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# Maximal Surfaces of Riemann Type in Lorentz–Minkowski Space $\mathbb{L}^3$

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## 1. Introduction and Statement of Results

In 1867, Riemann [19] found a 1-parameter family of complete minimal surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ that are fibered by circles and straight lines in parallel planes. Riemann also proved that these are the only surfaces (besides the catenoid) with this property. In 1870, Enneper [3] proved that if a minimal surface in $\mathbb{E}^3$ is foliated by pieces of circles, then the planes containing these circles are actually parallel and so the surface is a piece of either a Riemann example or a catenoid. Nowadays, we know more general uniqueness theorems for Riemann minimal examples (see e.g. [4; 10; 14]).

In this paper we deal with the same kind of questions for maximal spacelike surfaces in Lorentz–Minkowski 3-dimensional space $\mathbb{L}^3$. A smooth immersion of a surface in $\mathbb{L}^3$ is called *spacelike* if the induced metric on the surface is a Riemannian metric. A spacelike surface in $\mathbb{L}^3$ is maximal provided its mean curvature vanishes. Spacelike maximal surfaces in $\mathbb{L}^3$ represent a maximum for the area integral [1]. It is known that the only complete maximal spacelike surfaces are planes (see [1] and [2] for arbitrary dimension). Hence, it is natural to consider nonflat maximal spacelike immersions with singularities. These singularities correspond to either curves of points where the immersion is not spacelike or to isolated branch points.

Some properties of minimal surfaces in $\mathbb{E}^3$ have an analogous version for maximal spacelike surfaces in $\mathbb{L}^3$. For example, they admit a Weierstrass representation closely related to that of minimal surfaces in $\mathbb{E}^3$. As a matter of fact, there is a natural method of constructing maximal surfaces in $\mathbb{L}^3$ from minimal ones in $\mathbb{E}^3$, and vice versa.

Inspired by the works of Riemann and Enneper just cited, we classify maximal spacelike surfaces in $\mathbb{L}^3$ that are foliated by pieces of circles. Rotational maximal surfaces in $\mathbb{L}^3$ have been studied in [7]. As in the minimal case in $\mathbb{E}^3$, a maximal spacelike surface in $\mathbb{L}^3$ is foliated by circles in parallel planes if and only if a Shiffman-type function vanishes at any point of the surface. This function lies in the kernel of the Lorentzian Jacobi operator of the surface. In this work we prove

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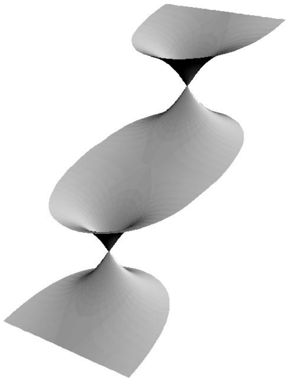

**Figure 1** A piece of a maximal surface that is bounded by two spacelike straight lines and containing two cone points of a Riemann-type surface in the family $\mathcal{R}$. The whole surface is invariant under the translation determined by a half of the vector joining orthogonally the two boundary straight lines.

a version of Shiffman's theorem (see [20]) for maximal spacelike annuli bounded by circles in parallel spacelike planes.

This paper is organized as follows. In Section 2, we introduce the concept of *circle* in $\mathbb{L}^3$ and recall the Weierstrass representation for spacelike maximal surfaces. In Section 3, we determine the family of spacelike maximal surfaces foliated by pieces of circles in parallel planes. To be more precise, we prove the following.

*Let $M$ be a spacelike maximal surface in $\mathbb{L}^3$. If $M$ is foliated by pieces of circles in parallel planes, then $M$ is one of the surfaces described in*:

*Theorem 1, if the planes are spacelike*;
*Theorem 2, if the planes are timelike*;
*Theorem 3, if the planes are lightlike.*

This space of maximal surfaces in $\mathbb{L}^3$ is related to a particular family of singly periodic minimal annuli in $\mathbb{E}^3$ with parallel embedded ends of Riemann type (see [10]). It includes, besides the catenoid, a 1-parameter family of singly periodic examples $\mathcal{R}$ (see Remark 3) foliated by circles in parallel spacelike planes whose set of singularities is mapped under the immersion on a discrete subset of $\mathbb{R}^3$. In particular, the foliating curves, except the singular ones, are spacelike. Therefore, these curves are either *complete* circles or *complete* straight lines, and any surface in $\mathcal{R}$ is like a Riemann minimal example in $\mathbb{R}^3$. See Figure 2 and Figure 3.

Since maximal spacelike surfaces in $\mathbb{L}^3$ are stable, we have obtained easily the following version of Shiffman's theorem.

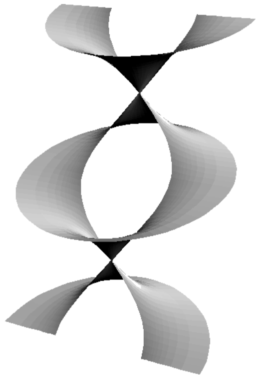

**Figure 2** A piece of a singly periodic Riemann-type surface whose level curves are circles or straight lines with two singular points in parallel spacelike planes. The immersion folds back at the singular points and so the foliation curves are *pieces* of circles or straight lines. The surface also contains singular cone points.

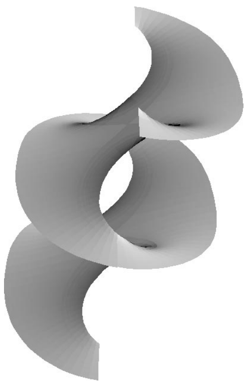

**Figure 3** A piece of a singly periodic Riemann-type surface whose level curves are pieces of circles or straight lines with two singular points in parallel spacelike planes. At these singular points, the immersion folds back. In this case there are no singular cone points.

> *A compact maximal spacelike annulus in* $\mathbb{L}^3$ *whose boundary consists of two circles in parallel spacelike planes is a piece of either a Lorentzian catenoid or a surface in the family* $\mathcal{R}$.

In fact, we prove a slightly more general version of this theorem for annuli with singularities of cone type. In Section 4 we prove the following Enneper-type result for maximal spacelike surfaces in $\mathbb{L}^3$.

*If a maximal spacelike surface in $\mathbb{L}^3$ is foliated by pieces of circles lying in a 1-parameter family of planes, then the planes in the family are actually parallel.*

Finally, we mention that some interesting results on constant mean curvature hypersurfaces foliated by spheres in different ambient spaces have been recently obtained by Jagy [5; 6]; see also [11; 12; 13; 16].

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## 2. Preliminaries

Throughout this paper, $\mathbb{L}^3$ will denote the 3-dimensional Lorentz–Minkowski space $(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$, where

$$\langle\cdot,\cdot\rangle = dx_1^2 + dx_2^2 - dx_3^3.$$

We will also denote the Euclidean metric in $\mathbb{R}^3$ by $\langle\cdot,\cdot\rangle_0 = dx_1^2 + dx_2^2 + dx_3^3$ and label $\mathbb{E}^3 = (\mathbb{R}^3, \langle\cdot,\cdot\rangle_0)$.

We say that a vector $\mathbf{v} \in \mathbb{R}^3 - \{\mathbf{0}\}$ is spacelike, timelike, or lightlike if and only if $\langle \mathbf{v}, \mathbf{v}\rangle$ is positive, negative, or zero (respectively). The vector $\mathbf{0}$ is spacelike by definition. A plane in $\mathbb{L}^3$ is spacelike, timelike, or lightlike if and only if its Euclidean unit normals are (resp.) timelike, spacelike, or lightlike. A curve is called spacelike, timelike, or lightlike if and only if the tangent vector at any point is spacelike, timelike, or lightlike. A surface in $\mathbb{L}^3$ is spacelike, timelike, or lightlike if and only if the tangent plane at any point is (resp.) spacelike, timelike, or lightlike.

### *2.1. Circles in $\mathbb{L}^3$*

We shall first determine which planar curves in $\mathbb{L}^3$ play the same role as circles in Euclidean space $\mathbb{E}^3$. To do this, it is necessary to describe the family of planar spacelike curves with nonzero constant curvature in $\mathbb{L}^3$. Let us examine the concept of nonzero curvature for a regular planar curve in $\mathbb{L}^3$. Consider $\Pi$ a plane in $\mathbb{L}^3$ and let $\alpha = \alpha(s)$ be a spacelike curve in $\Pi$, where $s$ denotes the arc-length parameter of $\alpha$ in $\mathbb{L}^3$. Let $\mathbf{t}(s) = \alpha'(s)$ be the unit tangent vector to $\alpha$. Since we want nonzero curvature, we assume that $\mathbf{t}'(s)$ never vanishes. The causal character of the plane $\Pi$ leads to three possibilities as follows.

*$\Pi$ is spacelike.* In this case, $(\Pi, \langle\cdot,\cdot\rangle)$ is a Riemannian plane and the definition of curvature is the Riemannian one. Hence, given an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ in $\Pi$, the curves in $\Pi$ with constant positive curvature $k$ are given by

$$\alpha(s) = \mathbf{c} + \frac{1}{k}(\cos(ks)\mathbf{e}_1 + \sin(ks)\mathbf{e}_2), \quad \mathbf{c} \in \Pi. \tag{1}$$

*$\Pi$ is timelike.* Since $\langle \mathbf{t}', \mathbf{t}\rangle = 0$, it follows that $\mathbf{t}'$ is a timelike vector in $(\Pi, \langle\cdot,\cdot\rangle)$. By definition, the curvature $\kappa$ of $\alpha$ is the number $\kappa = \sqrt{-\langle \mathbf{t}', \mathbf{t}'\rangle}$. Thus, if $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthogonal basis of $\Pi$ such that $\langle \mathbf{e}_1, \mathbf{e}_1\rangle = -\langle \mathbf{e}_2, \mathbf{e}_2\rangle = -1$, then the spacelike curves in $\Pi$ with constant curvature $k > 0$ are given by

$$\alpha(s) = \mathbf{c} + \frac{1}{k}(\cosh(ks)\mathbf{e}_1 + \sinh(ks)\mathbf{e}_2), \quad \mathbf{c} \in \Pi. \tag{2}$$

*$\Pi$ is lightlike.* Since $\mathbf{t}' \neq 0$ and $\mathbf{t}$ is not lightlike, the equation $\langle \mathbf{t}', \mathbf{t}\rangle = 0$ implies that $\mathbf{t}'$ is a lightlike vector in $\Pi$. Fix a *constant* lightlike field $\mathbf{n}$ on $\alpha$. Then $\mathbf{t}' = \kappa\mathbf{n}$. Obviously, the function $\kappa$ depends on the choice of $\mathbf{n}$. However, the fact that $\kappa$ is a nonzero constant does not depend on this choice. In other words, $\kappa$ is constant if and only if $\mathbf{t}'$ is a nonzero constant field on $\alpha$. Therefore, a spacelike curve with constant nonzero curvature $k \neq 0$ in $\Pi$ is given by

$$\alpha(s) = \mathbf{c} + s\mathbf{e}_1 + \frac{k}{2}s^2\mathbf{e}_2, \quad \mathbf{c} \in \Pi, \tag{3}$$

where $\mathbf{e}_2 = \mathbf{n}$ is a fixed *constant* lightlike field on $\Pi$ and $\mathbf{e}_1$ is a constant unit spacelike vector field on the curve $\alpha$.

These planar curves can be described from the Euclidean point of view as follows. Up to a linear isometry in $\mathbb{L}^3$, assume that $\Pi$ is the plane $\Pi_{\mathbf{v}} = \{x \in \mathbb{R}^3 : \langle x, \mathbf{v}\rangle_0 = 0\}$ and that $\mathbf{v}$ is one of the next vectors: $\mathbf{v} = (0,0,1)$, $\mathbf{v} = (1,0,0)$, or $\mathbf{v} = \frac{1}{\sqrt{2}}(1,0,-1)$.

