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MAXIMAL SLICES IN ANTI-DE SITTER SPACES

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Abstract. We prove the existence of maximal slices in anti-de Sitter spaces (ADS spaces) with small boundary data at spatial infinity. The main argument is carried out by implicit function theorem. We also get a necessary and sufficient condition for the boundary behavior of totally geodesic slices in ADS spaces. Moreover, we show that any isometric and maximal embedding of hyperbolic spaces into ADS spaces must be totally geodesic. Combined with this, we see that most of maximal slices obtained in this paper are not isometric to hyperbolic spaces, which implies that the Bernstein Theorem in ADS space fails.

1. Introduction. Finding a minimal surface with a given boundary data is an interesting problem in Riemannian geometry. In particular, the existence and regularity of minimal hypersurfaces with a prescribing asymptotic boundary at infinity in hyperbolic space $\boldsymbol{H}^n$ have been discussed in [2, 3, 10, 11], etc. On the other hand, we know that a maximal slice, which is a spacelike hypersurface of a Lorentzian manifold and a critical point of the induced area functional, plays an important role in General Relativity. It was used in the first proof of the positive mass theorem [12] and in the analysis of the Cauchy problem for asymptotically flat space-times. Many interesting results for the existence of compact maximal slices had been obtained in, e.g., [4, 5, 7]. For complete noncompact cases, we have known that there are entire solutions in asymptotically flat space-times (see [4]). It should be pointed out that a complete maximal hypersurface in Minkowski space must be totally geodesic, i.e., a hyperplane (see [6]). Anti-de Sitter (ADS) space is a Lorentzian manifold with negative constant sectional curvature, which plays in Lorentzian geometry a similar role as that played by the hyperbolic space $\boldsymbol{H}^n$ in Riemannian geometry. So, it is natural to study maximal slices in ADS spaces. Also, we note that all the time slices (level sets of the time function) are isometric to $\boldsymbol{H}^n$ and are totally geodesic, and hence are maximal. It may be of some interest in view of geometry to find maximal slices which are not totally geodesic. By assuming a global barrier condition in asymptotically ADS spaces, Akutagawa [1] proved the existence of entire maximal slices with certain decay of the height function at infinity. In ADS space case, he also showed that, if the height function of a maximal slice satisfies this decay condition at spatial infinity, the maximal slice must be a time slice (Proposition 3 in [1]).

In this paper, we obtain some maximal slices by implicit function theorem, which can be regarded as perturbations of time slices. These maximal slices are $C^{1,1}$ up to the boundary.

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We also obtain a necessary and sufficient condition for boundary values to be totally geodesic slices in ADS space. Moreover, we show that any isometric and maximal embedding of hyperbolic spaces into ADS space must be totally geodesic. Together with this, we see that most of maximal slices obtained in this paper are not isometric to hyperbolic spaces, which implies that the Bernstein Theorem in ADS space fails. Note that a similar problem in the setting of hyperbolic spaces have been studied in [13].

Indeed, a maximal slice in ADS space satisfies a second order PDE in $\boldsymbol{H}^n$ (see Equation (1)). Therefore, it is natural to consider the Dirichlet problem for maximal slices of ADS spaces with infinity boundary value on $\boldsymbol{H}^n$. We shall address this problem in our forthcoming papers.

This paper is organized as follows. In Section 2, we derive the equation satisfied by maximal slices and its corresponding linearized equation. In Section 3, we show that the linearized operator is an isomorphism between some weighted Hölder spaces. Hence, using implicit function theorem, we prove our main result Theorem 3.1. In Section 4, we prove a necessary and sufficient condition for boundary value to be totally geodesic slices in ADS spaces. We also show that any isometric and maximal embedding of $\boldsymbol{H}^n$ into ADS space is totally geodesic. By these facts we see that most of our solutions are not totally geodesic.

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2. Maximal slice equation in anti-de Sitter space. In this section, we will derive the maximal slice equation in anti-de Sitter space. Let us begin with some basic facts. Suppose

$$\boldsymbol{H}^n = \left(\boldsymbol{B}^n, \frac{4(dr^2 + r^2 d\sigma_0^2)}{(1-r^2)^2}\right)$$

where $\boldsymbol{B}^n$ is the unit ball in $\boldsymbol{R}^n$, r is the Euclidean distance to the center of the unit ball, and $d\sigma_0^2$ is the standard metric on $\boldsymbol{S}^{n-1}$. Let $\rho = -\log r$, we know the hyperbolic space can also be expressed as:

$$\boldsymbol{H}^n = (\boldsymbol{R}^+ \times \boldsymbol{S}^{n-1}, \sinh^{-2}\rho \cdot (d\rho^2 + d\sigma_0^2)),$$

Then $n+1$ dimensional anti-de Sitter space V can be expressed as a warped product of $\boldsymbol{R}$ and $\boldsymbol{H}^n$, namely, $V = (\boldsymbol{R} \times \boldsymbol{H}^n, ds^2)$, where $ds^2 = -\coth^2\rho dt^2 + \sinh^{-2}\rho(d\rho^2 + d\sigma_0^2)$. As is well-known, V is a vacuum solution of Einstein fields equations with a negative cosmological constant. We denote the canonical connection in V by $\bar{\nabla}$. Let M^n be a smooth spacelike hypersurface in V. The height function $u \in C^\infty(M)$ of M is the restriction of the time function t to M. Then M can be regarded as a graph over $\boldsymbol{H}^n$. In the following, we assume that $M = \{(x, u(x)) \mid x \in \boldsymbol{H}^n\}$, and u is defined on the whole V by requiring $\partial u/\partial t = 0$.

