . We have

$$g_1 f g_3 \leq h_1 f h_2 \leq g_2 f g_4$$

and both  $g_1 f g_3, g_2 f g_4$  belong to  $(x f y) + A(G)$ . Hence  $G_1$  is an  $l$ -subgroup of  $H$ .

**3.5. Proposition.**  $G_1$  is a generalized Dedekind completion of  $G$ .

*Proof.* According to 3.4, the condition (i) is valid. From the construction of  $H$  it follows immediately that  $c(A(G))$  is the Dedekind completion of  $A(G)$ , hence (ii) holds.

Let  $x$  and  $X$  be as in (iii),  $X = \{x_i\}$ . Denote  $X_1 = \{x_i - x\}$ . Then  $X_1 \subseteq A(G)$ . There is  $y \in x + A(G)$  such that  $y$  is an upper bound of  $X$ . Thus  $y - x$  is an upper bound of  $X_1$  in  $A(G)$ . Hence  $\sup X_1 = x_1$  exists in  $D(A(G)) = c(A(G))$ . Put  $x_0 = x_1 + x$ . We have

$$x_0 = \sup \{x_i - x\} + x = \sup \{x_i\} = \sup X$$

in  $H$  and clearly  $x_i \leq x_0 \leq y$  for each  $x_i \in X$ ; thus  $x_0 \in G_1$ . Hence (iii) is valid.

Let  $x_0 \in G_1$ . Then there are elements  $x, g_1, g_2 \in G$  such that  $g_i \in x + A(G)$  ( $i = 1, 2$ ) and  $g_1 \leq x_0 \leq g_2$ . At the same time we have  $x_0 \in H$ , hence there is a subset  $X \subseteq G$  with  $X \neq \emptyset$  such that

$$\sup X = x_0$$

holds in  $H$ . Put  $X = \{x_i\}$ ,  $Y = \{(x_i \vee g_1) \wedge g_2\}$ . Then  $\emptyset \neq Y \subseteq x + A(G)$  and

$$\sup Y = x_0$$

is valid in  $G_1$ . Therefore the condition (iv) holds. Hence  $G_1$  is a generalized Dedekind completion of  $G$ .

In [8], the question was proposed what relations exist between the maximal Dedekind completion  $M(G)$  and the generalized Dedekind completion  $D_1(G)$  of  $G$ . From 3.4 and 3.5 it follows that  $D_1(G)$  is an  $l$ -subgroup of  $M(G)$ . The following example shows that  $D_1(G)$  does not, in general, coincide with  $M(G)$ .

**3.6. Example.** There exists a linearly ordered group  $G$  such that  $D_1(G) = G \neq M(G)$ .

Let  $N$  and  $A$  be as in 3.3. For each  $n \in N$  let  $A_n = A$  and consider the lexicographic product

$$G_0 = \Gamma_{n \in N} A_n$$

(cf. [6]). For  $g \in G_0$  put  $s(g) = \{n \in N : g(n) \neq 0\}$ . Let  $G$  be the set of all  $g \in G_0$  such that the set  $s(g_0)$  is finite. Then  $G$  is an  $l$ -subgroup of  $G_0$ , thus  $G$  is linearly ordered. Obviously  $A(G) = \{0\}$  and hence according to [8], we have  $D_1(G) = G$ . Let  $g_0 \in G_0$  be such that  $g_0(n) = 1$  for each  $n \in N$ . Put

$$X = \{g \in G : g < g_0\}, \quad Y = \{g \in G : g > g_0\}, \quad Z = \{y - x : y \in Y, x \in X\}.$$

Let  $0 < u \in G$ . There is  $n_0 \in N$  such that  $u(n_0) > 0$  and  $u(n) = 0$  for each  $n \in N$  with  $b < n_0$ . Let  $x, y \in G_0$  be defined as follows:

$$x(n) = 1 \text{ for each } n < n_0 \text{ and } x(n) = 0 \text{ otherwise;}$$

$$y(n) = 1 \text{ for each } n \leq n_0, y(n_0 + 1) = 2, \text{ and } y(n) = 0 \text{ for } n > n_0 + 1.$$

Then  $x \in X$ ,  $y \in Y$  and  $y - x < u$ . Thus  $\inf Z = 0$ . Hence by 1.3 there exists  $h \in M(G)$  such that  $\sup X = h = \inf Y$  holds in  $M(G)$ . Assume that  $h$  belongs to  $G$ . Then  $\sup X = h$  is valid in  $G$ . But it is not difficult to verify that  $\sup X$  does not exist in  $G$ , which is a contradiction. Hence  $G \neq M(G)$ .

