International Journal of Algebra, Vol. 7, 2013, no. 12, 581 - 588
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ija.2013.3773

Maximal Graph of a Commutative Ring

Atul Gaur [1]

Department of Mathematics
University of Delhi, Delhi 110007, India
agaur@maths.du.ac.in

Arti Sharma [2]

Department of Mathematics
University of Delhi, Delhi 110007, India
anjanaarti@gmail.com

Abstract

Let R be a commutative ring with identity. Let G be a graph with vertices as elements of R, where two distinct vertices a and b are adjacent if and only if there is a maximal ideal of R containing both. In this paper we show that a ring R is finite if and only if clique number of the graph G (associated with R as above) is finite. We also have shown that for a semilocal ring R, the chromatic number of the graph G is equals to the clique number of G which turns out to be the maximum of the cardinalities of all the maximal ideals in R.

Mathematics Subject Classification: Primary 13C99, Secondary 05C99, 13M99

Keywords: Chromatic Number, Clique Number

1 *Introduction*

In 1988, Istvan Beck [3], first introduced the idea of associating a graph with a commutative ring with unity. The objective was to establish a connection

[1]The author was supported by a grant from University of Delhi, No. R& D/2012/917
[2]The author was supported by a grant from CSIR India, No. 09/045(1142)/2011-EMR-I

between Graph theory and Commutative ring theory. Beck presented the idea of coloring of commutative rings and thus produced some fundamental results.

Let R be a commutative ring with unity. In order to apply a graph structure to R, Beck considered R as a simple graph whose vertices are the elements of R such that two different elements x and y are adjacent if and only if $xy = 0$.

A subset $C = \{x_1, x_2, ..., x_n\}$ is called a clique provided $x_i x_j = 0$ for every $i \neq j$. If R contains a clique with n elements and every clique contains at most n element, we say that the clique number of R is n and denote it by $Clique(R)$. Let $\chi(R)$ denote the chromatic number of graph associated with R, that is, the minimal number of colors which can be assigned to the elements of R in such a way that no two adjacent elements have the same color. Note that $\chi(R) \geq Clique(R)$, for any ring R.

In [3], Beck conjectured that $\chi(R) = Clique(R)$, for every ring R. He established the conjecture for certain classes of rings, namely reduced rings and principal ideal rings. But unfortunately, it is not true in general. This was proved in 1993, when D.D. Anderson and M. Naseer presented a strong counter example (see, Theorem 2.1 in [1]) and hence disproved Beck's conjecture for general rings.

In 1995, Sharma and Bhatwadekar [8], introduced another graphical structure on R, which later came to be known as Comaximal graphs. In their graphical structure, R is a graph whose vertices are elements of R and two distinct vertices x and y are adjacent if and only if $Rx + Ry = R$.

With this definition, a very important theorem (see, Theorem 2.3, [8]) was proved, namely $\chi(R)$ is finite if and only if the ring R itself is finite. In this case, the chromatic number being equal to the clique number of R which is equals to the sum of number of maximal ideals of R and the number of unit elements of R.

Later, 2008, H.R. Maimani et al [13], further studied the graph structure defined by Sharma and Bhatwadekar and named such graph structures as "Comaximal Graphs" and denoted it by $\Gamma(R)$.

In this article, we are introducing another graphical structure associated with R. For any ring R, we associate a simple graph with vertices as the elements of R such that two different vertices x and y are adjacent if and only if $x, y \in \mathfrak{m}$, for some maximal ideal $\mathfrak{m}$ of R. We call this graph as a Maximal graph associated with R. It is easy to see that every maximal ideal in R forms a maximally complete subgraph of maximal graph associated with R. However, the converse may not be true.

2 *Clique number of rings*

Throughout this paper R denotes a commutative ring with unity. For any ideal I of R, $|I|$ denotes the cardinality of R, i.e. the number of elements in

I. If A and B are any two subsets of R then $|A| \leq |B|$ if there is one one map from A to B.

By $G(R)$, we denote the maximal graph (as explained in the introduction) of R. We recall the definition of clique in this context:

Definition 2.1 *A subset C of R is called a clique if every pair of distinct elements x and y of C is adjacent in the maximal graph of R, that is, if $x, y \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. If R contains a clique with n elements and every clique contains at most n element, we say that the clique number of R is n and denote it by $Clique(R)$. If no such n exists then $Clique(R)$ is defined to be ∞.*

Note that if R is a local ring with maximal ideal $\mathfrak{m}$ then $Clique(R) = |\mathfrak{m}|$. For a ring with two maximal ideals we have the following result.