Note that M is then a level set of $f(t,x) = t - u(x)$. By direct computation, we see that the future-directed unit normal vector N to M is given by

$$N = |\bar{\nabla} f|^{-1}[\bar{\nabla} f] = \frac{1}{\sqrt{1-\coth^2\rho|\nabla u|^2}}\left(\coth\rho\nabla u + \tanh\rho\frac{\partial}{\partial t}\right),$$

where and in the sequel, ∇, div and Δ are the gradient, divergence and Laplacian on $\boldsymbol{H}^n$, respectively.

Let $\Box$ be the wave operator in V, H_M be the mean curvature of M in V with respect to N. Then, by direct computation, we see that

$$\Box f = -\Delta u + \tanh\rho \frac{\partial u}{\partial \rho}\,.$$

On the other hand, we also have

$$\Box f = -NNf - H_M \cdot Nf\,.$$

Thus, we see that

$$H_M = \tanh\rho \operatorname{div}\left(\frac{\coth^2\rho\nabla u}{\sqrt{1-\coth^2\rho|\nabla u|^2}}\right).$$

If M is maximal, we have

$$\tanh\rho \operatorname{div}\left(\frac{\coth^2\rho\nabla u}{\sqrt{1-\coth^2\rho|\nabla u|^2}}\right) = 0\,,$$

which is equivalent to

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-\coth^2\rho|\nabla u|^2}}\right) - \frac{2\tanh\rho\ \partial u/\partial\rho}{\sqrt{1-\coth^2\rho|\nabla u|^2}} = 0\,. \tag{1}$$

From the structure of the equation, we see that if u_ε is a solution of the following equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-\varepsilon\coth^2\rho|\nabla u|^2}}\right) - \frac{2\tanh\rho\ \partial u/\partial\rho}{\sqrt{1-\varepsilon\coth^2\rho|\nabla u|^2}} = 0 \tag{2}$$

for some $\varepsilon > 0$, then $\sqrt{\varepsilon}u_\varepsilon$ is a solution of Equation (1).

In the following, we consider a family of operators defined by

$$\begin{aligned} F(u,\varepsilon) &:= \operatorname{div}\left(\frac{\nabla u}{\sqrt{1-\varepsilon\coth^2\rho|\nabla u|^2}}\right) - \frac{2\tanh\rho\ \partial u/\partial\rho}{\sqrt{1-\varepsilon\coth^2\rho|\nabla u|^2}} \\ &= 0\,. \end{aligned}$$

It is easy to see that

$$F(u,0) = \Delta u - 2\tanh\rho\frac{\partial u}{\partial\rho} = 0$$

is the linearized equation of (1) at its trivial solution $u = 0$.

For the purpose of further discussion, we need to consider the following Dirichlet problem:

$$\begin{cases} \boldsymbol{L}(u) := \Delta u - 2\tanh\rho\ \partial u/\partial\rho = 0\,, & \text{in } \boldsymbol{H}^n\,, \\ u|_{\boldsymbol{S}^{n-1}} = \varphi\,, \end{cases} \tag{3}$$

where φ is a smooth function defined on the infinity boundary of $\boldsymbol{H}^n$. Here and in the sequel, $\boldsymbol{S}^{n-1}$ is regarded as the infinity boundary of $\boldsymbol{H}^n$. Besides the above facts, we need to introduce the ball model for $\boldsymbol{H}^n$ which is denoted by $(\boldsymbol{D}^n, dS^2)$. Here, $\boldsymbol{D}^n$ is the unit ball in $\boldsymbol{R}^n$, and dS^2 is the standard hyperbolic metric which is defined to be:

$$dS^2 = \tau^{-2} \sum_{i=1}^{n} (dx^i)^2 ,$$

where $\tau(x) = (1 - |x|^2)/2$ and $\sum_{i=1}^n (dx^i)^2$ is the Euclidean metric. The relation between ρ and τ can be expressed as $\sinh \rho = \tau/r$, where $r(x) = |x|$ is the Euclidean distance from the origin. Hence Equations (1), (2) and (3) can be written as

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - ((1-\tau)/\tau)^2 |\nabla u|^2}}\right) + \frac{2\tau}{1-\tau} \frac{\sum_{i=1}^{n} x^i \dfrac{\partial u}{\partial x^i}}{\sqrt{1 - ((1-\tau)/\tau)^2 |\nabla u|^2}} = 0 ,$$

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - \varepsilon((1-\tau)/\tau)^2 |\nabla u|^2}}\right) + \frac{2\tau}{1-\tau} \frac{\sum_{i=1}^{n} x^i \dfrac{\partial u}{\partial x^i}}{\sqrt{1 - \varepsilon((1-\tau)/\tau)^2 |\nabla u|^2}} = 0 ,$$

and

$$(4) \qquad \begin{cases} \boldsymbol{L}(u) := \Delta u + \dfrac{2\tau}{1-\tau} \displaystyle\sum_{i=1}^{n} x^i \dfrac{\partial u}{\partial x^i} = 0 \quad \text{in } \boldsymbol{H}^n , \\ u|_{\boldsymbol{S}^{n-1}} = \varphi , \end{cases}$$

respectively.