**3.7. Lemma.** Let  $x_0 \in H$ ,  $X \subset G$ ,  $\sup X = x_0$ ,  $0 < g \in G$ . Then there is  $x \in X$  with  $x + g$  non  $\leq x_0$ .

Proof. Suppose that  $x + g \leq x_0$  for each  $x \in X$ . Put  $X = \{x_i\}$ . Then

$$x_0 < x_0 + g = \sup \{x_i\} + g = \sup \{x_i + g\} \leq x_0,$$

a contradiction.

**3.8. Proposition.** Let  $G$  be linearly ordered,  $A(G) \neq \{0\}$ . Then  $M(G) = D_1(G)$ .

Proof. Let  $0 < x_0 \in M(G)$ . There exists  $X \subset G$  such that  $\sup X = x_0$  is valid in  $M(G)$ . Choose  $0 < a \in A(G)$ . According to 3.7 there exists  $x \in X$  such that  $x + a$  non  $\leq x_0$ . Hence

$$x \leq x_0 < x + a$$

and thus  $x_0 \in G_1 = D_1(G)$ . Therefore  $(M(G))^+ \subseteq D_1(G)$  and this implies  $M(G) \subseteq D_1(G)$ . Thus  $M(G) = D_1(G)$ .

**3.9. Example.** There exists a lattice ordered group  $G'$  with  $A(G') \neq \{0\}$  such that  $D_1(G') = G' \neq M(G')$ .

Let  $G$  be as in 3.6 and let  $G_0$  be the additive group of all reals with the natural linear order. Put  $G' = G_0 \times G$ . According to [9], we have

$$D_1(G') = D_1(G_0) \times D_1(G) = G_0 \times G = G',$$

since  $A(G_0) = G_0$ ,  $D_1(G_0) = D(G_0) = G_0$  and  $A(G) = \{0\}$ . On the other hand, from 2.7 we infer

$$M(G') = M(G_0) \times M(G) = G_0 \times M(G) \neq G',$$

since  $M(G) \neq G$  (cf. 3.6).

A convex  $l$ -subgroup  $B$  of  $G$  is said to be a large  $l$ -subgroup of  $G$  if  $B \cap K \neq \{0\}$  for each convex  $l$ -subgroup  $K \neq \{0\}$  of  $G$ . In other wo. ds,  $B$  is large in  $G$  if for each  $0 < g \in G$  there exists  $0 < g_1 \in B$  with  $g_1 \leq g$ .

**3.10. Proposition.** Let  $G$  be a direct product of linearly ordered groups. Suppose that  $A(G)$  is a large  $l$ -subgroup of  $G$ . Then  $M(G) = D_1(G)$ .

Proof. Let  $G = \prod_{i \in I}^0 G_i$ , where all lattice ordered groups  $G_i$  are linearly ordered. Without loss of generality we can assume that  $G_i \neq \{0\}$  for each  $i \in I$ . Since  $A(G)$  is large in  $G$ ,  $A(G) \cap G_i \neq \{0\}$  for each  $i \in I$ . Clearly  $A(G_i) = A(G) \cap G_i$  and hence according to 3.8 we have  $M(G_i) = D_1(G_i)$  for each  $i \in I$ . Hence using 2.7 and [9], 2.17 we obtain

$$M(G) = \prod_{i \in I}^0 M(G_i) = \prod_{i \in I}^0 D_1(G_i) = D_1(G).$$



**3.10.1.** Open question: Let  $A(G)$  be a large  $l$ -subgroup of  $G$ . Is then  $M(G) = D_1(G)$ ?

**3.11. Proposition.** Let  $A(G) \neq 0$ . The following conditions for  $G$  are equivalent:

- (a) If  $X, Y$  are nonempty subsets of  $G$  such that (i)  $x < y$  for each  $x \in X$  and each  $y \in Y$ , and (ii)  $\inf \{y - x : x \in X, y \in Y\} = 0$ , then there are elements  $0 < a \in A(G)$  and  $g \in G$  such that for each  $x \in X$  and  $y \in Y$  we have  $x \vee g \leq y, x \leq y \wedge (g + a)$ .
- (b)  $M(G) = D_1(G)$ .