Proposition 2.2 *Let R be a ring with two maximal ideals, say $\mathfrak{m}_1$ and $\mathfrak{m}_2$ such that $|\mathfrak{m}_1| \geq |\mathfrak{m}_2|$. Then*

$$\chi(R) = Clique(R) = |\mathfrak{m}_1|.$$

Proof. First color the elements of $\mathfrak{m}_1$ using $|\mathfrak{m}_1|$ distinct colors. To all the units of R assign the same color as that of zero element. Now the elements of $\mathfrak{m}_2 \setminus \mathfrak{m}_1$ can be colored by using the colors used in the elements of $\mathfrak{m}_1 \setminus \mathfrak{m}_2$ as $|\mathfrak{m}_2 \setminus \mathfrak{m}_1| \leq |\mathfrak{m}_1 \setminus \mathfrak{m}_2|$. Therefore $\chi(R) = |\mathfrak{m}_1|$.

Now $Clique(R) \geq |\mathfrak{m}_1|$, since the elements of any maximal ideal in R forms a clique. If possible, suppose $Clique(R) > |\mathfrak{m}_1|$. Then there exist a clique C in R such that $|C| > |\mathfrak{m}_1|$. Therefore, C has an element, say α, which is not in $\mathfrak{m}_1$, that is, $\alpha \in \mathfrak{m}_2$ as α must be nonunit. Therefore, if $\beta \in C$ then $\alpha, \beta \in \mathfrak{m}_2$. Thus $C \subseteq \mathfrak{m}_2$, which is a contradiction. Therefore, $Clique(R) = |\mathfrak{m}_1|$. □

Theorem 2.3 *Let R be a semilocal ring with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ such that $|\mathfrak{m}_1| \geq |\mathfrak{m}_2| \geq \ldots \geq |\mathfrak{m}_n|$. Then $Clique(R) = |\mathfrak{m}_1|$.*

Proof. We apply induction on n. For $n = 1$, the result holds trivially. For $n = 2$, the result follows from Proposition 2.2. Now assume $n \geq 3$. Since elements of any maximal ideal forms a clique, $Clique(R) \geq |\mathfrak{m}_1|$. Suppose, if possible, that $Clique(R) > |\mathfrak{m}_1|$. Then there exists a subset A of R, containing nonunits only such that $|A| > |\mathfrak{m}_1|$ and the elements of A forms a clique in R. If A contains any element of $\mathfrak{m}_n \setminus \cup_{i=1}^{n-1} \mathfrak{m}_i$ then $A \subseteq \mathfrak{m}_n$, which contradicts that $|A| > \mathfrak{m}_1$. Therefore, $A \subseteq \cup_{i=1}^{n-1} \mathfrak{m}_i$.

Put $I = \cap_{i=1}^{n-1} \mathfrak{m}_i$ and $\bar{R} = R/I$. Note that $I \neq (0)$, by Proposition 1.11(ii) [2]. Also the elements of $\bar{A}$ forms a clique in $\bar{R}$. Since $\bar{R}$ is a ring with $n-1$ maximal ideals, by induction hypothesis, $Clique(\bar{R}) = |\bar{\mathfrak{m}}_1| \geq |\bar{A}|$. But this implies that $|\mathfrak{m}_1| \geq |A|$, a contradiction. Therefore, $Clique(R) = |\mathfrak{m}_1|$. □

Theorem 2.4 *Let R be a ring. Then $Clique(R)$ is finite if and only if R is a field or a finite ring.*

Proof. The sufficiency is obvious. We only need to prove the necessity. Suppose, if possible, that $Clique(R)$ is finite. As the elements of any maximal ideal forms a clique, we conclude that every maximal ideal of R is finite. Suppose, first, that R is not local. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two distinct maximal ideals of R. Then R is finite as $\mathfrak{m}_1 + \mathfrak{m}_2 = R$.

Suppose, now, that R is a local ring with maximal ideal $\mathfrak{m}$. If $\mathfrak{m} = \{0\}$, then R is a field. Now assume that $\mathfrak{m} \neq \{0\}$ and let $x \in \mathfrak{m} \setminus \{0\}$. Then Rx is a finite ideal of R. As $x \neq 0$, $ann(x) \neq R$ and therefore, $ann(x)$ is also finite. As $Rx \cong R/ann(x)$, R is finite. □

Note that if R_1 and R_2 are rings of finite clique numbers then $Clique(R_1 \times R_2)$ may be infinite. For example, if R_1 and R_2 are any two infinite fields then their clique numbers are finite. However, $Clique(R_1 \times R_2) = \infty$.