3. Existence of maximal slices, Weighted Hölder spaces and analysis of the linearized equation. In this section, we prove the existence of maximal slices in $n+1$ dimensional anti-de Sitter space V with certain boundary data at infinity. More specifically, we are going to show the following

THEOREM 3.1. *Given any $\varphi \in C^{4,\alpha}(\boldsymbol{S}^{n-1})$, there is a positive constant $\delta = \delta(\varphi) > 0$ such that for any $\varepsilon \in (0, \delta)$, the Dirichlet problem*

$$(5) \qquad \begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1 - ((1-\tau)/\tau)^2 |\nabla u|^2}}\right) + \dfrac{2\tau}{1-\tau} \dfrac{\sum_{i=1}^{n} x^i \dfrac{\partial u}{\partial x^i}}{\sqrt{1 - ((1-\tau)/\tau)^2 |\nabla u|^2}} \\ = 0 \quad \text{in } \boldsymbol{H}^n , \\ u|_{\boldsymbol{S}^{n-1}} = \sqrt{\varepsilon}\varphi \end{cases}$$

admits a solution $u \in C^2(\boldsymbol{D}^n)$ with $\|u\|_{C^2(\boldsymbol{D}^n)} \le C$, where C is a constant depending on φ.

REMARK 3.2. 1. In Theorem 3.1, we adopt the ball model for $\boldsymbol{H}^n$, and u is regarded as a function defined on $\boldsymbol{D}^n$.

2. The second fundamental form of our solution in Theorem 3.1 decays as $O(\tau^2)$ when τ goes to 0. We conjecture that the solution with the second fundamental form decaying faster than quadratic must be $\boldsymbol{H}^n$.

In order to prove Theorem 3.1, we show some basic estimates for the linearied equation by which we are able to show that the corresponding linear elliptic operator is a linear isomorphism between specific function spaces. To do this, let us first introduce certain weighted Hölder spaces defined on $\Omega \subset \boldsymbol{D}^n$, for $0 \le k \in \boldsymbol{Z}$ (for more details, see [8]). Let $C^k(\bar{\Omega})$ be the usual Banach spaces of k times continuously differential functions on $\bar{\Omega}$, and for $0 < \alpha < 1$ denote by $C^{k,\alpha}(\bar{\Omega})$ the subspace of functions whose k-th derivatives satisfy a uniform Hölder condition of order α, with the usual norms denoted by $\|\cdot\|_{k;\Omega}$, $\|\cdot\|_{k,\alpha;\Omega}$, respectively. Also, denote by $C^k(\Omega)$ and $C^{k,\alpha}(\Omega)$ the linear spaces of functions satisfying the corresponding estimates uniformly on compact subsets of Ω. For $s \in \boldsymbol{R}$ define

$$\|w\|^{(s)}_{k,0;\Omega} = \sum_{l=0}^{k} \sum_{|\gamma|=l} \|\tau^{-s+l} \partial^\gamma w\|_{L^\infty(\Omega)}\,,$$

where for any multi-index γ, $\partial^\gamma = \partial^{|\gamma|}/\partial x^\gamma$; and for $0 < \alpha < 1$ define

$$\|w\|^{(s)}_{k,\alpha;\Omega} = \|w\|^{(s)}_{k,0;\Omega} + \sum_{|\gamma|=k} \sup_{x,y\in\Omega} \left[\min(\tau^{-s+k+\alpha}(x), \tau^{-s+k+\alpha}(y)) \frac{|\partial^\gamma w(x) - \partial^\gamma w(y)|}{|x-y|^\alpha} \right].$$

Let $\Lambda^s_{k,\alpha;\Omega} = \{w \in C^{k,\alpha}(\Omega) \mid \|w\|^{(s)}_{k,\alpha;\Omega} < +\infty\}$, which is a Banach space. For $x \in \boldsymbol{H}^n$, let $B(x)$ be the open Euclidean ball with center x and radius $\tau(x)/3$. Then we have the following

LEMMA 3.3. *For any $\Omega' \subset \Omega \subset \boldsymbol{H}^n$, we have $\Lambda^s_{k,\alpha;\Omega} \subset \Lambda^s_{k,\alpha;\Omega'}$, and*

$$\|w\|^{(s)}_{k,\alpha;\Omega'} \le \|w\|^{(s)}_{k,\alpha;\Omega}$$

for any $w \in \Lambda^s_{k,\alpha;\Omega}$. Also, for any $\Omega_m \subset \Omega_{m+1} \subset \boldsymbol{H}^n$ and $\bigcup_m \Omega_m = \boldsymbol{H}^n$, we have

$$\|w\|^{(s)}_{k,\alpha;\boldsymbol{H}^n} \le \sup_m \|w\|^{(s)}_{k,\alpha;\Omega_m}$$

for any $w \in \Lambda^s_{k,\alpha;\boldsymbol{H}^n}$.

PROOF. By the definition, we see that for any $w \in \Lambda^s_{k,\alpha;\Omega}$,

$$\|\tau^{-s+l} \partial^\gamma w\|_{L^\infty(\Omega')} \le \|\tau^{-s+l} \partial^\gamma w\|_{L^\infty(\Omega)}\,,$$

and

$$\sup_{x,y\in\Omega'} \left[\min(\tau^{-s+k+\alpha}(x), \tau^{-s+k+\alpha}(y)) \frac{|\partial^\gamma w(x) - \partial^\gamma w(y)|}{|x-y|^\alpha} \right] \le \sup_{x,y\in\Omega} \left[\min(\tau^{-s+k+\alpha}(x), \tau^{-s+k+\alpha}(y)) \frac{|\partial^\gamma w(x) - \partial^\gamma w(y)|}{|x-y|^\alpha} \right].$$

