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MAXIMAL ELEMENTS FOR G_B-MAJORIZED MAPPINGS IN PRODUCT G-CONVEX SPACES AND APPLICATIONS (Ⅱ) *

DING Xie-ping (丁协平)
(Department of Mathematics, Sichuan Normal University,
Chengdu 610066, P.R.China)
(Contributed by DING Xie-ping)

Abstract: *By applying existence theorems of maximal elements for a family of G_B-majorized mappings in a product space of G-convex spaces, some coincidence theorem, Fan-Browder type fixed point theorem and some existence theorems of solutions for a system of minimax inequalities are proved under noncompact setting of G-convex spaces. These theorems improve and generalize many important known results in literature.*

Key words: maximal element; family of G_B-majorized mappings; coincidence theorem; minimax inequalities; product space of G-convex space

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Introduction

This paper is a continuum of the preceding paper of author[1]. For the concepts and notations of generalized convex (or G-convex) space, CG-convex space, the class $B(Y,X)$ of better admissible mappings, G_B-mappings, G_B-majorized mappings and the related notions, the reader may consult Ref.[1].

The main aim of this paper is to give some applications of the existence theorems of maximal elements obtained by author[1]. By applying our results in Ref.[1], we give some component version of coincidence theorem and Fan-Browder type fixed point theorems. Some existence theorems of solutions for a system of Ky Fan type minimax inequalities involving a family of G_B-majorized mappings defined on the product space of G-convex spaces are also given. These results improve and generalize many known results in literature.

1 Preliminaries

In order to prove our main results, we need the following existence theorems of maximal

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Biography: DING Xie-ping (1938 ~), Professor (E-mail: dingxip@sichu.edu.cn)

elements for a family of G_B-majorized mappings and G_B-mappings in Ref. [1]. For convenience, we state these theorems as follows.

Theorem 1 Let X be a topological space and I be any index set. For each $i \in I$, let (Y_i, Γ_i) be a CG-convex space and let $Y = \prod_{i \in I} Y_i$ such that (Y, Γ) is a CG-convex space defined as in Ref. [1]. Let $F \in B(Y, X)$ be a compact mapping such that for each $i \in I$,

(i) $A_i : X \to 2^{Y_i}$ is a G_B-majorized mapping;

(ii) $\bigcup_{i \in I}\{x \in X : A_i(x) \neq \phi\} = \bigcup_{i \in I}\text{int}\{x \in X : A_i(x) \neq \phi\}$.

Then there exists $\hat{x} \in X$ such that $A_i(\hat{x}) = \phi$ for exch $i \in I$.

Proof For each $x \in X$, let $I(x) = \{i \in I : A_i(x) \neq \phi\}$. Define $A : X \to 2^Y$ by

$$A(x) = \begin{cases} \bigcap_{i \in I(x)} \pi_i^{-1}(A_i(x)) & (\text{if } I(x) \neq \phi), \\ \phi & (\text{if } I(x) = \phi). \end{cases}$$

Then for each $x \in X$, $A(x) \neq \phi$ if and only if $I(x) \neq \phi$. By the proof of Theorem 2.5 in Ref. [1], we can show that $A : X \to 2^Y$ is a G_B-majorized mapping. By applying Corollary 2.4 in Ref. [1], there exists $\hat{x} \in X$ such that $A(\hat{x}) = \phi$ and so $I(\hat{x}) = \phi$. Hence we have $A_i(\hat{x}) = \phi$ for all $i \in I$.

Remark 1 Theorem 1 improves and generalizes Theorem 3 of Deguire *et al*.[2] in following ways: 1) From convex subsets of topological vector spaces to CG-convex spaces without linear structure; 2) From a family of L_S-majorized mappings to a family of G_B-majorized mappings.

The following results are Theorem 2.6 and Theorem 2.7 in Ref. [1].

