MAXIMAL NORMAL FUCHSIAN GROUPS

BY

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1. Let $\mathcal{L}$ denote the group of all conformal homeomorphisms of D, the unit disc. The elements of $\mathcal{L}$ are linear fractional transformations and so $\mathcal{L}$ is a topological group. We define a *Fuchsian group* Γ as a finitely-generated discrete subgroup of $\mathcal{L}$. Then Γ has a presentation of the following form.

$$\begin{aligned} &\text{Generators: } a_1, b_1, \cdots, a_\gamma, b_\gamma, x_1, \cdots, x_r, p_1, \cdots, p_s, h_1, \cdots, h_t \\ &\text{Relations: } x_1^{m_1} = \cdots = x_r^{m_r} = 1, \\ &\qquad \prod_{i=1}^{\gamma} [a_i, b_i] \prod_{j=1}^{r} x_j \prod_{k=1}^{s} p_k \prod_{l=1}^{t} h_l = 1 \end{aligned} \tag{1}$$

We say Γ has *signature* $(\gamma; m_1, \cdots, m_r; s; t)$ and any two groups of the same signature are isomorphic.

If $\varphi_\Gamma : D \rightarrow D/\Gamma$ denotes the orbit-map, then $\varphi_\Gamma(D)$ can be made into a Riemann surface with an associated ramification index $d_\Gamma : \varphi_\Gamma(D) \rightarrow N$ where N denotes the natural numbers [2, p. 4]. $\varphi_\Gamma(D)$ is obtained from a compact Riemann surface of genus γ by deleting s points and t discs. The x_i, in the presentation (1), correspond to elliptic elements of Γ and to those points q of $\varphi_\Gamma(D)$ such that $d_\Gamma(q) > 1$, the p_i to parabolic and the a_i, b_i, h_j to hyperbolic. The m_i are called the *periods* of Γ.

Following Greenberg [1], Γ is defined to be a *maximal Fuchsian group* if there does not exist a Fuchsian group Γ_0 such that $\Gamma \subset \Gamma_0$ and $[\Gamma_0 : \Gamma]$ is finite. We also define Γ to be a *maximal normal Fuchsian group* if there does not exist a Fuchsian group Γ_0 such that Γ is normal in Γ_0 and $[\Gamma_0 : \Gamma]$ is finite.

If Γ is a Fuchsian group with generators $\gamma_1, \gamma_2, \cdots, \gamma_n$, a topology is defined on the set of all isomorphisms $\tau : \Gamma \rightarrow \mathcal{L}$ by associating with τ the point $(\tau(\gamma_1), \tau(\gamma_2), \cdots, \tau(\gamma_n))$ of $\mathcal{L}^n$. On this space, define the equivalence relation $\tau \sim \tau'$ if there exists an angle-preserving homeomorphism t of D such that

$$\tau'(f) = t^{-1}\tau(f)t \quad \text{for all } f \in \Gamma.$$

This quotient space is denoted by $T(\Gamma)$. Let Max (Γ) and Max Normal (Γ) be the subspaces corresponding to those $\tau(\Gamma)$ which are maximal and maximal normal respectively. Note that Max $(\Gamma) \subset$ Max Normal (Γ). Greenberg [1] has shown that Max (Γ) is either empty or a dense subset of $T(\Gamma)$.

In this paper, we obtain necessary and sufficient criteria on the signature of Γ such that Max Normal $(\Gamma) = T(\Gamma)$. This is tantamount to obtaining criteria on the signature of Γ such that for at least one $[\tau] \in T(\Gamma)$, $\tau(\Gamma)$ is a normal subgroup of finite index in some Fuchsian group Γ_0.

Received September 23, 1968.

2. The elements of $\mathcal{L}$ may be considered as acting on the extended plane. Let $L(\Gamma)$ denote the set of limit points in the extended plane of a Fuchsian group Γ. If Γ is of finite index in Γ_0, then $L(\Gamma) = L(\Gamma_0)$. Also, for any Fuchsian group Γ, $L(\Gamma)$ is a subset of $C = \{z \mid |z| = 1\}$ of one of three types.