1. Let $\mathbf{v} = (1,0,0)$. Label $\mathbf{e}_1 = (1,0,0)$ and $\mathbf{e}_2 = (0,1,0)$. Then the spacelike curves in $\Pi_{\mathbf{v}}$ of curvature $k > 0$ are

$$\alpha(s) = \mathbf{c} + \frac{1}{k}(\cos(ks), \sin(ks), 0) \tag{4}$$

(see (1)), where $\mathbf{c} \in \Pi_{\mathbf{v}}$. These curves are Euclidean circles in horizontal planes.

2. Let $\mathbf{v} = (1,0,0)$. Label $\mathbf{e}_1 = (0,0,1)$ and $\mathbf{e}_2 = (0,1,0)$. Then, if $k > 0$, the curves (2) are given by

$$\alpha(s) = \mathbf{c} + \frac{1}{k}(0, \sinh(ks), \cosh(ks)), \tag{5}$$

where $\mathbf{c} \in \Pi_{\mathbf{v}}$. These curves are Euclidean hyperbolas in vertical planes.

3. Let $\mathbf{v} = \frac{1}{\sqrt{2}}(1,0,-1)$. If we choose the *fixed* basis of $\Pi_{\mathbf{v}}$ given by $\mathbf{e}_1 = (0,1,0)$ and $\mathbf{e}_2 = (1,0,1)$, then the curves in (3) are given by

$$\alpha(s) = \mathbf{c} + \left(\mu s + \frac{k}{2}s^2, s, \mu s + \frac{k}{2}s^2\right), \tag{6}$$

where $\mathbf{c} \in \Pi_{\mathbf{v}}$, $\mu \in \mathbb{R}$, and $\kappa \neq 0$. They are Euclidean parabolas $(x - c_1) = \frac{k}{2}(y - c_2)^2 + \mu(y - c_2)$, $c_1, c_2 \in \mathbb{R}$, in the plane $\Pi_{\mathbf{v}}$.

We may summarize as follows.

Definition 1. A *circle* in $\mathbb{L}^3$ is a planar curve with nonzero constant curvature.

There is another approach to the concept of circle. Let $l$ be a straight line in $\mathbb{L}^3$ and consider $G = \{R_\theta; \theta \in \mathbb{R}\}$, the 1-parameter group of linear isometries in $\mathbb{L}^3$ that leave $l$ pointwise fixed. This group is called the group of *rotations* with axis $l$. Let $p$ be a point lying in $\mathbb{L}^3 - l$, and consider the curve determined by the orbit of $p$ under the action of $G$. This planar curve has nonzero constant curvature, and it is contained in a plane orthogonal to $l$. Hence, an equivalent definition of circle in $\mathbb{L}^3$ is (see [12] for details) the following.

DEFINITION 2. A *curve* $\alpha$ in $\mathbb{L}^3$ is a circle if there exists a straight line $l$ in $\mathbb{L}^3$ such that $\alpha$ describes the nonlinear orbit of a point $p \in \mathbb{L}^3 - l$ under the action of the 1-parameter group of motions in $\mathbb{L}^3$ that fix pointwise $l$.

### 2.2. The Weierstrass Representation of Maximal Spacelike Surfaces in $\mathbb{L}^3$

We end this section with a few words about the Weierstrass representation of maximal spacelike surfaces in $\mathbb{L}^3$. Let $X: M \to \mathbb{L}^3$ be a spacelike maximal immersion of an orientable surface $M$ in 3-dimensional Lorentz–Minkowski space. The Gauss map $N$ of $X$ assigns to each point of $M$ a point of the spacelike surface $\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = -1\}$, which has constant intrinsic curvature $-1$ with respect to the induced metric. Note that $\mathbb{H}^2$ has two connected components, one on which $x_3 \geq 1$ and one on which $x_3 \leq -1$.

Throughout this paper, $\mathbb{C}^*$ denotes the extended complex plane $\mathbb{C} \cup \{\infty\}$. Define a stereographic projection $\sigma$ for $\mathbb{H}^2$ as follows:

$$\sigma : \mathbb{C}^* - \{|z| = 1\} \to \mathbb{H}^2; \quad z \to \left( \frac{-2\operatorname{Re}(z)}{|z|^2 - 1}, \frac{-2\operatorname{Im}(z)}{|z|^2 - 1}, \frac{|z|^2 + 1}{|z|^2 - 1} \right),$$

where $\sigma(\infty) = (0, 0, 1)$. Using isothermal parameters, $M$ has in a natural way a conformal structure, and up to a suitable choice of the orientation, the map $g \overset{\text{def}}{=} \sigma^{-1} \circ N$ is meromorphic.

Moreover, there exists a holomorphic 1-form $\eta$ on $M$ such that the 1-forms

$$\Phi_1 = \frac{i}{2}\eta(1 - g^2), \quad \Phi_2 = -\frac{1}{2}\eta(1 + g^2), \quad \Phi_3 = \eta g \tag{7}$$

are holomorphic on $M$ and without common zeroes. Furthermore, the 1-forms $\Phi_j$ $(j = 1, 2, 3)$ have no real periods, and the immersion $X$ is determined, up to a translation, by

$$X = \operatorname{Re} \int (\Phi_1, \Phi_2, \Phi_3). \tag{8}$$

The induced Riemannian metric $ds^2$ on $M$ is given by

$$ds^2 = |\Phi_1|^2 + |\Phi_2|^2 - |\Phi_3|^2 = \left( \frac{|\eta|}{2}(1 - |g|^2) \right)^2. \tag{9}$$

Conversely, let $M$, $g$, and $\eta$ be (resp.) a Riemann surface, a meromorphic map on $M$, and a holomorphic 1-form $\eta$ on $M$, such that: $|g(P)| \neq 1$ for all $P \in M$; and the 1-forms defined in (7) are holomorphic, have no common zeroes, and have no real periods. Then (8) defines a conformal spacelike maximal immersion of $M$ in $\mathbb{L}^3$, and its Gauss map is $\sigma \circ g$. If we allow that the set $\{|g| = 1\} \neq \emptyset$, we say that $X\colon M \to \mathbb{L}^3$ is a maximal spacelike immersion with singularities. We also say that $X(M)$ is a maximal spacelike surface with singularities in $\mathbb{L}^3$. In this case, the immersion $X$ is not regular at the nodal set of the harmonic function $\log(|g|)$.

We call $(M, \Phi_1, \Phi_2, \Phi_3)$ (or simply $(M, g, \eta)$) the *Weierstrass representation* of $X$ (see e.g. [7] for more details).

REMARK 1. The transformation $(M, \Phi_1, \Phi_2, \Phi_3) \to (M, i\Phi_1, i\Phi_2, \Phi_3)$ converts Weierstrass data of maximal spacelike surfaces in $\mathbb{L}^3$ into Weierstrass data of minimal surfaces in $\mathbb{R}^3$, and vice versa. For more details about theory of minimal surfaces, see [18].

Throughout this paper, we say that a maximal immersion in $\mathbb{L}^3$ is complete if and only if the corresponding minimal one in $\mathbb{R}^3$ is. If the set of singularities of a maximal immersion consists of cone points (see Definition 3), then this concept of completeness agrees with the natural one (that divergent curves have infinite length).

## 3. Existence of Maximal Surfaces of Riemann Type

In this section we classify the family of maximal spacelike surfaces in $\mathbb{L}^3$ that are foliated by pieces of circles in parallel planes. The main tool used is the Weierstrass representation of maximal spacelike surfaces in $\mathbb{L}^3$. At the end of the section, we will introduce the *Lorentzian* Shiffman-type functions on a maximal surface and then prove a version of Shiffman's theorem (see [20]) for maximal spacelike surfaces in $\mathbb{L}^3$.

Let $X\colon M \to \mathbb{L}^3$ be a spacelike conformal nonplanar maximal immersion of a Riemann surface $M$. We denote by $(\eta, g)$ the Weierstrass representation of $X$ and define $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ as in (7). Let $ds^2$ denote the Riemannian metric in $M$ induced by $X$ and $\langle \cdot, \cdot \rangle$. See equations (7), (8), and (9) for details.

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$, and denote by $\Pi_{\mathbf{v}}$ the plane $\{x \in \mathbb{R}^3 : \langle x, \mathbf{v}\rangle_0 = 0\}$. The vector $\mathbf{v}$ can be timelike, spacelike, or lightlike. Hence, and up to *linear isometries in* $\mathbb{L}^3$, we will assume that $\mathbf{v} = (0, 0, 1)$, $\mathbf{v} = (1, 0, 0)$, or $\mathbf{v} = \frac{1}{\sqrt{2}}(1, 0, -1)$.

Throughout this section, we suppose that $X(M)$ is foliated by curves of nonzero constant curvature (circles in $\mathbb{L}^3$) in parallel planes with normal vector $\mathbf{v}$ in $\mathbb{E}^3$. In case $\mathbf{v} = (0, 0, 1)$, this means that $X(M)$ is foliated by pieces of Euclidean circles in horizontal planes. However, in cases $\mathbf{v} = (1, 0, 0)$ and $\mathbf{v} = \frac{1}{\sqrt{2}}(1, 0, -1)$, it means that the surface $X(M)$ is foliated by pieces of Euclidean hyperbolas and parabolas, respectively. See the preceding section.

The following three theorems describe, up to linear isometries of $\mathbb{L}^3$, the Weierstrass representation of the immersion $X$. Since these results are local, we will suppose that $M$ is simply connected and that $\langle \mathrm{Re}(\Phi(P)), \mathbf{v}\rangle_0 + i\langle \mathrm{Im}(\Phi(P)), \mathbf{v}\rangle_0 \neq 0$ for all $P \in M$. Moreover, we will assume that the holomorphic function

$z = \langle X, \mathbf{v}\rangle_0 + i\langle X, \mathbf{v}\rangle_0{}^*$ is a conformal parameter on $M$, where $\langle X, \mathbf{v}\rangle_0{}^*$ is the harmonic conjugate of $\langle X, \mathbf{v}\rangle_0$.

### *3.1. Maximal Spacelike Surfaces Foliated by Pieces of Circles*

We shall prove the following theorem.

THEOREM 1. *If $X(M)$ is foliated by pieces of Euclidean circles in parallel planes with normal Euclidean vector $\mathbf{v} = (0,0,1)$, then, up to scaling and linear isometries in $\mathbb{L}^3$, the stereographic projection $g$ of the Gauss map of $X$ satisfies*:

1. $\frac{dg}{dz} = g$; *or*
2. $\left(\frac{dg}{dz}\right)^2 = g(g^2 + 2rg + 1)$, *where* $r \in \mathbb{R}$.