Thus, we see that

$$\|w\|^{(s)}_{k,\alpha;\Omega'} \le \|w\|^{(s)}_{k,\alpha;\Omega}\,,$$

and hence $\Lambda^s_{k,\alpha;\Omega} \subset \Lambda^s_{k,\alpha;\Omega'}$. On the other hand, for any $\varepsilon > 0$ there is $x \in \Omega_m$ such that

$$|\tau^{-s+l}\partial^\gamma w(x)| > \|\tau^{-s+l}\partial^\gamma w\|_{L^\infty(\boldsymbol{H}^n)} - \varepsilon\,,$$

and

$$|\tau^{-s+l}\partial^\gamma w(x)| \le \|\tau^{-s+l}\partial^\gamma w\|_{L^\infty(\Omega_m)}\,.$$

Hence we see that

$$\|\tau^{-s+l}\partial^\gamma w\|_{L^\infty(\boldsymbol{H}^n)} - \varepsilon \le \|\tau^{-s+l}\partial^\gamma w\|_{L^\infty(\Omega_m)}\,.$$

By similar arguments, we have

$$\begin{aligned}
&\sup_{x,y\in\boldsymbol{H}^n}\left[\min(\tau^{-s+k+\alpha}(x),\tau^{-s+k+\alpha}(y))\frac{|\partial^\gamma w(x)-\partial^\gamma w(y)|}{|x-y|^\alpha}\right]\\
&\quad\le \sup_{x,y\in\Omega_m}\left[\min(\tau^{-s+k+\alpha}(x),\tau^{-s+k+\alpha}(y))\frac{|\partial^\gamma w(x)-\partial^\gamma w(y)|}{|x-y|^\alpha}\right]+\varepsilon\,.
\end{aligned}$$

Thus, for any $\varepsilon > 0$ and sufficiently large m, we have

$$\|w\|^{(s)}_{k,\alpha;\boldsymbol{H}^n} \le \sup_m \|w\|^{(s)}_{k,\alpha;\Omega_m} + \varepsilon\,,$$

which implies that the conclusion is true. □

The following lemma is same as Lemma 3.1 in [8].

LEMMA 3.4. *For $x \in \Omega$, we have*

$$\|w\|^{(s)}_{k,\alpha;B(x)\cap\Omega} \le \|w\|^{(s)}_{k,\alpha;\Omega}$$

and

$$\|w\|^{(s)}_{k,\alpha;\Omega} \le C \sup_{x\in\Omega}\|w\|^{(s)}_{k,\alpha;B(x)\cap\Omega}\,,$$

where C depends only on k.

Let B denote the open Euclidean ball with center 0 and radius $1/3$, and for $x \in \boldsymbol{H}^n$ define $\psi_x : B \to B(x)$ by

$$y := \psi_x(z) = x + \tau(x)z\,. \tag{6}$$

If $y \in B(x)$, then

$$\frac{1}{10}\tau(x) \le \tau(y) \le 40\tau(x)\,. \tag{7}$$

Therefore, there exist a universal constant Λ_1 such that

$$\begin{aligned}
\Lambda_1^{-1}\tau^{-s+l}(x)\|\partial^\gamma w\|_{L^\infty(B(x)\cap\Omega)} &\le \|\tau^{-s+l}\partial^\gamma w\|_{L^\infty(B(x)\cap\Omega)}\\
&\le \Lambda_1\tau^{-s+l}(x)\|\partial^\gamma w\|_{L^\infty(B(x)\cap\Omega)}\,,
\end{aligned}$$

where Λ_1 depends only on s and l. Let $v(z) = w \circ \psi_x(z)$. Then we have $\partial^\gamma v/\partial z^\gamma = \tau^l(x)\partial^\gamma w/\partial y^\gamma$ for $|\gamma| = l$. So for any $y \in B(x) \cap \Omega$, we can show that

$$\tau^{-s+l}(x)\partial^\gamma w(y) = \tau^{-s}(x)\partial^\gamma v(z)\,.$$

Using (6) and (7), one can conclude that

$$\begin{aligned}\Lambda_1^{-1}\tau^{-s}(x)\|\partial^\gamma v\|_{L^\infty(\psi_x^{-1}(B(x)\cap\Omega))} &\le \|\tau^{-s+l}\partial^\gamma w\|_{L^\infty(B(x)\cap\Omega)} \\ &\le \Lambda_1\tau^{-s}(x)\|\partial^\gamma v\|_{L^\infty(\psi_x^{-1}(B(x)\cap\Omega))}\,,\end{aligned}$$

from which it follows that

$$\Lambda^{-1}\tau^{-s}(x)\|v\|_{k,\alpha;\psi_x^{-1}(B(x)\cap\Omega)} \le \|w\|^{(s)}_{k,\alpha;B(x)\cap\Omega} \le \Lambda\tau^{-s}(x)\|v\|_{k,\alpha;\psi_x^{-1}(B(x)\cap\Omega)}\,, \tag{8}$$

where Λ depends only on k , α and s.

Next, consider the following Dirichlet problem

$$\begin{cases} \boldsymbol{L}(u) = \Delta u - 2\tanh\rho\dfrac{\partial u}{\partial\rho} = \Delta u + 2\dfrac{\tau(y)}{(1-\tau(y))}\displaystyle\sum_{i=1}^n y^i\dfrac{\partial}{\partial y^i}u = \eta \quad \text{in } \boldsymbol{H}^n\,, \\ u|_{\boldsymbol{S}^{n-1}} = 0\,, \end{cases} \tag{9}$$

where $\eta \in \Lambda^s_{0,\alpha;\boldsymbol{H}^n}$ and s is to be determined later.