(a) $L(\Gamma)$ has at most two points.
(b) $L(\Gamma) = C$.
(c) $L(\Gamma)$ is a perfect subset of C [4, Ch 3].

We consider the three types separately. In case (a), Greenberg shows that Max (Γ) is empty [1, Theorem 3A] and one can easily check that Max Normal (Γ) is also empty.

Groups of type (b) are called Fuchsian groups of the *first kind* and we call groups of type (c) of the *second kind* although this term usually includes the groups of type (a). If F_Γ denotes a Fundamental region for Γ in D, $\mu(F_\Gamma)$ the hyperbolic area of F_Γ, and Γ is of the first kind then, in the presentation (1), $t = 0$ and

$$\mu(F_\Gamma) = 2\pi\{2(\gamma - 1) + \textstyle\sum_{i=1}^{r} (1 - 1/m_i) + s\} > 0 \tag{2}$$

If Γ is of the second kind then $t > 0$ and the area of its fundamental region is infinite. However, if a signature $(\gamma; m_1, \cdots, m_r; s; t)$ is given such that the inequality

$$2(\gamma - 1) + \textstyle\sum_{i=1}^{r} (1 - 1/m_i) + s + t > 0 \tag{3}$$

holds, then a Fuchsian group with that signature exists always provided, of course, that the presentation is consistent [4, Ch 7] and [2].

3. The presentation (1) of a Fuchsian group is obtained from a Fundamental region for the group, the generators being those elements which map F_Γ into a full neighbour and the relations being obtained from the copies of F_Γ which meet at a vertex, it being sufficient to consider one vertex out of a congruent set, [5], [4, Ch 7]. Thus $x_1, \cdots, x_r$ is a *complete system of elliptic representatives* (c.s.e.r.) by which we mean a set of elliptic elements such that

(i) every elliptic element of Γ is conjugate in Γ to some power of an x_i $(1 \leq i \leq r)$,
(ii) non-trivial power of x_i is conjugate in Γ to a power of x_j $(i \neq j)$ [6, p. 46].

In the same way $p_1, p_2, \cdots, p_s$ is a complete system of parabolic representatives (c.s.p.r.).

The $h_1, \cdots, h_t$ must be treated differently as, for example, the hyperbolic element a_1 is not conjugate to a power of an h_i. Let Ω denote *the component of the set of proper discontinuity of* Γ *containing* D, as defined in [2, p. 4].

Definition 1. *$h \in \Gamma$ is said to be an admitted hyperbolic element of Γ if $h(\neq 1)$ maps some component of $\Omega \cap C$ into itself.*

LEMMA 1. *If* Γ *has the presentation at* (1), *then every admitted hyperbolic element of* Γ *is conjugate to a power of some* h_i $(1 \leq i \leq t)$.

Proof. The action of Γ splits the set of components of $\Omega \cap C$ into distinct equivalence classes, corresponding to the number of holes in $\varphi_\Gamma(D)$ and the number of generators $h_1, h_2, \cdots, h_t$. If F_Γ is a canonical polygon for Γ with surface symbol [4, Ch 7]

$$a_1 b_1 a_1' b_1' \cdots a_\gamma b_\gamma a_\gamma' b_\gamma' x_1 x_1' \cdots x_r x_r' p_1 p_1' \cdots p_s p_s' c_1 f_1 c_1' \cdots c_t f_t c_t'$$

where $f_1, f_2, \cdots, f_t$ denote 'sides' of the polygon which lie on C, which gives rise to the presentation (1), then h_i is the hyperbolic generator which maps over the side c_i. Let E_i denote the equivalence class containing the component α_i containing the 'side' f_i $(i = 1, 2, \cdots, t)$. Let h be an admitted hyperbolic element and let $h\alpha = \alpha$ where α is a component of E_j. So $\alpha = t\alpha_j$ and $t^{-1}ht\alpha_j = \alpha_j$. Thus $t^{-1}ht(f_j) \in \alpha_j$. Since no point of α_j is a limit point, there are a finite number of copies of F_Γ, abutting on α_j, say $F_\Gamma = u_0(F_\Gamma)$, $u_1(F_\Gamma), \cdots, u_n(F_\Gamma) = t^{-1}ht(F_\Gamma)$ such that $u_i(F_\Gamma)$ is a full neighbour of $u_{i+1}(F_\Gamma)$. Now $u_1 = h_j^{\pm 1}$ and similarly $u_{i+1} = h_j^{\pm 1}u_i$. Thus $t^{-1}ht = h_j^{\pm n}$ and $h = th_j^{\pm n t^{-1}}$.

