Hokkaido Mathematical Journal Vol. 35 (2006) p. 13–40

Maximal surfaces with singularities in Minkowski space

Masaaki Umehara and Kotaro Yamada

(Received December 8, 2003; Revised April 6, 2005)

Abstract. We shall investigate maximal surfaces in Minkowski 3-space with singularities. Although the plane is the only complete maximal surface without singular points, there are many other complete maximal surfaces with singularities and we show that they satisfy an Osserman-type inequality.

Key words: maximal surface, front, cuspidal edge, swallowtail, singularity.

Introduction

It is well-known that the only complete maximal (mean curvature zero) space-like surface in the Minkowski 3-space L^3 is the plane, and it is also well-known that any maximal surface can be locally lifted to a null holomorphic immersion into $\boldsymbol{C}^3$ (see, for example, [K1] or [McN]). However, the projection of a null holomorphic immersion to L^3 might not be regular. We shall call such surfaces *maxfaces*, and show that this class of generalized surfaces is a rich object to investigate global geometry.

This is somewhat parallel to the case of flat surfaces in hyperbolic 3-space, in which the only complete non-singular examples are the horosphere or hyperbolic cylinders. But if one considers flat (wave) fronts (namely, projections of Legendrian immersions), there are many complete examples and interesting global properties. See [KoUY3] and [KRSUY] for details.

It should be remarked that Osamu Kobayashi ([K1, K2]) gave a Weierstrass-type representation formula for maximal surfaces and investigated such surfaces with conelike singularities. Using the holomorphic representation, Estudillo and Romero [ER] defined a class of maximal surfaces with singularities in more general type, and investigated criteria for such surfaces to be a plane. Recently, Imaizumi [I2] studied the asymptotic behavior of maxfaces, and Imaizumi-Kato [IK] gave a classification of maxfaces of genus zero with at most three ends. On the other hand,

---

2000 Mathematics Subject Classification : Primary 53A10, Secondary 53A40, 32S05.

Lopez-Lopez-Souam [LLS] classified maximal surfaces that are foliated by circles, which includes a Lorentzian correspondence of Riemann's minimal surface. Fernàndez-López-Souam [FLS] investigated the moduli space of maximal graphs over the space-like plane with a finite number of conelike singularities.

The Lorentzian Gauss map g of nonsingular maximal surface is a map into upper or lower connected component of the two-sheet hyperboloid in L^3. By the stereographic projection from $(1,0,0)$ of the hyperboloid to the plane, the Lorentzian Gauss map g can be expressed as a meromorphic function into $\boldsymbol{C}\cup\{\infty\}\setminus\{\zeta\in\boldsymbol{C}\,;\,|\zeta|=1\}$. The singular set of a maxface corresponds to the set $\{|g|=1\}$, and g can be extended meromorphically on the singular set. We shall prove in Section 4 that a complete maxface $f\colon M^2\to L^3$ satisfies the following Osserman-type inequality (The definition of completeness is given in Section 4.)

$$2\deg g \geq -\chi(M^2) + (\text{number of ends}),$$

and equality holds if and only if all ends are properly embedded, where

$$g\colon\ M^2 \longrightarrow S^2 = \boldsymbol{C}\cup\{\infty\}$$

is the Lorentzian Gauss map and $\deg g$ is its degree as a map to S^2.

We also give examples for which equality is attained (Section 4). Moreover, applying the results on singularities of wave fronts in [KRSUY], we can investigate singularities of maxfaces and give a criterion for a given singular point to be locally diffeomorphic to cuspidal edges or swallowtails in terms of the Weierstrass data (Section 3).

It should be remarked that Kim and Yang [KY] very recently constructed complete a genus one maxface of two catenoidal ends. Recently Ishikawa and Machida [IM] showed that generic singular points of surfaces of constant Gaussian curvature in the Euclidean 3-space and generic singular points of improper affine spheres in the 3-dimensional affine space are both cuspidal edges and swallowtails. On the other hand, Fujimori [Fu] and Lee and Yang [LY] investigate space-like surfaces with singularities of mean curvature one in the de Sitter space. (See also a forthcoming paper [FRUYY].) In contrast to space-like maximal surfaces, time-like minimal surfaces are related to Lorentz surfaces and a partial differential equation of hyperbolic type, see Inoguchi-Toda [IT].

The authors thank Wayne Rossman, Shoichi Fujimori and the referee

for valuable comments.

1. Preliminaries

The Minkowski 3-space L^3 is the 3-dimensional affine space $\boldsymbol{R}^3$ with the inner product

$$\langle\ ,\ \rangle := -(dx^0)^2+(dx^1)^2+(dx^2)^2, \tag{1.1}$$

where (x^0,x^1,x^2) is the canonical coordinate system of $\boldsymbol{R}^3$. An immersion $f\colon M^2 \to L^3$ of a 2-manifold M^2 into L^3 is called *space-like* if the induced metric

$$ds^2 := f^*\langle\ ,\ \rangle = \langle df, df\rangle$$

is positive definite on M^2. Throughout this paper, we assume that M^2 is orientable. (If M^2 is non-orientable, we consider the double cover.) Then without loss of generality, we can regard M^2 as a Riemann surface and f as a conformal immersion.

The (Lorentzian) unit normal vector ν of a space-like immersion $f\colon M^2 \to L^3$ is perpendicular to the tangent plane, and $\langle \nu,\nu\rangle = -1$ holds. Moreover, it can be regarded as a map

$$\nu\colon M^2 \longrightarrow H^2_\pm = H^2_+ \cup H^2_-, \tag{1.2}$$

where

$$H^2_+ := \bigl\{\nu = (\nu^0,\nu^1,\nu^2) \in L^3 \mid \langle \nu,\nu\rangle = -1,\ \nu^0 > 0\bigr\},$$
$$H^2_- := \bigl\{\nu = (\nu^0,\nu^1,\nu^2) \in L^3 \mid \langle \nu,\nu\rangle = -1,\ \nu^0 < 0\bigr\}.$$

The map $\nu\colon M^2 \to H^2_\pm$ is called the *Gauss map* of f. A space-like immersion $f\colon M^2 \to L^3$ is called *maximal* if and only if the mean curvature function vanishes identically. The composition of the Gauss map to the stereographic projection $\pi\colon H^2_\pm \to \boldsymbol{C}\cup\{\infty\}$ from the north pole $(1,0,0)$ is expressed by

$$g := \pi\circ\nu = -\frac{\partial f^0}{\partial f^1 - \sqrt{-1}\,\partial f^2} \tag{1.3}$$

which is a meromorphic function when $f=(f^0,f^1,f^2)$ is maximal. We also call g the *Gauss map* of f. Since ν is valued on the set $H^2_\pm$, $|g| \neq 1$ holds on M^2. The original Gauss map ν of the maximal surface as in (1.2) is rewritten by

$$\nu = \frac{1}{1-|g|^2}\bigl(-(1+|g|^2), 2\operatorname{Re} g, 2\operatorname{Im} g\bigr). \tag{1.4}$$

A holomorphic map

$$F = (F^0, F^1, F^2)\colon M^2 \longrightarrow \boldsymbol{C}^3$$

of a Riemann surface M^2 to $\boldsymbol{C}^3$ is called a *Lorentzian null map* if

$$\langle dF, dF\rangle = -(dF^0)^2 + (dF^1)^2 + (dF^2)^2 = 0$$

holds on M^2, where we denote by $\langle\ ,\ \rangle$ the complexification of the Lorentzian metric (1.1). Let $g\colon M^2 \to \boldsymbol{C} \cup \{\infty\}$ be a Gauss map of the conformal spacelike maximal immersion f, then the holomorphic map

$$F := \frac{1}{2}\int_{z_0}^{z} \bigl(-2g, (1+g^2), \sqrt{-1}(1-g^2)\bigr)\omega$$

is a Lorentzian null map defined on the universal cover $\widetilde{M}^2$ of M^2, and $f = F + \overline{F}$ holds, where ω is a holomorphic 1-form on M^2 given by

$$\omega := \partial f^1 - \sqrt{-1}\,\partial f^2.$$

Moreover

$$ds^2 = (1-|g|^2)^2\,\omega\bar{\omega} \tag{1.5}$$

holds (see [K1]). Let ds^2_{Hyp} be the hyperbolic metric on $\boldsymbol{C} \cup \{\infty\} \setminus \{\zeta \in \boldsymbol{C}\,;\, |\zeta| = 1\}$:

$$ds^2_{\mathrm{Hyp}} = \frac{4\,d\zeta\,d\bar{\zeta}}{(1-|\zeta|^2)^2}.$$

Then we have

Lemma 1.1 *The pull-back of the metric ds^2_{Hyp} by the Gauss map g satisfies*

$$K_{ds^2}\,ds^2 = g^* ds^2_{\mathrm{Hyp}} = \frac{4\,dg\,d\bar{g}}{(1-|g|^2)^2}, \tag{1.6}$$

where K_{ds^2} is the Gaussian curvature of ds^2. In particular, the Gaussian curvature of maximal surface in L^3 is non-negative.