3 *Beck's conjecture for finite rings*

In [8], Sharma and Bhatwadekar proved the Beck's conjecture for comaximal graphs of finite rings. In this section we establish the Beck's conjecture for maximal graphs of finite rings.

Definition 3.1 *By a coloring of R, we mean a coloring of the maximal graph $G(R)$ of R. In other words, a coloring of R is an assignment of colors to the vertices of R, one color to each vertex, so that the adjacent vertices are assigned distinct colors. The chromatic number of R is the minimum number of colors required for the coloring of R and is denoted by $\chi(R)$. The corresponding coloring we call as a minimal coloring.*

Proposition 3.2 *Let R be a ring. Then $\chi(R) = 1$ if and only if R is a field.*

Proof. Obviously if R is a field then $\chi(R) = 1$. For necessity, note that if $\chi(R) = 1$, then (0) is the only proper ideal of R and hence R is a field. □

Proposition 3.3 *Let R be a ring. Then $\chi(R) = 2$ if and only if either $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[x]/(x^2)$ or $R \cong \mathbb{Z}_2[x]/(x^2 - x)$.*

Proof. The sufficiency is obvious. For necessity, suppose $\chi(R) = 2$. Since the elements of a maximal ideal forms a clique, we conclude that every maximal ideal in R has at most two elements. Now by Proposition 3.2, R is not a field.

Therefore every maximal ideal in R has exactly two elements and hence is a principal ideal. Thus if I is any nonzero proper ideal of R then I is a maximal ideal and hence is principal. Therefore R is a principal ideal ring.

Let x be a nonzero nonunit in R. Since $R/ann(x) \cong Rx$ and $ann(x) \neq R$, we have $|R| \leq 4$. As R is not a field, $|R| = 4$. Now if $char(R) = 4$ then $R \cong \mathbb{Z}_4$ and if $char(R) = 2$ then $R \cong \mathbb{Z}_2[x]/(x^2)$ or $R \cong \mathbb{Z}_2[x]/(x^2 - x)$. □

Lemma 3.4 *Let R be a semilocal ring and let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ be maximal ideals of R such that $|\mathfrak{m}_1| \geq |\mathfrak{m}_2| \geq \ldots \geq |\mathfrak{m}_n|$. Then in any minimal coloring of R the elements of $\mathfrak{m}_n$ have the colors already used in $\cup_{i=1}^{n-1}\mathfrak{m}_i$.*

Proof. Assume the contrary. Then there is a coloring of R in which there exist an element $a \in \mathfrak{m}_n \setminus \cup_{i=1}^{n-1}\mathfrak{m}_i$ for whom the assigned color is not used in $\cup_{i=1}^{n-1}\mathfrak{m}_i$. As $|\mathfrak{m}_1 \setminus (\mathfrak{m}_1 \cap \mathfrak{m}_n)| \geq |\mathfrak{m}_n \setminus (\mathfrak{m}_1 \cap \mathfrak{m}_n)|$, there are elements in $\mathfrak{m}_1 \setminus (\mathfrak{m}_1 \cap \mathfrak{m}_n)$, having assigned colors which are not used in colors assigned to elements of $\mathfrak{m}_n \setminus (\mathfrak{m}_1 \cap \mathfrak{m}_n)$. Let $\alpha \in \mathfrak{m}_1 \setminus (\mathfrak{m}_1 \cap \mathfrak{m}_n)$ be such an element. Then α and a cannot be adjacent as $\alpha \notin \mathfrak{m}_n$ and $a \notin \cup_{i=1}^{n-1}\mathfrak{m}_i$. Therefore, if we assign to a the same color as that of α, we still have a coloring. But this contradicts that we have a coloring on R. □

Sharma and Bhatwadekar [8], proved the Beck's conjecture for Comaximal graph of finite rings. It was shown that $\chi(R)$ is finite if and only if the ring R itself is finite and in this case the chromatic number being equal to the clique number of R which is equals to the sum of number of maximal ideals of R and the number of unit elements of R.

Beck's conjecture for maximal graph of finite rings and semilocal rings is also true as we have shown in the next two theorems.

Theorem 3.5 *Let R be a semilocal ring and let $\mathfrak{m}$ be a maximal ideal of R of the largest cardinality. Then $\chi(R) = Clique(R) = |\mathfrak{m}|$.*

Proof. $Clique(R) = |\mathfrak{m}|$, follows from Theorem 2.3. For $\chi(R)$, by Lemma 3.4, we may assume that R is a ring with two maximal ideals $\mathfrak{m}_1 = \mathfrak{m}$ and $\mathfrak{m}_2$ such that $|\mathfrak{m}_1| > |\mathfrak{m}_2|$. Now the result follows from Proposition 2.2.

