MAXIMAL SETS OF INTEGERS WITH DISTINCT DIVISORS

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Abstract. A set of positive integers is said to have the *distinct divisor property* if there is an injective map that sends every integer in the set to one of its proper divisors. In 1983, P. Erdős and C. Pomerance showed that for every $c > 1$, a largest subset of $[N, cN]$ with the distinct divisor property has cardinality $\sim \delta(c)N$, for some constant $\delta(c) > 0$. They conjectured that $\delta(c) \sim c/2$ as $c \to \infty$. We prove their conjecture. In fact we show that there exist positive absolute constants D_1, D_2 such that $D_1 \le c^\beta(c/2 - \delta(c)) \le D_2$ where $\beta = \log 2/\log(3/2)$.

1. Introduction

Let S denote a set of positive integers and $\tau : S \to \mathbb{N}$ be defined so that $\tau(s)$ is a proper divisor of s (that is, $\tau(s)$ divides s and $\tau(s) < s$). The ensemble (S, τ) is said to have the '*distinct divisor property*' if τ is injective, that is, if the $\tau(s)$ are different for different values of s. We will also say that S has the distinct divisor property if there exists a τ, as above, such that (S, τ) has the distinct divisor property.

Let $c > 1$ denote a real number and N a large natural number. Let S be a subset of $[N, cN]$ with the distinct divisor property such that, of all subsets of $[N, cN]$ having distinct divisors, S has maximal cardinality. If c is fixed and N tends to infinity then P. Erdős and C. Pomerance, [1], have shown that

$$|S| = (\delta(c) + o(1))N$$

where $\delta(c)$ is a continuous increasing function of c. As c tends to 1 they established that

$$\delta(c) = c - 1 + o(1).$$

In this note we are concerned with the behaviour of $\delta(c)$ as c tends to infinity. Division by 2 clearly invests the set of even integers in $[N, cN]$ with the distinct divisor property; hence $\delta(c) \ge (c-1)/2$. Also, since a proper divisor of an integer less than cN is less than $cN/2$ clearly $\delta(c) \le c/2$. Erdős and Pomerance conjectured that this latter upper bound is actually the truth for large c. In other words they conjectured that as c tends to infinity

$$\delta(c) = \frac{c}{2} + o(1).$$

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We prove this and more by finding the exact order of magnitude for $c/2 - \delta(c)$ as $c \to \infty$.

Theorem 1. *There exist positive absolute constants D_1 and D_2 such that*

$$\frac{D_1}{c^\beta} \le \frac{c}{2} - \delta(c) \le \frac{D_2}{c^\beta},$$

where $\beta = \log 2/\log(3/2) = 1.7095\ldots$.

We realise Theorem 1 as the sum of the following two Propositions, which are proved by two very different arguments.

Proposition 2. *Let k denote the greatest integer not exceeding $\log c/\log(3/2)$. Suppose S is a subset of the integers in $[N, cN]$ and that (S, τ) satisfies the distinct divisor property. Then*

$$\frac{cN}{2} - |S| \ge \frac{N}{2^{k+2}} + O(k).$$

Proposition 3. *Suppose $c > 2$. There exists a subset, S, of integers in $[N, cN]$ and a map τ such that (S, τ) obeys the distinct divisor property and with*

$$\frac{cN}{2} - |S| \ll \frac{N}{c^\beta}.$$

All implied constants are absolute; that is they are independent of c and N. The restriction to $c > 2$ in Proposition 3 is obviously harmless. The presence of the constant β is best explained by noting that it is the minimum value of the function $\log p_i/\log(p_{i+1}/p_i)$ (where p_i denotes the ith smallest prime).

We thank Professor A. Granville to whom our present exposition is largely due. An earlier version of this note proved the weaker result $\delta(c) = c/2 + o(1)$. We are grateful to the referee, Professor C. Pomerance, who, by simplifying our earlier proof, helped clarify the situation and motivated us to strengthen our result.