Theorem 2 Let X be a topological space, K be a nonempty compact subset of X and I be any index set. For each $i \in I$, let (Y_i, Γ_i) be a G-convex space and $Y = \prod_{i \in I} Y_i$ be a G-convex space defined as in Ref. [1]. Let $F \in B(Y, X)$ and for each $i \in I$, $A_i : X \to 2^{Y_i}$ be a G_B-mapping such that

(i) For each $i \in I$ and $N_i \in \mathscr{F}(Y_i)$, there exists a nonempty compact G-convex subset L_{N_i} of Y_i containing N_i and for each $x \in X \setminus K$, there exists $i \in I$ satisfying $L_{N_i} \cap A_i(x) \neq \phi$.

Then there exists $\hat{x} \in K$ such that $A_i(\hat{x}) = \phi$, for each $i \in I$.

Theorem 3 Let X be a paracompact topological space and I be any index set. For each $i \in I$, let (Y_i, Γ_i) be a G-convex space and $Y = \prod_{i \in I} Y_i$ be a G-convex space defined as in Ref. [1]. Let $F \in B(Y, X)$ be such that for each $i \in I$, $A_i : X \to 2^{Y_i}$ is a G_B-majorized mapping. Suppose that there exists a nonempty compact subset K of X and for each $i \in I$ and $N_i \in \mathscr{F}(Y_i)$, there is a compact G-convex subset L_{N_i} of Y_i containing N_i such that for each $x \in X \setminus K$, there exists $i \in I$ satisfying $L_{N_i} \cap A_i(x) \neq \phi$. Then there exists an $\hat{x} \in X$ such that $A_i(\hat{x}) = \phi$ for all $i \in I$.

2 Fixed Points and Coincidence Points in Product Spaces

As application of Theorems 1, 2 and 3, we have the following component version of coincidence theorem and Fan-Browder type fixed point theorem in the product space of G-convex spaces.

Theorem 4 Let X be a topological space, K be a nonempty compact subset of X and I be any index set. For each $i \in I$, let (Y_i, Γ_i) be a G-convex space and $Y = \prod_{i \in I} Y_i$ be a G-convex space defined as in Ref.[1]. Let $F \in B(Y, X)$ and for each $i \in I$, $A_i : X \to 2^{Y_i}$ be G-convex valued. Suppose

(i) For each $i \in I$ and $y_i \in Y_i$, $A_i^{-1}(y_i)$ is compactly open in X,

(ii) For each $i \in I$ and $N_i \in \mathscr{F}(Y_i)$, there exists a nonempty compact G-convex subset L_{N_i} of Y_i containing N_i and for each $x \in X \setminus K$, there exists $i \in I$ satisfying $L_{N_i} \cap A_i(x) \neq \phi$,

(iii) For each $x \in K$, there exists $i \in I$ such that $A_i(x) \neq \phi$.

Then there exist $i_0 \in I$ and $(\hat{x}, \hat{y}) \in (X, Y)$ such that $\hat{x} \in F(\hat{y})$ and $\hat{y}_{i_0} = \pi_{i_0}(\hat{y}) \in A_{i_0}(\hat{x})$. Moreover if $F = S$ is a single-valued continuous mapping, then we have $\hat{y}_{i_0} = \pi_{i_0}(\hat{y}) \in A_{i_0}(S(\hat{y}))$.

Proof By condition (iii), the conclusion of Theorem 2 does not holds. By Theorem 2, there exist $i_0 \in I$ such that $A_{i_0}: X \to 2^{Y_i}$ does not satisfy the condition (a) in the definition of a G_B-mapping, i.e., there exists $N \in \mathscr{F}(Y)$ such that $F(\Gamma(N)) \cap (\bigcap_{y \in N}(A_{i_0}^{-1}(\pi_{i_0}(y))) \neq \phi$. It follows that there exists $(\hat{x}, \hat{y}) \in X \times Y$ such that $\hat{y} \in \Gamma(N)$, $\hat{x} \in F(\hat{y})$, and $\hat{x} \in \bigcap_{y \in N} A_{i_0}^{-1}(\pi_{i_0}(y))$. Hence we have $\pi_{i_0} \subset A_{i_0}(\hat{x})$. Since $A_{i_0}(\hat{x})$ is G-convex, we have

$$\hat{y}_{i_0} = \pi_{i_0}(\hat{y}) \in \pi_{i_0}(\Gamma(N)) = \Gamma_{i_0}(\pi_{i_0}(N)) \subset A_{i_0}(\hat{x}).$$