Remark 1.2 *The only complete maximal space-like immersion is a plane.* This classical fact is easily proved as follows: Without loss of generality, we may assume M^2 is connected and simply-connected. Moreover, we may assume the Gauss map is valued in H^2_-, that is $|g| < 1$. Suppose M^2 is biholomorphic to the unit disk D^2. Since $(1-|g|^2)^2\omega\bar{\omega} < \omega\bar{\omega}$, the metric $\omega\bar{\omega}$ is a complete flat metric on D^2, which is impossible. So M^2 is conformally equivalent to $\boldsymbol{C}$, then g is constant. This implies that the image of f is a plane.

2. Maxfaces

Definition 2.1 A smooth map $f\colon M^2 \to L^3$ of an oriented 2-manifold M^2 into L^3 is called a *maximal map* if there exists an open dense subset $W \subset M^2$ such that $f|_W$ is a maximal immersion. A point p where ds^2 degenerates is called a *singular point* of f.

Definition 2.2 Let $f\colon M^2 \to L^3$ be a maximal map which gives a maximal immersion on $W \subset M^2$, and $p \in M^2 \setminus W$ a singular point. Then p is called an *admissible singular point* if

(1) On a neighborhood U of p, there exists a C^1-differentiable function $\beta\colon U \cap W \to \boldsymbol{R}_+$ such that the Riemannian metric $\beta\, ds^2$ on $U \cap W$ extends to a C^1-differentiable Riemannian metric on U, and

(2) $df(p) \neq 0$

hold. A maximal map f is called a *maxface* if all singular points are admissible.

The condition "$df(p) \neq 0$" is equivalent to "$\operatorname{rank} df = 1$" at the singular point p.

Proposition 2.3 *Let M^2 be an oriented 2-manifold and $f\colon M^2 \to L^3$ a maxface which is a maximal immersion on an open dense subset $W \subset M^2$. Then there exists a complex structure of M^2 which satisfies the following:*

(1) *$f|_W$ is conformal with respect to the complex structure.*

(2) *There exists a holomorphic Lorentzian null immersion $F\colon \widetilde{M}^2 \to \boldsymbol{C}^3$ such that $f \circ \pi = F + \overline{F}$, where $\pi\colon \widetilde{M}^2 \to M^2$ is the universal cover of M^2.*

The holomorphic null immersion F as above is called the *holomorphic lift* of the maxface f.

Proof of Proposition 2.3. Since the induced metric $ds^2 = f^*\langle\ ,\ \rangle$ gives a Riemannian metric on W, it induces a complex structure on W. So it is sufficient to construct a complex coordinate on a neighborhood of an admissible singular point which is compatible to the complex structure on W.

Let p be an admissible singular point of f and U a sufficiently small neighborhood of p. By definition, there exists a function β on $U \cap W$ such that $\beta\, ds^2$ extends to a C^1-differentiable Riemannian metric on a neighborhood U. We assume U is simply connected. Then there exists a positively oriented orthonormal frame field $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ with respect to $\beta\, ds^2$ which is C^1-differentiable on U. Using this, we can define a C^1-differentiable almost complex structure J on U such that

$$J(\boldsymbol{e}_1) = \boldsymbol{e}_2, \qquad J(\boldsymbol{e}_2) = -\boldsymbol{e}_1. \tag{2.1}$$

Since ds^2 is conformal to $\beta\, ds^2$ on W, J is compatible to the complex structure on W induced by ds^2. There exists a C^1-differentiable decomposition

$$(T^*M^2)^{\boldsymbol{C}} = (T^*M^2)^{(1,0)} \oplus (T^*M^2)^{(0,1)}$$

with respect to J. Since f is C^∞-differentiable, df is a smooth $\boldsymbol{R}^3$-valued 1-form. So we can take the $(1,0)$-part ζ of df with respect to this decomposition. Then ζ is a C^1-differentiable $\boldsymbol{C}^3$-valued 1-form which is holomorphic on W with respect to the complex structure (2.1). In particular $d\zeta$ vanishes on W. Moreover, since W is an open dense subset, $d\zeta = 0$ holds on U.

As we assumed that U is simply connected, the Poincaré lemma implies that there exists a C^1-differentiable map $F_U \colon U \to \boldsymbol{C}^3$ such that $dF_U = \zeta$.

Since the point p is an admissible singularity, $\zeta + \bar{\zeta} = df(p) \neq 0$ on M^2. In particular $\zeta \neq 0$, and at least one component of $\zeta = dF_U$ does not vanish at p. If we write $F_U = (F^0, F^1, F^2)$, we can choose $j = 0,\ 1,\ 2$ such that $dF^j(p) \neq 0$. Using this F^j, we define a function $z = F^j \colon U \to \boldsymbol{C} = \boldsymbol{R}^2$. Then, z gives a coordinate system on M^2 on a neighborhood of p, because $dF^j(p) \neq 0$. Since $z = F^j$ is a holomorphic function $U \cap W$, it gives a complex analytic coordinate around p compatible with respect to that of $U \cap W$. (If k is another suffix such that $dF^k(p) \neq 0$, then $w = F^k$ gives also a local complex coordinate system compatible with respect to z. In fact,

$$\frac{dw}{dz} = \frac{dF^k}{dF^j} = \frac{\zeta^k}{\zeta^j}$$

is holomorphic on $U \cap W$, and satisfies the Cauchy-Riemann equation on U, since ζ^k, ζ^j are C^1-differentiable and $U \cap W$ is open dense in U.)

Since p is arbitrary fixed admissible singularity, the complex structure of W extends across each singular point p, In particular, ∂f is holomorphic whole on M^2 and there exists a holomorphic map $F\colon \widetilde{M}^2 \to \boldsymbol{C}^3$ such that $dF = \partial f$. Since $\partial(F+\overline{F}) = dF = \partial f$, $F+\overline{F}$ differs $f\circ\pi$ by a constant. So we may take F such that $F+\overline{F} = f\circ\pi$. Since $f|_W$ is a maximal immersion, F is a Lorentzian null holomorphic immersion on $\pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open dense subset, F is a Lorentzian null map on $\widetilde{M}^2$. Moreover, since $df(p) \neq 0$ at each admissible singular point, we have

$$dF(q) = \partial(f\circ\pi)(q) = \partial f(p) \neq 0 \qquad (q \in \pi^{-1}(p))$$

which implies that F is an immersion whole on $\widetilde{M}^2$. $\square$

Conversely, a Lorentzian null immersion $F\colon M^2 \to \boldsymbol{C}^3$ gives a maxface $f = F + \overline{F}$, if it defines a maximal immersion on an open dense subset. More precisely, we have:

Proposition 2.4 *Let M^2 be a Riemann surface and $F\colon M^2 \to \boldsymbol{C}^3$ a holomorphic Lorentzian null immersion. Assume*

$$-|dF^0|^2+|dF^1|^2+|dF^2|^2 \tag{2.2}$$

does not vanish identically. Then $f = F + \overline{F}$ is a maxface. The set of singularities of f is points where (2.2) *vanishes.*

Proof. If (2.2) does not vanish identically, the set

$$W = \{-|dF^0|^2+|dF^1|^2+|dF^2|^2 \neq 0\}$$

is open dense in M^2. Since F is Lorentzian null,

$$\begin{aligned}
-|dF^0|^2 + |dF^1|^2 + |dF^2|^2 &= -|(dF^0)^2| + |dF^1|^2 + |dF^2|^2 \\
&= -|(dF^1)^2 + (dF^2)^2| + |dF^1|^2 + |dF^2|^2 \\
&\geq -(|dF^1|^2 + |dF^2|^2) + |dF^1|^2 + |dF^2|^2 \\
&= 0.
\end{aligned}$$

Then it holds that

$$-|dF^0|^2+|dF^1|^2+|dF^2|^2 > 0 \qquad \text{on } W.$$

In particular, $f = F + \overline{F}$ determines a conformal maximal immersion of W

into L^3 with induced metric

$$ds^2 = 2\big(-|dF^0|^2 + |dF^1|^2 + |dF^3|^2\big).$$

On the other hand, since F is an immersion, $dF \neq 0$. Then

$$|dF^0|^2 + |dF^1|^2 + |dF^2|^2 > 0.$$

Hence if we set

$$\beta := \frac{|dF^0|^2 + |dF^1|^2 + |dF^2|^2}{-|dF^0|^2 + |dF^1|^2 + |dF^2|^2}$$

on W, β is a positive function on W such that

$$\beta\, ds^2 = 2\big(|dF^0|^2 + |dF^1|^2 + |dF^2|^2\big)$$

can be extended to a Riemannian metric on M^2. This completes the proof. $\square$

Remark 2.5 Even if F is a holomorphic Lorentzian null immersion, $f = F + \overline{F}$ might not be a maxface. In fact, for the Lorentzian null immersion

$$F = (z, z, 0) \colon \boldsymbol{C} \longrightarrow \boldsymbol{C}^3,$$

$f = F + \overline{F}$ degenerates on whole $\boldsymbol{C}$.