Thus $h_1, h_2, \cdots, h_t$ is a complete set of admitted hyperbolic representatives (c.s.a.h.r.).

Thus the signature of a Fuchsian group is dependent upon its systems of elliptic, parabolic and admitted hyperbolic representatives, and we now obtain results in this direction for normal subgroups of finite index in a given group. First note the following result which follows by consideration of fixed points [6, p 16].

LEMMA 2. *If* Γ *has presentation* (1) *and* a *is an elliptic, parabolic or admitted hyperbolic generator, then* $ta^rt^{-1} = a^s$ *implies that* $r \equiv s$ (mod $o(a)$) *and* t *is a power of* a.

In [3] Knopp and Newman prove

THEOREM 1. *Let* Γ *be normal in* Γ_0 *and of finite index* μ, *and* $p_1, p_2, \cdots, p_s$ *be a c.s.p.r. for* Γ_0. *Suppose that* p_i *is of exponent* r_i *modulo* Γ, $1 \leq i \leq s$. *Then a c.s.p.r. for* Γ *contains* $\mu \sum_{i=1}^{s} 1/r_i$ *members.*

The proof only uses the fact that $p_1, p_2, \cdots, p_s$ is a complete system of representative for the class of parabolic elements and Lemma 2. It can thus immediately be applied to a c.s.a.h.r. in Γ_0 to obtain the number of elements in a c.s.a.h.r. for Γ. In the case of elliptic elements, we must entertain the possibility that the exponent r_i of x_i modulo Γ is, in fact, equal to the order of x_i. In this case, $x_i^{r_i} = 1$ and so, corresponding to x_i, there are no elliptic representatives in Γ. Thus the number of elements in a c.s.e.r. for Γ would be

$\mu \sum_{i=1, r_i \neq o(x_i)}^{r} 1/r_i$. These elliptic representatives are conjugates of $x_i^{r_i}$ which have order $o(x_i)/r_i$. Thus

COROLLARY 1. *Theorem 1 is true with parabolic replaced by admitted hyperbolic.*

COROLLARY 2. *Theorem 1 is true with parabolic replaced by elliptic and the sum restricted to those i such that* $o(p_i)/r_i > 1$.

These results enable us to compute part of the signature of a normal subgroup Γ of finite index in Γ_0 in terms of the finite index and the signature of Γ_0. It remains to obtain the genus of Γ. If Γ is of index μ in Γ_0 and F_{Γ_0} is a fundamental region for Γ_0, then μ copies of F_{Γ_0}, corresponding to the coset representatives of Γ in Γ_0, form a fundamental region for Γ [4, p 257]. When Γ_0, and hence Γ, are of the first kind we can use the hyperbolic area formula (2) to compute the genus of Γ since,

$$|\Gamma_0/\Gamma| = \mu(F_\Gamma)/\mu(F_{\Gamma_0}). \tag{4}$$

4. In this section we obtain a result akin to (4) for groups of the second kind, using the results and notation of Heins [2]. For Γ of the second kind, let Ω be as in §3. Let ψ_Γ denote the orbit-mapping $\psi_\Gamma : \Omega \to \psi_\Gamma(\Omega)$ such that $\psi_\Gamma(\Omega)$ is a Riemann surface and $\delta_\Gamma : \psi_\Gamma(\Omega) \to N$ the ramification index. Since Γ is finitely-generated, $\psi_\Gamma(\Omega)$ is conformally equivalent to a compact Riemann surface less a finite number of points and $\{q \mid \delta_\Gamma(q) > 1\}$ is finite.