*Proof.* Consider the conformal parameter $z = \langle X, \mathbf{v}\rangle_0 + i\langle X, \mathbf{v}\rangle_0{}^* = X_3 + iX_3^*$. From (7), (8), and (9), we have:

$$X(P) = \operatorname{Re}\left(\int^P \left(\frac{i}{2}\left(\frac{1}{g} - g\right) dz,\ -\frac{1}{2}\left(\frac{1}{g} + g\right) dz, dz\right)\right); \tag{10}$$

$$ds^2 = \left(\frac{1 - |g|^2}{2|g|}\right)^2 |dz|^2.$$

Let $\alpha(t)$ be a curve in $(M, ds^2)$ parameterized by the arc length such that $X_3(\alpha(t)) = C$ (i.e., $\operatorname{Re}(z(\alpha(t))) = C$), where $C$ is a constant. For the sake of simplicity, write $z(t) = z(\alpha(t))$, $X(t) = X(\alpha(t))$, and $g(t) = g(\alpha(t))$. Then we have $\frac{|1-|g|^2|}{2|g|}\left|\frac{dz}{dt}\right| = 1$, and since $\operatorname{Re}\left(\frac{dz}{dt}\right) = 0$ we deduce that $\frac{dz}{dt} = \pm\frac{2i|g|}{1-|g|^2}$. Up to the change $g \to 1/g$ (which corresponds to the linear isometry in $\mathbb{L}^3$ defined by the symmetry with respect to the plane $x_1 = 0$), $\Phi_3$ does not change, and we can assume that

$$\frac{dz}{dt} = \frac{2i|g|}{|g|^2 - 1}. \tag{11}$$

Since $X$ is spacelike and $\mathbf{v} = (0,0,1)$, the vectors $\frac{dX}{dt}$ and $\frac{d^2X}{dt^2}$ are spacelike. Hence, from (10) and (11), it is not hard to check that the planar curvature $k(t)$ of the curve $X(t)$ is given by

$$k(t) = \sqrt{\left\langle \frac{d^2X}{dt^2}, \frac{d^2X}{dt^2}\right\rangle} = \sqrt{\left\langle \frac{d^2X}{dt^2}, \frac{d^2X}{dt^2}\right\rangle_0} = \operatorname{Im}\left(\frac{d\ \log(g)}{dt}\right).$$

Since $\Phi_3 = dz$, this 1-form does not vanish at any point in $M$, and the same holds for the map $g$. Thus, up to a choice of the branch, the map $\log(g)$ is holomorphic and well-defined on $M$. For simplicity, we write $u = \operatorname{Re}(\log(g))$ and $v = \operatorname{Im}(\log(g))$.

At this point, we introduce the new parameter $s(t)$ determined (up to an additive constant) by the equation $\frac{ds}{dt} = \frac{1}{\sinh(u(\alpha(t))}$. Observe that equation (11) gives

$\frac{dz(\alpha(s))}{ds} = i$ and so, for any constant $C$, we can choose $s(t) = \mathrm{Im}(z(t))$. On the other hand, it is clear that

$$k(s) \stackrel{\mathrm{def}}{=} k(t(s)) = \frac{dv/ds}{\sinh(u)},$$

and since $k(s)$ is constant we have

$$\frac{d^2 v}{ds^2} - \frac{du}{ds}\frac{dv}{ds}\coth(u) = 0.$$

If we define

$$\mathcal{S}_1 = \mathrm{Im}\left(\left(\frac{d(\log(g))}{dz}\right)^2 \frac{3|g|^2-1}{2(|g|^2-1)} - \left(\frac{d^2 g}{dz^2}\right)\frac{1}{g}\right) \tag{12}$$

and take into account that $s(t) = \mathrm{Im}(z(t))$, we obtain

$$\mathcal{S}_1 = 0. \tag{13}$$

In particular, the function $\mathrm{Im}(h_1)$ is harmonic, where

$$h_1 = \left(2\left(\frac{d^2 g}{dz^2}\right)\frac{1}{g} - 3\left(\frac{d(\log(g))}{dz}\right)^2\right)|g|^2,$$

and thus

$$\frac{h_1}{|g|^2} - \lambda_1 - \frac{\mu}{g} = 0, \tag{14}$$

where $\lambda_1 \in \mathbb{R}$ and $\mu \in \mathbb{C}$. In particular, $\mathrm{Im}(h_1) = -\mathrm{Im}(\bar{\mu} g)$. On the other hand, labeling

$$h_2 = 2\left(\frac{d^2 g}{dz^2}\right)\frac{1}{g} - \left(\frac{d(\log(g))}{dz}\right)^2,$$

from (13) we deduce that $\mathrm{Im}(h_2) = \mathrm{Im}(h_1)$ and thus

$$h_2 + \bar{\mu} g + \lambda_2 = 0, \tag{15}$$

where $\lambda_2 \in \mathbb{R}$.

From equations (14), (15), and their derivatives, it is not hard to see that

$$\lambda_1 = \lambda_2, \qquad \left(\frac{dg}{dz}\right)^2 + \frac{g}{2}(\bar{\mu} g^2 + 2\lambda_1 g + \mu) = 0.$$

Up to a rotation about the $x_3$-axis (which is a linear isometry in $\mathbb{L}^3$ that substitutes $g$ for $\theta g$, $|\theta| = 1$), we can suppose that $\mu \in \mathbb{R}$ and $\mu \leq 0$. Furthermore, up to a homothety in $\mathbb{L}^3$ (which corresponds to a homothetical change of variable $z \to kz$, $k \in \mathbb{R}$), we can put $\mu = -2$, provided that $\mu \neq 0$. This leads to case 2 of the theorem. In case $\mu = 0$, we deduce that $\lambda_1 \neq 0$ (recall that $X$ is nonplanar); hence we can assume that $\lambda_1 = \pm 1$. Taking into account that $s(t) = \mathrm{Im}(z(t))$, $k(s) =$ constant $\neq 0$, and the preceding expression for $k(t(s))$, we deduce that $\lambda_1 = 1$. Up to the change $g \to 1/g$ once again if necessary, we obtain case 1. This concludes the proof. □

REMARK 2. Following the proof of Theorem 1, it is easy to see that $X(M)$ is foliated by pieces of straight lines in planes parallel to $\Pi_{\mathbf{v}}$, $\mathbf{v} = (0, 0, 1)$, if and only if $\mu = 0$ and $\lambda_1 = -1$ $\left(\text{i.e., } \frac{dg}{dz} = ig\right)$. In this case, $X(M)$ is a piece of the complete maximal spacelike surface with Weierstrass data

$$\left(\mathbb{C}^*,\ \Phi_1 = \frac{1-g^2}{2g^2}\,dg,\ \Phi_2 = \frac{i(1+g^2)}{2g^2}\,dg,\ \Phi_3 = -i\frac{1}{g}\,dg\right).$$

Up to the change of variables $g = e^{iu}$, these meromorphic data determine, following (8), a well-defined maximal spacelike surface with singularities in the $u$-plane. This surface is the one associated to the helicoid according to Remark 1.

Let us now determine the surfaces arising from cases 1 and 2 in Theorem 1. In case 1 and from (10), we derive that $X(M)$ is a piece of the complete maximal spacelike surface with singularities determined by the Weierstrass data

$$\left(\mathbb{C}^*,\ \Phi_1 = \frac{i}{2}\frac{1-g^2}{g^2}\,dg,\ \Phi_2 = -\frac{1}{2}\frac{1+g^2}{g^2}\,dg,\ \Phi_3 = \frac{1}{g}\,dg\right).$$

This rotational surface is the one associated, following Remark 1, to the catenoid with vertical normal vector at the ends.

To discuss case 2, consider the compact Riemann surface $\bar{N}_r = \{(u, w) : w^2 = u(u^2 + 2ru + 1)\}$ and observe that, except in the degenerate case $r^2 = 1$, this surface is homeomorphic to a torus. When $r^2 = 1$, $\bar{N}_r$ is the Riemann sphere. Assume that $r^2 \neq 1$. Then define the Weierstrass data

$$N_r = \bar{N}_r - (u^{-1}(0) \cup u^{-1}(\infty)), \quad g = u, \quad \eta g = du/w,$$

with $\Phi_j$ $(j = 1, 2, 3)$ as in (7). However, these meromorphic data do not define a maximal immersion because the 1-forms $\Phi_j$ have real periods on certain homology curves (see (8)). Indeed, first observe that the 1-form $\Phi_1$ is exact. Let $l$ denote a closed real interval in the $u$-plane whose limit points lie in $u(w^{-1}(0)) \cup \{\infty\}$ and $w\big|_l \leq 0$, and let $\gamma$ be any closed curve in $N_r$ in the same homology class of the lift to $\bar{N}_r$ of $l$. If $\gamma'$ is any closed curve with $\gamma' \neq m\gamma$, $m \in \mathbb{Z}$ (e.g., a curve in the same class of homology of the lift of a closed real interval $l'$ in the $u$-plane as before, but satisfying $w\big|_l \geq 0$), then it is easy to see that the period $\int_{\gamma'} \Phi_j$ does not vanish for $j = 2, 3$. Furthermore, if $p\colon \tilde{N}_r \to \bar{N}_r$ is the covering that satisfies $p_*(\mathcal{H}_1(\tilde{N}_r, \mathbb{Z})) = \{m\gamma : m \in \mathbb{Z}\}$, then the lift to $\tilde{N}_r$ of the Weierstrass data just listed gives new Weierstrass data without real periods and also determines a complete singly periodic maximal spacelike surface with singularities in $\mathbb{L}^3$. Since $\tilde{N}_r$ is conformally equivalent to $\mathbb{C}^*$, the lift $M_r$ of $N_r$ to $\tilde{N}_r$ is conformally diffeomorphic to $\mathbb{C}^*$ minus infinitely many points, which are just the lifts of the two ends of $N_r$. Since $dz = du/w$, it is clear from (10) that $X(M)$ is a piece of this surface.

If $r^2 < 1$ then the associated minimal surfaces, according to Remark 1, have been recently studied in [9] and [8].

In the degenerated case we have $r = \pm 1$, and up to the change $g \to v^2$ it is straightforward to check that $X(M)$ is a piece of the maximal spacelike surface with singularities associated to the following Weierstrass data:

$$\left(\mathbb{C}-\left\{0, i\sqrt{\pm 1}, -i\sqrt{\pm 1}\right\},\ \Phi_1 = -i\,\frac{v^2-(\pm 1)}{v^2}\,dv,\right.$$
$$\left.\Phi_2 = -\frac{v^4+1}{v^2(v^2+(\pm 1))}\,dv,\ \Phi_3 = \frac{2}{v^2+(\pm 1)}\,dv\right).$$

Note that in case $r=-1$, the periods of $\Phi_j$ $(j=1,2,3)$ are imaginary and so the immersion given in (8) is well-defined on $M_{-1}=\mathbb{C}-\{0,1,-1\}$. In case $r=1$, this does not occur and the immersion is well-defined in a suitable covering $M_1$ of $\mathbb{C}-\{0,i,-i\}$.

REMARK 3. If $r\le 1$, it is not hard to see that any level curve $x_3=C$ with $C\in\mathbb{R}$ is either noncompact or nonspacelike (i.e., it contains singular points lying in the set $\{|g|=1\}$). See Figures 2 and 3. For the 1-parameter family of surfaces $\mathcal{R}=\{M_r : r>1\}$, the set $\{|g|=1\}$ has infinitely many connected components in $M_r$, all of them homeomorphic to $\mathbb{S}^1$, and the image under the immersion of this set of singularities is a discrete subset of $\mathbb{L}^3$. Indeed, any connected component $c$ of $\{|g|=1\}$ is a lift to $M_r$ of one of the two closed curves in $N_r$ defined by $|u|=1$. These two curves are pointwise invariant under the antiholomorphic transformation in $N_r$ given by $T(u,w)=(1/\bar{u}, \bar{w}/\bar{u})$ that satisfies $T^*(\Phi_j)=-\overline{\Phi_j}$ for $j=1,2,3$. Hence, it is not hard to conclude that the points of $c$ are mapped under the immersion on the same point $P_c\in\mathbb{L}^3$ and that the maximal surface is symmetric with respect to $P_c$ (see Definition 3). In particular, $c=x_3^{-1}(x_3(P_c))$ and $c\subset\{|g|=1\}$ are the only singular level curves; the other level curves $x_3=C$, $C\notin\{x_3(P_c) : c\subset\{|g|=1\}\}$, are spacelike (i.e., they are either complete circles or straight lines). See Figure 1.

### 3.2. *Maximal Spacelike Surfaces Foliated by Pieces of Hyperbolas*

Let us study the case $\mathbf{v}=(1,0,0)$. In this case, the metric induced by $\langle\cdot,\cdot\rangle$ on $\Pi_{\mathbf{v}}$ is given by $dx_2^2-dx_3^2$.