For maximal surfaces, an analogue of the Weierstrass representation formula is known (see [K1]). Summing up, we have:

Theorem 2.6 (Weierstrass-type representation for maxfaces) *Let M^2 be a Riemann surface and $f\colon M^2 \to L^3$ a maxface. Then there exists a meromorphic function g and a holomorphic 1-form ω on M^2 such that*

$$f = \operatorname{Re}\int_{z_0}^{z} \big(-2g, (1+g^2), \sqrt{-1}(1-g^2)\big)\omega, \tag{2.3}$$

where $z_0 \in M^2$ is a base point. Conversely, let g and ω be a meromorphic function and a holomorphic 1-form on M^2 such that

$$(1+|g|^2)^2\, \omega\bar{\omega} \tag{2.4}$$

is a Riemannian metric on M^2 and $(1-|g|^2)^2$ does not vanish identically. Suppose

$$\operatorname{Re}\oint_{\gamma} \big(-2g, (1+g^2), \sqrt{-1}(1-g^2)\big)\omega = 0 \tag{2.5}$$

for all loops γ on M^2. Then (2.3) *defines a maxface $f\colon M^2 \to L^3$. The set of singular points of f is given by $\{p \in M^2\,;\, |g(p)| = 1\}$.*

Definition 2.7 We set

$$d\sigma^2 := (1+|g|^2)^2|\omega|^2 = 2(|dF^0|^2+|dF^1|^2+|dF^2|^2), \tag{2.6}$$

and call it the *lift-metric* of the maxface f, where $F = (F^0, F^1, F^2)$ is the holomorphic lift.

The metric $(1/2)d\sigma^2$ is nothing but the pull-back of the canonical Hermitian metric on $\boldsymbol{C}^3$ by the holomorphic lift F. We call a pair (g, ω) in Theorem 2.6 the *Weierstrass data* of the maxface f. As seen in (1.4), g is the Gauss map on regular points of f. We also call $g\colon M^2 \to \boldsymbol{C} \cup \{\infty\}$ the *Gauss map* of the maxface f.

Denote by $K_{d\sigma^2}$ the Gaussian curvature of the lift-metric $d\sigma^2$. Then, by (2.6), we have

$$(-K_{d\sigma^2})\, d\sigma^2 = \frac{4dg\, d\bar{g}}{(1+|g|^2)^2}. \tag{2.7}$$

The right-hand side is the pull-back of the Fubini-Study metric of $P^1(\boldsymbol{C})$ by the Gauss map $g\colon M^2 \to \boldsymbol{C} \cup \{\infty\} = P^1(\boldsymbol{C})$.

Remark 2.8 In [ER], Estudillo and Romero defined a notion of *generalized maximal surfaces* as follows: Let M^2 be a Riemann surface and $f\colon M^2 \to L^3$ a differentiable map. Then f is called a generalized maximal surface if (1) $\varphi := \partial f/\partial z$ is holomorphic, (2) $-(\varphi^0)^2 + (\varphi^1)^2 + (\varphi^2)^2 = 0$, and (3) $-|\varphi^0|^2 + |\varphi^1|^2 + |\varphi^2|^2$ is not identically zero. Singular points of such a surface is either (A) an isolated zero of φ (a "branch point") or (B) a point where $|g| = 1$. Propositions 2.3 and 2.4 implies that a maxface in our sense is a generalized maximal surface without singular points of type (A).

3. Singularities of maxfaces

In the previous section, we defined maxfaces as surfaces with singularities. So it is quite natural to investigate which kind of singularities appear on maxfaces. We note that $\{(x, y, z) \in \boldsymbol{R}^3\,;\, x^2 = y^3\}$ is the cuspidal edge, and $\{(x, y, z) \in \boldsymbol{R}^3\,;\, x = 3u^4+u^2v,\ y = 4u^3+2uv,\ z = v\}$ is the swallowtail. We shall prove the following:

Theorem 3.1 *Let U be a domain of the complex plane $(\boldsymbol{C}, z)$ and $f\colon U \to L^3$ a maxface with the Weierstrass data $(g, \omega = \hat{\omega}\, dz)$, where $\hat{\omega}$ is a holomorphic function on U. Then*
(1) *A point $p \in U$ is a singular point if and only if $|g(p)| = 1$.*
(2) *The image of f around a singular point p is locally diffeomorphic to a cuspidal edge if and only if*

$$\operatorname{Re}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0 \qquad \text{and} \qquad \operatorname{Im}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0$$

hold at p, where $\omega(z) = \hat{\omega}(z)dz$ and $' = d/dz$.
(3) *The image of f around a singular point p is locally diffeomorphic to a swallowtail if and only if*

$$\frac{g'}{g^2\hat{\omega}} \in \boldsymbol{R} \setminus \{0\} \qquad \text{and} \qquad \operatorname{Re}\left\{\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right\} \neq 0$$

hold at p.

In [KRSUY], a criterion for a singular point on a wave front in $\boldsymbol{R}^3$ to be a cuspidal edge or a swallowtail is given. We shall recall it and prove the theorem as an application of it: We identify the unit cotangent bundle of the Euclidean 3-space $\boldsymbol{R}^3$ with $\boldsymbol{R}^3 \times S^2 = \{(x, \boldsymbol{n})\,;\, x \in \boldsymbol{R}^3,\ \boldsymbol{n} \in S^2\}$, then

$$\xi := n_1\, dx^1 + n_2\, dx^2 + n_3\, dx^3 \qquad \bigl(x = (x^1, x^2, x^3),\ \boldsymbol{n} = (n_1, n_2, n_3)\bigr)$$

gives a contact form and a map

$$L = (f_L, \boldsymbol{n})\colon\ U\ (\subset \boldsymbol{R}^2) \longrightarrow \boldsymbol{R}^3 \times S^2$$

is called a *Legendrian* if the pull-back of the contact form ξ vanishes, that is $(f_L)_u$ and $(f_L)_v$ are both perpendicular to $\boldsymbol{n}$, where $z = u + \sqrt{-1}\,v$. If L is a Legendrian immersion, the projection f_L of L into $\boldsymbol{R}^3$ is called a (*wave*) *front*.

Now let $L = (f, \boldsymbol{n})\colon U \to \boldsymbol{R}^3 \times S^2$ be a Legendrian immersion. A point $p \in U$ where f is not an immersion is called a *singular point* of the front f.

By definition, there exists a smooth function λ on U such that

$$f_u \times f_v = \lambda\, \boldsymbol{n} \tag{3.1}$$

where $\times$ is the Euclidean vector product of $\boldsymbol{R}^3$. A singular point $p \in U$ is called *non-degenerate* if $d\lambda$ does not vanish at p. We assume p is a non-degenerate singular point. Then there exists a regular curve around the point p

$$\gamma = \gamma(t)\colon\ (-\varepsilon, \varepsilon) \longrightarrow U$$

(called the *singular curve*) such that $\gamma(0) = p$ and the image of γ coincides with the set of singular points of f around p. The tangential direction of $\gamma(t)$ is called the *singular direction*. On the other hand, a non-zero vector $\eta \in TU$ such that $df(\eta) = 0$ is called the *null direction*. For each point $\gamma(t)$, the null direction $\eta(t)$ determined uniquely up to scalar multiplications. We recall the following

Proposition 3.2 ([KRSUY]) *Let $p = \gamma(0) \in U$ be a non-degenerate singular point of a front $f\colon U \to \boldsymbol{R}^3$.*

(1) *The germ of the image of the front at p is locally diffeomorphic to a cuspidal edge if and only if $\eta(0)$ is not proportional to $\dot{\gamma}(0)$, where $\dot{\gamma} = d\gamma/dt$.*

(2) *The germ of the image of the front at p is locally diffeomorphic to a swallowtail if and only if $\eta(0)$ is proportional to $\dot{\gamma}(0)$ and*

$$\frac{d}{dt}\det\bigl(\dot{\gamma}(t), \eta(t)\bigr)\bigg|_{t=0} \neq 0.$$

Now, we identify the Minkowski space L^3 with the affine space $\boldsymbol{R}^3$, and denote by $\langle\ ,\ \rangle_{\mathrm{Euc}}$ the Euclidean metric of $\boldsymbol{R}^3$. To prove Theorem 3.1, we prepare the following:

Lemma 3.3 *Let $f\colon M^2 \to L^3 \simeq \boldsymbol{R}^3$ be a maxface with Weierstrass data (g, ω). Then f is a projection of a Legendrian map $L\colon M^2 \to \boldsymbol{R}^3 \times S^2$. Moreover, f is a front on a neighborhood of p, and p is a non-degenerate singular point if and only if*

$$\operatorname{Re}\left(\frac{g'}{g^2\hat{\omega}}\right)\bigg|_p \neq 0, \tag{3.2}$$

where $\omega = \hat{\omega}\,dz$.