Let $\chi : D \to \Omega$ define D as a universal covering surface of Ω and let $\bar{\Gamma}$ be the group of conformal automorphisms of D leaving $\psi_\Gamma \circ \chi$ invariant. Then $\varphi_{\bar{\Gamma}}(D)$ is conformally equivalent to $\psi_\Gamma \circ \chi(D)$ so that $\tau \circ \varphi_{\bar{\Gamma}} = \psi_\Gamma \circ \chi$ where τ is the conformal mapping. Now $\bar{\Gamma}$ is of the first kind and let $d_{\bar{\Gamma}}$ denote the ramification index of $\varphi_{\bar{\Gamma}}(D)$. Then τ is such that $\tau \circ d_{\bar{\Gamma}} = \delta_\Gamma$ so that the ramification indices of $\varphi_{\bar{\Gamma}}$ and $\varphi_\Gamma(\Omega)$ agree at corresponding points.

Also $\psi_\Gamma(\Omega)$ is the double of $\psi_\Gamma(D)$ and the number of deleted neighbourhoods of point-like boundary elements of $\psi_\Gamma(\Omega)$ is twice the number of such boundary elements of $\psi_\Gamma(D)$. Hence $\varphi_{\bar{\Gamma}}(D)$ has this number of boundary elements which will be the number of parabolic generators of $\bar{\Gamma}$. The genus of $\psi_\Gamma(\Omega)$ will be $2\gamma + (t-1)$, if Γ has presentation (1) with $t > 0$. We thus have

THEOREM 2. *If* Γ *has signature* $(\gamma{:}m_1, m_2, \cdots, m_r; s; t)$ *where* $t > 0$, $\bar{\Gamma}$ *has the signature* $(2\gamma + (t-1); m_1, m_1, m_2, m_2, \cdots, m_r, m_r; 2s; 0)$.

THEOREM 3. *If* Γ *is normal in* Γ_0 *and of index* μ *where* Γ, Γ_0 *are groups of the second kind, then* $\bar{\Gamma}$ *is a subgroup of* $\bar{\Gamma}_0$ *of index* μ, *where* $\bar{\Gamma}$, $\bar{\Gamma}_0$ *are defined as above.*

Proof. Since the set of discontinuity is the same for Γ and Γ_0 we have $\pi : \psi_\Gamma(\Omega) \to \psi_{\Gamma_0}(\Omega)$ such that

$$\pi \circ \psi_\Gamma = \psi_{\Gamma_0}.$$

If τ, τ_0 denote the conformal mappings defined, as above, for Γ, Γ_0 respectively then

$$(\tau_0^{-1} \circ \pi \circ \tau) \circ \varphi_{\bar{\Gamma}} = \tau_0^{-1} \circ \pi \circ (\psi_\Gamma \circ \chi) = \tau_0^{-1} \circ (\psi_{\Gamma_0} \circ \chi) = \varphi_{\bar{\Gamma}_0}.$$

Thus $\bar{\pi} = \tau_0^{-1} \circ \pi \circ \tau$ is such that $\bar{\pi} \circ \varphi_{\bar{\Gamma}} = \varphi_{\bar{\Gamma}_0}$. Let us denote the orbit of $x \in D$ under $\bar{\Gamma}$ by $x^{\bar{\Gamma}}$. Thus $\bar{\pi}(x^{\bar{\Gamma}}) = x^{\bar{\Gamma}_0}$ $x \in D$. Let $\gamma \in \bar{\Gamma}$. Thus $x^{\bar{\Gamma}_0} = \bar{\pi}(x^{\bar{\Gamma}}) = \bar{\pi}((x^\gamma)^{\bar{\Gamma}}) = (x^\gamma)^{\bar{\Gamma}_0}$ for every $x \in D$. So for any $x \in D$, $x^\gamma = x^{\gamma_0(x)}$ where $\gamma_0(x) \in \bar{\Gamma}_0$. Let x, y be distinct points of D such that $d(x, y) < \varepsilon/2$, where d denotes the hyperbolic metric, and x, y are not fixed points of $\bar{\Gamma}$ or $\bar{\Gamma}_0$. Suppose

$$x^\gamma = x^{\gamma_0(x)}, \qquad y^\gamma = y^{\gamma_0(y)}.$$

Since d is invariant with respect to elements of $\mathfrak{L}$, $d(x, y) = d(x^\gamma, y^\gamma) = d(x, y^{\gamma_0(y)\gamma_0(x)^{-1}}) < \varepsilon/2$.