We adapt the Weierstrass representation $(g,\eta)$ of $X$ to this new frame as follows. Define

$$g_h = -i\,\frac{g+1}{g-1}, \qquad \eta_h = \frac{2}{(-ig_h+1)^2}\,\eta, \tag{16}$$

and observe that

$$\Phi_1 = g_h\eta_h, \quad \Phi_2 = \tfrac{1}{2}(1-g_h^2)\eta_h, \quad \Phi_3 = -\tfrac{1}{2}(1+g_h^2)\eta_h. \tag{17}$$

Recall that in this case $dz=\Phi_1$.

THEOREM 2. *If $X(M)$ is foliated by pieces of Euclidean hyperbolas in parallel planes with normal Euclidean vector $\mathbf{v}=(1,0,0)$, then, up to scaling and linear isometries in $\mathbb{L}^3$, the meromorphic map $g_h$ defined in* (16) *of $X$ satisfies*:

1. $\left(\frac{dg_h}{dz}\right)^2 = g_h$;
2. $\left(\frac{dg_h}{dz}\right)^2 = \pm g_h(g_h-1)$;
3. $\frac{dg_h}{dz} = ig_h$; *or*
4. $\left(\frac{dg_h}{dz}\right)^2 = g_h(g_h^2+2rg_h+r_0)$, *where $r\in\mathbb{R}$ and $r_0\in\{1,-1\}$.*

*Proof.* Note that $\langle X, \mathbf{v}\rangle_0 = X_1$ and so, from (16), (8), and (9), we have:

$$X(P) = \operatorname{Re}\left(\int^P \left(dz, \frac{1}{2}\left(\frac{1}{g_h} - g_h\right) dz, -\frac{1}{2}\left(\frac{1}{g_h} + g_h\right) dz\right)\right); \tag{18}$$

$$ds^2 = \left(\frac{\operatorname{Im}(g_h)}{|g_h|}\right)^2 |dz|^2.$$

Let $\alpha(t)$ be a curve in $(M, ds^2)$ parameterized by the arc length such that $X_1(\alpha(t)) = C$ (i.e., $\operatorname{Re}(z(\alpha(t))) = C$), where $C$ is a constant. We write $z(t) = z(\alpha(t))$, $X(t) = X(\alpha(t))$, and $g_h(t) = g_h(\alpha(t))$. Then we have $\frac{|\operatorname{Im}(g_h)|}{|g_h|}\left|\frac{dz}{dt}\right| = 1$, and since $\operatorname{Re}\left(\frac{dz}{dt}\right) = 0$ we deduce that $\frac{dz}{dt} = \pm i \frac{|g_h|}{\operatorname{Im}(g_h)}$. Up to the change $g_h \to -g_h$ (which corresponds to the linear isometry in $\mathbb{L}^3$ determined by the reflection about the $x_1$-axis), $\Phi_1$ does not change, and we can assume that

$$\frac{dz}{dt} = i \frac{|g_h|}{\operatorname{Im}(g_h)}. \tag{19}$$

Since $X$ is spacelike and $\mathbf{v} = (1, 0, 0)$, the vectors $\frac{dX}{dt}$ and $\frac{d^2X}{dt^2}$ are spacelike and timelike, respectively. Therefore, from (18) and (19), the planar curvature $k(t)$ of the curve $X(t)$ is given by

$$k(t) = \sqrt{-\left\langle \frac{d^2X}{dt^2}, \frac{d^2X}{dt^2}\right\rangle} = \operatorname{Re}\left(\frac{d\log(g_h)}{dt}\right).$$

Because $\Phi_1 = dz$, the map $g_h$ never vanishes on $M$. Thus, up to a choice of the branch, the map $\log(g_h)$ is holomorphic and well-defined on $M$. For simplicity, we write $u = \operatorname{Re}(\log(g_h))$ and $v = \operatorname{Im}(\log(g_h))$.

At this point, we introduce the new parameter $s(t)$ determined (up to an additive constant) by the equation $\frac{ds}{dt} = \frac{|g_h|}{\operatorname{Im}(g_h)}$. Observe that $\frac{dz(\alpha(s))}{ds} = i$; hence we can choose $s(t) = \operatorname{Im}(z(t))$ for any constant $C$. On the other hand, it is clear that

$$k(s) \overset{\text{def}}{=} k(t(s)) = \frac{du/ds}{\sin(v)},$$

and since $k(s)$ is constant we have

$$\frac{d^2u}{ds^2} - \frac{du}{ds}\frac{dv}{ds}\cot(v) = 0.$$

If we define

$$\mathcal{S}_2 = \operatorname{Re}\left(\left(\frac{d(\log(g_h))}{dz}\right)^2\left(1 - \frac{i\operatorname{Re}(g_h)}{2\operatorname{Im}(g_h)}\right) - \left(\frac{d^2g_h}{dz^2}\right)\frac{1}{g_h}\right) \tag{20}$$

and take into account that $s(t) = \operatorname{Im}(z(t))$, it follows that

$$\mathcal{S}_2 = 0. \tag{21}$$

In particular, the function $\operatorname{Im}(h_1)$ is harmonic, where

$$h_1 = \left( \left( \frac{d^2 g_h}{dz^2} \right) \frac{1}{g_h} - \frac{1}{2} \left( \frac{d(\log(g_h))}{dz} \right)^2 \right) \overline{g_h},$$

and thus

$$\frac{h_1}{g_h} + \lambda_1 g_h - \mu = 0, \tag{22}$$

where $\lambda_1 \in \mathbb{R}$ and $\mu \in \mathbb{C}$. In particular, $\mathrm{Im}(h_1) = -\mathrm{Im}(\bar{\mu} g_h)$. On the other hand, labeling

$$h_2 = -\frac{d^2 g_h}{dz^2} + \frac{3}{2} \left( \frac{d(\log(g_h))}{dz} \right)^2 g_h,$$

from (21) we deduce that $\mathrm{Im}(h_2) = -\mathrm{Im}(h_1)$ and thus

$$h_2 - \bar{\mu} g_h + \lambda_2 = 0, \tag{23}$$

where $\lambda_2 \in \mathbb{R}$.

From equations (22), (23), and their derivatives, it is not hard to see that

$$\left( \frac{dg_h}{dz} \right)^2 = g_h(-\lambda_1 g_h^2 + 2\mu g_h - \lambda_2), \quad \mu \in \mathbb{R}.$$

Since $X$ is nonplanar, $\lambda_1 = \lambda_2 = 0$ implies that $\mu \neq 0$. In this case, and up to scaling in $\mathbb{L}^3$, we can suppose $\mu = \pm\frac{1}{2}$. Taking into account that $s(t) = \mathrm{Im}(z(t))$, $k(s) = \text{constant} \neq 0$ and the preceding expression for $k(s)$, we deduce that $\mu = -\frac{1}{2}$. Up to the change $g_h \to 1/g_h$ if necessary, we obtain case 3 of the theorem. Assume now that $|\lambda_1| + |\lambda_2| \neq 0$. Up to the change $g_h \to 1/g_h$, which corresponds to the linear isometry in $\mathbb{L}^3$ given by a symmetry with respect to the plane $x_2 = 0$, we can suppose that $\lambda_2 \neq 0$. Moreover, note that the change $g_h \to \pm e^l g_h$ ($l \in \mathbb{R}$) is associated to the linear isometry $R$ in $\mathbb{L}^3$ given by

$$R(x_1, x_2, x_3) = \big(x_1, \pm(\cosh(l)x_2 + \sinh(l)x_3), \pm(\cosh(l)x_3 + \sinh(l)x_2)\big).$$

Therefore, up to this kind of rigid motion in $\mathbb{L}^3$ and homotheties, we can suppose that $\lambda_1 = 1$ and $\lambda_2 = \pm 1$, provided that $\lambda_1 \neq 0$ also (case 4). Analogously, $\lambda_1 = 0$ and $\lambda_2 \neq 0$ lead to case 1 (if $\mu = 0$) and case 2 (if $\mu \neq 0$) of the theorem. This concludes the proof. □

REMARK 4. Following the proof of Theorem 2, it is easy to see that $X(M)$ is foliated by pieces of straight lines in parallel planes to $\Pi_{\mathbf{v}}$, $\mathbf{v} = (1, 0, 0)$, if and only if $\mu = \frac{1}{2}$ and $\lambda_1 = \lambda_2 = 0$ $\big($i.e., $\frac{dg_h}{dz} = g_h\big)$. In this case, $X(M)$ is a piece of the complete maximal spacelike surface with singularities determined by the following Weierstrass data:

$$\left( \mathbb{C}^*,\ \Phi_1 = \frac{1}{g_h}\, dg_h,\ \Phi_2 = \frac{1 - g_h^2}{2g_h^2}\, dg_h,\ \Phi_3 = -\frac{1 + g_h^2}{2g_h^2}\, dg_h \right).$$

These meromorphic data determine, following (8), a well-defined maximal surface. This surface is the one associated to the helicoid (viewed in $\mathbb{E}^3$ with horizontal axis) following Remark 1. See Figure 4.

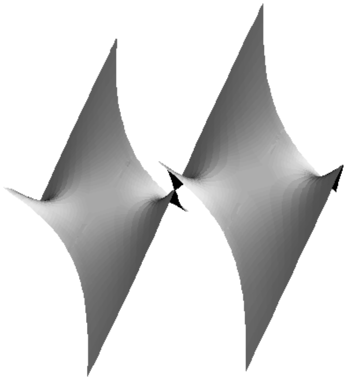

**Figure 4** A piece of a translational invariant maximal surface foliated by hyperbolas in parallel timelike planes.

Let us determine the surfaces arising from the four cases of Theorem 2. In case 1, and doing the change $g_h \to u^2$, it is easy to see that $X(M)$ is a piece of the complete maximal spacelike surface with singularities determined by the Weierstrass data

$$\left(\mathbb{C}^*,\ \Phi_1 = 2\,du,\ \Phi_2 = \frac{1-u^4}{u^2}\,du,\ \Phi_3 = -\frac{1+u^4}{u^2}\right).$$

Since the three 1-forms are exacts, the maximal immersion given by (8) is well-defined.

In case 2, and up to linear isometries in $\mathbb{L}^3$, we have two possibilities: either $\frac{dg_h}{dz} = \sqrt{g_h(g_h-1)}$ or $\frac{dg_h}{dz} = i\sqrt{g_h(g_h-1)}$. Suppose first that $\frac{dg_h}{dz} = \sqrt{g_h(g_h-1)}$. Up to the change $g_h \to \frac{1}{1-u^2}$ and from (18), we obtain that $X(M)$ is a piece of the maximal spacelike surface with singularities associated to the Weierstrass data

$$\left(\mathbb{C}-\{1,-1\},\ \Phi_1 = \frac{2}{1-u^2}\,du,\ \Phi_2 = \frac{(2-u^2)u^2}{(u^2-1)^2}\,du,\ \Phi_3 = \frac{u^4-2u^2+2}{(u^2-1)^2}\,du\right).$$

The three 1-forms have no real periods, so the immersion $X$ in (8) is well-defined. If $\frac{dg_h}{dz} = i\sqrt{g_h(g_h-1)}$ then we obtain the conjugate Weierstrass data, and the immersion is well-defined in a suitable covering of $\mathbb{C}-\{1,-1\}$.

In case 3, and using (18), we have the following Weierstrass data:

$$\left(\mathbb{C}-\{0\},\ \Phi_1 = \frac{i}{g_h}\,dg_h,\ \Phi_2 = \frac{i(1-g_h^2)}{2g_h^2}\,dg_h,\ \Phi_3 = -\frac{i(1+g_h^2)}{2g_h^2}\,dg_h\right).$$

The lift of these meromorphic data to the holomorphic universal covering $\mathbb{C}$ leads to a rotational maximal surface. The associated minimal surface, according to Remark 1, is a catenoid with horizontal normal vector at the ends.