Proof. Let $z = u + \sqrt{-1}\,v$ be a complex coordinate of M^2 around p and write $\omega = \hat{\omega}\,dz$, where $\hat{\omega}$ is a holomorphic function in z. Then (2.3) implies

$$f_z = \frac{1}{2}\bigl(-2g, 1 + g^2, \sqrt{-1}(1 - g^2)\bigr)\hat{\omega},$$

$$f_{\bar z} = \frac{1}{2}\bigl(-2\bar g, 1+\bar g^2, -\sqrt{-1}(1-\bar g^2)\bigr)\bar{\hat\omega}.$$

Thus, we have

$$f_u \times f_v = -2\sqrt{-1}\, f_z \times f_{\bar z} = (|g|^2-1)|\hat\omega|^2(1+g\bar g, 2\operatorname{Re} g, 2\operatorname{Im} g),$$

where $\times$ is the Euclidean vector product of $\boldsymbol{R}^3$. Let

$$\boldsymbol{n} := \frac{1}{\sqrt{(1+|g|^2)^2+4|g|^2}}\,(1+|g|^2, 2\operatorname{Re} g, 2\operatorname{Im} g). \tag{3.3}$$

Then $\boldsymbol{n}$ is the Euclidean unit normal vector of f, that is $\langle df(X), \boldsymbol{n}\rangle_{\mathrm{Euc}} = 0$ for all $X \in TM^2$, where $\langle\ ,\ \rangle_{\mathrm{Euc}}$ is the Euclidean inner product.

From now on, we assume $|g(p)| = 1$, and hence $\omega(p) \neq 0$. At the singular point p, we have

$$\begin{aligned} df &= \frac{1}{2}\bigl(-2g, (1+g^2), \sqrt{-1}(1-g^2)\bigr)\hat\omega\, dz \\ &\quad + \frac{1}{2}\bigl(-2\bar g, (1+\bar g^2), -\sqrt{-1}(1-\bar g^2)\bigr)\bar{\hat\omega}\, d\bar z \\ &= \frac{1}{2}\left(-2, \frac{1}{g}+g, \sqrt{-1}\left(\frac{1}{g}-g\right)\right) g\omega \\ &\quad + \frac{1}{2}\left(-2, \frac{1}{\bar g}+\bar g, -\sqrt{-1}\left(\frac{1}{\bar g}-\bar g\right)\right)\bar g\bar\omega \\ &= (-1, \operatorname{Re} g, \operatorname{Im} g)(g\omega + \bar g\bar\omega) \end{aligned}$$

because $\bar g(p) = 1/g(p)$. In particular,

$$\eta = \frac{\sqrt{-1}}{g\hat\omega}, \tag{3.4}$$

gives the null-direction at p, where we identify T_pM^2 with $\boldsymbol{R}^2$ and $\boldsymbol{C}$ as

$$\zeta = a + \sqrt{-1}\, b \in \boldsymbol{C} \leftrightarrow (a,b) \in \boldsymbol{R}^2 \leftrightarrow a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} \leftrightarrow \zeta\frac{\partial}{\partial z} + \bar\zeta\frac{\partial}{\partial \bar z}. \tag{3.5}$$

On the other hand, we have

$$d\boldsymbol{n}(p) = \frac{\sqrt{-1}}{2\sqrt{2}}\left(\frac{dg}{g} - \frac{d\bar g}{\bar g}\right)\bigl(0, \operatorname{Im} g, -\operatorname{Re} g\bigr).$$

If $dg(p) = 0$ then $(f, \boldsymbol{n})$ is not an immersion at p because $d\boldsymbol{n}(p) = 0$. So we may assume $dg(p) \neq 0$. Then the null direction of $d\boldsymbol{n}$ at p is proportional to

$$\mu = \overline{\left(\frac{g'}{g}\right)} \qquad \left({}' = \frac{d}{dz}\right) \tag{3.6}$$

under the identification with (3.5). On the other hand, f is a front on a neighborhood at p if and only if $\langle df, df\rangle_{\mathrm{Euc}} + \langle d\boldsymbol{n}, d\boldsymbol{n}\rangle_{\mathrm{Euc}}$ is positive definite, that is η in (3.4) and μ in (3.6) are linearly independent, or equivalently

$$0 \neq \det(\mu, \eta) = \mathrm{Im}(\bar{\mu}\eta) = \mathrm{Im}\,\frac{g'}{g}\frac{\sqrt{-1}}{g\hat{\omega}}.$$

Then we have the first part of the conclusion. On the other hand, the function λ as in (3.1) is calculated as

$$\lambda = \langle f_u \times f_v, \boldsymbol{n}\rangle_{\mathrm{Euc}} = (|g|^2 - 1)|\hat{\omega}|^2\sqrt{(1+|g|^2)^2 + 4|g|^2}. \tag{3.7}$$

Since $d\sigma^2$ in (2.6) is a Riemannian metric, g must have pole at p if $\omega(p) = 0$ for $p \in M^2$. In this case, f is an immersion at p since $|g(p)| \neq 1$. Hence it is sufficient to consider the case $\omega(p) \neq 0$. At a singular point p, we have

$$d\lambda(p) = 2\sqrt{2}\,|\hat{\omega}|^2\,(\bar{g}\,dg + g\,d\bar{g}) = 2\sqrt{2}\,|\hat{\omega}|^2\left(\frac{dg}{g} + \frac{d\bar{g}}{\bar{g}}\right)$$

because $|g(p)| = 1$. Hence $d\lambda(p) \neq 0$ if and only if $dg(p) \neq 0$. If (3.2) holds at p, p is non-degenerate because $dg(p) \neq 0$. $\square$

Remark 3.4 At a singular point p such that $g'/(g^2\hat{\omega})(p) = 0$, f is not a front. For example, for a maxface f defined by the Weierstrass data $g = e^z$ and $\omega = \sqrt{-1}\,dz$, the pair $(f(z), \boldsymbol{n}(z))$ is not an immersion into $\boldsymbol{R}^3 \times S^2$ at $z = 0$.

Proof of Theorem 3.1. We have already shown (1) in the proof of Lemma 3.3. Assume $\mathrm{Re}(g'/(g^2\hat{\omega})) \neq 0$ holds at a singular point p. Then f is a front and p is a non-degenerate singular point. Since the singular set of f is characterized by $g\bar{g} = 1$, the singular curve $\gamma(t)$ with $\gamma(0) = p$ satisfies $g(\gamma(t))\overline{g(\gamma(t))} = 1$. Differentiating this, we get

$$\mathrm{Re}\left(\frac{g'}{g}\dot{\gamma}\right) = 0 \qquad \left({}' = \frac{d}{dz},\ \dot{} = \frac{d}{dt}\right).$$

This implies that $\dot{\gamma}$ is perpendicular to $\overline{g'/g}$, that is, proportional to

$\sqrt{-1}\,\overline{(g'/g)}$. Hence we can parametrize γ as

$$\dot\gamma(t) = \sqrt{-1}\,\overline{\left(\frac{g'}{g}\right)}\big(\gamma(t)\big)$$

under the identification as in (3.5). On the other hand, the null direction is given as (3.4). Then by Proposition 3.2, the germ of the image of the front at p is locally diffeomorphic to a cuspidal edge if and only if $\det(\dot\gamma,\eta)\neq 0$. Here,

$$\det(\dot\gamma,\eta) = \operatorname{Im}\bar{\dot\gamma}\eta = -\operatorname{Im}\sqrt{-1}\,\frac{g'}{g}\,\frac{\sqrt{-1}}{g\hat\omega}.$$

Then we have (2).

Next, we assume $\operatorname{Im}(g'/(g^2\hat\omega)) = 0$ holds at the singular point p. In this case,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\det(\dot\gamma,\eta) &= \operatorname{Im}\left(\left(\frac{g'}{g^2\hat\omega}\right)'\frac{d\gamma}{dt}\right) = -\operatorname{Re}\left(\frac{g'}{g^2\hat\omega}\right)'\overline{\left(\frac{g'}{g}\right)}\\ &= -\left|\frac{g'}{g}\right|^2\operatorname{Re}\left\{\frac{g}{g'}\left(\frac{g'}{g^2\hat\omega}\right)'\right\}.\end{aligned}$$

Thus, the second part of Proposition 3.2 implies (3). $\square$

4. Complete Maxfaces

Firstly, we define completeness and finiteness of total curvature for maxfaces:

Definition 4.1 Let M^2 be a Riemann surface. A maxface $f\colon M^2\to L^3$ is *complete* (resp. *of finite type*) if there exists a compact set $C\subset M^2$ and a symmetric 2-tensor T on M^2 such that T vanishes on $M^2\setminus C$ and ds^2+T is a complete metric (resp. a metric of finite total Gaussian curvature) on M^2, where ds^2 is the pull-back of the Minkowski metric by f.

Later (Theorem 4.6), we shall show that complete maxfaces are always of finite type.

Remark 4.2 As seen in Lemma 1.1, the Gaussian curvature of ds^2 is non-negative wherever ds^2 is non-degenerate. Then the total curvature of ds^2+T is well-defined as a real number or $+\infty$. (The total curvature of ds^2 itself is not well-defined because (1.6) diverges on the singular set $\{|g|=1\}$.

In fact, the only complete maxface of finite total curvature (in the sense of improper integral) is the plane ([ER, Theorem 5.2]).