Furthermore, if $F = S$ is a single-valued continuous, then $S \in B(Y, X)$ and so we have $\hat{y}_{i_0} \in A_{i_0}(S(\hat{y}))$.

Remark 2 1) Theorem 4 improves and generalizes Theorem 6 and Theorem 9 of Deguire *et al*.[2] in following ways: (a) From the product space of nonempty convex subsets of topological vector spaces to the product space of G-convex spaces; (b) From S being a single-valued continuous mapping to $F \in B(Y, X)$.

2) Theorem 4, in turn, generalizes Theorem 5.1 of Deguire and Lassonde[3], Theorem 3.1 of Deguire[4], and the corresponding results of Ben-El-Mechaiekh *et al*.[5,6], Lassonde[7,8], Tarafdar[9,10], Ding and Tarafdar[11,12] and Ding[13,14] in several aspects.

3) We wish to point out that the conclusion of Theorem 4 could not guarantee in general that $\hat{y}_i \in A_i(\hat{x})$ for each $i \in I$ as we can only assure that there exist $i_0 \in I$ and $(\hat{x}, \hat{y}) \in X \times Y$ such that $\hat{x} \in F(\hat{y})$ and $\hat{y}_{i_0} \in A_{i_0}(\hat{x})$.

Theorem 5 Let I be any index set. For each $i \in I$, let (X_i, Γ_i') and (Y_i, Γ_i) be G-convex spaces, and $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$ be the product G-convex spaces defined as in Ref.[1]. For each $i \in I$, let $A_i: X \to 2^{Y_i}$ and $B_i: Y \to 2^{X_i}$ be set-valued mappings with G-convex values. Suppose there exist a nonempty compact subset K of X and a nonempty compact subset L of Y such that

(i) For each $i \in I$ and $(x_i, y_i) \in X_i \times Y_i$, $A_i^{-1}(y_i)$ and $B_i^{-1}(x_i)$ are both compactly open in X and Y respectively;

(ii) For each $i \in I$, $M_i \in \mathscr{F}(X_i)$ and $N_i \in \mathscr{F}(Y_i)$, there exist nonempty compactly G-convex subsets L_{M_i} of X_i containing M_i and nonempty compactly G-convex subset L_{N_i} of Y_i

containing N_i and for each $(x,y) \in (X \times Y) \setminus (K \times L)$, there exists $i \in I$ such that $A_i(x) \cap L_{N_i} \neq \phi$ and $B_i(y) \cap L_{M_i} \neq \phi$;

(iii) For each $(x,y) \in K \times L$, there exists $i \in I$ such that $A_i(x) \neq \phi$ and $B_i(y) \neq \phi$. Then there exist $i_0 \in I$ and $(\hat{x},\hat{y}) \in X \times Y$ such that $\hat{y}_{i_0} \in A_{i_0}(\hat{x})$ and $\hat{x}_{i_0} \in B_{i_0}(\hat{y})$.

Proof Let $C = K \times L$, then C is a nonempty compact subset of $X \times Y$. Clearly, for each $i \in I$, $Y_i \times X_i$ and $Y \times X = \prod_{i \in I}(Y_i \times X_i)$ are also G-convex spaces. By (ii), for each $i \in I$ and $N_i \times M_i \in \mathscr{F}(Y_i \times M_i)$, there exists a nonempty compactly G-convex subset $L_{N_i} \times L_{M_i}$ of $Y \times X$ containing $N_i \times M_i$. Define $F: Y \times X \to 2^{(X \times Y)}$ by $F(y,x) = \{(x,y)\}$, then we have $F \in B(Y \times X, X \times Y)$. Define $W_i: X \times Y \to 2^{(Y_i \times X_i)}$ by