LEMMA 3.5. *Suppose $u \in C^2(\boldsymbol{H}^n) \cap \Lambda^s_{0,0;\boldsymbol{H}^n}$ is a solution for* (9) *with $\eta \in \Lambda^s_{k,\alpha;\boldsymbol{H}^n}$. Then we have*

$$\|u\|^{(s)}_{k+2,\alpha;\boldsymbol{H}^n} \le C(\|\eta\|^{(s)}_{k,\alpha;\boldsymbol{H}^n} + \|u\|^{(s)}_{0,0;\boldsymbol{H}^n})\,,$$

where $C = C(k,\alpha)$.

PROOF. It is easy to see that (9) is equivalent to

$$\begin{cases} \tau^2(y)\Delta_0 u + \tau(y)\left(n-2+\dfrac{2}{(1-\tau(y))}\right)\displaystyle\sum_{i=1}^n y^i\dfrac{\partial}{\partial y^i}u = \eta \quad \text{in } \boldsymbol{D}^n\,, \\ u|_{S^{n-1}} = 0\,, \end{cases} \tag{10}$$

where Δ_0 is the standard Laplacian for $\boldsymbol{D}^n \subset \boldsymbol{R}^n$. Suppose $v(z) = u \circ \psi_x(z)$ for each $z \in B$. Then (10) becomes

$$\begin{cases} \dfrac{\tau^2(y)}{\tau^2(x)}\Delta_0 v + \dfrac{\tau(y)}{\tau(x)}\left(n-2+\dfrac{2}{(1-\tau(y))}\right)\displaystyle\sum_{i=1}^n y^i\dfrac{\partial}{\partial z^i}v = \eta \quad \text{in } B(0)\,, \\ u|_{S^{n-1}} = 0\,. \end{cases} \tag{11}$$

Let B' and $B'(x)$ denote the open Euclidean balls with center 0 and radius $1/4$ and with center x and radius $\tau(x)/4$, respectively. Since $1/100 \le \tau^2(y)/\tau^2(x) \le 160$ when $y \in B(x)$ and $n \le n-2+2/(1-\tau(y)) \le n+2$, it follows that (11) is uniformly elliptic on B. Hence by the standard Schauder theory ([9]), we have

$$\|v\|_{k+2,\alpha;B'(0)} \le C(\|\eta\circ\psi_x\|_{k,\alpha;B(0)} + \|v\|_{0,0;B(0)})\,,$$

where C depends only on k and α. Choose $\Omega' \subset \Omega$ such that $B(x) \subset \Omega$ for any $x \in \Omega'$. Applying (8) and Lemma 3.4, we obtain

$$\begin{aligned}
\|u\|^{(s)}_{k+2,\alpha;\Omega'} &\le C \sup_{x\in\Omega'} \tau^{-s}(x)\|u\circ\psi_x\|_{k+2,\alpha;\psi_x^{-1}(B'(x)\cap\Omega')} \\
&\le C \sup_{x\in\Omega'} \tau^{-s}(x)\|u\circ\psi_x\|_{k+2,\alpha;B'(0)} \\
&\le C \sup_{x\in\Omega'} \tau^{-s}(x)(\|\eta\circ\psi_x\|_{k,\alpha;B(0)} + \|v\|_{0,0;B(0)}) \\
&\le C(\|\eta\|^{(s)}_{k,\alpha;\Omega} + \|u\|^{(s)}_{0,0;\Omega}) \\
&\le C(\|\eta\|^{(s)}_{k,\alpha;\boldsymbol{H}^n} + \|u\|^{(s)}_{0,0;\boldsymbol{H}^n}) .
\end{aligned}$$

Therefore the lemma follows from Lemma 3.3. □

PROPOSITION 3.6. *Given any $\eta \in \Lambda^s_{0,\alpha;\boldsymbol{H}^n}$ for $0 < s < n+1$, there exists $u \in C^2(\boldsymbol{H}^n) \cap \Lambda^s_{0,0;\boldsymbol{H}^n}$ satisfying* (9), *and $\|u\|^{(s)}_{0,0;\boldsymbol{H}^n} \le C\|\eta\|^{(s)}_{0,\alpha;\boldsymbol{H}^n}$ with C depending on s. Moreover, $u \in \Lambda^s_{2,\alpha;\boldsymbol{H}^n}$ with*

$$\|u\|^{(s)}_{2,\alpha;\boldsymbol{H}^n} \le C\|\eta\|^{(s)}_{0,\alpha;\boldsymbol{H}^n} ,$$

where C is a constant depending only on s and α.

PROOF. Let $\{\Omega_m\}_{m=1}^\infty$ be an exhausting sequence of domains such that $\Omega_m \subset \Omega_{m+1}$ and $\bigcup_m \Omega_m = \boldsymbol{H}^n$. Let w_m be a solution for the following equation:

$$\begin{cases} \boldsymbol{L}(w_m) = \eta & \text{in } \Omega_m , \\ w_m|_{\partial\Omega_m} = 0 . \end{cases}$$

Note that $w_m \in C^{2,\alpha}(\bar{\Omega}_m)$, since $\eta \in C^{0,\alpha}(\bar{\Omega}_m)$ ([9]).

