MAXIMAL FIELDS DISJOINT FROM FINITE SETS

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Let F be a subfield of an algebraically closed field A of characteristic 0, S a finite subset of A disjoint from F, K a subfield of A containing F and maximal with respect to disjointness from S, L a finite extension of K, and $G=G(L/K)$ the group of automorphisms of L/K. Quigley [4] and McCarthy [1] obtained precise information about G in the case where S has one or two elements, respectively (they handled the characteristic p case also). Theorem 1 of this paper shows that there is some restriction on G in the general case. In particular (Theorem 2), G is solvable if S has at most twenty elements.

LEMMA. *If r is a positive integer, then there is a finite set Π of primes such that if G is a finite group containing at most r maximal subgroups, then either*

(i) *G is cyclic of order $p_1^{n_1} \cdots p_k^{n_k}$, p_i prime, $k \leqq r$, or*

(ii) *G is a Π-group.*

PROOF. Let G contain at most r maximal subgroups H_i, and let the index of $H_{i_1} \cap \cdots \cap H_{i_s}$ in G be denoted by $n_{i_1 \cdots i_s}$. If G is cyclic, then conclusion (i) clearly holds. Assume that G is not cyclic. We may count elements as follows:

$$\begin{aligned} o(G) = {} & [o(H_1)] + [o(H_2) - o(H_1 \cap H_2)] \\ & + [o(H_3) - o(H_1 \cap H_3) - o(H_2 \cap H_3) \\ & \qquad + o(H_1 \cap H_2 \cap H_3)] + \cdots . \end{aligned}$$

Dividing by $o(G)$, we get

$$\begin{aligned} 1 = {} & 1/n_1 + (1/n_2 - 1/n_{12}) \\ & + (1/n_3 - 1/n_{13} - 1/n_{23} + 1/n_{123}) + \cdots . \end{aligned}$$

We may take $n_1 \leqq n_2 \leqq \cdots$. Also each expression surrounded by parentheses is positive since the H_i are distinct maximal subgroups. Moreover, each such expression is less than or equal to its first term. It follows readily that $n_1 \leqq r$. Suppose inductively that $n_{i_1} \cdots n_{i_s}$ have all been chosen for $i_1 < \cdots < i_s < t$. The corresponding sum equals $1-a$, say. Then $n_t \leqq r/a$, and $n_{i_1 \cdots t} \leqq n_{i_1} \cdots n_t$. Thus, by induction, there are only a finite number of choices for the $n_{ijk \cdots}$.

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Now the intersection $\Phi(G)$ of all H_i is normal, and there are only a finite number of possibilities for $o(G/\Phi(G))$. Let Π be the set of primes dividing such orders. By a theorem of Suzuki [5, p. 347], if $p | o(G)$ then $p | o(G/\Phi(G))$. Hence G is a Π-group.

THEOREM 1. *Let r be a positive integer and Π the set of primes guaranteed by the lemma. Let F be a field of characteristic 0, A an algebraically closed field containing F, S a subset of A disjoint from F and containing exactly r elements, and K a subfield maximal such that $K \cap S = \varnothing$ and $F \subset K \subset A$. Then there is a set S' of r or fewer primes such that, if L is a finite extension of K, then $G(L/K)$ is either*

(i) *cyclic of order $p_1^{n_1} \cdots p_k^{n_k}$, $p_i \in \Pi \cup S'$, $k \leqq r$, or*

(ii) *a Π-group.*

PROOF. K exists by Zorn. By [1, Lemma 1], A/K is algebraic. The field K has at most r minimal extensions in A, for each such extension contains an element of S. Let M be the smallest subfield of A which is normal over K and which contains L and these minimal extensions. Then M is finite Galois over K. $G(M/K)$ has at most r maximal subgroups by the fundamental theorem of Galois theory. By the lemma, $G(M/K)$ is cyclic of order $p_1^{n_1} \cdots p_k^{n_k}$, $k \leqq r$, or a Π-group. Let S' be the set of all $p_i \notin \Pi$ that occur as L varies. If S' contains distinct primes $p_1, \cdots, p_{r+1}$, then there are finite Galois extensions M_i such that $p_i | o(G(M_i/K))$. The composite of the M_i is then a finite Galois extension M such that either $G(M/K)$ is cyclic with order divisible by more than r primes, or is noncyclic and not a Π-group. In either case this is a contradiction. Hence the theorem is true for L replaced by M. In case (i), it is clear that $G(L/K)$ also has the form (i). In the other case, we use the easily proved result that[1]

$$G(L/K) \cong N(G(M/L))/G(M/L)$$

where the normalizer is taken in $G(M/K)$. Since subgroups and factor groups of Π-groups are Π-groups, $G(L/K)$ is a Π-group.

For small values of r, the set Π in the lemma and Theorem 1 may be determined from known theorems. In fact, it follows from [2] that for $r=1, 2, 3, 4$, the set Π may be taken as $\varnothing$, $\varnothing$, $\{2\}$, and $\{2, 3\}$, respectively.

Pazderski [3] has shown that if a finite group G contains fewer than 21 maximal subgroups, then G is solvable. Hence we have

THEOREM 2. *If, in Theorem 1, $r < 21$, then $G(L/K)$ is solvable.*

[1] This fact was pointed out to me by Robert Gordon.

Bibliography

1. P. J. McCarthy, *Maximal fields disjoint from certain sets*, Proc. Amer. Math. Soc. **18** (1967), 347–351.

2. G. A. Miller, *Maximal subgroups of a finite group*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 212–214.

3. Gerhard Pazderski, *Über maximale Untergruppen endlicher Gruppen*, Math. Nachr. **26** (1963/1964), 307–319.

4. Frank Quigley, *Maximal subfields of an algebraically closed field not containing a given element*, Proc. Amer. Math. Soc. **13** (1962), 562–566.

5. M. Suzuki, *On the lattice of subgroups of finite groups*, Trans. Amer. Math. Soc. **70** (1951), 345–371.

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