Lemma 4.3 *If a maxface $f\colon M^2 \to L^3$ is complete (resp. of finite type), then the lift-metric $d\sigma^2$ is complete (resp. a metric of finite total absolute curvature) on M^2.*

Proof. Let (g, ω) be the Weierstrass data of f and T a symmetric 2-tensor as in Definition 4.1. Then by (1.5) and (2.6), we have $ds^2+T \le d\sigma^2$ outside the compact set C. Thus, if $ds^2 + T$ is complete, so is $d\sigma^2$.

We denote the Gaussian curvature of the metric $d\sigma^2$ by $K_{d\sigma^2}$. Then we have

$$(-K_{d\sigma^2})\, d\sigma^2 = \frac{4\, dg\, d\bar{g}}{(1+|g|^2)^2} \le \frac{4\, dg\, d\bar{g}}{(1-|g|^2)^2} = K_{ds^2}\, ds^2 \qquad \text{on } M^2 \setminus C \tag{4.1}$$

because of (1.6) and (2.7). Thus, if $ds^2 + T$ is of finite total curvature, the total absolute curvature of $d\sigma^2$ is finite. $\square$

Our definition of 'completeness' of maxface is rather restrictive: In fact the universal covering of complete maxface might not be complete since the singular set might not be compact on the universal cover. The following 'weak completeness' seems useful in some cases.

Definition 4.4 A maxface $f\colon M^2 \to L^3$ is *weakly complete* if the lift-metric $d\sigma^2$ is a complete metric.

By Lemma 4.3, completeness implies weakly completeness. However, the converse is not true. For example, let

$$F := \int \big(-2\sqrt{-1}\, z, 1-z^2, \sqrt{-1}(1+z^2)\big) \frac{dz}{(z^2-1)^2}.$$

Then F is a Lorentzian null immersion of the universal cover of $\boldsymbol{C} \cup \{\infty\} \setminus \{-1, 1\}$ into $\boldsymbol{C}^3$, and $f = F + \overline{F}$ gives a maxface defined on $\boldsymbol{C} \cup \{\infty\} \setminus \{-1, 1\}$. Though the lift-metric

$$d\sigma^2 = \frac{(|z|^2+1)^2}{|z^2-1|^2}\, dz\, d\bar{z}$$

is complete on $\boldsymbol{C} \cup \{\infty\} \setminus \{-1, 1\}$, the induced metric

$$ds^2 = \frac{(|z|^2-1)^2}{|z^2-1|^2}\, dz\, d\bar{z}$$

is not. In fact, the set of singularities (degenerate points of ds^2) is the set $\{|z|=1\}$ which accumulates at $z=\pm 1$.

Proposition 4.5 *Let $f\colon M^2 \to L^3$ be a weakly complete maxface. Suppose that the lift-metric $d\sigma^2$ has finite absolute total curvature. Then the Riemann surface M^2 is biholomorphic to a compact Riemann surface $\overline{M}^2$ excluding a finite number of points $\{p_1,\dots,p_n\}$. Moreover, the Weierstrass data (g,ω) of f can be extended meromorphically on $\overline{M}^2$.*

Proof. By our assumptions, the lift-metric $d\sigma^2$ is a complete metric of finite absolute total curvature. Moreover, by (4.1), the Gaussian curvature of $d\sigma^2$ is non-positive. Hence by Theorem A.1 in Appendix (or by Theorem 9.1 in [O]), M^2 is biholomorphic to $\overline{M}^2 \setminus \{p_1,\dots,p_n\}$.

Identifying $\boldsymbol{C}\cup\{\infty\}$ with the unit sphere S^2, the total absolute curvature of $d\sigma^2$ is nothing but the area of the image of the Gauss map $g\colon M^2 \to S^2$ counting multiplicity. Hence if $d\sigma^2$ is a metric of finite total curvature, g cannot have an essential singularity at $\{p_j\}$. Finally, we shall prove that p_j is at most a pole of ω. If $g(p_j)\neq\infty$, there exists a neighborhood U of p_j in $\overline{M}^2$ such that $|g|$ is bounded on U. In this case,

$$d\sigma^2 = (1+|g|^2)^2\,\omega\bar{\omega} \le k\,\omega\bar{\omega}$$

holds on U, where k is a positive constant, and hence $\omega\bar{\omega}$ is complete at p_j. Then by Lemma 9.6 of [O], ω must have a pole at p_j.

On the other hand, if $g(p_j)=\infty$, $d\sigma^2 \le k(g^2\omega)\overline{(g^2\omega)}$ holds on a neighborhood of p_j, where k is a positive constant. Hence $g^2\omega$ has a pole at p_j. □

We call the points $p_1,\dots,p_n$ in Proposition 4.5 the *ends* of the maxface f. For a weakly complete maxface of finite total absolute curvature with respect to $d\sigma^2$, the Gauss map g is considered as a holomorphic map $g\colon \overline{M}^2 \to \boldsymbol{C}\cup\{\infty\}$.

Theorem 4.6 *If a maxface $f\colon M^2 \to L^3$ is complete, then it is of finite type.*

Remark 4.7 This assertion is essentially different from the case of minimal surfaces in the Euclidean 3-space. There are many complete minimal surfaces with infinite total curvature like as a helicoid. The main difference is that the Gaussian curvature of maximal surfaces are non-negative while that of minimal surfaces are non-positive.

Proof. Since the Gaussian curvature of f is nonnegative, Theorem 13 of Huber [H] implies that M^2 is diffeomorphic to $\overline{M}^2 \setminus \{p_1, \dots, p_n\}$, where $\overline{M}^2$ is a compact Riemann surface and $\{p_1, \dots, p_n\}$ is a finite subset in $\overline{M}^2$. Moreover, a modification of Theorem 15 in Huber [H] yields that M^2 is biholomorphic to $\overline{M}^2 \setminus \{p_1, \dots, p_n\}$. (See the introduction of Li [Li] and also the Appendix.) We fix an end p_j arbitrary, and take a small coordinate neighborhood (U, z) with the origin p_j. Without loss of generality, we may assume that there are no singular points on $U \setminus \{p_j\}$, and thus we may also assume that $|g| < 1$ holds on $U \setminus \{p_j\}$ for the Gauss map g. (In fact, g changes to $1/g$ if we move the position of the stereographic projection to the south pole.) By the Great Picard theorem, g has at most pole at $z = p_j$.

We now suppose $|g(p_j)| = 1$ for an end p_j. Since $g \colon \overline{M}^2 \to \boldsymbol{C} \cup \{\infty\}$ is holomorphic at p_j, we can take a complex coordinate z on $\overline{M}^2$ such that $z(0) = p_j$ and $g(z) = a + z^k$, where a is a complex number with $|a| = 1$ and k is a positive integer. In this coordinate, the set $\{z\,;\, |g(z)|^2 = (a + z^k)(\bar{a} + \bar{z}^k) = 1\}$ accumulates at the end $z = 0$. Thus the singular set of f is non-compact, which contradicts to completeness. Hence we have $|g(p_j)| \neq 1$. Then there exists a positive number ε (< 1) such that $|g|^2 < 1 - \varepsilon$ holds on U. In this case, the Gaussian curvature K_{ds^2} (resp. $K_{d\sigma^2}$) of ds^2 (resp. $d\sigma^2$) satisfies

$$K_{ds^2}\, ds^2 = \frac{4dg\, d\bar{g}}{(1 - |g|^2)^2} \leq \left(\frac{2}{\varepsilon} - 1\right)^2 \frac{4\, dg\, d\bar{g}}{(1 + |g|^2)^2} = \text{const.}(-K_{d\sigma^2})\, d\sigma^2. \tag{4.2}$$

Since p_j is a pole of g, $d\sigma^2$ has finite total curvature on U. Hence ds^2 is of finite type at the end p_j. □

Corollary 4.8 *A maxface $f \colon M^2 \to L^3$ is complete if and only if f is weakly complete of finite total curvature with respect to the lift-metric $d\sigma^2$ and $|g(p_j)| \neq 1$ holds for each end $p_1, \dots, p_n$.*

Proof. Let $f \colon M^2 \to L^3$ be a complete maxface. Then f is of finite type by the previous theorem. By Lemma 4.3, f is weakly complete whose total absolute curvature of the lift-metric is finite, and get the conclusion.

Conversely, we let $f \colon \overline{M}^2 \setminus \{p_1, \dots, p_n\} \to L^3$ be an weakly complete maxface whose total absolute curvature of the lift-metric is finite, and assume $|g(p_j)| \neq 1$ for $j = 1, \dots, n$. Fix an end p_j and assume $|g(p_j)| < 1$.

Then we can take a neighborhood U_j such that $|g|^2 < 1-\varepsilon$ holds on U_j, where $\varepsilon \in (0,1)$ is a constant. In this case,

$$ds^2 = (1-|g|^2)^2\omega\bar{\omega} \geq \varepsilon^2\omega\bar{\omega} \geq \frac{\varepsilon^2}{4}(1+|g|^2)^2\omega\bar{\omega} = \frac{\varepsilon^2}{4}d\sigma^2$$

holds on U_j. Since $d\sigma^2$ is complete at p_j, so is ds^2. $\square$

Remark 4.9 In the case of $|g(p_j)| = 1$, the unit normal vector ν tends to a null (light-like) vector at the end. Imaizumi [I2] investigated the asymptotic behavior of such ends.