*Proof.* Suppose that (a) holds and  $x_0 \in M(G)$ . According to 1.3 there are subsets  $X, Y$  of  $G$  such that (i) and (ii) are valid and  $\sup X = x_0 = \inf Y$ . From (a) we obtain  $g \leq x_0 < g + a$  and hence  $x_0 \in G_1 = D_1(G)$ . Thus  $M(G) \subseteq D_1(G)$  and therefore  $M(G) = D_1(G)$ .

Conversely, suppose that  $M(G) = D_1(G)$  and let  $X, Y$  be subsets of  $G$  fulfilling (i) and (ii). According to 1.3 there is  $x_0 \in M(G)$  with  $\sup X = x_0 = \inf Y$ . By the assumptions we have  $x_0 \in D_1(G)$  and hence there are elements  $g \in G$  and  $0 < a \in A(G)$  such that  $g \leq x_0 < g + a$ . Hence for each  $x \in X$  and each  $y \in Y$  we have  $g \vee x \leq y, x \leq y \wedge (g + a)$ . Thus (a) is valid.

A lattice ordered group  $G$  is said to be generalized complete [9] if  $D_1(G) = G$ .

**3.12. Proposition.** If  $G$  is  $M$ -complete, then it is generalized complete.

*Proof.* Suppose that  $G$  is  $M$ -complete. From 1.4 it follows that each convex  $l$ -subgroup of  $G$  is  $M$ -complete. Hence  $A(G)$  is  $M$ -complete and so, being archimedean, it is complete. Therefore, according to [8],  $G$  is generalized complete.

In [10] it was shown that in each lattice ordered group  $G$  (that need not be commutative), the greatest convex generalized complete  $l$ -subgroup  $d_1(G)$  exists. From 3.12 we obtain immediately:

**3.12.1. Corollary.** Let  $G$  be an abelian lattice ordered group. Then  $m(G) \subseteq d_1(G)$ .

**3.12.2. Example.** There exists an abelian lattice ordered group  $G$  such that  $m(G) \subset d_1(G)$ . Let  $G$  be as in Example 3.6. Then  $D_1(G) = G$ , hence  $G$  is generalized complete and so  $d_1(G) = G$ . It is not hard to verify that  $m(G) = \{0\}$ .

#### 4. SOME FURTHER PROPERTIES OF $M(G)$

Let  $G$  be an abelian lattice ordered group. In this section some relations between  $G$  and  $H = M(G)$  will be investigated.

**4.1. Proposition.**  $G$  is a dense  $l$ -subgroup of  $H$ .

Proof. For each  $0 < x_0 \in H$  there exists  $\emptyset \neq X \subseteq G$  such that  $\sup X = x_0$  holds in  $H$ , hence for each  $0 \neq x \in X$  we have  $0 < |x| \leq x_0, |x| \in G$ .

It is easy to verify that if  $X = \{x_i\} \subset H, x_0 \in H, \sup \{x_i\} = x_0$ , then for each positive integer  $n$  we have  $\sup \{nx_i\} = nx_0$ , and dually.

**4.2. Proposition.** *If  $G$  is divisible, then  $H$  is divisible.*

Proof. Suppose that  $G$  is divisible. It suffices to verify that for each positive integer  $n$  and for each  $0 < x_0 \in H$  there exists  $y_0 \in H$  with  $ny_0 = x_0$ .