Thus $d(y, y^{\gamma_0(y)\gamma_0(x)^{-1}}) < \varepsilon$. But $\bar{\Gamma}_0$ acts discontinuously on D and does not fix y. Therefore $\gamma_0(y) = \gamma_0(x)$ and $x^{\gamma_0(x)\gamma^{-1}} = x$ and $y^{\gamma_0(x)\gamma^{-1}} = y$, i.e. $\gamma_0(x)\gamma^{-1}$ fixes two points of D. Thus $\gamma = \gamma_0(x) \in \bar{\Gamma}_0$ and we have proved that $\bar{\Gamma} \subset \bar{\Gamma}_0$. The following diagram is commutative

$$\begin{array}{ccc} \varphi_{\bar{\Gamma}}(D) & \xrightarrow{\ \tau\ } & \psi_\Gamma(\Omega) \\ \downarrow{\scriptstyle \bar{\pi}} & & {\scriptstyle \pi}\downarrow \\ \varphi_{\bar{\Gamma}_0}(D) & \xrightarrow{\ \tau_0\ } & \psi_{\Gamma_0}(\Omega) \end{array}$$

Let $\mu' = [\bar{\Gamma}_0 : \bar{\Gamma}]$, so that the inverse image of each point of $\varphi_{\bar{\Gamma}_0}(D)$ under $\bar{\pi}^{-1}$ contains μ' points while the inverse image of each point of $\psi_{\Gamma_0}(\Omega)$ contains μ points. But τ, τ_0 are homeomorphisms, so that $\mu = \mu'$.

5. We now aim to determine for which signatures does there exist a group Γ such that Γ is normal and of finite index in some other Fuchsian group Γ_0. Thus without loss we can assume that the index is a prime p.

The periods in the signature of a group can be considered as unordered as a re-ordering of the elliptic (or parabolic or admitted hyperbolic) generators merely defines an automorphism of the group. In the signature, we use $m^{(p)}$ to denote that the period m is repeated p times.

THEOREM 4. *Let Γ_0 have signature $(\gamma; m_1, \cdots, m_r; s; t)$ with presentation (1) and Γ be normal in Γ_0 of index p, a prime. Suppose that*

(a) *$x_1, \cdots, x_\delta$ have order p*

(b) *$x_1, \cdots, x_\alpha$ $(\alpha \geq \delta)$ have exponent* $> 1 \bmod \Gamma$

(c) *$p_1, \cdots, p_\beta$ have exponent* $> 1 \bmod \Gamma$

(d) *$h_1, \cdots, h_\mu$ have exponent* $> 1 \bmod \Gamma$.

Then Γ has the signature

$$(p\gamma + (p-1)(\alpha + \beta + \mu - 2)/2; m_{\delta+1}/p, \cdots, m_\alpha/p, m_{\alpha+1}^{(p)}, \cdots, m_r^{(p)};$$
$$\beta + p(s - \beta); \mu + p(t - \mu)).$$

Proof. All exponents will be 1 or p. From Theorem 1 the number of parabolic generators of Γ is

$$p \sum_{i=1}^{\beta} 1/p + p \sum_{i=\beta+1}^{s} 1 = \beta + p(s - \beta).$$

From Corollary 1 to Theorem 1, the number of admitted hyperbolic generators will be $\mu + p(t - \mu)$. From Corollary 2 the number of elliptic generators is $(\alpha - \delta) + p(r - \alpha)$ with corresponding periods m_i/p $(i = \delta + 1, \cdots, \alpha)$ and m_i $(i = \alpha + 1, \cdots, r)$ the latter being repeated p times. Each elliptic generator x of order m contributes $(1 - 1/m)$ to the area formula (2). Thus we can without loss, include the δ trivial elliptic generators of Γ without altering the area formula. Let g be the genus of Γ.