To prove the inequality mentioned in Introduction, we first investigate the behavior of the holomorphic lift around a single end.

Proposition 4.10 *Let $\Delta^* = \{z \in \boldsymbol{C}\,;\, 0 < |z| < 1\}$ and $f\colon \Delta^* \to L^3$ be a maxface such that an end 0 is complete, and denote by F the holomorphic lift of it. Then dF has a pole at 0 of order at least 2.*

Proof. Since f is complete, the lift-metric

$$d\sigma^2 = 2\left(\left|\frac{dF^0}{dz}\right|^2 + \left|\frac{dF^1}{dz}\right|^2 + \left|\frac{dF^2}{dz}\right|^2\right) dz\,d\bar{z}$$

is a complete metric at the origin. Then at least one of dF^j/dz ($j = 0,\,1,\,2$) has a pole at $z = 0$. We assume dF/dz has a pole of order 1 at $z = 0$. Then dF/dz is expanded as

$$\frac{dF}{dz} = \frac{1}{z}(a^0, a^1, a^2) + O(1),$$

where $O(1)$ denotes the higher order terms. Since $f = F + \overline{F}$ is well-defined on a neighborhood of $z = 0$, the residue of dF at $z = 0$ must be real (see (2.5)): $(a^0, a^1, a^2) \in \boldsymbol{R}^3$. On the other hand, by the nullity of F, we have

$$-(a^0)^2 + (a^1)^2 + (a^2)^2 = 0. \tag{4.3}$$

Here, by (1.3),

$$g(0) = -\lim_{z\to 0}\frac{dF^0}{dF^1 - \sqrt{-1}\,dF^2} = -\frac{a^0}{a^1 - \sqrt{-1}\,a^2}.$$

Then by (4.3), $|g(0)|^2 = (a^0)^2/\{(a^1)^2 + (a^2)^2\} = 1$. This is a contradiction, because of Theorem 4.8. Hence dF has a pole of order at least 2. $\square$

Theorem 4.11 (Osserman-type inequality) *Let $\overline{M}^2$ be a compact Riemann surface and $f\colon \overline{M}^2 \setminus \{p_1, \dots, p_n\} \to L^3$ a complete maxface. Then the Gauss map $g\colon \overline{M}^2 \to \boldsymbol{C} \cup \{\infty\}$ satisfies*

$$2 \deg g \geq -\chi(M^2) + n = -\chi(\overline{M}^2) + 2n,$$

and equality holds if and only if all ends are properly embedded, that is, there exists a neighborhood U_j of each end p_j such that $f|_{U_j \setminus \{p_j\}}$ is an embedding.

Proof. By (2.7), we have

$$\deg g = \frac{1}{4\pi} \int_{M^2} (-K_{d\sigma^2}) dA_{d\sigma^2}.$$

By a rigid motion in L^3, we may assume $g(p_j) \neq \infty$ $(j = 1, \dots, n)$. Since

$$d\sigma^2 = (1 + |g|^2)^2\, \omega\bar{\omega}$$

and ω has at least pole of order 2, the inequality follows from the proof of the original Osserman inequality for the metric $d\sigma^2$. (See Theorem 9.3 in [O], or [Fa]).

As we assumed $g(p_j) \neq \infty$ $(j = 1, \dots, n)$, the equality holds if and only if ω has a pole of order exactly 2 at each end. Assume ω has pole of order 2 at p_j and take a coordinate z around p_j such that $z(p_j) = 0$. Without loss of generality, we may assume $g(0) = 0$. By a direct calculation, we have an expansion of $f(z)$ as

$$f(z) = \frac{a}{r}(\cos\theta, \sin\theta, 0) + c \log r\,(0, 0, 1) + O(1) \qquad (z = re^{\sqrt{-1}\,\theta})$$

around $z = 0$, where $a \in \boldsymbol{R} \setminus \{0\}$ and $c \in \boldsymbol{R}$ are constants (see [KoUY1] and also [I2]). If $c \neq 0$ (resp. $c = 0$), the end is asymptotic to the end of the Lorentzian catenoid as in Example 5.1 (resp. the plane), which is embedded. Conversely, if ω has pole of order more than 2 at p_j, a similar argument to that of Jorge-Meeks [JM] or [S] concludes that the end is not embedded. (A good reference is [KoUY1].) $\square$

5. Examples

We shall first introduce two classical examples.

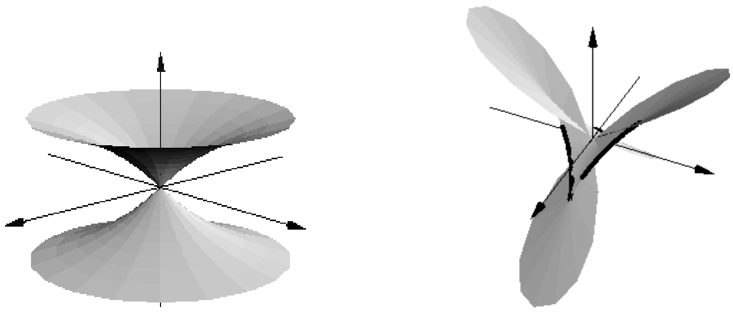
Example 5.1 (Lorentzian catenoid) Rotating a curve $x^1 = a\sinh(x^0/a)$ $(a \neq 0)$ in the x^0x^1-plane around the x^0-axis, we have a surface of revolution

$$f\colon S^1 \times \boldsymbol{R} \ni (\theta, t) \longmapsto a\bigl(t, \cos\theta \sinh t, \sin\theta \sinh t\bigr) \in L^3.$$

Then one can see that f gives a maximal immersion on $S^1 \times \boldsymbol{R} \setminus \{0\}$, hence f is a maximal map in the sense of Definition 2.1, and $S^1 \times \{0\}$ is the set of singularities of f. Since the induced metric is represented as

$$ds^2 = a^2 \sinh^2 t\,(dt^2 + d\theta^2), \tag{5.1}$$

$\operatorname{cosech}^2 t\, ds^2 = a^2(dt^2 + d\theta^2)$ extends smoothly across on the singularities. Hence f is a maxface. Moreover, it can be easily seen that f is complete. The Weierstrass representation of f is given as follows: Let $M^2 = \boldsymbol{C} \setminus \{0\}$ and $g = z$, $\omega = a\,dz/z^2$. Then (g, ω) gives the Lorentzian catenoid (5.1). The set of singularities is $\{|z| = 1\}$ and its image by f is the origin in L^3 at which the image of f is tangent to the light-cone (see Figure 1 left). Such a singularity is called a *conelike singularity*, which was first investigated in [K2]. See also [FLS] and [I1].



the Lorentzian catenoid   the Lorentzian Enneper surface: the singular set is shown in the black line.

Fig. 1. Examples 5.1 and 5.2

Example 5.2 (Lorentzian Enneper surface) Let $M^2 = \boldsymbol{C}$ and $(g, \omega) = (z, dz)$. Then there exists the maxface $f\colon \boldsymbol{C} \to L^3$ with Weierstrass data (z, dz). The set of the singularities is the unit circle $\{|z| = 1\}$. The points ± 1, $\pm\sqrt{-1}$ are swallowtails and the points $e^{\sqrt{-1}\frac{\pi}{4}}$, $e^{\sqrt{-1}\frac{3}{4}\pi}$, $e^{\sqrt{-1}\frac{5}{4}\pi}$, $e^{\sqrt{-1}\frac{7}{4}\pi}$ are neither cuspidal edges nor swallowtails. In fact, these four points are cuspidal crosscaps (see [FSUY]). The singular points other than these 8 points are cuspidal edges (see Figure 1, right).

To produce further examples, we consider a relationship between maxfaces and minimal surfaces in the Euclidean space $\boldsymbol{R}^3$, and shall give a method transferring minimal surfaces to maxfaces.

Let $f\colon M^2 \to L^3$ be a maxface and $F = (F^0, F^1, F^2)\colon \widetilde{M}^2 \to \boldsymbol{C}^3$ its holomorphic lift, where $\widetilde{M}^2$ is the universal cover of M^2. Set

$$F_0 := \bigl(F_0^1, F_0^2, F_0^3\bigr) = \bigl(F^1, F^2, \sqrt{-1}\,F^0\bigr).$$

Since F is a Lorentzian null immersion, F_0 is an (Euclidean) null immersion, that is,

$$(dF_0^1)^2 + (dF_0^2)^2 + (dF_0^3)^2 = 0.$$

Hence

$$f_0 = F_0 + \overline{F_0} \tag{5.2}$$

is a conformal minimal immersion of $\widetilde{M}^2$ into the Euclidean 3-space $\boldsymbol{R}^3$.

Definition 5.3 A minimal immersion $f_0\colon \widetilde{M}^2 \to \boldsymbol{R}^3$ as in (5.2) is called the *companion* of the maxface f.

The companion of the Lorentzian catenoid (resp. the Lorentzian Enneper surface) as in Examples 5.1 and 5.2 is the helicoid (resp. the Enneper surface).