$$W_i(x,y) = A_i(x) \times B_i(y) \qquad (\forall (x,y) \in X \times Y).$$

Then it is easy to check that all conditions of Theorem 4 are satisfied. By Theorem 4, there exist $i_0 \in I$ and $(\hat{y},\hat{x}) \in Y \times X$ such that

$$(\hat{y}_{i_0},\hat{x}_{i_0}) \in W_{i_0}(F(\hat{y},\hat{x})) = W_{i_0}(\hat{x},\hat{y}) = A_{i_0}(\hat{x}) \times B_{i_0}(\hat{y})$$

and hence we obtain $\hat{y}_{i_0} \in A_{i_0}(\hat{x})$ and $\hat{x}_{i_0} \in B_{i_0}(\hat{y})$.

Remark 3 Theorem 5 generalizes Theorem 10 of Deguire *et al*.[2] and Theorem 4.3 of Deguire and Lassonde[3] from convex subsets of topological vector spaces to G-convex spaces.

3 System of Minimax Inequalities

In this section, by applying our existence theorems of maximal elements, some existence theorems of solutions for a system of Ky Fan type minimax inequalities (see Ky Fan[15,16]) will be proved under much weaker assumptions. In particular, our results improve and generalize well-known Ky Fan maximal inequality in Refs. [2,15,16] to the product space of noncompact G-convex spaces without linear structure. Before we study the system of Ky Fan type minimax inequalities for a family of real valued functions, we need the following definitions.

Let X be a topological space and I be any index set. For each $i \in I$, let (Y_i, Γ_i) be a G-convex space and $Y = \prod_{i \in I} Y_i$ be the G-convex space defined as in Ref. [1]. Let $F \in B(Y,X)$ be the family of admissible mappings and for each $i \in I$, $f_i: X \times Y_i \to \mathbf{R}$ be a real valued function.

1) For each given $x \in X$, the function $y_i \mapsto f_i(x,y_i)$ is said to be G-quasiconcave if for any $\lambda \in \mathbf{R}$, the set $\{y_i \in Y_i : f_i(x,y_i) > \lambda\}$ is G-convex,

2) $f_i(x,y_i)$ is said to be G_B-majorized if the following conditions are satisfied: for each given $\lambda \in \mathbf{R}$, if there exists $(x,y_i) \in X \times Y_i$ such that $f_i(x,y_i) > \lambda$, then there exist a nonempty open neighborhood $N(x)$ of x in X and a real valued function $f_{i,x}: X \times Y_i \to \mathbf{R}$ such that

(a_1) $f_i(z,y_i) \leqslant f_{i,x}(z,y_i)$ for all $(z,y_i) \in N(x) \times Y_i$;

(a_2) For each $y_i \in Y_i$, $z \mapsto f_{i,x}(z,y_i)$ is lower semicontinuous on each compact subset of X;

(a_3) For each $N \in \mathscr{F}(Y)$, $y \in \Gamma(N)$ and $z \in F(y)$, $f_i(z,\pi_i(y)) \leqslant \lambda$ implies that there exists $y' \in N$ such that $f_{i,x}(z, \pi_i(y')) \leqslant \lambda$.

Theorem 6 Let X be a topological space and I be any index set. For each $i \in I$, let (Y_i, Γ_i) be a CG-convex space and let $Y = \prod_{i \in I} Y_i$ such that (Y, Γ) is a CG-convex space defined as in Ref. [1]. Suppose that $F \in B(Y, X)$ is a compact mapping and for each $i \in I$, the function f_i: $X \times Y_i \to \mathbf{R}$ satisfies that for each $y_i \in Y_i$, $x \mapsto f_i(x, y_i)$ is lower semicontinuous on X. Then at least one of the following statement holds:

