Journal of Artificial Intelligence Research 69 (2020) 613-655 Submitted 09/2019; published 10/2020

Maximin Share Allocations on Cycles

Zbigniew Lonc ZBLONC@MINI.PW.EDU.PL
Warsaw University of Technology
Koszykowa 75
00-662 Warszawa Poland

Miroslaw Truszczynski MIREK@CS.UKY.EDU
Department of Computer Science
University of Kentucky
Lexington, KY 40506-0633 USA

Abstract

The problem of fair division of indivisible goods is a fundamental problem of resource allocation in multi-agent systems, also studied extensively in social choice. Recently, the problem was generalized to the case when goods form a graph and the goal is to allocate goods to agents so that each agent's bundle forms a connected subgraph. For the maximin share fairness criterion, researchers proved that if goods form a tree, an allocation offering each agent a bundle of at least her maximin share value always exists. Moreover, it can be found in polynomial time. In this paper we consider the problem of maximin share allocations of goods on a cycle. Despite the simplicity of the graph, the problem turns out to be significantly harder than its tree version. We present cases when maximin share allocations of goods on cycles exist and provide in this case results on allocations guaranteeing each agent a certain fraction of her maximin share. We also study algorithms for computing maximin share allocations of goods on cycles.

1. Introduction

Fair allocation of indivisible goods is a fundamental problem in social choice (Brams & Taylor, 1996; Bouveret et al., 2016). In such a problem we have a set of elements, referred to as *goods*, and a collection of agents each with her own utility function on all subsets (called *bundles*) of the set of goods. The utility functions are commonly assumed to be additive and so it is enough to specify their values on individual goods only. The objective of the problem is to assign to agents disjoint subsets of goods so that to satisfy some fairness criteria. Among the most commonly studied ones are *proportionality* and *envy-freeness*,[1] adapted to the case of indivisible goods from the problem of fair allocation of divisible goods or *cake-cutting* (Brams & Taylor, 1996; Procaccia, 2016), as well as the recently proposed *maximin* share fairness criterion (Budish, 2011; Bouveret & Lemaître, 2016), which we formally define later in the paper. For each criterion, it is of interest to identify classes of instances when fair allocations exist, to establish the complexity of deciding the existence of fair allocations, and to design algorithms for computing them.

In this paper, we focus on the maximin share criterion (Budish, 2011). Informally, we ask each of n agents to partition the set of goods into n bundles, such that the least-valued bundle is maximized. The *maximin share* for this agent is then equal to the value of this least-valued bundle.

1. *Proportionality* requires that if n agents are to be allocated goods, each agent must receive a bundle valued at least $1/n$ of the value of all goods for that agent; *envy-freeness* requires that each agent values her bundle at least as much as the bundle assigned to any other agent.

An allocation of bundles of goods to agents satisfies the *maximin share* fairness criterion if each agent receives a bundle of value greater than or equal to the maximin share for this agent. In the few years since the criterion was first proposed it has received substantial attention. The key reason is that it is easier to achieve than proportionality and, even more so, than envy-freeness (envy-free allocations are proportional and proportional allocations are maximin share ones).

However, the criterion turns out to be highly non-trivial to analyze. First, while it is easy to see that envy-free and proportional allocations do not always exist, it is not at all clear whether the same is true for the less restrictive maximin criterion. We now know it is, but it took a few years before the first examples showing that maximin share allocations are not guaranteed to exist were found (Procaccia & Wang, 2014; Kurokawa et al., 2018). In fact, counterexamples are quite intricate in their structure, and they seem to be exceedingly rare — Bouveret and Lemaître (2016) did not find a single one among thousands of randomly generated instances. Further, the complexity of deciding the existence of maximin share allocations has not yet been fully determined (Bouveret & Lemaître, 2016).

To get around the difficulty of constructing maximin allocations, researchers proposed to relax the maximin share criterion by requiring that the value of each agent's share in an allocation be at least equal to some positive fraction of her maximin share. Procaccia and Wang (2014) proved that an allocation guaranteeing agents at least 2/3 of their maximin share always exists (2/3-approximating maximin share allocation), and that it can be found in polynomial time if the number of agents is constant (not part of the input). Both the result and the algorithm are based on deep combinatorial insights. Building on that work, Amanatidis et al. (2017) proved that for every constant ϵ, $0 < \epsilon < 2/3$, there is a polynomial-time algorithm for finding a 2/3-approximating maximin share allocation (with the number of agents a part of the input). Barman and Murthy (2017) and Garg et al. (2019) proposed simpler and easier to analyze polynomial time algorithms constructing allocations guaranteeing agents at least 2/3 of their maximin share. A major breakthrough was obtained by Ghodsi et al. (2018), who proved the existence of a 3/4-approximating maximin share allocation and showed that for every constant ϵ, $0 < \epsilon < 3/4$, a $(3/4 - \epsilon)$-approximating maximin share allocation can be found in polynomial time. These results were further improved on by Garg and Taki (2019). They presented a polynomial-time algorithm that finds a 3/4-approximating maximin share allocation, as well as algorithms that, under some additional assumptions, break the barrier of the 3/4 fraction.