2. Proof of Proposition 2

We partition the interval $(N, cN]$ into the sets $B_1 \cup B_2 \cup \cdots \cup B_{k+1}$ where $B_j = ((2/3)^j cN, (2/3)^{j-1} cN]$ for $j = 1, 2, \ldots, k$, and $B_{k+1} = (N, (2/3)^k cN]$. Similarly we partition $[1, cN/2]$, where the potential divisors lie, into intervals $A_1 \cup A_2 \cup \cdots \cup A_{k+2}$, where $A_i = ((2/3)^i cN/2, (2/3)^{i-1} cN/2]$ for $i = 1, 2, \ldots, k$, with $A_{k+1} = (N/2, (2/3)^k cN/2]$ and $A_{k+2} = (1, N/2]$. Note that if $s \in B_j$ then any proper divisor of s must lie in some interval A_i with $i \ge j$; moreover, if that divisor lies in A_j, then it must be $s/2$, since any other proper divisor is $\le s/3 \le (2/3)^{j-1} cN/3 = (2/3)^j cN/2$ and thus belongs to A_i for some $i > j$.

Now $[cN/2] - |S| = [cN/2] - |\tau(S)|$ counts the number of integers in $[1, cN/2]$ that do not belong to $\tau(S)$. We obtain a lower bound for this quantity by only counting, for each i, those integers $n \in A_i$ which do not belong to $\tau(S)$, and which are divisible by 2^{i-1}. Thus

$$[cN/2] - |S| \ge \sum_{i=1}^{k+2} \left(\#\{n \in A_i :\ 2^{i-1} | n\} - \#\{s \in S : \tau(s) \in A_i,\ 2^{i-1} | \tau(s)\}\right).$$

As we saw above, if $\tau(s) \in A_i$ then $s \in B_j$ for some $j \le i$. Suppose that 2^{i-1} divides $\tau(s)$. We claim that 2^j divides s, which follows if $j = i$ since 2^{j-1} divides $\tau(s) = s/2$; and which follows if $j < i$ since then 2^j divides 2^{i-1}, which divides $\tau(s)$, which divides s. Therefore

$$\sum_{i=1}^{k+2} \#\{s \in S : \tau(s) \in A_i,\ 2^{i-1}|\tau(s)\} \le \sum_{j=1}^{k+1} \#\{s \in B_j :\ 2^j|s\}$$

(noting that, since τ is injective, no value of s gets counted twice in the argument above). Now $\#\{n \in A_i :\ 2^{i-1}|n\} = \#\{n \in B_i :\ 2^i|n\} + O(1)$, so substituting this into the two displays above, we get

$$[cN/2] - |S| \ge \#\{n \in A_{k+2} :\ 2^{k+1}|n\} + O(k) = N/2^{k+2} + O(k).$$

3. Proof of Proposition 3

We wish to construct a 'big' set S of integers s in $[N, cN]$ with the distinct divisors $\tau(s)$; since τ is injective, this is equivalent to constructing a 'big' set $R = \tau(S) \subset [1, cN/2]$, such that for each $n \in R$, there exists some distinct proper multiple $\tau^{-1}(n)$, of n, in $[N, cN]$. In fact we shall select $\tau^{-1}(n) = np(n)$ for some prime $p(n)$, which we choose as follows: For $n = [cN/2]$ let $p([cN/2]) = 2$. For $n = [cN/2] - 1, [cN/2] - 2, \ldots, 1$ we define $p(n)$ to be the *largest* prime p for which

i) $N < np \le cN$, and
ii) $np \ne n'p(n')$ for any $n' > n$, with $n' \le [cN/2]$,

provided such a prime p exists, otherwise we let $p(n) = 0$ (and then $n \notin R$). We note that $|S| = |R|$ is exactly the number of integers $n \le cN/2$ for which $p(n) \ne 0$; and thus

$$[cN/2] - |S| = \#\{n \le cN/2 :\ p(n) = 0\}. \tag{1}$$

For each prime p_k, we define the set of integers

$$\mathcal{I}_k = \{p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} :\ \alpha_k \ge 1,\ \prod_{j=1}^{k} (p_{j+1}/p_j)^{\alpha_j} > c/2\}.$$

Lemma. *If $p(n) = 0$ for some integer $n \le cN/2$, then there exists k such that $n \le N/p_k$, and $\mathcal{I}_k$ contains a divisor d of n.*