Let  $n$  be positive integer and let  $0 < x_0 \in H$ . There exist subsets  $X, Y$  of  $G^+$  such that  $\sup X = x_0 = \inf Y$  holds in  $H$  and  $\inf Z = 0$ , where  $Z = \{y - x : y \in Y, x \in X\}$ . Let  $X = \{x_i\}, Y = \{y_j\}$ . Denote  $X_1 = \{(1/n)x_i\}, Y_1 = \{(1/n)y_j\}$ . Then  $x_1 < y_1$  for each  $x_1 \in X_1$  and each  $y_1 \in Y_1$ . Put  $Z_1 = \{y_1 - x_1 : x_1 \in X_1, y_1 \in Y_1\}$ . For each  $z_1 \in Z_1$  we have  $0 < z_1$  and there is  $z \in Z$  with  $z_1 = (1/n)z$ ; from this and from  $\inf Z = 0$  we infer that  $\inf Z_1 = 0$ . Thus according to 1.3 there exists  $y_0 \in H$  such that

$$\sup X_1 = y_0 = \inf Y_1.$$

**4.3. Proposition.** *Let  $G$  be a vector lattice. Then  $H$  is a vector lattice.*

Proof. Let  $0 < x_0 \in H$  and let  $r$  be a positive real. Let  $X, Y$  and  $Z$  be as in 4.2. Put  $X_1 = \{rx_i\}, Y_1 = \{ry_j\}, Z_1 = \{y_1 - x_1 : x_1 \in X_1, y_1 \in Y_1\}$ . We have  $y_1 > x_1$  for each  $x_1 \in X_1$  and each  $y_1 \in Y_1$ . Suppose that  $\inf Z_1 = 0$  fails to hold in  $G$ . Then there is  $0 < u \in G$  such that  $u \leq z_1$  for each  $z_1 \in Z_1$ . Hence  $0 < (1/r)u \leq (1/r)z_1$  for each  $z_1 \in Z_1$  and thus  $(1/r)u \leq z$  for each  $z \in Z$ , which is a contradiction. Thus  $\inf Z_1 = 0$ . Hence there is  $y_0 \in H$  with

$$\sup X_1 = y_0 = \inf Y_1.$$

Now we define  $rx_0$  by putting  $rx_0 = y_0$ . If  $t_0$  is any element of  $H$ , then we put  $rt_0 = rt_0^- - rt_0^+$  and  $(-r)t_0 = -(rt_0)$ . It is a routine to verify that under this definition of multiplication of elements of  $H$  by reals,  $H$  turns out to be a vector lattice such that  $G$  is a vector sublattice of  $H$ .

Let  $K$  be a convex  $l$ -subgroup of  $G$  with the property that  $g > k$  for each  $k \in K$  provided  $0 < g \in G \setminus K$ . Under this assumption  $G$  is called a lexicographic extension of  $K$  and we write  $G = \langle K \rangle$ . If, moreover,  $G \neq K$ , then  $G$  is said to be a proper lexicographic extension of  $K$ . From  $G = \langle K \rangle$  it follows that the factor  $l$ -group  $G/K$  is linearly ordered and that, for  $g_1, g_2 \in G$  with  $g_1 + K \neq g_2 + K$ , the relation  $g_1 + K < g_2 + K$  is valid in  $G/K$  if and only if  $g_1 < g_2$  holds.

**4.4. Proposition.** *Let  $K$  be a convex  $l$ -subgroup of  $G, \{0\} \neq K \neq G$ , such that  $G = \langle K \rangle$ . Then  $H = \langle c(K) \rangle$  and  $c(K) \neq H$ .*

Proof. Since  $G \neq K$ , there exists  $g \in G^+ \setminus K$ . Then  $g > k$  for each  $k \in K$ , thus  $g$  does not belong to  $c(K)$  and hence  $c(K) \neq H$ . Let  $x_0 \in H \setminus c(K)$ . Put

$$X = \{x \in G : x \leq x_0\}, \quad Y = \{y \in G : y \geq x_0\}.$$

If  $X \cap K \neq \emptyset \neq Y \cap K$ , then  $x_0 \in c(K)$ , a contradiction. Assume that  $Y \cap K = \emptyset$ . Then  $y > k$  for each  $y \in Y$  and each  $k \in K$ . Hence  $K \subseteq X$ . Now we distinguish two cases.