Case A. Γ and Γ_0 are of the first kind so $t = 0$ and from equation (4)

$$\mu(F_\Gamma) = p\mu(F_{\Gamma_0}) \tag{5}$$

$$\mu(F_\Gamma) = 2\pi\{2(g-1) + \textstyle\sum_{i=1}^{\alpha} (1 - p/m_i) + p \sum_{i=\alpha+1}^{r} (1 - 1/m_i) + \beta + p(s - \beta)\}$$

Substituting in (5) gives

$$g = p\gamma + (p - 1)(\alpha + \beta - 2)/2.$$

Case B. Γ and Γ_0 are of the second kind so $t > 0$. From Theorem 2, $\bar{\Gamma}_0$ has the signature

$$(2\gamma + (t - 1); m_1^{(2)}, \cdots, m_r^{(2)}; 2s; 0)$$

and $\bar{\Gamma}$ has the signature

$$(2\gamma + (\mu + p(t - \mu) - 1); m_{\delta+1}^{(2)}/p, \cdots, m_\alpha^{(2)}/p, m_{\alpha+1}^{(2p)}, \cdots, m_r^{(2p)}; \\ 2(\beta + p(s - \beta)); 0)$$

and from Theorem 3 and equation (4)

$$\mu(F_{\bar{\Gamma}}) = p\mu(F_{\bar{\Gamma}_0})$$

since $\bar{\Gamma}$, $\bar{\Gamma}_0$ are of the first kind. Substituting in this equation we obtain

$$g = p\gamma + (p - 1)(\alpha + \beta + \mu - 2)/2.$$

We note that Γ being normal in Γ_0 of index p places certain restrictions on α, β, μ. Thus there exists a homomorphism of Γ_0 onto Z_p if and only if

(a) $\alpha + \beta + \mu$ is even if $p = 2$
(b) $\alpha + \beta + \mu \neq 1$
(c) $\gamma > 0$ if $\alpha + \beta + \mu = 0$.

This follows since such a homomorphism exists if and only if Z_p is a factor group of Γ_0/Γ_0' where Γ_0' is the first derived group of Γ_0.

Provided the inequalities (2) and (3) are satisfied for the integers of a given signature, we have pointed out that there exists a Fuchsian group of that signature. Further, if the inequality is satisfied for Γ, it will be satisfied for

Γ_0 from (4) for groups of the first kind and Theorem 3 and (4) for groups of the second kind. Thus we have the converse to the above theorem.

THEOREM 5. *If σ is the signature of a Fuchsian group of the first or second kind, which can be written in the form given at the end of the statement of Theorem 4 for some p, α, β, μ where α, β, μ satisfy conditions* (a), (b), (c) *above, then there exists a group Γ with signature σ and a group Γ_0 such that Γ is normal in Γ_0 of index p.*

6. The remainder of the paper is devoted to obtaining this result in a manageable form. To this end, we adopt the following notation.

Let $\sigma = (\gamma; m_1, m_2, \cdots, m_r; s; t)$ be the signature of a Fuchsian group. Define the equivalence relation on the periods of σ by $m_i \sim m_j$ if $m_i = m_j$. Let the q equivalence classes contain $n_1, n_2, \cdots, n_q$ periods respectively. For a fixed prime p define

$$k(\sigma, p) = \sum_{i=1}^{q} [n_i/p] \tag{6}$$

where $[a]$ denotes the largest integer in $[a]$. Also define

$$l(\sigma, p) = r - pk(\sigma, p) \tag{7}$$

so that $l(\sigma, p)$ is the number of periods which do not fall into sets containing p equal periods. We can assume that the periods of σ are ordered such that the first p are equal, the next p are equal, and so on up to the $pk(\sigma, p)'$-th period.

Now consider the parabolic generators of σ. Define

$$s(\sigma, p) = \text{least non-negative residue} \equiv s \pmod{p} \tag{8}$$

and in the same way define $t(\sigma, p)$ for the admitted hyperbolic generators. From Theorem 5, we see that if σ is to be the signature of a group Γ which is to be normal in Γ_0 of index p, then Γ_0 must have at least $l(\sigma, p)$ elliptic-generators of periods $pm_{pk(\sigma,p)+1}, \cdots, pm_r$ whose exponents are p mod Γ, at least $s(\sigma, p)$ parabolic generators whose exponents are p mod Γ and at least $t(\sigma, p)$ parabolic generators whose exponents are p mod Γ. Finally, define

$$n(\sigma, p) = l(\sigma, p) + s(\sigma, p) + t(\sigma, p). \tag{9}$$

LEMMA 3. *Let $\sigma = (\gamma; m_1, m_2, \cdots, m_r; s; t)$ and let*

$$n(\sigma, 2) = 2n'(\sigma, 2) + \varepsilon(\sigma, 2) \quad \text{where } \varepsilon(\sigma, 2) = 0 \text{ or } 1.$$

Then there exists a group Γ with signature σ and a group Γ_0 such that Γ is of index 2 in Γ_0 if and only if $\gamma \geq n'(\sigma, 2) + \varepsilon(\sigma, 2) - 1$.