We now complete the proof of Proposition 3, postponing the proof of the Lemma:

Proof of Proposition 3. Using the Lemma we have

$$\#\{n \le cN/2 :\ p(n) = 0\} \le \sum_{k\ge1} \sum_{d\in\mathcal{I}_k} \#\{n \le N/p_k :\ d|n\} \le \sum_{k\ge1} \frac{N}{p_k} \sum_{d\in\mathcal{I}_k} \frac{1}{d}. \tag{2}$$

By definition, we have that

$$\sum_{d\in\mathcal{I}_k} \frac{1}{d} \le \sum_{\alpha_k\ge1} \frac{1}{p_k^{\alpha_k}} \sum_{\alpha_{k-1}\ge0} \frac{1}{p_{k-1}^{\alpha_{k-1}}} \cdots \sum_{\alpha_2\ge0} \frac{1}{p_2^{\alpha_2}} \sum_{\alpha_1\ge A_1} \frac{1}{2^{\alpha_1}}, \tag{3}$$

where $(3/2)^{A_1} > (c/2)/\prod_{j=2}^{k}(p_{j+1}/p_j)^{\alpha_j} \geq (c/2)/(5/3)^{(\alpha_2+\alpha_3+\cdots+\alpha_k)}$, from the definition of the set $\mathcal{I}_k$, since $p_{j+1}/p_j \leq 5/3$ when $j \geq 2$. Therefore, setting $\gamma = 2^{\log(5/3)/\log(3/2)} \approx 2.39471$, we get

$$\sum_{\alpha_1 \geq A_1} \frac{1}{2^{\alpha_1}} \ll \frac{1}{2^{A_1}} \ll c^{-\beta}\gamma^{\alpha_2+\alpha_3+\cdots+\alpha_k}.$$

Substituting this into (3) gives

$$\sum_{d\in\mathcal{I}_k} \frac{1}{d} \ll c^{-\beta} \sum_{\alpha_k \geq 1} \left(\frac{\gamma}{p_k}\right)^{\alpha_k} \sum_{\alpha_{k-1}\geq 0} \left(\frac{\gamma}{p_{k-1}}\right)^{\alpha_{k-1}} \cdots \sum_{\alpha_2 \geq 0} \left(\frac{\gamma}{p_2}\right)^{\alpha_2}$$
$$= c^{-\beta}\frac{\gamma}{p_k}\prod_{i=2}^{k}\left(1-\frac{\gamma}{p_i}\right)^{-1} \ll c^{-\beta}\frac{1}{p_k}\prod_{3\leq p\leq p_k}\left(1-\frac{1}{p}\right)^{-\gamma} \ll c^{-\beta}\frac{(\log p_k)^{\gamma}}{p_k},$$

using Mertens' theorem that $\prod_{p\leq x}(1-1/p) \asymp 1/\log x$ (see [2] for example). Substituting this estimate into (2), and that estimate back into (1), gives

$$cN/2 - |S| = \#\{n \leq N/2 :\ p(n) = 0\} \ll c^{-\beta} N \sum_{k\geq 1} \frac{(\log p_k)^{\gamma}}{p_k^2} \ll N/c^{\beta}.$$

Finally we return to the

Proof of the Lemma. We must have $n \leq N/2$ for, if $cN/2 \geq n > N/2$ then $p = 2$ satisfies i) $N < 2n \leq cN$, and ii) $2n \neq n'p(n')$ for any $n' > n$, since $n'p(n') \geq 2n' > 2n$, so that $p(n) \geq 2$.

Let p_{k_0} be the least prime exceeding N/n; by Bertrand's postulate $p_{k_0} \leq 2N/n < cN/n$ (since $c > 2$), and so $N < np_{k_0} \leq cN$. However $p(n) = 0$, which means that np_{k_0} cannot satisfy (ii) above; in other words, there must exist an integer $n_1 > n$ such that $np_{k_0} = n_1 p_{k_1}$ (where we define k_1 so that $p_{k_1} = p(n_1)$). We note that $p_{k_0} > p_{k_1}$ (since $n_1 > n$ and $np_{k_0} = n_1 p_{k_1}$), so that $p_{k_1} \leq N/n$ and thus $n \leq N/p_{k_1}$.