(a) Suppose that there is  $x \in X$  such that  $x > k$  for each  $k \in K$ . Since  $x_0 = \sup X$ , we have  $x_0 > k$  for each  $k \in K$  and thus  $x_0 > x_1$  for each  $x_1 \in c(K)$ .

(b) Suppose that no  $x \in X$  exceeds the whole set  $K$ . Then for each  $x \in X$ , either  $x \in K$  or  $x < k$  for each  $k \in K$ . From this and from  $K \subseteq X$  we obtain

$$x_0 = \sup K.$$

There exists  $0 < k \in K$ . Put  $K = \{k_i\}$ . Then in  $H$  we have  $k + K = K$  and

$$x_0 < k + x_0 = k + \bigvee k_i = \bigvee (k + k_i) = x_0$$

which is a contradiction.

The case  $X \cap K \neq \emptyset$  is analogous. We have verified that  $H = \langle c(K) \rangle$  is valid.

**4.5. Proposition.** *Let  $K$  be a convex  $l$ -subgroup of  $G$ ,  $\{0\} \neq K \neq G$ ,  $G = \langle K \rangle$ . Then the linearly ordered groups  $G/K$  and  $H/c(K)$  are isomorphic.*

*Proof.* According to 4.4 we have  $H = \langle c(K) \rangle$ , hence  $H/c(K)$  is a linearly ordered group. For  $g \in G$  we denote

$$\varphi(g + K) = g + c(K).$$

Let  $g_1, g_2 \in G$ . If  $g_1 + K = g_2 + K$ , then  $g_1 + c(K) = g_2 + c(K)$ , hence  $\varphi$  is a mapping of the set  $G/K$  into  $H/c(K)$ . Suppose that  $g_1 + c(K) = g_2 + c(K)$ ; hence  $g_1 - g_2 \in c(K)$  and thus  $g_1 - g_2 \in K$ . Therefore  $\varphi$  is a monomorphism. Obviously  $\varphi$  is regular with respect to the operation  $+$ . If  $g_1 + K < g_2 + K$  in  $G/K$ , then  $g_1 < g_2$  and hence  $g_1 + c(K) < g_2 + c(K)$ ; conversely, if  $g_1 + c(K) < g_2 + c(K)$ , then  $g_1 + K < g_2 + K$ . Hence  $\varphi$  is an isomorphism of the linearly ordered group  $G/K$  into  $H/c(K)$ .

Let  $x_0, X$  and  $Y$  be as in the proof of 4.4. Put

$$\bar{X} = \{g + K : g \in G, (g + K) \cap X \neq \emptyset\}$$

and let  $\bar{Y}$  be defined analogously. There exists  $0 < k \in K$ . If  $X$  is a join of some classes  $g + K$ , then  $x + k \in X$  for each  $x \in X$  and thus  $x + k \leq x_0$  for each  $x \in X$ , which contradicts 3.7. Hence there is  $g \in G$  such that

$$(g + K) \cap X \neq \emptyset, \quad g + K \not\subseteq X.$$

Thus there is  $g_1 \in g + K$  such that  $g_1 \in X$ . If  $g' \in G$ ,  $g' + K < g + K$ , then  $g' + K \subset X$ . If  $g'' \in G$ ,  $g'' + K > g + K$ , then  $g''$  cannot belong to  $X$ , since  $g'' \in X$

would imply  $g + K \subset X$ , which is a contradiction. Hence

$$X = [(g + K) \cap X] \cup X',$$

where  $X'$  is the join of all  $g' + K$  with  $g' + K < g + K$ . Thus  $g'' + K \subset Y$  for each  $g'' + K > g + K$ . Similarly as we did for  $X$  we can now verify that  $Y$  cannot be a join of some classes  $g'' + K$  with  $g'' \in G$ . From this we infer that  $(g + K) \cap Y \neq \emptyset$ . Thus there is  $g_2 \in g + K$  with  $g_2 \in Y$ . Then we have

$$g_1 - g \leq x_0 - g \leq g_2 - g$$

and  $g_1 - g, g_2 - g \in K$ , thus  $x_0 - g \in c(K)$  and so  $x_0 \in g + c(K)$ ,  $x_0 + c(K) = g + c(K)$ . Hence  $\varphi$  is an epimorphism. This completes the proof.