Set $\phi = \tau^s$. Then we have

$$\begin{aligned}
\boldsymbol{L}(\phi) &= -s(2s-n+2)\tau^{s+1} + s(s-n-1)\tau^s + (2/(1-\tau))s\tau^{s+1} \\
&\le -s(2s-n-2)\tau^{s+1} + s(s-n-1)\tau^s ,
\end{aligned}$$

since $2/(1-\tau) \le 4$ and $s > 0$. For $0 \le s < n+1$, it is easy to check that

$$\boldsymbol{L}(\phi) = \Delta\phi - \frac{2\tau(y)(1-2\tau(y))}{1-\tau(y)}\frac{\partial}{\partial\tau}\phi \le -\delta\phi$$

for some constant $\delta > 0$ depending only on s. On the other hand, we have $|\eta| \le C\tau^s$, where $C = \|\eta\|^{(s)}_{0,\alpha;\boldsymbol{H}^n}$, since $\eta \in \Lambda^s_{0,\alpha;\boldsymbol{H}^n}$. We choose a constant $C_1 = C/\delta$ such that

$$\begin{cases} \boldsymbol{L}(w_m) \ge \boldsymbol{L}(C_1\phi) & \text{in } \Omega_m , \\ (C_1\phi - w_m)|_{\partial\Omega_m} \ge 0 . \end{cases}$$

By the maximum principle, we obtain $w_m \le C_1\tau^s$. By the same argument, we may get the lower bound of w_m. Hence, $|w_m| \le C_1\tau^s$. Therefore, w_m converges to a function $u \in$

$C^2(\boldsymbol{H}^n)\cap\Lambda^s_{0,0;\boldsymbol{H}^n}$, which solves (9), and we have $\|u\|^{(s)}_{0,0;\boldsymbol{H}^n}\le C\|\eta\|^{(s)}_{0,\alpha;\boldsymbol{H}^n}$, where C depends only on s. By Lemma 3.5, we know that $u\in\Lambda^s_{2,\alpha;\boldsymbol{H}^n}$ with

$$\|u\|^{(s)}_{2,\alpha;\boldsymbol{H}^n}\le C\|\eta\|^{(s)}_{0,\alpha;\boldsymbol{H}^n},$$

where $C=C(s,\alpha)$. □

Now, by Lemma 3.5 and Proposition 3.6, we have the following

THEOREM 3.7. *The operator* $\boldsymbol{L}:\Lambda^s_{2,\alpha;\boldsymbol{H}^n}\to\Lambda^s_{0,\alpha;\boldsymbol{H}^n}$ *defined in* (9) *is an isomorphism, where* $0<s<n+1$.

COROLLARY 3.8. *For any* $\varphi\in C^{4,\alpha}(\boldsymbol{S}^{n-1})$, *Dirichlet problem* (3) (*or* (4)) *has a solution.*

PROOF. We use the cylindrical coordinate system (ρ,θ). We extend φ as $\varphi(\rho,\theta)=\varphi(\theta)$ for $\theta\in\boldsymbol{S}^{n-1}$ and small ρ. Then, let $f(\rho,\theta)\in C^{2,\alpha}(\boldsymbol{H}^n)$ such that for some small ρ,

$$f(\rho,\theta)=\varphi+\frac{1}{2(n-1)}\rho^2\Delta_{\boldsymbol{S}^{n-1}}\varphi,$$

where $\Delta_{\boldsymbol{S}^{n-1}}$ is the Laplacian operator on $\boldsymbol{S}^{n-1}$. Putting f into the left side of (3), one can see that

$$\begin{aligned}\boldsymbol{L}(f)&=\sinh^2\rho/(n-1)\Delta_{\boldsymbol{S}^{n-1}}\varphi-((n-2)\sinh\rho\cosh\rho+2\tanh\rho)/(n-1)\\&\quad\cdot\rho\Delta_{\boldsymbol{S}^{n-1}}\varphi+\sinh^2\rho\Delta_{\boldsymbol{S}^{n-1}}\varphi+\rho^2\sinh^2\rho/(2(n-2))\Delta^2_{\boldsymbol{S}^{n-1}}\varphi\\&=O(\rho^4)=O(\tau^4)\quad\text{as }\ \tau\to0.\end{aligned}$$

Because $\boldsymbol{L}(f)$ is $C^{0,\alpha}$ in any compact subset and behaves like τ^4 near boundary, we conclude that $\boldsymbol{L}(f)\in\Lambda^s_{0,\alpha;\boldsymbol{H}^n}$ for any $s\le4$. Then the corollary follows from Theorem 3.7. □

Now, we are in a position to prove Theorem 3.1. By Corollary 3.8, (4) has a solution u satisfying $u-f\in\Lambda^s_{2,\alpha;\boldsymbol{H}^n}$ for some $s\in(0,4)$, where f is given as in the proof of Corollary 3.8. Since (4) is a linear equation, we can multiply φ by a suitable constant such that the corresponding solution u satisfies

$$v=\frac{1}{\sqrt{1-((1-\tau)/\tau)^2|\nabla u|^2}}<+\infty,$$

that is, u is spacelike. Define

$$\Xi_A=\{w\in\Lambda^2_{2,\alpha;\boldsymbol{H}^n}\,;\ 1/\sqrt{1-((1-\tau)/\tau)^2|\nabla(w+u)|^2}<A<+\infty\}\subset\Lambda^2_{2,\alpha;\boldsymbol{H}^n}.$$