The discussion of case 4 is similar to that of case 2 of Theorem 1. Consider the compact Riemann surface $\bar{M}_0 = \{(u,w) : w^2 = u(u^2+2ru+r_0)\}$ and observe that, except in the degenerate case $r^2 = r_0 = 1$, this surface is homeomorphic to a torus. If $r^2 = r_0 = 1$, then $\bar{M}_0$ is the Riemann sphere. Assume that either $r^2 \neq 1$ or $r_0 \neq 1$. Then define the Weierstrass data

$$\big(M_0 = \bar{M}_0 - (u^{-1}(0) \cup u^{-1}(\infty)),\ g_h = u,\ \eta_h g_h = du/w\big),$$

with $\Phi_j$ $(j = 1, 2, 3)$ as in (17). However, these meromorphic data do not define a maximal immersion because the 1-forms $\Phi_j$ have real periods on certain homology curves (see (8)). Indeed, first observe that, from (18) and taking into account $dz = du/w$, if $r_0 = 1$ (resp. $r_0 = -1$) then $\Phi_2$ (resp. $\Phi_3$) is exact. Let $l$ denote a closed real interval in the $u$-plane whose limit points lie in $u(w^{-1}(0)) \cup \{\infty\}$ and $w\big|_l \leq 0$, and let $\gamma$ be any closed curve in $M_0$ in the same homology class of the lift to $\bar{M}_0$ of $l$. If $\gamma'$ is any closed curve with $\gamma' \neq m\gamma$, $m \in \mathbb{Z}$ (e.g., a curve in the same class of homology of the lift of a closed real interval $l'$ in the $u$-plane as before, but satisfying $w\big|_l \geq 0$), then it is easy to see that the period $\int_{\gamma'} \Phi_j$ does not vanish, where $j = 1, 3$ (if $r_0 = 1$) or $j = 1, 2$ (if $r_0 = -1$). Furthermore, if $p\colon \tilde{M}_0 \to \bar{M}_0$ is the covering that satisfies $p_*(\mathcal{H}_1(\tilde{M}_0, \mathbb{Z})) = \{m\gamma : m \in \mathbb{Z}\}$, then the lift to $\tilde{M}_0$ of the Weierstrass data displayed previously gives new Weierstrass data without real periods and also determines a complete singly periodic maximal surface with singularities in $\mathbb{L}^3$. Since $\tilde{M}_0$ is conformally equivalent to $\mathbb{C}^*$, the lift of $M_0$ is conformally diffeomorphic to $\mathbb{C}^*$ minus infinitely many points, which are just the lifts of the two ends of $M_0$. It is clear that $X(M)$ is a piece of this surface.

If $r^2 < 1$ and $r_0 = 1$, then Remark 1 leads (as in Section 3.1) to the examples in [9] and [8] (but from a different point of view in $\mathbb{E}^3$).

In the degenerated case we have $r = \pm 1$ and $r_0 = 1$, and up to the change $g_h \to v^2$ it is straightforward to check that $X(M)$ is a piece of the maximal spacelike surface with singularities associated to the following Weierstrass data:

$$\Bigg(\mathbb{C} - \left\{0, i\sqrt{(\pm 1)}, -i\sqrt{(\pm 1)}\right\},\ \Phi_1 = \frac{2}{v^2 + (\pm 1)}\, dv,$$

$$\Phi_2 = -\frac{v^2 - (\pm 1)}{v^2}\, dv,\ \Phi_2 = -\frac{v^4 + 1}{(v^2 + (\pm 1))v^2}\, dv\Bigg).$$

Note that in case $r = -1$, the periods of $\Phi_j$ $(j = 1, 2, 3)$ are imaginary and so the immersion given in (8) is well-defined. In case $r = 1$, this does not occur and the immersion is well-defined in a suitable covering of $\mathbb{C} - \{0, i, -i\}$.

### 3.3. *Maximal Spacelike Surfaces Foliated by Pieces of Parabolas*

Finally, we study the case $\mathbf{v} = \frac{1}{\sqrt{2}}(1, 0, -1)$. For convenience, we introduce the following frame in $\mathbb{E}^3$:

$$\mathbf{e}_1 = (0, 1, 0), \quad \mathbf{e}_2 = \frac{1}{\sqrt{2}}(1, 0, 1), \quad \mathbf{e}_3 = \mathbf{v} = \frac{1}{\sqrt{2}}(1, 0, -1).$$

We label $y_j = \langle \mathbf{e}_j, \cdot \rangle_0$, $j = 1, 2, 3$, as the three coordinate functions associated to this frame. Write $Y_j = \langle \mathbf{e}_j, X \rangle_0$, $j = 1, 2, 3$. Since $X = \sum_{j=1}^{3} Y_j \mathbf{e}_j$, for simplicity and in what follows we write $X = (Y_1, Y_2, Y_3)$. Note that the metric induced by $\langle \cdot, \cdot \rangle$ on $\Pi_{\mathbf{v}}$ (i.e., on the plane $y_3 = 0$) is given by $dy_1^2$.

The adapted Weiertrass representation of $X$ is given by $\Psi_j(\xi) = \frac{\partial Y_j}{\partial \xi}\, d\xi$ $(j = 1, 2, 3)$, where $\xi$ is a conformal parameter on $M$. In other words,

$$\Psi_1 = \Phi_2, \quad \Psi_2 = \frac{1}{\sqrt{2}}(\Phi_1 + \Phi_3), \quad \Psi_3 = \frac{1}{\sqrt{2}}(\Phi_1 - \Phi_3).$$

From (7), it is easy to see that

$$\Psi_1^2 + 2\Psi_2\Psi_3 = 0.$$

Since $dz = \Psi_3$, defining $g_p = \Psi_1/dz$ yields

$$\Psi_1 = g_p\, dz, \quad \Psi_2 = -\frac{1}{2} g_p^2\, dz, \quad \Psi_3 = dz. \tag{24}$$

THEOREM 3. *If $X(M)$ is foliated by pieces of Euclidean parabolas in parallel planes with normal Euclidean vector $\mathbf{v} = \frac{1}{\sqrt{2}}(1, 0, -1)$, then, up to scaling and linear isometries in $\mathbb{L}^3$, the meromorphic map $g_p$ defined in* (24) *of $X$ satisfies*:

1. $\left(\frac{dg_p}{dz}\right)^2 = g_p$; *or*
2. $\left(\frac{dg_p}{dz}\right)^2 = \pm g_p(g_p - r)$; $r \in \mathbb{R}$; *or*
3. $\frac{dg_p}{dz} = i$.

*Proof*. From (24), (8), and (9), we have:

$$X(P) = \operatorname{Re}\left(\int^P \left(g_p\, dz, -\frac{g_p^2}{2}\, dz, dz\right)\right); \tag{25}$$

$$ds^2 = (\operatorname{Im}(g_p))^2 |dz|^2.$$

Let $\alpha(t)$ be a curve in $(M, ds^2)$ parameterized by the arc length such that $Y_3(\alpha(t)) = C$ (i.e., $\operatorname{Re}(z(\alpha(t))) = C$), where $C$ is a constant. We write $z(t) = z(\alpha(t))$, $X(t) = X(\alpha(t))$, and $g_p(t) = g_p(\alpha(t))$. Then we have $|\operatorname{Im}(g_p)|\left|\frac{dz}{dt}\right| = 1$, and since $\operatorname{Re}\left(\frac{dz}{dt}\right) = 0$ we deduce that $\frac{dz}{dt} = \pm\frac{i}{\operatorname{Im}(g_p)}$. Up to the change $g_p \to -g_p$ (which corresponds to the linear isometry in $\mathbb{L}^3$ determined by the symmetry with respect to the plane $y_1 = 0$), we can assume that

$$\frac{dz}{dt} = -\frac{i}{\operatorname{Im}(g_p)}. \tag{26}$$

Since $X$ is spacelike and $\mathbf{v} = \frac{1}{\sqrt{2}}(1, 0, -1)$, the vectors $\frac{dX}{dt}$ and $\frac{d^2X}{dt^2}$ are spacelike and lightlike, respectively. Therefore, the planar curvature of the curve $X(t)$ is constant if and only if the function

$$k(t) = \sqrt{\left\langle \frac{d^2X}{dt^2}, \frac{d^2X}{dt^2}\right\rangle_0} = -\operatorname{Re}\left(\frac{dg_p}{dt}\right)$$

is constant. For simplicity, we write $u = \operatorname{Re}(g_p)$ and $v = \operatorname{Im}(g_p)$.

At this point, we introduce the new parameter $s(t)$ determined (up to an additive a constant) by the equation: $\frac{ds}{dt} = -\frac{1}{\operatorname{Im}(g_p)}$. Observe that equation (26) gives

$\frac{dz(\alpha(s))}{ds} = i$ and so we can choose $s(t) = \mathrm{Im}(z(t))$ for any constant $C$. On the other hand, it is clear that

$$k(s) \stackrel{\text{def}}{=} k(t(s)) = \frac{du}{ds}\frac{1}{v};$$

since $k(s)$ is constant, we have

$$\frac{d^2u}{ds^2} - \frac{du}{ds}\frac{dv}{ds}\frac{1}{v} = 0.$$

If we define

$$\mathcal{S}_3 = \mathrm{Re}\left(\frac{d^2g_p}{dz^2} + \left(\frac{dg_p}{dz}\right)^2 \frac{1}{\overline{g_p} - g_p}\right) \tag{27}$$

and take into account that $s(t) = \mathrm{Im}(z(t))$, we have

$$\mathcal{S}_3 = 0. \tag{28}$$

In particular, the function $\mathrm{Im}(h_1)$ is harmonic, where $h_1 = \overline{g_p}\frac{d^2g_p}{dz^2}$, and thus

$$\frac{d^2g_p}{dz^2} - \lambda_1 g_p - \mu = 0, \tag{29}$$

where $\lambda_1 \in \mathbb{R}$ and $\mu \in \mathbb{C}$. Hence, we deduce that $\mathrm{Im}(h_1) = -\mathrm{Im}(\bar{\mu} g_p)$. On the other hand, labeling

$$h_2 = g_p \frac{d^2g_p}{dz^2} - \left(\frac{dg_p}{dz}\right)^2,$$

from (28) we deduce $\mathrm{Im}(h_2) = \mathrm{Im}(h_1)$ and thus

$$h_2 + \bar{\mu} g_h + \lambda_2 = 0, \tag{30}$$

where $\lambda_2 \in \mathbb{R}$.