The lift-metric of a maxface f as in (2.6) is the induced metric of the companion f_0, and the Gauss map g_0 of f_0 is represented as

$$g_0 = -\sqrt{-1}\,g, \tag{5.3}$$

where g is the Gauss map of f. Moreover, by Lemma 4.3 and by Theorem 4.6, $d\sigma^2$ is a complete metric on M^2 with finite total curvature if f is complete.

By definition of F_0, there exists representations $\rho_j\colon \pi_1(M^2) \to \boldsymbol{R}$ ($j =$ 1, 2, 3) such that

$$F_0 \circ \tau = F_0 + \bigl(\sqrt{-1}\,\rho_1(\tau), \sqrt{-1}\,\rho_2(\tau), \rho_3(\tau)\bigr) \qquad \bigl(\tau \in \pi_1(M^2)\bigr) \tag{5.4}$$

holds, where τ in the left-hand side is considered as a deck transformation on $\widetilde{M}^2$.

Conversely, we should like to construct a complete maxface via the complete minimal surfaces of finite total curvature:

Proposition 5.4 *Let M^2 be a Riemann surface and $\widetilde{M}^2$ the universal cover of it. Assume a null holomorphic immersion $F_0\colon \widetilde{M}^2 \to \boldsymbol{C}^3$ satisfies the following conditions.*

(1) *There exists representations ρ_j $(j = 1, 2, 3)$ such that* (5.4) *holds for each $\tau \in \pi_1(M^2)$.*

(2) *If we set $dF_0 = (\varphi_0^1, \varphi_0^2, \varphi_0^3)$, the function $-|\varphi_0^3|^2 + |\varphi_0^1|^2 + |\varphi_0^2|^2$ does not vanish identically.*

Then there exists a maxface $f\colon M^2 \to L^3$ whose companion is $f_0 = F_0 + \overline{F_0}$. Moreover, if the induced metric of f_0 defines a complete metric of finite total curvature on M^2, then f is a complete maxface if and only if

$$|g(p_j)| \neq 1 \qquad \left(g := \sqrt{-1}\frac{\partial f_0^3}{\partial f_0^1 - \sqrt{-1}\,\partial f_0^2} \right) \qquad (j = 1, \dots, n), \tag{5.5}$$

where $M^2 = \overline{M}^2 \setminus \{p_1, \dots, p_n\}$ with compact Riemann surface $\overline{M}^2$ and $\{p_1, \dots, p_n\}$ is the set of ends.

Proof. Let

$$\varphi = \left(-\sqrt{-1}\,\varphi_0^3, \varphi_0^1, \varphi_0^2\right) \qquad \text{and} \qquad F := \int_{z_0}^{z} \varphi,$$

where $z_0 \in \widetilde{M}^2$ is a base point. Then by the condition (1), $f = F + \overline{F}$ is well-defined on M^2. Moreover, by (2), $\langle dF, d\overline{F} \rangle$ is not identically 0. Hence f is a maxface. Suppose now that f_0 is complete and of finite total curvature. Then, the lift-metric $d\sigma^2$ is complete and of finite total curvature. If $|g(p_j)| \neq 1$ for $j = 1, \dots, n$, f is complete by Theorem 4.8. □

Example 5.5 (Lorentzian Chen-Gackstatter surface) We set

$$\overline{M}^2 = \{(z, w) \in \boldsymbol{C}^2 \cup \{\infty, \infty\}\,;\, w^2 = z(z^2 - a^2)\},$$

where a is a positive real number, and set

$$\varphi_0 := \frac{1}{2}\left(\left(\frac{z}{w} - B^2\frac{w}{z}\right), \sqrt{-1}\left(\frac{z}{w} + B^2\frac{w}{z}\right), 2B\right) dz,$$

where

$$B^2 = \int_0^a \frac{x\,dx}{\sqrt{x(a^2-x^2)}} \bigg/ \int_0^a \frac{(a^2-x^2)\,dx}{\sqrt{x(a^2-x^2)}}.$$

Then

$$f_0 := F_0 + \overline{F_0}, \qquad \left(F_0 = \int_{z_0}^{z} \varphi_0\right)$$

gives a complete minimal immersion f_0 of $M^2 := \overline{M}^2 \setminus \{\infty\}$ into $\boldsymbol{R}^3$, which is called *Chen-Gackstatter surface* ([CG]).

Since the third component of φ_0 is an exact form, F_0^3 is well-defined on M^2. In particular, (5.4) holds for $\rho_3 = 0$.

Moreover, $g = Bw/z$ in (5.5) tends to 0 as $z \to \infty$. Hence the corresponding maxface $f\colon M^2 \to L^3$ given by Proposition 5.4 is complete, which is called the *Lorentzian Chen-Gackstatter surface*.

Example 5.6 (Minimal surfaces which admits a Lopez-Ros deformation) Let $f\colon M^2 \to \boldsymbol{R}^3$ be a complete conformal minimal immersion of finite total curvature. Then there exists a null holomorphic lift $F\colon \widetilde{M}^2 \to \boldsymbol{C}^3$ such that $f = F + \overline{F}$. Then we have a representation $\rho\colon \pi_1(M^2) \to \boldsymbol{R}^3$ such that

$$F \circ \tau = F + \sqrt{-1}\,\rho(\tau) \qquad \bigl(\tau \in \pi_1(M^2)\bigr). \tag{5.6}$$

Then f is called *a minimal surface which admits the Lopez-Ros deformation* if $\rho(\pi_1(M^2))$ is contained in a 1-dimensional subspace of $\boldsymbol{R}^3$. In this case, by a suitable rotation of the surface, we may assume that

$$\rho\bigl(\pi_1(M^2)\bigr) \subset \boldsymbol{R}\bigl(0,0,1\bigr). \tag{5.7}$$

We set

$$dF = \frac{1}{2}\bigl((1-g^2), \sqrt{-1}(1+g^2), 2g\bigr)\omega.$$

For each non-zero real number λ, replacing Weierstrass data (g,ω) by $(\lambda g, \omega/\lambda)$, the new minimal immersion

$$f_\lambda = F_\lambda + \overline{F_\lambda}, \qquad dF_\lambda = \frac{1}{2}\bigl((1-\lambda^2 g^2), \sqrt{-1}(1+\lambda^2 g^2), 2g\bigr)\omega$$

also gives a conformal minimal immersion of M^2 because of (5.7). In particular, f_λ is complete and of finite total curvature. The 1-parameter family $\{f_\lambda\}$ is called a *Lopez-Ros deformation of f* [LR]. Then one can easily check that all of $\sqrt{-1}\,F_\lambda$ satisfy (5.4) with $\rho_1 = \rho_2 = 0$. Moreover, except for only finite many values of λ, the condition $|g(p_j)| \neq 1$ holds ($j = 1, \dots, n$), where $p_1, \dots, p_n$ are ends of the immersion f. Thus we can construct complete maxface $\hat{f}_\lambda$ from $\sqrt{-1}\,F_\lambda$ except for at most finitely many values of λ. Remark that f_λ and $f_{-\lambda}$ are congruent with each other. The number of $\lambda > 0$ such that $\hat{f}_\lambda$ is not complete is not exceed the number of the ends n.

Many examples of minimal surfaces which admit Lopez-Ros deformation are known [Lo, KaUY, Mc]. So we have uncountably many examples of complete maxfaces.

Example 5.7 The Jorge-Meeks surface is a complete minimal surface in $\boldsymbol{R}^3$ with n catenoidal ends. Such a surface is realized as an immersion

$$f_0 \colon \boldsymbol{C} \cup \{\infty\} \setminus \{1, \zeta, \dots, \zeta^{n-1}\} \longrightarrow \boldsymbol{R}^3 \qquad (\zeta = e^{2\pi i/n})$$

with Weierstrass data

$$g_0 = z^{n-1}, \qquad \omega_0 = \frac{dz}{(z^n - 1)^2}.$$

It can be easily checked that, there exists a maxface f whose minimal companion is the Jorge-Meeks' surface. Here, g in (5.5) is z^{n-1}, $|g| = 1$ holds on each end. Hence the maxface f is not complete but weakly complete. As pointed out in Remark 4.9, Imaizumi [I2] investigated weakly complete maxface, and introduced a notion of *simple ends* for an end p_j satisfying $|g(p_j)| = 1$. Imaizumi and Kato [IK] classified weakly complete maxfaces of genus zero with at most 3 simple ends.

A. A consequence of Huber's theorem.

This appendix was prepared for the forthcoming paper Fujimori, Rossman, Umehara, Yamada and Yang [FRUYY], but the other authors allow to put it in this paper. We shall show that the following assertion is a simple consequence of Huber's theorem [H, Theorem 13].

Theorem A.1 *Let (M^2, ds^2) be a complete Riemannian 2-manifold and K the curvature function. Suppose*

$$\int_{M^2} (-K_-)\, dA < \infty, \tag{A.1}$$

where

$$K_- := \min(K, 0).$$

Then there exists a compact Riemann surface $\overline{M}^2$ and finite points $p_1, \dots, p_n \in \overline{M}^2$ such that M^2 is bi-holomorphic to $\overline{M}^2 \setminus \{p_1, \dots, p_n\}$.