□

Unfortunately, we have not found any example of a ring where $\chi(R) > Clique(R)$. The lack of such counter examples together with the above theorem motivates the following conjecture:

Conjecture. For any ring R, $\chi(R) = Clique(R)$.

Corollary 3.6 *Let R be a semilocal ring and let $\mathfrak{m}$ be a maximal ideal of R of the largest cardinality. Then for any ideal $I \subseteq \mathfrak{m}$,*

$$\chi(R) = |I|\chi(R/I).$$

Proof. Follows immediately from Theorem 3.5. □

Theorem 3.7 *Let R be a ring. Then $\chi(R)$ is finite if and only if either R is a field or a finite ring. Also, in this case $\chi(R) = Clique(R) = |\mathfrak{m}|$, where $\mathfrak{m}$ is a maximal ideal of R of the largest cardinality.*

Proof. First assume $\chi(R)$ is finite. Then $Clique(R)$ is finite as $\chi(R) \geq Clique(R)$. This implies that either R is a field or a finite ring, by Theorem 2.4.

Conversely, assume that R is a field or a finite ring. Then R is semilocal. Now the result follows from Theorem 3.5.

□

4 *Isomorphism*

We begin this section with the following definition:

Definition 4.1 *For any subset $A \subseteq R$, the subgraph $G(A)$ is said to be complete if all the elements of A are adjacent to each other. The subgraph $G(A)$ is said to be maximally complete if it is a maximal element of the set of all complete subgraphs of $G(R)$.*

Proposition 4.2 *Let $\mathfrak{m}$ be a maximal ideal of R. Then $G(\mathfrak{m})$ is a maximally complete subgraph of $G(R)$ consisting of nonunits. Converse may not be true.*

Proof. If possible, suppose $G(\mathfrak{m} \cup \{x\})$ forms a complete graph for any nonunit $x \in R \setminus \mathfrak{m}$. Then $\mathfrak{m} + Rx = R$ and hence $y + rx = 1$, for some $y \in \mathfrak{m}$ and $r \in R$. But this contradicts the fact that x and y are adjacent. Therefore, $G(\mathfrak{m})$ is a maximally complete subgraph of $G(R)$.

For the converse, take $R = k_1 \times k_2 \times k_3$, where k_1, k_2, k_3 are fields. Put $A = (\{0\} \times \{0\} \times k_3) \cup (\{0\} \times k_2 \times \{0\}) \cup (k_1 \times \{0\} \times \{0\})$. Then $G(A)$ is maximally complete subgraph of $G(R)$ consisting of nonunits, however, A is not even an ideal in R. □

We now recall the following definition from [4].

Definition 4.3 *Any two graphs $\mathcal{A}$ and $\mathcal{B}$ are said to be isomorphic to each other, denoted by $\mathcal{A} \cong \mathcal{B}$, if there is one to one correspondence between their vertices such that the incidence relationship is preserved, that is, vertices v_1 and v_2 of $\mathcal{A}$ are adjacent (in $\mathcal{A}$) if and only if the corresponding vertices are adjacent in $\mathcal{B}$.*

The question of isomorphism of comaximal graphs of two given rings implies isomorphism of given rings itself was discussed in [6] and it was shown that in general, such two rings may not be isomorphic. The same is true for maximal graph also as we have the following example.

Example 4.4 *Let $A = \mathbb{Z}_9$ and $B = \mathbb{Z}_3[x]/(x^2)$. Note that the maximal graphs of $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. However, A and B are not isomorphic.*

Proposition 4.5 *Let R and S be semilocal rings. Let $\phi : G(R) \to G(S)$ be graph isomorphisms, where $G(R)$ and $G(S)$ denotes the maximal graph of R and S respectively. Then for every maximal ideal $\mathfrak{m}$ of R, $\phi(\mathfrak{m})$ is a maximal ideal in S. In particular, R and S have same number of maximal ideals.*

Proof. Note that if ϕ is a isomorphism then $\phi^{-1} : G(S) \to G(R)$ is also an isomorphism. Therefore, it is enough to show that $\phi(\mathfrak{m})$ contains a maximal ideal in S.