From Equations (29), (30), and their derivatives, it is not hard to see that

$$\left(\frac{dg_p}{dz}\right)^2 = \lambda_1 g_p^2 + 2\mu g_p + \lambda_2, \quad \mu \in \mathbb{R}.$$

Because $X$ is nonplanar, $\lambda_1 = \mu = 0$ implies that $\lambda_2 \neq 0$. In this case, and up to scaling in $\mathbb{L}^3$, we can suppose $\lambda_2 = \pm 1$. Taking into account that $s(t) = \mathrm{Im}(z(t))$, $k(s) = \text{constant} \neq 0$ and the preceding expression for $k(s)$, we deduce that $\lambda_2 = -1$. Up to the change $g_p \to -g_p$ if necessary, we obtain case 3 of the theorem. Assume now that $|\lambda_1| + |\mu| \neq 0$. The change $g_p \to g_p + l$, $l \in \mathbb{R}$, is associated to the linear isometry $R$ in $\mathbb{L}^3$ given by

$$R(y_1, y_2, y_3) = \left(y_1 + l y_3, -l y_1 + y_2 - \frac{l^2}{2} y_3, y_3\right).$$

Hence, up to this kind of rigid motion in $\mathbb{L}^3$ and homotheties, we can suppose that $\lambda_2 = 0$ and $\lambda_1 = \pm 1$, provided that $\lambda_1 \neq 0$ (case 2). When $\lambda_1 = 0$ and $\mu \neq 0$, we also take into account the change $g_p \to -g_p$ (which corresponds to a symmetry with respect to the plane $y_1 = 0$) to obtain case 1. This concludes the proof. $\square$

REMARK 5. Following the proof of Theorem 3, it is easy to see that $X(M)$ is foliated by pieces of straight lines in parallel planes to $\Pi_{\mathbf{v}}$, $\mathbf{v} = \frac{1}{\sqrt{2}}(1, 0, -1)$, if and only if $\lambda_1 = \mu = 0$ and $\lambda_2 = 1$ $\big($i.e., $\frac{dg_p}{dz} = 1\big)$. In this case, and after the change $g \to -\sqrt{2}u$, we have that $X(M)$ is a piece of the complete maximal spacelike surface with singularities associated to the following Weierstrass data:

$$\big(\mathbb{C},\ \Phi_1 = (u^2 - 1)\,du,\ \Phi_2 = 2u\,du,\ \Phi_3 = (u^2 + 1)\,du\big).$$

These meromorphic data determine, following (8), a well-defined maximal spacelike surface with singularities. It is the surface associated, according to Remark 1, to the Enneper surface (viewed with horizontal normal vector at the end in $\mathbb{E}^3$). See Figure 5.

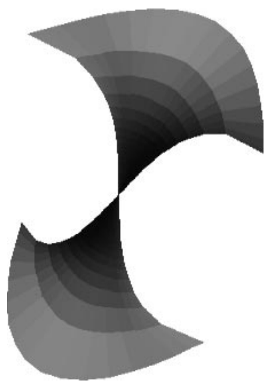

**Figure 5** A piece of a maximal surface foliated by parabolas in parallel lightlike planes.

Let us now determine the surfaces arising from the three cases of Theorem 3. In case 1, doing the change $g \to \sqrt{2}u^2$ and up to scaling, it is easy to see from (25) and the definition of $\Psi_j$ $(j = 1, 2, 3)$ that $X(M)$ is a piece of the complete maximal spacelike surface with singularities associated to the Weierstrass data

$$(\mathbb{C},\ \Phi_1 = (1 - u^4)\,du,\ \Phi_2 = 2u^2\,du,\ \Phi_3 = -(1 + u^4)\,du).$$

The three 1-forms are exact, so the maximal immersion given by (8) is well-defined.

In case 2, and up to linear isometries in $\mathbb{L}^3$, we have two possibilities: either $\frac{dg}{dz} = \sqrt{g(g-r)}$ or $\frac{dg}{dz} = i\sqrt{g(g-r)}$. Suppose first that $\frac{dg}{dz} = \sqrt{g(g-r)}$. We distinguish two new cases: $r \neq 0$ and $r = 0$. Consider $r \neq 0$. Up to the change $g \to \frac{ru^2}{u^2-1}$, and from (25) and the definition of $\Psi_j$ $(j = 1, 2, 3)$, we get that $X(M)$ is a piece of the maximal spacelike surface with singularities associated to the Weierstrass data

$$\left(\bar{\mathbb{C}} - \{1, -1\},\ \Phi_1 = \frac{-2 + 4u^2 - 2u^4 + r^2u^4}{\sqrt{2}(u^2 - 1)^3}\, du,\right.$$
$$\left.\Phi_2 = -\frac{2ru^2}{(u^2 - 1)^2}\, du,\ \Phi_3 = \frac{2 - 4u^2 + 2u^4 + r^2u^4}{\sqrt{2}(u^2 - 1)^3}\, du\right).$$

If $r = 0$ then, up to the change $g \to \sqrt{2}u$ and scaling, the meromorphic data are

$$\left(\mathbb{C} - \{0\},\ \Phi_1 = \frac{1 - u^2}{u}\, du,\ \Phi_2 = 2\, du,\ \Phi_3 = -\frac{1 + u^2}{u}\, du\right).$$

In both cases, there are no real periods and the maximal immersion given in (8) is well-defined. Concerning the second possibility $\frac{dg}{dz} = i\sqrt{g(g - r)}$, a similar argument leads to the conjugate of the previous surfaces. In this case, the immersion defined in (8) has real periods and so is well-defined in a suitable covering of $\bar{\mathbb{C}} - \{1, -1\}$ (if $r \neq 0$) or $\mathbb{C} - \{0\}$ (if $r = 0$).

In case 3, using the change $g \to i\sqrt{2}u$, (25) and the definition of $\Psi_j$ $(j = 1, 2, 3)$, we obtain the following Weierstrass data:

$$(\mathbb{C},\ \Phi_1 = (1 + u^2)\, du,\ \Phi_2 = -2iu\, du,\ \Phi_3 = (u^2 - 1)\, du).$$

This rotational maximal surface in $\mathbb{L}^3$ is associated, following Remark 1, to the Enneper surface, but viewed in $\mathbb{E}^3$ with the horizontal normal vector at its end.

### *3.4. Shiffman Type Functions and Maximal Spacelike Annuli Bounded by Circles*

We conclude Section 3 by introducing the Shiffman-type functions on a maximal spacelike surface in $\mathbb{L}^3$.

Let $X\colon M \to \mathbb{L}^3$ be a maximal spacelike immersion, and let $(\eta, g)$ denote the Weierstrass representation of $X$ (see (7)). Recall that $g$ is the stereographic projection of the Gauss map of $X$.

Let $\xi$ be any conformal parameter on $M$, and let $\Delta_{\xi\bar{\xi}}$ denote the Laplacian

$$\frac{\partial}{\partial \xi}\frac{\partial}{\partial \bar{\xi}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Then, the functions $\mathcal{S}_j$ $(j = 1, 2, 3)$ given in (12), (20), and (27) satisfy the equation

$$\Delta_{\xi\bar{\xi}}\mathcal{S}_j - \frac{2}{(1 - |g|^2)^2}\left|\frac{dg}{d\xi}\right|^2 \mathcal{S}_j = 0.$$

In other words, these functions lie in the kernel of the Jacobi operator $\Delta - |\sigma|^2$ on $M$. As usual, $|\sigma|$ represents the norm of the second fundamental form of $X$ in $\mathbb{L}^3$, and $\Delta$ is the Laplacian associated to the induced metric on $M$.

Furthermore, we have proved in Theorems 1, 2, and 3 that a nonruled maximal spacelike surface in $\mathbb{L}^3$ is foliated in parallel planes by pieces of circles, hyperbolas, or parabolas if and only if the function $\mathcal{S}_1$, $\mathcal{S}_2$, or $\mathcal{S}_3$ (respectively) vanishes on the surface. It is natural to call these three functions as the Lorentzian Shiffman functions. See [20].

On the other hand, it will be interesting to note that the function $\mathcal{S}_1$ has a good behavior around singularities of *cone* type. Let us explain the details, starting with the following definition.

Let $X\colon M \to \mathbb{L}^3$ be a maximal spacelike immersion with singularities, and label $(\eta, g)$ as its Weierstrass data, following (7). Suppose that the set of singularities $\{|g| = 1\}$ in $M$ contains a connected component $c$ that is homeomorphic to $\mathbb{S}^1$. In particular, the stereographic projection $g$ of the Gauss map of $X$ has no branch points on $c$. Assume also that there is an antiholomorphic involution $T\colon M \to M$ satisfying

(i) $T^*(\Phi_j) = -\overline{\Phi_j},\ j = 1, 2, 3$;
(ii) $T$ fixes pointwise $c$.

It is clear from (8) that $X(c)$ is a point, $P_c$, in $\mathbb{L}^3$, and that $X(M)$ is invariant under the reflection about this point.

DEFINITION 3. We call $P_c$ a *cone point* of $X(M)$. We also say that $X(M)$ has a *singularity of cone type* at $P_c$.

It is not hard to prove the following lemma.

LEMMA 1. *Let $P_c$ be a cone point in $X(M)$ associated to the curve of singularities $c$ in $M$. Then, the Shiffman function $\mathcal{S}_1$ extends in a differentiable way to $c$.*

*Proof*. Let $z$ be a conformal parameter around an arbitrary point in $c$ such that $dz = \Phi_3$. Without loss of generality, we can suppose that $z \circ T = -\bar{z}$, and so locally $c$ becomes the curve $\mathrm{Re}(z) = 0$. Moreover, equation $g \circ T = 1/\bar{g}$ gives that $g = e^{f(z)}$, where $f(-\bar{z}) = -\overline{f(z)}$.

From (12), the function $\mathcal{S}_1$ extends to $c$ if and only if

$$G(z) \stackrel{\text{def}}{=} \mathrm{Im}\left(\left(\frac{d(\log(g))}{dz}\right)^2 \frac{1}{|g|^2 - 1}\right)$$

extends to $c$. To see this, take $z_0 \in i\mathbb{R}$ and observe that

$$f(z) = \sum_{m=0}^{\infty} a_m(z_0)(z - z_0)^m,$$

where $\mathrm{Re}(a_{2j}(z_0)) = 0$, and $\mathrm{Im}(a_{2j+1}(z_0)) = 0$ for $j \in \mathbb{Z}$. Since the meromorphic function $g$ has no ramification points on $c$, it follows that $a_1(z_0) \neq 0$ for all $z_0 \in c$.

On the other hand,

$$|g(z)| - 1 = e^{\mathrm{Re}(f(z))} - 1 = \mathrm{Re}(f(z))\tilde{H}_1(\mathrm{Re}(f(z))),$$

where $\tilde{H}_1(w) = (e^w - 1)/w$ for $w \in \mathbb{C}$. In particular, $\tilde{H}_1(0) = 1$. Taking into account the preceding Taylor series expansion of $f$, we deduce that $\mathrm{Re}(f(z)) = \mathrm{Re}(z)G_1(z)$, where $G_1$ is a suitable differentiable function around $c$. Moreover, using that $a_1(z_0) \neq 0$ for all $z_0 \in c$, we infer that $\left|\left(G_1\big|_c\right)\right| \geq \varepsilon > 0$. Thus,

$$|g(z)| - 1 = \mathrm{Re}(z)H_1(z),$$

where $\left|\left(H_1\big|_c\right)\right| \geq \varepsilon' > 0$. Furthermore, a similar argument gives that

$$\mathrm{Im}\left(\left(\frac{d(\log(g))}{dz}\right)^2\right) = \mathrm{Re}(z)H_2(z),$$

where $H_2$ is differentiable around $c$. We conclude that, around $c$,

$$G(z) = \frac{H_2(z)}{H_1(z)(1+|g|(z))}$$

is a differentiable function. □

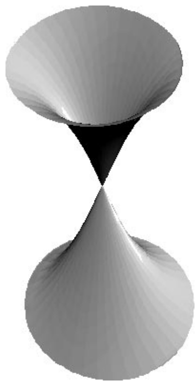

**Figure 6** A maximal spacelike annulus bounded by two circles in parallel spacelike planes and containing a cone point.

Looking at the expression of Jacobi operator, it is straightforward to check that maximal spacelike surfaces are stable in a strong sense; that is, the first eigenvalue of this operator on any compact domain is positive. As a consequence of this fact and Lemma 1, we can prove the following version of Shiffman's theorem for maximal spacelike annuli with singularities of cone type in $\mathbb{L}^3$.

Let $S$ denote a slab determined by two spacelike planes $\Pi_1$, $\Pi_2$ in $\mathbb{L}^3$.