Obviously, Ξ_A is a nonempty open set of $\Lambda^2_{2,\alpha;\boldsymbol{H}^n}$, since $0\in\Xi_A$. Define an operator

$$H(\cdot,\cdot):(-1,+1)\times\Xi_A\to\Lambda^2_{0,\alpha;\boldsymbol{H}^n}$$

by

$$
\begin{aligned}
H(\varepsilon, w) :&= \operatorname{div}\left(\frac{\nabla(w+u)}{\sqrt{1-\varepsilon((1-\tau)/\tau)^2|\nabla(w+u)|^2}}\right) \\
&\qquad + \frac{2\tau}{1-\tau}\frac{\displaystyle\sum_{i=1}^{n} x^i \frac{\partial(w+u)}{\partial x^i}}{\sqrt{1-\varepsilon((1-\tau)/\tau)^2|\nabla(w+u)|^2}} \\
&= \frac{\Delta(w+u)}{\sqrt{1-\varepsilon((1-\tau)/\tau)^2|\nabla(w+u)|^2}} \\
&\qquad + \left\langle \nabla \frac{1}{\sqrt{1-\varepsilon((1-\tau)/\tau)^2|\nabla(w+u)|^2}}, \nabla(w+u)\right\rangle \\
&\qquad + \frac{2\tau}{1-\tau}\frac{\displaystyle\sum_{i=1}^{n} x^i \frac{\partial(w+u)}{\partial x^i}}{\sqrt{1-\varepsilon((1-\tau)/\tau)^2|\nabla(w+u)|^2}} .
\end{aligned}
$$

From Corollary 3.8, we have $H(0,0)=0$. By direct computation, we see that H is a smooth operator, and for any $h \in \Lambda^2_{2,\alpha;\boldsymbol{H}^n}$,

$$
\frac{\partial}{\partial t} H(0, th)\bigg|_{t=0} = \Delta h + 2\frac{\tau(y)}{1-\tau(y)}\sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} h .
$$

It follows that the map $\partial H(0,th)/\partial t|_{t=0} = \boldsymbol{L} : \Lambda^2_{2,\alpha;\boldsymbol{H}^n} \to \Lambda^2_{0,\alpha;\boldsymbol{H}^n}$ is an isomorphism from Theorem 3.7. Now, by the implicit function theorem (cf. [9]), we can conclude that (5) has a solution whose difference by u is in $\Lambda^2_{2,\alpha;\boldsymbol{H}^n}$ and boundary data is given by small $\sqrt{\varepsilon}\varphi$. Thus we finish proving Theorem 3.1.

4. Boundary behavior of totally geodesic slices of ADS spaces. In this section, we show that any isometric and maximal embedding of $\boldsymbol{H}^n$ into ADS space is totally geodesic, and moreover, we give a sufficient and necessary condition for the boundary value of the height function for totally geodesic slices. Combined with Theorem 3.1, we know that the Bernstein Theorem in ADS space-time fails. By standard arguments, we have

PROPOSITION 4.1. *If a hyperbolic space is isometrically immersed in the anti-de Sitter space as its maximal hypersurface, it must be totally geodesic.*

Now, we are in a position to study the boundary behavior of totally geodesic slices of ADS space V. For simplicity, we only consider the case that $\dim V = 4$.

Let $\boldsymbol{R}^5_2$ be 5 dimensional semi-Euclidean space, that is, it is a vector space with the inner product $\langle X, Y\rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 - x_5y_5$, where $X = (x_1, x_2, x_3, x_4, x_5)$ and $Y = (y_1, y_2, y_3, y_4, y_5)$. Denote the connection in $\boldsymbol{R}^5_2$ by $\tilde{\nabla}$. It is well-known that $V' = \{X \in \boldsymbol{R}^5_2 : \langle X, X\rangle = -1\}$ is a totally umbilical hypersurface of $\boldsymbol{R}^5_2$, and its universal covering space is 4 dimensional anti-de Sitter space V introduced in Section 2. For simplicity, V' is still called as anti-de Sitter space.

In the following, we adopt so called sausage coordinate for the anti-de Sitter space V', namely, any $X = (x_1, x_2, x_3, x_4, x_5) \in V'$ can be expressed by

$$\begin{cases} x_1 = 2r(1-r^2)^{-1}\sin\theta\cos\phi\,, \\ x_2 = 2r(1-r^2)^{-1}\sin\theta\sin\phi\,, \\ x_3 = 2r(1-r^2)^{-1}\cos\theta\,, \\ x_4 = (1+r^2)(1-r^2)^{-1}\cos t\,, \\ x_5 = (1+r^2)(1-r^2)^{-1}\sin t\,, \end{cases} \tag{12}$$

where angular coordinates have their usual range, while $0 \le r < 1$. In this coordinates, the Lorentz metric of V is

$$ds^2 = -\left(\frac{1+r^2}{1-r^2}\right)^2 dt^2 + \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2)\,,$$

thus, t can be viewed as a time function in V'. For any slice in V', we can define its height function by restriction t on it. Let M' be a slice of V'. Then its height function u' can be regarded as a function on $\boldsymbol{H}^3$, which is still denoted by u. In the sequel, we always assume that u is at least continuous at the infinity boundary of $\boldsymbol{H}^3$. Thus, we may define

$$w(\theta, \phi) = \lim_{r\to 1} u(r, \theta, \phi)\,,$$

hence, w is a function on $\boldsymbol{S}^2$.

Let us adopt ball model for $\boldsymbol{H}^3$, then for any point in V can be expressed as (t, r, θ, ϕ), where $r \in [0, 1]$. Now we can define

$$\Pi : V \mapsto V'\,,$$

as

$$\Pi(t, r, \theta, \phi) = (x_1, x_2, x_3, x_4, x_5)\,,$$

where $(x_1, x_2, x_3, x_4, x_5)$ is defined by (12). Then it is easy to see that $\Pi : V \mapsto V'$ is covering map.