**4.6. Lemma.** *Let  $\{P_i\}$  ( $i \in I$ ) be an upper-directed system of convex  $l$ -subgroups of  $G$ . Then  $\bigcup_{i \in I} c(P_i) = c(\bigcup_{i \in I} P_i)$ .*

The proof is a routine and so it will be omitted. From 4.6 and from 1.11 (ii) we obtain

**4.6.1. Corollary.** *Let  $\{P_i\}$  ( $i \in I$ ) be an upper-directed system of convex  $l$ -subgroups of  $G$ . Then  $\bigcup_{i \in I} M(P_i) = M(\bigcup_{i \in I} P_i)$ .*

A subset  $B = \{g_i\}_{i \in I}$  of  $G$  is said to be a basis for  $G$  if  $B$  is a maximal disjoint subset of  $G$  and the interval  $[0, g_i]$  of  $G$  is a chain for each  $i \in I$ .

**4.7. Proposition.** *Let  $B = \{g_i\}_{i \in I}$  be a basis for  $G$ . Then  $B$  is a basis for  $H$ .*

*Proof.* Let  $i \in I$  and let  $A_i$  be the interval in  $H$  with the endpoints  $0, g_i$ . Suppose that  $A_i$  fails to be a chain. Then there are elements  $0 < x_0$  and  $0 < y_0$  in  $A_i$  such that  $x_0 \wedge y_0 = 0$ . Hence there are elements  $x_1, y_1$  in  $G$  such that  $0 < x_1 \leq x_0$ ,  $0 < y_1 \leq y_0$ . Then  $x_1 \wedge y_1 = 0$  and both  $x_1, y_1$  belong to the interval  $A'_i$  in  $G$  with the endpoints  $0$  and  $g_i$ ; since  $A'_i$  is a chain, we have a contradiction. Hence  $A_i$  is a chain for each  $i \in I$ . Let  $0 < z_0 \in H$ . There is  $z_1 \in G$  with  $0 < z_1 \leq z_0$ . Further (since  $B$  is maximal disjoint in  $G$ ), there exists  $i \in I$  with  $0 < g_i \wedge z_1$ . Hence  $0 < g_i \wedge z_0$ . Thus  $B$  is a basis for  $H$ .

Let us consider the following condition for  $G$  (cf. [3]):

(F) Each disjoint subset of  $G$  that is upper bounded in  $G$  is finite.

**4.8. Proposition.** *Suppose that  $G$  fulfils (F). Then  $H$  fulfils (F).*

*Proof.* Let  $0 < x_0 \in H$ . Assume that there exists an infinite disjoint system  $\{x_i\}$  ( $i \in I$ ) in  $H$  such that  $x_0$  exceeds all  $x_i$ . There is  $y \in G$  with  $x_0 \leq y$ . For each  $i \in I$  there is  $g_i \in G$  with  $0 < g_i \leq x_i$ . Hence  $\{g_i\}$  ( $i \in I$ ) is an infinite disjoint system in  $G$  and  $y$  exceeds each  $g_i$ ; this is a contradiction.

**4.9. Definition.** (Cf. [3].) A lattice ordered group  $G$  is said to be a *lexicographic sum* of the system  $\Gamma^0 = \{A_i^0\} (i \in I_0)$  of its convex  $l$ -subgroups if there exists an ordinal  $\beta$  and convex  $l$ -subgroups  $A^\alpha$  of  $G$  for each  $\alpha$  with  $0 \leq \alpha < \beta$  such that the following conditions are fulfilled:

- (i)  $A^0 = \sum^0 A_i^0 (i \in I_0)$ ,  $A^{\alpha_1} \subseteq A^{\alpha_2}$  whenever  $0 \leq \alpha_1 < \alpha_2$  and  $\alpha_1 < \alpha_2$ ;
- (ii)  $\bigcup_{\alpha < \beta} A^\alpha = G$ ;
- (iii) for each ordinal  $\alpha$  with  $0 < \alpha < \beta$  there exists a system  $\Gamma^\alpha = \{A_i^\alpha\} (i \in I_\alpha)$  of convex  $l$ -subgroups of  $G$  with  $A^\alpha = \sum^0 A_i^\alpha (i \in I)$  such that