COROLLARY 1. *Let $A$ be a compact maximal spacelike annulus in $\mathbb{L}^3$ whose set of singularities consists of a finite (possibly empty) set of cone points. Suppose that $A$ is bounded by a circle or a cone point in $\Pi_1$ and by a circle or a cone point in $\Pi_2$. Then the intersection of $A$ by a plane contained in $S$ is either a circle or a cone point. Therefore, $A$ is a piece of either a Lorentzian catenoid or a surface in the family $\mathcal{R}$.*

*Proof*. Up to a linear isometry in $\mathbb{L}^3$, we can suppose that the Euclidean normal vector of $\Pi_1$ and $\Pi_2$ is $(0, 0, 1)$. We write $X : M \to \mathbb{L}^3$ as the maximal immersion such that $A = X(M)$, and we label $c_1$ and $c_2$ as the two boundary closed curves in $\partial M$.

First, note that the case $X(M)$ bounded by two cone points $P_{c_1}$ and $P_{c_2}$ is impossible. Otherwise, by successive reflections about cone points, we obtain a complete maximal spacelike annulus $\tilde{X} : \tilde{M} \to \mathbb{L}^3$, with infinitely many cone singularities,

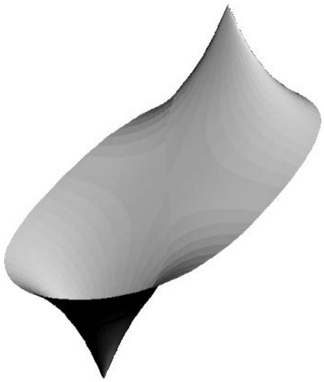

**Figure 7** A translational invariant fundamental piece of a Riemann-type example in $\mathcal{R}$. It is a spacelike annulus bounded by two cone points and with one end asymptotic to a plane. This annulus is a graph over any spacelike foliation plane.

such that $\tilde{X}(\tilde{M})$ is invariant under a translation. The quotient of $\tilde{M}$ under the holomorphic transformation induced by this translation gives a torus $T$, and the Weierstrass data $(\Phi_1, \Phi_2, \Phi_3)$ of $\tilde{X}$ can be induced on this torus. Furthermore, $\Phi_j$ is holomorphic, and so $\Phi_j = \lambda_j \tau_0$ for $j = 1, 2, 3$, where $\lambda_j \in \mathbb{C}$ and $\tau_0$ is a nonzero holomorphic 1-form on $T$. Since $\Phi_1^2 + \Phi_2^2 - \Phi_3^2 = 0$ and the associated maximal immersion is singly periodic, it is not hard to see that $\lambda_j = r_j \lambda$, where $r_j \in \mathbb{R}$, $\lambda \in \mathbb{C}$, and $r_1^2 + r_2^2 - r_3^2 = 0$. In particular, $\tilde{X}(\tilde{M})$ lies in a lightlike straight line in $\mathbb{L}^3$, which is absurd.

Hence, we can suppose that at least one of the boundary curves $c_1, c_2$ is mapped under $X$ onto a circle. After a reflection about a boundary cone point (if it exists), we can suppose that, in fact, both curves $c_1$ and $c_2$ determine circles and so the set of singularities lies in the interior of $M$.

Since $A$ is an annulus with boundary lying in horizontal planes, basic Morse theory or complex analysis implies that the third coordinate function of the immersion has no critical points. Thus, there are no points in $M$ with vertical normal vector and so the stereographic projection of the Gauss map $g$ (see (7)) has neither zeroes nor poles. Looking at (12) and taking into account Lemma 1, we infer that $\mathcal{S}_1$ is well-defined on $M$. As mentioned previously, $\mathcal{S}_1$ lies in the kernel of the Jacobi operator $\Delta - |\sigma|^2$ on $M$. Furthermore, since $X(c_1)$ and $X(c_2)$ are circles in the planes $x_3 = C$ with $C \in \mathbb{R}$, we deduce that $\mathcal{S}_1$ vanishes on $\partial M$.

On the other hand, if we denote by $ds^2$ the induced metric on $M$, then the metric $ds_0^2 = \frac{1}{(|g|-1)^2}\, ds^2$ has no singular points; that is, it is a Riemannian metric on $M$ that is conformal to $ds^2$. Labeling $\Delta_0$ as the Laplacian associated to $ds_0^2$ yields that $\Delta = \frac{1}{(|g|-1)^2}\Delta_0$, where $\Delta$ is the Laplacian associated to $ds^2$. Furthermore, it is clear that

$$\Delta_0 \mathcal{S}_1 - q\mathcal{S}_1 = 0,$$

where $q = |\sigma|^2(|g|-1)^2 > 0$. Since the first eigenvalue of the operator $\Delta_0 - q$ on $M$ is positive and $\mathcal{S}_1$ lies in its kernel, we have that $\mathcal{S}_1$ vanishes at any point of $M$—

that is, the surface is foliated by circles or cone points in horizontal planes. These kinds of surfaces were classified in Theorem 1. Finally, Remark 3 implies that $A$ is a piece of either a catenoid or a surface lying in the family $\mathcal{R}$. See Figure 7. $\square$

REMARK 6. Recall that the plane is the only spacelike maximal graph in $\mathbb{L}^3$ (in fact, the only complete spacelike maximal surface in $\mathbb{L}^3$). However, any surface in $\mathcal{R}$ has a translational fundamental piece consisting of a spacelike annulus with two boundary cone points and an interior end that is asymptotic to a plane. It can be proved that this piece is in fact a graph on the plane $x_3 = 0$, as occurs with a half-catenoid.

Therefore, the space of spacelike graphs in $\mathbb{L}^3$ with a finite number of cone points is nontrivial, and its study could be an interesting problem.

## 4. Maximal Surfaces Foliated by Pieces of Circles

Consider a spacelike maximal surface foliated by pieces of circles. In this section we show that the planes of the foliation are actually parallel (see Theorem 4). To do this, we use a similar technique to that developed for the Euclidean case in [15, pp. 85–86]. The main dificulty in Lorentz–Minkowski space lies in the causal character of the foliation planes: spacelike, timelike, and lightlike. As a matter of fact, each case needs a different discussion.

Let $M$ be an oriented maximal spacelike surface in $\mathbb{L}^3$ and consider $X = X(u, v)$ a local system of coordinates in $M$. We write $X_u = \frac{\partial X}{\partial u}$ and $X_v = \frac{\partial X}{\partial v}$. As the mean curvature of $X$ vanishes, we have

$$E[X_u, X_v, X_{vv}] - 2F[X_u, X_v, X_{uv}] + G[X_u, X_v, X_{uu}] = 0, \tag{31}$$

where

$$E = \langle X_u, X_u\rangle, \quad F = \langle X_u, X_v\rangle, \quad G = \langle X_v, X_v\rangle$$

are the coefficients of the first fundamental form with respect to $\{X_u, X_v\}$ and where $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ denotes the determinant of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ (see e.g. [17] or [21, Chap. 7]). On the other hand, since $X$ is spacelike,

$$W^2 = EG - F^2 = \langle X_u, X_u\rangle\langle X_v, X_v\rangle - \langle X_u, X_v\rangle^2 > 0. \tag{32}$$

Let us assume that $M$ is foliated by pieces of circles in $\mathbb{L}^3$; in other words, assume that $M$ is generated by a 1-parameter family of pieces of circles, each of which is contained in a plane of $\mathbb{L}^3$. Denote by $u$ the parameter of this family. We distinguish the following three cases.

*Case 1: the planes are spacelike.* Choose an orthogonal basis $\{\mathbf{e}_1(u), \mathbf{e}_2(u)\}$ in each $u$-plane. Then the surface can be parameterized by

$$X(u, v) = \mathbf{c} + r(\cos v\mathbf{e}_1 + \sin v\mathbf{e}_2), \quad u \in I,\ v \in J \tag{33}$$

(see (1)), where $I$ and $J$ are real intervals, $\mathbf{c} = \mathbf{c}(u)$ belongs to the $u$-plane, and $r = r(u) > 0$ is a smooth function.

*Case 2: the planes are timelike.* Let $\{\mathbf{e}_1(u), \mathbf{e}_2(u)\}$ be an orthogonal basis in each $u$-plane, with $-\langle\mathbf{e}_1, \mathbf{e}_1\rangle = \langle\mathbf{e}_2, \mathbf{e}_2\rangle = 1$. Following (2), we have

$$X(u,v) = \mathbf{c} + r(\cosh v\mathbf{e}_1 + \sinh v\mathbf{e}_2), \quad u \in I,\ v \in J, \tag{34}$$

where $\mathbf{c}$, $I$, $J$, and $r$ are as given in Case 1.

*Case 3: the planes are lightlike.* Following (3),

$$X(u,v) = \mathbf{c} + v\mathbf{e}_1 + rv^2\mathbf{e}_2, \quad u \in I,\ v \in J, \tag{35}$$

where $I$, $J$, $\mathbf{c}$, and $r \neq 0$ are as before and where, for each $u$, $\mathbf{e}_1(u)$ and $\mathbf{e}_2(u)$ are vectors in the $u$-plane such that $\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = 1$ and $\langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 0$.

In Cases 1 and 2, let $\mathbf{N}(u)$ denote the unit orthogonal vector to the $u$-plane in $\mathbb{R}^3$ (the lightlike Case 3 merits a different treatment). Notice that $\mathbf{N}(u)$ is not a lightlike vector and that $\mathbf{N}(u)$ does not belong to the $u$-plane.

Let us explain the global strategy. Reasoning by contradiction, suppose that the planes containing the pieces of circles are not parallel. This means that $\mathbf{N}'(u) \neq 0$ for $u$ in some real interval, and hence the curve $C$ having $\mathbf{N}(u)$ as unit tangent field is not a straight line. We will construct a moving frame adapted to the foliation, one that actually comes from the Frenet frame of the curve $C$; we will express (33) and (34) in terms of this frame. Later, we shall compute (31). We will obtain either a real trigonometric polynomial or just a polynomial in one variable that vanishes in some interval of $\mathbb{R}$. The fact that the coefficients of this polynomial vanish will give the contradiction.

Theorem 4. *Let $M$ be a maximal spacelike surface in the Lorentz–Minkowski space $\mathbb{L}^3$ foliated by pieces of circles. Then the planes containing these pieces of circles must be parallel.*

*Proof.* As just mentioned, the proof is by contradiction. We say that a set $I \subset \mathbb{R}$ is spacelike, timelike, or lightlike if and only if, for any $u \in I$, the associated plane in the foliation is (resp.) spacelike, timelike, or lightlike. In the next three subsections we shall prove that the planes in the foliation are parallel on spacelike, timelike, and lightlike intervals. Since the set of points of $\mathbb{R}$ whose corresponding plane in the foliation is spacelike (resp., timelike) is open, it is not hard to see that the union of the spacelike, timelike, and lightlike open intervals is an open and dense subset of $\mathbb{R}$. Because the map that takes every $u$ on its corresponding plane in the foliation is continuous, we deduce that the foliation must be by circles in parallel planes.

Hence, we can split the proof into three parts.