Now, we have

THEOREM 4.2. *Let M be a maximal slice in V. Then M is totally geodesic if and only if there are constants w_0, A, B, C with $A^2 + B^2 + C^2 < 1$ such that*

$$f(\theta, \phi) = A\sin\theta\cos\phi + B\sin\theta\sin\phi + C\cos\theta\,, \tag{13}$$

where $f = \cos(w + w_0)$.

REMARK 4.3. We would like to point out that $p = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ can be regarded as a point on the standard $\boldsymbol{S}^2 \subset \boldsymbol{R}^3$, and each coordinate component is a first eigenfunction of the Laplacian on $\boldsymbol{S}^2$.

PROOF. Suppose M is a totally geodesic spacelike silce in V. Hence $M' = \Pi(M) \subset V'$ can be viewed also as a spacelike submanifold in $\boldsymbol{R}^5_2$. We take an orthogonal frame field $\{e_1, e_2, e_3, e_4, e_5\}$ for $\boldsymbol{R}^5_2$ such that e_1, e_2, e_3, e_4 and e_1, e_2, e_3 are tangent vectors of V' and M', respectively. Denote the position vector of V' by X. We may assume that $X = e_5$. Noting that M' is totally geodesic in V', and V' is totally umbilical in $\boldsymbol{R}^5_2$, we get

$$\langle \tilde{\nabla}_{e_i} e_4, e_j \rangle = \langle \tilde{\nabla}_{e_i} e_4, e_5 \rangle = 0\,,$$

for $i, j = 1, 2, 3$, where $\tilde{\nabla}$ is the connection in $\boldsymbol{R}^5_2$. Thus, we conclude that $e_4 \,|_{M'} = a$, where $a = (a_1, a_2, a_3, a_4, a_5) \in \boldsymbol{R}^5_2$ is a constant vector with $\langle a, a \rangle = -1$. Furthermore, one have

$$\langle X|_{M'}, a \rangle = \langle X|_M, e_4 \rangle = 0\,, \tag{14}$$

i.e., M' is the intersection of V' and a hyperplane $\Pi_a := \{x \in \boldsymbol{R}^5_2; \langle x, a \rangle = 0\}$.

By (14), we obtain

$$\begin{aligned} 0 = \langle X|_{M'}, a \rangle = a_1 \frac{2r}{1-r^2} \sin\theta\cos\phi + a_2 \frac{2r}{1-r^2} \sin\theta\sin\phi \\ + a_3 \frac{2r}{1-r^2}\cos\theta - a_4 \frac{1+r^2}{1-r^2}\cos t - a_5 \frac{1+r^2}{1-r^2}\sin t\,. \end{aligned}$$

Letting $r \to 1$, we have

$$\cos(t + w_0) = A\sin\theta\cos\phi + B\sin\theta\sin\phi + C\cos\theta\,,$$

or equivalently,

$$f(\theta, \phi) = A\sin\theta\cos\phi + B\sin\theta\sin\phi + C\cos\theta\,,$$

where $A = a_1/\sqrt{a_4^2 + a_5^2}$, $B = a_2/\sqrt{a_4^2 + a_5^2}$, $C = a_3/\sqrt{a_4^2 + a_5^2}$ and $\cos w_0 = a_4/\sqrt{a_4^2 + a_5^2}$.

Conversely, if M is a maximal slice in V, and its boundary data satisfies (13), then we choose two constants a_4, a_5 with $a_4^2 + a_5^2 > 1$ and

$$\cos w_0 = \frac{a_4}{\sqrt{a_4^2 + a_5^2}}\,, \quad -\sin w_0 = \frac{a_5}{\sqrt{a_4^2 + a_5^2}}\,.$$

Let

$$a_1 = \sqrt{a_4^2 + a_5^2}A, \quad a_2 = \sqrt{a_4^2 + a_5^2}B, \quad a_3 = \sqrt{a_4^2 + a_5^2}C\,.$$

Set $a = (a_1, a_2, a_3, a_4, a_5) \in \boldsymbol{R}^5_2$ and $L = \{(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta, \cos w, -\sin w);\ 0 < \theta < \pi, 0 < \phi < 2\,\pi\}$. By direct computation, we see that $L \subset \Pi_a$. Hence it is the boundary of $\Pi_a \cap V'$, which is a totally geodesic slice of V'. Hence, we can lift it up to V and get a totally geodesic slice in V which is denoted by M''. In particular, it is maximal, and hence, its height function satisfies Equation (1). By maximality of M, we see that the height function of M also satisfies the same equation, and they are equal at the infinity boundary of $\boldsymbol{H}^3$. Thus, by the maximum principle, we see that they are equal on $\boldsymbol{H}^3$, which implies M is totally geodesic. This completes the proof of the theorem.

As a corollary, we have

COROLLARY 4.4. *Let M be totally geodesic slice in V. Then there is a constant w_0 on $\boldsymbol{S}^2$ with*

$$f^2 + |\nabla^{\boldsymbol{S}^2} f|^2 = C\,,$$

where $f = \cos(w + w_0)$, $\nabla^{\boldsymbol{S}^2}$ is the connection on $\boldsymbol{S}^2$ and C is a constant.

Combining this fact with Theorem 3.1, we see that the Bernstein Theorem in V fails.

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