1) For each $\lambda \in \mathbf{R}$, there exists $\hat{x} \in X$ such that $\sup_{i \in I} \sup_{y_i \in Y_i} f_i(\hat{x}, y_i) \leqslant \lambda$,

2) There exist $i \in I$, $\lambda \in \mathbf{R}$, $N \in \mathscr{F}(Y)$ such that

$$\bigcap_{y \in N} \{x \in F(\Gamma(N)) : f_i(x, \pi_i(y)) > \lambda\} \neq \phi.$$

Proof If the statement 2) is false, then for any $\lambda \in \mathbf{R}$, $i \in I$ and $N \in \mathscr{F}(Y)$, we have

$$F(\Gamma(N)) \cap (\bigcap_{y \in N} \{x \in X : f_i(x, \pi_i(y)) > \lambda\}) = \phi.$$

For each $i \in I$, define a mapping A_i: $X \to 2^{Y_i}$ by

$$A_i(x) = \{y_i \in Y_i : f_i(x, y_i) > \lambda\} \quad (\forall x \in X).$$

Hence we have

$$F(\Gamma(N)) \cap (\bigcap_{y \in N} A_i^{-1}(\pi_i(y))) = \phi.$$

By the lower semicontinuity of the function $x \mapsto f_i(x, y_i)$, we have that for each $y_i \in Y_i$, $A_i^{-1}(y_i) = \{x \in X : f_i(x, y_i) > \lambda\}$ is open on X. Hence for each $i \in I$, A_i is a G_B-mapping and condition (ⅱ) of Theorem 1 is also satisfied. By Theorem 1, there exists $\hat{x} \in X$ such that $A_i(\hat{x}) = \phi$ for all $i \in I$. Hence we have

$$\sup_{i \in I} \sup_{y_i \in Y_i} f_i(\hat{x}, y_i) \leqslant \lambda,$$

i.e., the statement 1) holds.

Theorem 7 Let X be a topological space and I be any index set. For each $i \in I$, let (Y_i, Γ_i) be a CG-convex space and let $Y = \prod_{i \in I} Y_i$ such that (Y, Γ) is a CG-convex space defined as in Ref. [1]. Suppose that $F \in B(Y, X)$ is a compact mapping and for each $i \in I$, the function f_i: $X \times Y_i \to \mathbf{R}$ satisfies

(ⅰ) For each $x \in X$, $y_i \mapsto f_i(x, y_i)$ is G-quasiconcave,

(ⅱ) For each $y_i \in Y_i$, $x \mapsto f_i(x, y_i)$ is lower semicontinuous on X.

Then we have

(A) For any $\lambda \in \mathbf{R}$, at least one of the following statement holds:

1) There exists $\hat{x} \in X$ such that $\sup_{i \in I} \sup_{y_i \in Y_i} f_i(\hat{x}, y_i) \leqslant \lambda$,

2) There exist $i \in I$, $N \in \mathscr{F}(Y)$, $\hat{y} \in \Gamma(N)$ and $\hat{x} \in F(\hat{y})$ such that $f_i(\hat{x}, \pi_i(\hat{y})) > \lambda$.

(B) The following minimax inequality holds:

$$\inf_{x \in X} \sup_{i \in I} \sup_{y_i \in Y_i} f_i(x, y_i) \leqslant \sup_{i \in I} \sup_{N \in \mathscr{F}(Y)} \sup_{x \in F(y)} \sup_{y \in \Gamma(N)} f_i(x, \pi_i(y)).$$

Proof (A) For any $i \in I$ and $\lambda \in \mathbf{R}$, define A_i: $X \to 2^{Y_i}$ by

$$A_i(x) = \{y_i \in Y_i : f_i(x, y_i) > \lambda\} \qquad (\forall x \in X).$$

Then by (ⅰ), for each $x \in X$, $A_i(x)$ is G-convex and for each $y_i \in Y_i$, the set $A_i^{-1}(y_i) =$