Proof. We must choose α, β, μ to satisfy Theorem 5. The number s' of parabolic generators of Γ_0 satisfies $s = \beta + 2(s' - \beta)$ so that $s' = (s + \beta)/2$. Thus β is an integer ≥ 0 with the same parity as s, so set $\beta = 2\beta' + s(\sigma, 2)$. Note that β is bounded above by s. Similarly $\mu = 2\mu' + t(\sigma, 2)$. Since the parity of β, μ is determined by σ, we must choose α so that $\alpha + \beta + \mu$ is even. From Theorems 4 and 5, a group Γ with the periods of σ will be normal in Γ_0 and of index 2 if and only if Γ_0 has periods $\{n_i\}$ of the following form:

$$n_i = m_{2i-1} \quad (i = 1, 2, \cdots, b) \text{ where } 0 \leq b \leq k(\sigma, 2)$$

$$n_{i-b} = 2m_i \quad (i = 2b + 1, \cdots, 2k(\sigma, 2))$$

$$n_{i-b} = 2m_i \quad (i = 2k(\sigma, 2) + 1, \cdots, r)$$

$$n_{i-b} = 2 \quad (i = r + 1, \cdots, r + 2\alpha' + \varepsilon(\sigma, 2)) \text{ where } \alpha' \geq 0.$$

The generators corresponding to the period n_i $(i > b)$ will all have exponent 2 mod Γ and so

$$\alpha = 2(k(\sigma, 2) - b) + l(\sigma, 2) + 2\alpha' + \varepsilon(\sigma, 2)$$

and $\alpha + \beta + \mu$ is even.

It remains to determine the possible genera. Let Γ_0 have genus g and so, by Theorem 4,

$$\gamma = 2g + \tfrac{1}{2}(\alpha + \beta + \mu - 2)$$

$$= 2g + n'(\sigma, 2) + (k(\sigma, 2) - b) + \alpha' + \varepsilon(\sigma, 2) - 1 + \beta' + \mu'$$

Now g, $k(\sigma, 2) - b$, α', β', μ' all take non-negative values and g and α' are unbounded. Provided condition (c) is satisfied, i.e. in all but a finite number of cases, we can choose $g = k(\sigma, 2) - b = \beta' = \mu' = 0$ and $\alpha' \geq 0$, giving $\gamma \geq n'(\sigma, 2) + \varepsilon(\sigma, 2) - 1$. In the finite number of exceptional cases, we find that the criteria for the existence of Γ_0 is the same inequality.

LEMMA 4. *Let* $\sigma = (\gamma; m_1, m_2, \cdots, m_r; s; t)$. *Then there exists a group* Γ *with signature* σ *and a group* Γ_0 *such that* Γ *is normal in* Γ_0 *of index* p *where* p *is a prime* > 2 *if and only if*

$$[2\gamma/(p - 1)] \geq (2\gamma + n(\sigma, p) - 2)/p \tag{10}$$

and

$$[2\gamma/(p - 1)] \neq (2\gamma - 1)/p. \tag{11}$$

Proof. Using the same notation and argument as Lemma 3, we must have $\beta = p\beta' + s(\sigma, p)$ and $\mu = p\mu' + t(\sigma, p)$. Also a group Γ with the periods of σ will be normal in Γ_0 of index p if and only if Γ_0 has periods $\{n_i\}$ of the form:

$$n_i = m_{pi-(p-1)} \quad (i = 1, 2, \cdots, b) \text{ where } 0 \leq b \leq k(\sigma, p)$$

$$n_{i-(p-1)b} = pm_i \quad (i = pb + 1, \cdots, pk(\sigma, p))$$

$$n_{i-(p-1)b} = pm_i \quad (i = pk(\sigma, p) + 1, \cdots, r)$$

$$n_{i-(p-1)b} = p \quad (i = r + 1, \cdots, r + \alpha') \text{ where } \alpha' \geq 0$$

provided b, α', β', μ' can be chosen such that $\alpha + \beta + \mu \neq 1$.