(a) if  $\alpha$  is non-limit,  $\alpha = \gamma + 1$ , and if  $A_i^\alpha \in \Gamma^\alpha$ , then either  $A_i^\alpha$  equals to some  $l$ -subgroup belonging to  $\Gamma^\gamma$ , or there exists a subset  $I$  of  $I_\gamma$  with  $\text{card } I > 0$  and a convex  $l$ -subgroup  $A$  of  $G$  such that

$$A = \sum^0 A_i^\gamma (i \in I)$$

and  $A_i^\alpha$  is a proper lexicographic extension of  $A$ ;

(b) if  $\alpha$  is a limit ordinal and  $A_i^\alpha \in \Gamma^\alpha$ , then there exists a system  $\{A_{i(\gamma)}^\alpha\} (\gamma < \alpha)$  such that  $A_{i(\gamma)}^\alpha$  belongs to  $\Gamma^\gamma$  for each  $\gamma < \alpha$ ,  $A_{i(\gamma_1)}^{\gamma_1} \subseteq A_{i(\gamma_2)}^{\gamma_2}$  whenever  $\gamma_1 < \gamma_2$  and  $A_i^\alpha = \bigcup_{\gamma < \alpha} A_{i(\gamma)}^\gamma$ .

Now let us again suppose that  $G$  fulfils (F). Let  $G \neq \{0\}$ . It is easy to verify that then  $G$  possesses a basis  $\{b_i\} (i \in I_0)$  and for each  $i \in I_0$  there exists a largest convex linearly ordered subgroup  $A_i^0$  of  $G$  containing  $b_i$ . The following theorem has been proved by Conrad [3]:

**4.10. Theorem.** *Under the above notation,  $G$  is a lexicographic sum of the system  $\{A_i^0\} (i \in I_0)$ .*

From 4.8, 4.10 and 4.7 we obtain:

**4.11. Theorem.** *Let  $G \neq \{0\}$ . Suppose that  $G$  fulfils (F) and let  $\{b_i\} (i \in I_0)$  be a basis for  $G$ . For each  $i \in I_0$  let  $B_i^0$  be the largest convex linearly ordered subgroup of  $H$  containing  $b_i$ . Then  $H$  is a lexicographic sum of the system  $B_i^0 (i \in I_0)$ .*

A more detailed description of the representation of  $H$  as a lexicographic sum of linearly ordered groups is contained in the following theorem. Let  $G$  be as in 4.10. Hence there is an ordinal  $\beta$  and there are systems  $A^\alpha (\alpha < \beta)$  of convex  $l$ -subgroups of  $G$  such that the conditions (i)–(iii) from 4.9 are fulfilled. Then the following assertion is valid:

**4.12. Theorem.** *For each ordinal  $\alpha$  with  $0 < \alpha < \beta$ ,  $c(A^\alpha)$  is a convex  $l$ -subgroup of  $H$  such that the following conditions are fulfilled:*

- (i<sub>1</sub>)  $c(A^0) = \sum^0 c(A_i^0) (i \in I)$  and  $c(A^{\alpha_1}) \subseteq c(A^{\alpha_2})$  whenever  $0 \leq \alpha_1 < \alpha_2$  and  $\alpha_1 < \alpha_2$ ;
- (ii<sub>1</sub>)  $\bigcup_{\alpha < \beta} c(A^\alpha) = H$ ;
- (iii<sub>1</sub>) for each ordinal  $\alpha$  with  $0 \leq \alpha < \beta$  we have  $c(A^\alpha) = \sum^0 c(A_i^\alpha) (i \in I)$  and