$\{x \in X: f_i(x, y_i) > \lambda\}$ is open in X by (ii). Now assume that the statement (A) 2) is not true. We claim that A_i is a G_B-mapping. Indeed, if A_i is not a G_B-mapping, then there exists $N \in \mathscr{F}(Y)$ such that

$$F(\Gamma(N)) \cap \left(\bigcap_{y \in N} A_i^{-1}(\pi_i(y))\right) \neq \phi.$$

It follows that there exist $\hat{y} \in \Gamma(N)$, $\hat{x} \in F(\hat{y})$ such the $\pi_i(N) \subset A_i(\hat{x})$. Since $A_i(\hat{x})$ is G-convex, we have $\Gamma_i(\pi_i(N)) \subset A_i(\hat{x})$. Noting that $\hat{y} \in \Gamma(N) = \prod_{i \in I} \Gamma_i(\pi_i(N))$, we obtain

$$\pi_i(\hat{y}) \in \Gamma_i(\pi_i(N)) \subset A_i(\hat{x}),$$

i.e., $f_i(\hat{x}, \pi_i(\hat{y})) > \lambda$. This contradicts the assumption that (A) 2) does not hold. Therefore A_i is a G_B-mapping. Noting that each $A_i^{-1}(y_i)$ is open in X, condition (ii) of Theorem 1 is also satisfied. By Theorem 1, there exists $\hat{x} \in X$ such that $A_i(\hat{x}) = \phi$ for all $i \in I$. Hence $f_i(\hat{x}, y_i) \leqslant \lambda$ for all $i \in I$ and $y_i \in Y_i$, so that the statement (A) 1) holds.

(B) Let $\lambda_0 = \sup_{i \in I} \sup_{N \in \mathscr{F}(Y)} \sup_{x \in F(y)} \sup_{y \in \Gamma(N)} f_i(x, \pi_i(y))$, then the statement (A) 2) is false, so that the statement (A) 1) must hold, i.e., there exists $\hat{x} \in X$ such that

$$\inf_{x \in X} \sup_{i \in I} \sup_{y_i \in Y_i} f_i(x, y_i) \leqslant \sup_{i \in I} \sup_{N \in \mathscr{F}(Y)} \sup_{x \in F(y)} \sup_{y \in \Gamma(N)} f_i(x, \pi_i(y)).$$

Remark 4 Theorems 6 and 7 improve and generalize Theorem 11 of Deguire *et al*.[2] in several aspects.

Theorem 8 Let X be a topological space, K be a nonempty compact subset of X and I be any index set. For each $i \in I$, let (Y_i, Γ_i) be a G-convex space and $Y = \prod_{i \in I} Y_i$ be a G-convex space defined as in Ref. [1]. Let $F \in B(Y, X)$ and for each $i \in I$, $f_i: X \times Y_i \to \mathbf{R}$ be such that

(i) For each $x \in X$, $y_i \mapsto f_i(x, y_i)$ is G-quasiconcave,

(ii) For each $y_i \in Y_i$, $x \mapsto f_i(x, y_i)$ is lower semicontinuous on X,

(iii) For each $N_i \in \mathscr{F}(Y_i)$, there exists a nonempty compact G-convex subset L_{N_i} of Y_i containing N_i, and for each $\lambda \in \mathbf{R}$ and $x \in X \setminus K$, there exist $i \in I$ and $y_i \in L_{N_i}$ satisfying $f_i(x, y_i) > \lambda$.

Then we have

(A) For each $\lambda \in \mathbf{R}$, at least one of the following statement holds:

1) There exists $\hat{x} \in K$ such that $\sup_{i \in I} \sup_{y_i \in Y_i} f_i(\hat{x}, y_i) \leqslant \lambda$, or

2) There exist $i \in I$, $N \in \mathscr{F}(Y)$, $\hat{y} \in \Gamma(N)$ and $\hat{x} \in F(\hat{y})$ such that $f_i(\hat{x}, \pi_i(\hat{y})) > \lambda$.