Let Γ_0 have genus g. Then by Theorem 4

$$\gamma = pg + \tfrac{1}{2}(p-1)m \tag{12}$$

where
$$m = \alpha + \beta + \mu - 2$$
$$= n(\sigma, p) - 2 + \alpha' + p(k(\sigma, p) - b + \beta' + \mu')$$

so that
$$m \geq n(\sigma, p) - 2.$$

If the only solution of (12) in the range $g \geq 0, m \geq n(\sigma, p) - 2$, gives $m = -1$, then $\alpha + \beta + \mu = 1$. In the remaining cases, as in Lemma 3 with the exception of a finite number, we obtain the possible values of γ by taking $k(\sigma, p) - b = \beta' = \mu' = 0$ and $g \geq 0$, $\alpha' \geq 0$. The general solution of the linear diophantine equation (12) is given by $g = \gamma - \frac{1}{2}(p-1)y, m = py - 2\gamma$.

We require that $\gamma - \frac{1}{2}(p-1)y \geq 0$, $py - 2\gamma \geq n(\sigma, p) - 2$ and that $py - 2\gamma = -1$ does not give the unique solution of (12) i.e. that there exists an integer y such that

$$2\gamma/(p-1) \geq y \geq (2\gamma + n(\sigma, p) - 2)/2$$

and that $y = (2\gamma - 1)/p$ is not the unique solution of these inequalities. These conditions are equivalent to (10) and (11) and in the finite number of exceptional cases the same criteria are obtained.

7. If we substitute $p = 2$ in (10), it reduces to the inequality of Lemma 3 and (11) becomes trivial. Thus Lemma 4 can be taken to include all primes. From our definitions in §1, Max Normal $(\Gamma) = T(\Gamma)$ if and only if, for all p, either

$$[2\gamma/(p-1)] < (2\gamma + n(\sigma, p) - 2)/p$$

or $[2\gamma/(p-1)] = (2\gamma - 1)/p$. Of course, since Γ is finitely-generated, we need only investigate these for a finite number of primes. Indeed, if $\gamma \geq 2$, $2\gamma + n(\sigma, p) - 2 > 0$. Thus if $p - 1 > 2\gamma$, the inequality always holds. If $\gamma = 1$, $2\gamma + n(\sigma, p) - 2 > 0$ unless $n(\sigma, p) = 0$. Thus the inequality holds for all primes $p > 3$ except, perhaps, those such that $r \equiv s \equiv t \equiv 0 \pmod{p}$. Similarly for $\gamma = 0$. The equation is invalid in the cases $\gamma = 0, 1$.

THEOREM 6. *Let* $\sigma = (\gamma; m_1, m_2, \cdots, m_r; s; t)$ *be the signature of a Fuchsian group* Γ *of the first or second kind as defined in* §2. *Then*

$$\text{Max Normal}(\Gamma) = T(\Gamma)$$

if and only if either

$$[2\gamma/(p-1)] < (2\gamma + n(\sigma, p) - 2)/p \quad \text{or} \quad [2\gamma/(p-1)] = (2\gamma - 1)/p$$

holds

(a) *for all primes* $p \leq 2\gamma + 1$ *if* $\gamma > 1$

(b) *for all primes* $p \leq 3$ *or such that* $r \equiv s \equiv t \equiv 0 \pmod{p}$ *if* $\gamma = 1$

(c) *for all primes* p *such that* $r, s, t \equiv 0, 1, 2 \pmod{p}$ *if* $\gamma = 0$ *where* $[a]$ *denotes the largest integer in* a *and* $n(\sigma, p)$ *is defined at* (9).

Remark. It had been conjectured independently that if the periods of σ were co-prime in pairs then max $(\Gamma) = T(\Gamma)$. A study of the above result in such a situation and the fact that Max $(\Gamma) \subset$ Max Normal (Γ) shows that this is false.

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