(a) if  $\alpha$  is non-limit,  $\alpha = \gamma + 1$ , and if  $A_i^\alpha \in \Gamma^\alpha$ , then either  $c(A_i^\alpha)$  equals to some  $c(A_j^\gamma)$  with  $j \in I_\gamma$ , or there is a subset  $I$  of  $I_\gamma$  with  $\text{card } I > 0$  and a convex  $l$ -subgroup  $A_1$  of  $H$  such that

$$A_1 = \sum^0 c(A_i^\gamma) \quad (i \in I)$$

and  $c(A_i^\alpha)$  is a proper lexicographic extension of  $A_1$ ;

(b) if  $\alpha$  is a limit ordinal and  $i \in I_\alpha$ , then there exists a system  $\{c(A_{i(\gamma)}^\gamma)\}$  ( $\gamma < \alpha$ ) with  $i(\gamma) \in I_\gamma$  for each  $\gamma < \alpha$ ,  $c(A_{i(\gamma_1)}^{\gamma_1}) \subseteq c(A_{i(\gamma_2)}^{\gamma_2})$  whenever  $\gamma_1 < \gamma_2$  and  $c(A_i^\alpha) = \bigcup_{\gamma < \alpha} c(A_{i(\gamma)}^\gamma)$ .

In particular,  $H$  is a lexicographic sum of the system  $c(A_i^0)$  ( $i \in I$ ) and all  $c(A_i^0)$  are linearly ordered. Moreover, if  $A$  and  $A_1$  are as in (iii) of 4.9 or in (iii<sub>1</sub>), respectively, then  $c(A_i^\alpha)/A_1$  is isomorphic with  $A_i^\alpha/A$ .

Proof. Obviously all  $c(A^\alpha)$  and all  $c(A_i^\alpha)$  are convex  $l$ -subgroups of  $H$ . According to 1.11 we have  $c(A^\alpha) = M(A^\alpha)$ ,  $c(A_i^\alpha) = M(A_i^\alpha)$ . Now (i<sub>1</sub>) follows from (i) and 2.8. The assertion (ii<sub>1</sub>) is a consequence of (ii) and 4.6 (because  $H = c(G)$ ). From (iii), 2.8, 4.4 and 4.6 we obtain that (iii<sub>1</sub>) is valid. Hence according to 4.9,  $H$  is a lexicographic sum of the system  $c(A_i^0)$  ( $i \in I_0$ ). If  $A$  and  $A_1$  are as in 4.9 (iii) or in (iii<sub>1</sub>), respectively, then by 4.5 the linearly ordered groups  $c(A_i^\alpha)/A_1$  and  $A_i^\alpha/A$  are isomorphic. Since a maximal Dedekind completion of a linearly ordered group is linearly ordered, all  $c(A_i^0) = M(A_i^0)$  are linearly ordered.

Hence  $H$  is constructed from the system  $\{c(A_i^0)\}$  ( $i \in I_0$ ) by the same steps (using the operations of the direct sum and the lexicographic extension) as  $G$  is constructed from the system  $\{A_i^0\}$  ( $i \in I_0$ ).

#### References

- [1] G. Birkhoff: Lattice theory, third edition, Providence 1967.
- [2] P. Conrad: Lattice ordered groups, Tulane University 1970.
- [3] P. Conrad: Some structure theorems for lattice ordered groups. Trans. Amer. Math. Soc. 99 (1961), 212—240.
- [4] Š. Černák: Completely subdirect products of lattice ordered groups, Acta fac. rer. nat. Univ. Comen., Mathem., 1971, 121—128.
- [5] C. J. Everett: Sequence completion of lattice moduls. Duke Math. J. 11 (1944), 109—119.
- [6] Л. Фукс: Частично упорядоченные алгебраические системы, Москва 1965.
- [7] J. Jakubík: Radical classes and radical mappings of lattice ordered groups. Symposia mathem. 31 (1977), 451—477.
- [8] J. Jakubík: Archimedean kernel of a lattice ordered group. Czech. Math. J. 28 (1978), 140—154.
- [9] J. Jakubík: Generalized Dedekind completion of a lattice ordered group. Czech. Math. J. 28 (1978), 294—311.
- [10] F. Šik: Über subdirekte Summen geordneter Gruppen. Czech. Math. J. 10 (1960), 400—424.

Author's address: 040 01 Košice, Švermova 5, ČSSR (Katedra matematiky stroj. fakulty VŠT).