We now construct a useful sequence of integers $n_1, n_2, n_3, \ldots, n_m \in R$ (for some m); we show how to determine n_{j+1} from n_j:

Let k_j be defined by the relation $p_{k_j} = p(n_j)$.

- If $n_j p_{k_j+1} > cN$ then let $m = j$, and the sequence is terminated.
- If $n_j p_{k_j+1} \leq cN$ then there must exist an integer $n_{j+1} > n_j$ for which $n_j p_{k_j+1} = n_{j+1}p(n_{j+1})$ (else $p(n_j) \geq p_{k_j+1}$ by definition).

Since $n_{j+1}p_{k_{j+1}+1} > n_{j+1}p_{k_{j+1}} = n_j p_{k_j+1}$, we see that $n_1 p_{k_1+1} < n_2 p_{k_2+1} < n_3 p_{k_3+1} < \ldots$ forms an increasing sequence of integers, and so we will eventually find an integer m for which $n_m p_{k_m+1} > cN$.

We have seen that $n < n_1 < n_2 < \cdots < n_m$, and thus $p_{k_0} > p_{k_1} \geq p_{k_2} \geq p_{k_3} \geq \cdots \geq p_{k_m}$ (since $n_{j+1} > n_j$ and $n_j p_{k_j+1} = n_{j+1}p_{k_{j+1}}$ imply that $p_{k_j+1} > p_{k_{j+1}}$, and thus $p_{k_j} \geq p_{k_{j+1}}$). Now $n_{j+1} = (p_{k_j+1}/p_{k_{j+1}})n_j$; iterating this gives

$$n_j = \left(\frac{p_{k_{j-1}+1}}{p_{k_j}}\right)\left(\frac{p_{k_{j-2}+1}}{p_{k_{j-1}}}\right)\cdots\left(\frac{p_{k_1+1}}{p_{k_2}}\right)\left(\frac{p_{k_0}}{p_{k_1}}\right)n. \tag{4}$$

Define $d = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} = p_{k_m} p_{k_{m-1}} \dots p_{k_2} p_{k_1}$ where $k = k_1$. We show that d divides n by proving that $p_i^{\alpha_i}$ divides n for each i: Let j be the largest integer for which $k_j = i$. Then $p_{k_j} p_{k_{j-1}} \dots p_{k_1} = p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \dots p_k^{\alpha_k}$. Moreover $i \le k_{j-1} < k_{j-1} + 1 \le k_{j-2} + 1 \le \dots \le k_1 + 1 \le k_0$, and so p_i is coprime with $p_{k_{j-1}+1} p_{k_{j-2}+1} \dots p_{k_1+1} p_{k_0}$. Now, $p_i^{\alpha_i}$ divides $p_{k_j} p_{k_{j-1}} \dots p_{k_1}$, which is a divisor of $(p_{k_{j-1}+1} p_{k_{j-2}+1} \dots p_{k_1+1} p_{k_0})n$ by (4), since n_j is an integer; and so $p_i^{\alpha_i}$ divides n.

To complete the proof of the Lemma we need to show that $d \in \mathcal{I}_k$, which we do by taking (4) with $j = m$, multiplying it by $p_{k_{m+1}}$ and rearranging, to get

$$\prod_{i=1}^{k} \left(\frac{p_{i+i}}{p_i} \right)^{\alpha_i} = \left(\frac{p_{k_m+1}}{p_{k_m}} \right) \left(\frac{p_{k_{m-1}+1}}{p_{k_{m-1}}} \right) \dots \left(\frac{p_{k_1+1}}{p_{k_1}} \right) = \frac{n_m p_{k_m+1}}{n p_{k_0}} > \frac{cN}{2N} = \frac{c}{2},$$

using the fact that $p_{k_0} \le 2N/n$, by Bertrand's postulate.

References

[1] P. Erdős and C. Pomerance, *An analogue of Grimm's problem of finding distinct prime factors of consecutive integers.* Util. Math. **24** (1983), 45-65.

[2] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers.* Fourth edition; New York; Oxford University Press, 1960.