This assertion was pointed out in Li [Li] without proof. Here, we shall give a proof for a help for readers. To prove the assertion we use the following well-known fact in Huber's paper

Fact A.1 (Huber [H, Theorem 13]) *Let (M^2, ds^2) be a complete Riemannian 2-manifold such that* (A.1) *holds. Then M^2 is diffeomorphic to $\overline{M}^2 \setminus \{p_1, \dots, p_n\}$, where $\overline{M}^2$ is a compact 2-manifold.*

Moreover, the following assertion is known:

Fact A.2 (Blanc-Fiala [BF], Huber [H, Theorem 15]) *Let (M^2, ds^2) be a complete Riemannian 2-manifold such that* (A.1) *holds. Then M^2 is parabolic.*

This assertion firstly proved by Blanc and Fiala when M^2 is simply connected. To prove our theorem, we apply the above fact for this simply connected case.

Proof of the Theorem A.1. By Fact A.1, M^2 is diffeomorphic to $\overline{M}^2 \setminus \{p_1, \dots, p_n\}$. We fix an end p_j. There exists a coordinate neighborhood $(U_j; u, v)$ of $\overline{M}^2$ such that p_j corresponds to the origin and the boundary ∂U_j is a simple closed C^∞-regular curve. Then by uniformization theorem of annuli (see Ahlfors-Sario, Riemann surface (Princeton), I4D, II3B), $(U_j \setminus \{p_j\}, ds^2|_{U_j \setminus \{p_j\}})$ is conformally equivalent to

$$\Delta(r) := \{z \in \boldsymbol{C}\,;\, r < |z| < 1\}$$

where $r \in [0, 1)$. Then ds^2 can be considered as a metric defined on $\Delta(r)$. Let $\lambda\colon \Delta(r) \to [0, 1]$ be a C^∞-function such that
(1) $\lambda(z) = 0$ when $|z| \le (r+1)/2$
(2) $\lambda(z) = 1$ when $|z| \ge (r+2)/3$
Then we can define a new metric $d\sigma^2$ by

$$d\sigma^2 = (1-\lambda)ds^2 + \lambda \frac{4dz\, d\bar{z}}{(1+|z|^2)^2}.$$

Since this metric is constant Gaussian curvature 1 when $|z| > 1$ and can be extended at $z = \infty$, and we obtain a complete simply connected Riemannian manifold

$$\big(\{z \in \boldsymbol{C}\cup\{\infty\}\,;\,|z|>r\}, d\sigma^2\big)$$

Moreover, the integral of the negative part of the curvature function of $d\sigma^2$ is finite. So we can apply Fact A.2 and can conclude that $(\{z \in \boldsymbol{C}\cup\{\infty\}\,;\,|z|>r\}, d\sigma^2)$ is parabolic, namely conformally equivalent to $\boldsymbol{C}$, which implies that $\big(\{z\in\boldsymbol{C}\,;\,r<|z|<(r+1)/2\},\, ds^2|_{\{z\in\boldsymbol{C}\,;\,r<|z|<(r+1)/2\}}\big)$ is conformally equivalent to a punctured disc. Thus we must conclude that $r=0$. $\square$

References

[BF] Blanc Ch. and Fiala F., *Le type d'une surface et sa courbure totale*, Comment. Math. Helv., **14** (1941–42), 230–233.

[CG] Chen C.C. and Gackstatter F., *Elliptische und hyperelliptische Funktionen und vollständige Minimalflächen von Enneperschen Typ*, Math. Ann., **259** (1982), 359–369.

[CY] Cheng S.Y. and Yau S.T., *Maximal spacelike hypersurfaces in the Lorenz-Minkowski spaces*, Ann. of Math., **104** (1976), 407–419.

[ER] Estudillo F.J.M. and Romero A., *Generalized maximal surfaces in Lorentz-Minkowski space* $\mathbb{L}^3$, Math. Proc. Camb. Phil. Soc., **111** (1992), 515–524.

[Fa] Fang Y., *The Minding formula and its applications*, Arch. Math., **72** (1999), 473–480.

[FSUY] Fujimori S., Saji K., Umehara M. and Yamada K., *Cuspidal cross caps and singularities of maximal surfaces*, `math. DG/0510366`, preprint.

[Fu] Fujimori S., *Spacelike CMC-*1 *surfaces with elliptic ends in de Sitter space*, `math. DG/0408036`, to appear in Hokkaido Math. J.

[FLS] Fernández, I., López F.J. and Souam R., *The space of complete embedded maximal surfaces with isolated singularities in* 3*-dimensional Lorentz-Minkowski space* $\mathbb{L}^3$, Math. Ann. **332** (2005), 605–643.

[FRUYY] Fujimori S., Rossman W., Umehara M., Yamada K. and Yang S.-D., *Spacelike constant mean curvature one surfaces in de Sitter* 3*-space*, in preparation.

[H] Huber A., *On subharmonic functions and differential geometry in large*, Comment. Math. Helv., **32** (1957), 13–72.

[I1] Imaizumi T., *Maximal surfaces with conelike singularities of finite type*, Kobe J. Math., **18** (2001), 51–60.

[I2] Imaizumi T., *Maximal surfaces with simple ends*, Kyushu J. Math., **58** (2004), 59–70.

[IK] Imaizumi T. and Kato S., in preparation.

[IM] Ishikawa G. and Machida Y., *Singularities of improper affine spheres and surfaces of constant Gaussian curvature*, `math. DG/0502154`, to appear in International J. Math.

[IT] Inoguchi J. and Toda M., *Timelike minimal surfaces via loop groups*, Acta. Appl. Math. **83** (2004), 313-355.

[JM] Jorge L.P.M. and Meeks W.H., III, *The topology of complete minimal surfaces of finite total curvature*, Topology, **22** (1983), 203–221.

[K1] Kobayashi O., *Maximal surfaces in the* 3*-dimensional Minkowski space* L^3, Tokyo J. Math., **6** (1983), 297–309.

[K2] Kobayashi O., *Maximal surfaces with conelike singularities*, J. Math. Soc. Japan, **36** (1984), 609–617.

[KRSUY] Kokubu M., Rossman W., Saji K., Umehara M. and Yamada K., *Singularities of flat fronts in hyperbolic* 3*-space*, Pacific J. Math. **221** (2005), 303–351.

[KaUY] Kato S., Umehara M. and Yamada K., *An inverse problem of the flax of minimal surfaces*, Indiana Univ. Math. J., **46** (1997), 529–560.

[KoUY1] Kokubu M., Umehara M. and Yamada K., *Minimal surfaces that attain equality in the Chern-Osserman inequality*, Contemporary of Mathematics, **308** (2002), 223–228.

[KoUY2] Kokubu M., Umehara M. and Yamada K., *An elementary proof of Small's formula for null curves in* $\mathrm{PSL}(2, \boldsymbol{C})$ *and an analogue for Legendrian curves in* $\mathrm{PSL}(2, \boldsymbol{C})$, Osaka J. of Math., **40** (2003), 687–715.

[KoUY3] Kokubu M., Umehara M. and Yamada K., *Flat fronts in hyperbolic* 3*-space*, Pacific J. of Math., **216** (2004), 149–175.

[KY] Kim Y.W. and Yang S.-D., *Elliptic catenoids in Lorentz-Minkowski three-space with an arbitrary number of handles*, in preparation.

[LLS] Lopez F.J., Lopez R. and Souam R., *Maximal surfaces of Riemann type in Lorentz-Minkowski space* L^3, Michigan Math. J., **47** (2000), 469–497.

[Lo] Lopez F.J., *New complete genus zero minimal surfaces with embedded parallel ends*, Proc. of the Amer. Math. Soc., **112** (1991), 539–544.

[Li] Li P., *Complete surfaces of at most quadratic area growth*, Comment. Math. Helv., **72** (1997), 67–71.

[LY] Lee S.-W. and Yang S.-D., *A spinor representation for spacelike surfaces of constant mean curvature* -1 *in de Sitter three space*, to appear in Osaka J. Math.

[LR] Lopez F.J. and Ros A., *On embedded complete minimal surfaces of genus zero*, J. Differential Geometry, **33** (1991), 293–300.

[Mc] McCune C., *Rational minimal surface*, Q. Math., **52** (2001), 329–354.

[McN] McNertney L.V., *One-parameter families of surfaces with constant curvature in Lorentz* 3*-space*, thesis, Brown University, 1980.

[O] Osserman R., *A Survey of Minimal Surfaces*, Dover Publications, 1986.

[S] Schoen R., *Uniqueness, symmetry and embeddedness of minimal surfaces*, J. Differential Geometry, **18** (1983), 791–809.

M. Umehara
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043, Japan
E-mail: umehara@math.wani.osaka-u.ac.jp

K. Yamada
Faculty of Mathematics
Kyushu University
Higashi-ku, Fukuoka 812-8581, Japan
E-mail: kotaro@math.kyushu-u.ac.jp