Choose $a \in \mathfrak{m}$ such that a does not belongs to any other maximal ideal in R. Obviously, $\phi(a)$ cannot be a unit in S. Let $\phi(a) \in \mathfrak{n}$, for some maximal ideal $\mathfrak{n}$ in S. Note that $\mathfrak{m} \setminus \{a\}$ is precisely the set of all elements in R which are adjacent to a in $G(R)$ and as the incidence relationship is preserved by ϕ, $\phi(\mathfrak{m} \setminus \{a\})$ is precisely the set of all elements in S which are adjacent to $\phi(a)$ in $G(S)$. Thus $\mathfrak{n} \subseteq \phi(\mathfrak{m})$. □

The following theorem was proved in [6], for comaximal graphs. We are proving here the same for maximal graphs.

Theorem 4.6 *Let $\{(R_i, \mathfrak{m}_i) \mid i = 1, \ldots, m\}$ and $\{(S_j, \mathfrak{n}_j) \mid j = 1, \ldots, n\}$ be two families of finite local rings. Let $R = R_1 \times R_2 \ldots \times R_m$ and $S = S_1 \times S_2 \ldots \times S_n$. If the maximal graph of R and S are isomorphic then $m = n$ and there is a permutation σ on the set $\{1,\ 2,\ \ldots,\ m\}$ such that $|R_i/\mathfrak{m}_i| = |S_{\sigma(i)}/\mathfrak{n}_{\sigma(i)}|$ for every $i = 1,\ 2,\ \ldots,\ m$ and hence $R_i/\mathfrak{m}_i \cong S_{\sigma(i)}/\mathfrak{n}_{\sigma(i)}$.*

In particular, if the maximal graph of R and S are isomorphic and each R_i is a finite field, then each S_j is also a finite field and $R_i \cong S_{\sigma(i)}$ for every $i = 1,\ 2,\ \ldots,\ m$, and thus $R \cong S$.

Proof. Let $\mathcal{M}_i = R_1 \times \ldots \times R_{i-1} \times \mathfrak{m}_i \times R_{i+1} \times \ldots \times R_m$ and $\mathcal{N}_j = S_1 \times \ldots \times S_{i-1} \times \mathfrak{n}_j \times S_{i+1} \times \ldots \times S_m$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Then $\mathcal{M}_i$'s and $\mathcal{N}_j$'s, are precisely the maximal ideals of R and S respectively. Therefore, by Proposition 4.5, we have $n = m$ and for all i, $\phi(\mathcal{M}_i) = \mathcal{N}_{\sigma(i)}$ for some permutation σ on $\{1, 2, \ldots, n\}$, where $\phi : G(R) \to G(S)$ is the given isomorphism between maximal graphs of R and S.

As $|R| = |S|$, this implies that $|R/\mathcal{M}_i| = |S/\mathcal{N}_{\sigma(i)}|$ and hence $|R_i/\mathfrak{m}_i| = |S_{\sigma(i)}/\mathfrak{n}_{\sigma(i)}|$. Since $R_i/\mathfrak{m}_i$ and $S_{\sigma(i)}/\mathfrak{n}_{\sigma(i)}$ are finite fields having the same cardinality, we conclude that $R_i/\mathfrak{m}_i \cong S_{\sigma(i)}/\mathfrak{n}_{\sigma(i)}$.

In particular, if $G(R) \cong G(S)$ and each R_i is a finite field, then the Jacobson radical of R is zero and so the Jacobson radical of S is zero. Therefore, for all i, $\mathfrak{n}_i = (0)$, that is, S_i is also a finite field and $R_i \cong S_{\sigma(i)}$ and hence $R \cong S$. $\square$

Acknowledgements

The authors thank Alok K. Maloo, whose comments have improved this paper.

References

[1] D.D. Anderson, and M. Naseer, Beck's Coloring of a Commutative Ring, *J. Algebra* **159** (1993), 500-514.

[2] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra,* Addison Wesley Publishing Company, 1969.

[3] I. Beck, Coloring of Commutative Rings, *J. Algebra* **116** (1988), 208-226.

[4] N. Deo, *Graph Theory with Applications to Engineering and Computer Science,* Prentice Hall of India Private Limited, 2008.

[5] H.R. Maimani, M.R. Pournaki, and S. Yassemi, Zero-Divisor Graph with Respect to an Ideal, *Comm. Algebra* **34** (2006), 923-929.

[6] H.R. Maimani, M. Salimi, A. Sattari and S. Yassemi, Comaximal graph of commutative rings, *J. Algebra* **319** (2008), 1801-1808.

[7] S.M. Moconja and Z.Z. Petrovic, On the structure of comaximal graphs of commutative rings with identity *Bull. Aust. Math. Soc.* **83** (2011), 11-21.

[8] P.K. Sharma and S.M. Bhatwadekar, A Note on Graphical Representation of Rings, *J. Algebra* **176** (1995), 124-127.

Received: July 25, 2013