(B) The following minimax inequality holds:

$$\inf_{x \in K} \sup_{i \in I} \sup_{y_i \in Y_i} f_i(x, y_i) \leqslant \sup_{i \in I} \sup_{N \in \mathscr{F}(Y)} \sup_{x \in F(y)} \sup_{y \in \Gamma(N)} f_i(x, \pi_i(y)).$$

Proof By using Theorem 2 and the similar argument as in the proof of Theorem 7, it is easy to show that the conclusion of Theorem 8 holds.

Remark 5 Theorem 8 generalizes Theorem 12 of Deguire *et al*.[2] in several aspects.

Theorem 9 Let X be a paracompact topological space and I be any index set. For each

$i \in I$, let (Y_i, Γ_i) be a G-convex space and $Y = \prod_{i\in I} Y_i$ be a G-convex space defined as in Ref. [1]. Let $F \in B(Y, X)$ be such that for each $i \in I$,

(i) $f_i: X \times Y_i \to \mathbf{R}$ is G_B-majorized,

(ii) Suppose that there exist a nonempty compact subset K of X and for each $i \in I$ and $N_i \in \mathcal{F}(Y_i)$, there is a compact G-convex subset L_{N_i} of Y_i containing N_i such that for any $\lambda \in \mathbf{R}$ and for each $x \in X \setminus K$, there exist $i \in I$ and $y_i \in L_{N_i}$ satisfying $f_i(x, y_i) > \lambda$.

Then we have

(A) For each $\lambda \in \mathbf{R}$, at least one of the following statement holds:

1) There exists $\hat{x} \in K$ such that $\sup_{i\in I} \sup_{y_i\in Y_i} f_i(\hat{x}, y_i) \leqslant \lambda$, or

2) There exist $i \in I$, $N \in \mathcal{F}(Y)$, $\hat{y} \in \Gamma(N)$ and $\hat{x} \in F(\hat{y})$ such that $f_i(\hat{x}, \pi_i(\hat{y})) > \lambda$.

(B) The following minimax inequality holds:

$$\inf_{x\in K} \sup_{i\in I} \sup_{y_i\in Y_i} f_i(x, y_i) \leqslant \sup_{i\in I} \sup_{N\in\mathcal{F}(Y)} \sup_{x\in F(y)} \sup_{y\in\Gamma(N)} f_i(x, \pi_i(y)).$$

Proof (A) For any $i \in I$ and $\lambda \in \mathbf{R}$, define $A_i: X \to 2^{Y_i}$ by

$$A_i(x) = \{y_i \in Y_i : f_i(x, y_i) > \lambda\} \quad (\forall x \in X).$$

Since $f_i(x, y_i)$ is G_B-majorized, for each given $x \in X$, if $A_i(x) \neq \phi$, then there exist an open neighborhood $N(x)$ of x and $f_{i,x}: X \times Y_i \to \mathbf{R}$ having properties $(a_1) \sim (a_3)$. Now define a mapping $A_{i,x}: X \to 2^{Y_i}$ by

$$A_{i,x}(z) = \{y_i \in Y_i : f_i(x, y_i) > \lambda\}.$$

Then we have

(a) $A_i(z) \subset A_{i,x}(z)$ for each $z \in N(x)$ by (a_1),

(b) For each $y_i \in Y_i$, $A_{i,x}^{-1}(y_i)$ is compactly open in X by (a_2).

If the statement (A) 2) does not holds, then for any $i \in I$, $N \in \mathcal{F}(Y)$, $y \in \Gamma(N)$ and $z \in F(y)$, we have $f_i(z, \pi_i(y)) \leqslant \lambda$. By (a_3), there exists $y' \in N$ such that $f_{i,x}(z, \pi_i(y')) \leqslant \lambda$. Hence $z \notin \bigcap_{y\in N} \{z \in X : f_{i,x}(z, \pi_i(y) > \lambda\} = \bigcap_{y\in N} A_{i,x}^{-1}(\pi_i(y))$. It follows that for each $i \in I$ and $N \in \mathcal{F}(Y)$, $F(\Gamma(N)) \cap (\bigcap_{y\in N} A_{i,x}^{-1}(\pi_i(y))) = \phi$. Therefore $A_{i,x}$ is a G_B-majorant of A_i at x and A_i is G_B-majorized. By the condition (ii), it is easy to see that all conditions of Theorem 3 are satisfied. By Theorem 3, there exists $\hat{x} \in K$ such that $A_i(\hat{x}) = \phi$ for all $i \in I$. Hence we have $f_i(\hat{x}, y_i) \leqslant \lambda$ for all $i \in I$ and $y_i \in Y_i$, i.e., the statement (A) 1) holds. The proof of statement (B) is the same as that of Theorem 7.