*1: The surface is foliated by pieces of circles in spacelike planes*

In this case, $\mathbf{N}(u)$ is a timelike vector: $\langle \mathbf{N}(u), \mathbf{N}(u) \rangle = -1$. Consider $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the Frenet frame of the integral curve $C$ ($\mathbf{N} = \mathbf{t}$). Write the velocity vector $\mathbf{c}'$ as

$$\mathbf{c}' = \alpha\mathbf{t} + \beta\mathbf{n} + \gamma\mathbf{b}, \tag{36}$$

where $\alpha, \beta, \gamma$ are smooth functions of $u$. Up to a change of coordinates in (33) given by a translation on $v$, and taking into account that $\mathbf{n}$ and $\mathbf{b}$ are spacelike, we can put

$$X(u, v) = \mathbf{c} + r \cos v\mathbf{n} + r \sin v\mathbf{b}.$$

Up to signs, let $\kappa$ and $\sigma$ be respectively the curvature and the torsion of the curve $C$. Notice that $\kappa \neq 0$ because $\mathbf{N}' \neq 0$, where the prime symbol denotes the derivative with respect to $u$. Moreover, $\mathbf{t}'$ is a spacelike vector. Frenet equations for $C$ are:

$$\mathbf{t}' = \kappa\mathbf{n};$$

$$\mathbf{n}' = \kappa\mathbf{t} + \sigma\mathbf{b};$$

$$\mathbf{b}' = -\sigma\mathbf{n}.$$

From (31), a straightforward computation yields

$$0 = a \cos 3v + b \sin 3v + c \cos 2v + d \sin 2v + e \cos v + f \sin v + g, \tag{37}$$

where

$$a = -\tfrac{1}{2}r^3\kappa(r^2\kappa^2 - \beta^2 + \gamma^2),$$

$$b = r^3\kappa\beta\gamma,$$

$$c = \tfrac{1}{2}r^3(-5r\kappa^2\alpha + 6r'\kappa\beta + r\kappa'\beta - r\kappa\beta').$$

From $a = b = 0$, we deduce that $\gamma = 0$ and $\beta = \pm r\kappa$. Moreover, $c = 0$ implies that $\alpha = \pm r'$. Therefore, it is not hard to see that $W = 0$, a contradiction. Thus, $\kappa = 0$ and $C$ is a straight line.

*2: The surface is foliated by pieces of circles in timelike planes*

In this situation, $\mathbf{N}$ is a unit spacelike vector: $\langle \mathbf{N}(u), \mathbf{N}(u)\rangle = 1$. Let $\mathbf{t} = \mathbf{N}$ the unit tangent vector of $C$. Since we are assuming the planes are not parallel, $\mathbf{N}' = \mathbf{t}' \neq 0$. Moreover, it is clear that $\langle \mathbf{t}', \mathbf{t}\rangle = 0$. We distinguish three possibilities as follows.

*First Case:* $\langle \mathbf{t}', \mathbf{t}'\rangle > 0$. Let $\mathbf{n}$ be the unit spacelike vector field along $C$ such that $\mathbf{t}' = \kappa\mathbf{n}$ for some smooth function $\kappa \neq 0$. Take $\mathbf{b} = \mathbf{t} \wedge \mathbf{n}$, where $\wedge$ stands for the cross product in $\mathbb{L}^3$. Notice that $\langle \mathbf{b}, \mathbf{b}\rangle = -1$. Up to a change of coordinates by translations on $v$, we can write (34) as

$$X(u, v) = \mathbf{c} + r \sinh v\mathbf{n} + r \cosh v\mathbf{b}.$$

Here, Frenet equations are

$$\mathbf{t}' = \kappa\mathbf{n},$$

$$\mathbf{n}' = -\kappa\mathbf{t} + \sigma\mathbf{b},$$

$$\mathbf{b}' = \sigma\mathbf{n}.$$

The formula (31) can now be written as

$$0 = a \cosh 3v + b \sinh 3v + c \cosh 2v + d \sinh 2v + e \cosh v + f \sinh v + g,$$

where, with the same notation of (36),

$$
\begin{aligned}
a &= -r^3\kappa\beta\gamma,\\
b &= \tfrac{1}{2}r^3\kappa(-r^2\kappa^2+\beta^2+\gamma^2),\\
c &= \tfrac{1}{2}r^3(5r\kappa^2\alpha-6r'\kappa\beta-r\kappa'\beta+r\kappa\beta'),\\
d &= \tfrac{1}{2}r^3(r\kappa'\gamma-r\kappa\gamma'-6r'\kappa\gamma).
\end{aligned}
$$

From $a=0$, we have that $\beta\gamma=0$. We then reason as follows.

(i) Suppose $\beta=0$. Then $b=0$ implies that $\gamma^2=r^2\kappa^2$, and from $c=0$ we get that $\alpha=0$. Finally, $d=0$ yields $r'=0$. Then $W^2=-r^4\kappa^2<0$, which is a contradiction.

(ii) Suppose $\gamma=0$. Then $b=0$ gives $\beta=\pm r\kappa$, and $c=0$ implies that $\alpha=\pm r'$. We deduce that $W=0$, a contradiction.

*Second Case:* $\langle \mathbf{t}', \mathbf{t}'\rangle<0$. Let $\mathbf{n}$ be the unit timelike vector field along $C$ such that $\mathbf{t}'=\kappa\mathbf{n}$. The parameterization of $X$ is given by

$$X(u,v)=\mathbf{c}+r\cosh v\mathbf{n}+r\sinh v\mathbf{b},$$

where $\mathbf{b}=\mathbf{t}\wedge\mathbf{n}$. The corresponding Frenet equations are

$$
\begin{aligned}
\mathbf{t}' &= \kappa\mathbf{n},\\
\mathbf{n}' &= \kappa\mathbf{t}+\sigma\mathbf{b},\\
\mathbf{b}' &= \sigma\mathbf{n}.
\end{aligned}
$$

Hence, we can write (31) as

$$0=a\cosh 3v+b\sinh 3v+c\cosh 2v+d\sinh 2v+e\cosh v+f\sinh v+g,$$

where

$$
\begin{aligned}
a &= -\tfrac{1}{2}r^3\kappa(r^2\kappa^2-\beta^2-\gamma^2),\\
b &= -r^3\kappa\beta\gamma,\\
c &= \tfrac{1}{2}r^3(-5r\kappa^2\alpha+6r'\kappa\beta+r\kappa'\beta-r\kappa\beta'),\\
d &= \tfrac{1}{2}r^3(-r\kappa'\gamma+r\kappa\gamma'-6r'\kappa\gamma).
\end{aligned}
$$

From $b=0$, we deduce $\beta\gamma=0$.

(i) Suppose $\beta=0$. From $a=0$, it follows that $\gamma^2=r^2\kappa^2$, and using that $c=0$ we obtain $\alpha=0$. Hence, $d=0$ implies $r'=0$. Therefore, the coefficient $e$ can now be computed easily as $-3r^5\kappa^3=0$, a contradiction.

(ii) Therefore, $\gamma=0$. Taking into account that $a=0$, we have $\beta=\pm r\kappa$. Moreover, $c=0$ gives $\alpha=\pm r'$. With these data, one can check that $W=0$, which is absurd.

*Third Case:* $\langle \mathbf{t}', \mathbf{t}'\rangle=0$. Since we are assuming that the foliation planes are not parallel, it follows that $\mathbf{t}'\neq 0$. Hence $\mathbf{t}'$ is a lightlike vector lying in the $u$-plane. Let $\mathbf{n}=\mathbf{t}'$. For each $u$, let $\mathbf{b}$ be the unique vector orthogonal to $\mathbf{t}$ such that

$$\langle \mathbf{b}, \mathbf{b} \rangle = 0 \quad \text{and} \quad \langle \mathbf{b}, \mathbf{n} \rangle = 1.$$

In fact, from (34) we have

$$\mathbf{n} = \lambda(u)(-\mathbf{e}_1 + \mathbf{e}_2), \qquad \mathbf{b} = \frac{1}{\lambda(u)}(\mathbf{e}_1 + \mathbf{e}_2),$$

where $\lambda$ is a differentiable function of $u$. We may choose as a new parameter $-e^v/2\lambda(u)$ instead of $v$; then, still denoting the new parameter by $v$, (34) becomes

$$X(u, v) = \mathbf{c} + rv\mathbf{n} - \frac{r}{2v}\mathbf{b},$$

with $r, v \neq 0$. The Frenet equations are now

$$\mathbf{t}' = \mathbf{n},$$

$$\mathbf{n}' = \sigma \mathbf{n},$$

$$\mathbf{b}' = -\mathbf{t} - \sigma \mathbf{b}.$$

Equation (31) gives

$$0 = a\frac{1}{v^6} + b\frac{1}{v^5} + c\frac{1}{v^4} + d\frac{1}{v^3} + e\frac{1}{v^2},$$

where

$$\begin{aligned}
a &= -\frac{r^5}{4} + \frac{r^3\beta^2}{4}, \\
b &= \frac{r^3}{4}(-5r\alpha + 6r'\beta - r\beta'), \\
c &= -2r^3\alpha^2 - \frac{r^3\alpha\beta'}{2} - \frac{r^3\sigma\alpha\beta}{2} + \frac{r^3\alpha'\beta}{2} + \frac{3r^3r'^2}{2} \\
&\quad + \frac{r^4r'\sigma}{2} - \frac{r^4r''}{2} - \frac{3r^3\beta\gamma}{2} + r^2r'\alpha\beta.
\end{aligned}$$

From $a = 0$ we have $\beta = \pm r$. Moreover, $b = 0$ implies that $\alpha = \pm r'$, and from $c = 0$ we infer that $\gamma = 0$. Hence, it is not hard to check that $W = 0$, which is a contradiction.

*3: The surface is foliated by pieces of circles in lightlike planes*

With a change of notation, (35) becomes

$$X(u, v) = \mathbf{c} + v\mathbf{n} + rv^2\mathbf{t},$$

where $\mathbf{c}, \mathbf{t}, \mathbf{n}$ lie in the $u$-plane, $r > 0$, $v \in I$, $\langle \mathbf{t}, \mathbf{t} \rangle = \langle \mathbf{n}, \mathbf{t} \rangle = 0$, and $\langle \mathbf{n}, \mathbf{n} \rangle = 1$. For each $u$, let $\mathbf{b}(u)$ be the unique lightlike vector orthogonal to $\mathbf{n}$ such that $\langle \mathbf{t}, \mathbf{b} \rangle = 1$ and $[\mathbf{t}, \mathbf{n}, \mathbf{b}] = 1$. Note that, since a lightlike plane is the orthogonal plane to any lightlike vector that belongs to it, the planes are parallel if and only if the field $\mathbf{t}$ is constant (i.e., doesn't depend on the parameter $u$). We may assume that $\mathbf{t}' = \kappa\mathbf{n}$. Indeed, it is easily seen that one can always determine a function $\mu$ of $u$ such that the field $\tilde{\mathbf{t}} = \mu\mathbf{t}$ satisfies the preceding requirement (accordingly, one must change $r$ into $\tilde{r} = r/\mu)$. Since we are assuming that the planes are non-parallel, we have $\kappa \neq 0$. The corresponding Frenet equations are

$$\mathbf{t}' = \kappa\mathbf{n},$$
$$\mathbf{n}' = \sigma\mathbf{t} - \kappa\mathbf{b},$$
$$\mathbf{b}' = -\sigma\mathbf{n}.$$

The vector field $\mathbf{c}'$ is given in (36). The hypothesis $H = 0$ means here that

$$0 = av^4 + bv^3 + cv^2 + dv + e,$$

where

$$a = 7r\kappa^2(2r^2\gamma - r'),$$

$$b = 16r^2\kappa^2\beta + 4rr'\kappa\gamma - 2r^2\kappa\gamma' - r'\kappa' + 2r^2\kappa'\gamma - 8r\kappa^2\sigma + r''\kappa - 16r^3\kappa\gamma^2,$$

$$c = -2r\kappa\beta' - 9r\kappa^2\alpha + 2r\kappa'\beta - 20r^2\kappa\beta\gamma + r'\gamma' - 3r'\kappa\beta + 4rr'\gamma^2 - r''\gamma \\ - \kappa'\sigma + 11r\kappa\sigma\gamma + \kappa\sigma'.$$

From $a = 0$, it follows that $r' = 2r^2\gamma$. The equation $b = 0$ gives $\sigma = 2r\beta$. A computation of $c$ with these data gives $\alpha = 0$. Therefore, it is easy to prove that $W = 0$, which is absurd. □

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