Remark 6 Theorem 9 improves and generalizes Theorem 14 of Deguire *et al*.[2] in several aspects. We can also generalize Corollaries 15 ~ 18 of Deguire *et al*.[2] to the product space of G-convex spaces without linear structure. We omit it.

References:

[1] DING Xie-ping. Maximal elements for G_B-majorized mappings in product G_B-convex spaces and applications (I)[J]. *Applied Mathematics and Mechanics* (*English Edition*) 2003, **24**(6): 659 - 672.

[2] Deguire P, Tan K K, Yuan X Z. The study of maximal elements, fixed points for L_S-majorized mappings and their applications to minimax and variational inequalities in product topological spaces [J]. *Nonlinear Anal*, 1999, **37**(8): 933 - 951.

[3] Deguire P, Lassonde M. Familles sélectantes[J]. *Topol Methods Nonlinear Anal*, 1995, **5**: 261 - 269.

[4] Deguire P. Brewder-Fan fixed point theorem and relatd results[J]. *Discuss Math Differential Incl*, 1995, **15**: 149 - 162.

[5] Ben-El-Mechaiekh H, Deguire P, Granas A. Points fixes et coincidences pour les applications multivoques(applications de Ky Fan)[J]. *C R Acad Sci Paris*, 1982, **295**: 337 - 340.

[6] Ben-El-Mechaiekh H, Deguire P, Granas A. Points fixes et coincidences pour les applications multivoques (applications de ϕ and ϕ^*) [J]. *C R Acad Sci Paris*, 1982, **295**: 381 - 384.

[7] Lassonde M. On the use of KKM multifunctions in fixed point theory and related topics[J]. *J Math Anal Appl*, 1983, **97**: 151 - 201.

[8] Lassdone M. Fixed point for Kakutani factorizable multifunctions[J]. *J Math Anal Appl*, 1990, **152**: 46 - 60.

[9] Tarafdar E. A fixed points theorem and equilibrium points of abstract economics[J]. *J Math Econom*, 1991, **20**(2): 211 - 218.

[10] Tarafdar E. Fixed point theorems in *H*-spaces and equilibrium points of abstract economics[J]. *J Austral Math Soc, Ser A*, 1992, **53**: 252 - 260.

[11] DING Xie-ping, Tarafdar E. Some coincidence theorems and applications[J]. *Bull Austral Math Soc*, 1994, **50**: 73 - 80.

[12] DING Xie-ping, Tarafdar E. Fixed point theorem and existence of equilibrium points of noncompact abstract economics[J]. *Nonlinear World*, 1994, **1**: 319 - 340.

[13] DING Xie-ping. New H-KKM theorems and their applications to geometric property, coincidence theorems, minimax inequality and maximal elements[J]. *Indian J Pure Appl Math*, 1995, **26**(1): 1 - 19.

[14] DING Xie-ping. Fixed points, minimax inequalities and equilibria of noncompact abstract economics [J]. *Taiwanese J Math*, 1998, **2**(1): 25 - 55.

[15] Fan Ky. A minimax inequality and applications[A]. In: Shisha O Ed. *Inequality* Ⅲ [C]. New York: Academic Press, 1972.

[16] Fan Ky. Some properties of convex sets related to fixed point theorems[J]. *Math Ann*, 1984, **266**: 519 - 537.