We study maximin share allocations of indivisible goods in the setting proposed by Bouveret et al. (2017). In the original problem, there are no restrictions on sets of goods that can be allocated to agents. This ignores important practical constraints that may make some sets highly undesirable. For instance, goods may be rooms and labs in a building to be allocated to research groups (Bouveret et al., 2017), or plots of land to be consolidated (King & Burton, 1982). In such cases, legal sets of goods that could be allocated to an agent might be required to form connected subgraphs in some graph describing the neighborhood relation among goods (offices spanning segments of a hall, plots forming contiguous areas of land).

Bouveret et al. (2017) studied envy-free, proportional and maximin share allocations for that setting obtaining several interesting complexity and algorithmic results. Our paper extends their study for the maximin share criterion. In a striking positive result, Bouveret et al. (2017) proved that maximin share allocations of goods on trees always exist and can be found in polynomial time. In our work we look beyond trees and show that as soon as the underlying graph has a single cycle, the picture becomes much more complicated. Our main contributions are as follows.

1. We show that for goods on a cycle, the maximin share value for an agent can be computed in polynomial time. This is in contrast to the widely studied case when the goods form a complete graph, in which computing maximin share is NP-hard. In two cases, when $m \leq 2n$ and when agents can be grouped into a constant number of types (agents are of the same type if they have the same utility function), our result allows us to construct a polynomial-time algorithm for computing maximin share allocations of m goods (on cycles) to n agents or determining that such allocations do not exist.

2. We show that the problem to decide the existence of maximin share allocations of goods on an arbitrary graph is in the complexity class Δ_2^P. For complete graphs, the case equivalent to the original problem, this result improves on the result by Bouveret and Lemaître (2016), who proved the problem belongs to the class Σ_2^P (recall that $\Delta_2^P \subseteq \Sigma_2^P$, and that the inclusion is proper, unless the polynomial hierarchy collapses). We further improve our result for cycles and more generally, unicyclic graphs,[2] by showing that for such graphs the existence of a maximin share allocation is in the class NP (again, recall that $\mathrm{NP} \subseteq \Delta_2^P$, and that the inclusion is proper, unless the polynomial hierarchy collapses).[3]

3. We obtain approximation results on the existence of allocations of goods on a cycle that *guarantee* every agent a specified fraction of her maximin share value. While it is easy to show that for any number of agents there are allocations guaranteeing each agent at least $1/2$ her maximin share, showing that one can guarantee a higher proportion of maximin share values is a non-trivial problem. We show that the proportion can be improved to $(\sqrt{5}-1)/2$ (> 0.618). Further improvements are possible if we limit the number of agents or the number of types of agents. In particular, we show that for the three-agent case, there are allocations guaranteeing each agent $5/6$ of her maximin share; for an arbitrary number of agents of up to three types, there are allocations guaranteeing each agent $3/4$ of her maximin share; and for any number of agents of $t \geq 4$ types there are allocations guaranteeing each agent $t/(2t-2)$ of her maximin share. In each case, these allocations can be found in polynomial time. Moreover, the approximation ratios 5/6 and 3/4 for the cases of three agents and any number of agents of up to three types, respectively, are tight, that is, they cannot be improved. Our approximation results are summarized in Figure 1.

We note that our focus in this paper is on the class of cycles, with some results extending to *unicyclic graphs*. Clearly, this class of graphs is narrow. Nevertheless, it naturally arises from the original motivation proposed by Bouveret et al. (2017) (imagine a quadrangular building with a hall going all around), and for several reasons deserves a careful study.

First, cycles form the simplest class of graphs where allocations guaranteeing each agent a maximin share value (*mms-allocations*, for short) may not exist and where interesting mathematical questions and non-trivial results emerge. Second, it seems that examples of agents, defined via their utility functions on goods on a cycle, for which mms-allocations do not exist are rare and hard to find. This is a surprising phenomenon. Third, deciding whether an mms-allocation for cycles exists is a problem whose complexity seems to be hard to determine. We show in the paper that the problem is in NP (this, in itself, is by no means trivial), but we cannot prove its completeness nor can we show a polynomial-time algorithm that would solve it. In fact, we expect that finding a reduction to show hardness must be non-trivial and, if such reduction existed, it would have to

2. A graph is *unicyclic* if it has exactly one cycle.

3. The complexity classes mentioned here are well known (Garey & Johnson, 1979; Papadimitriou, 1993); we recall their definitions in Section 4, where the results reported here are proved.

		fraction of maximin share value guaranteed for goods on a cycle	
number of agents	number of agent types	lower bound	upper bound
arbitrary	arbitrary	0.618	3/4
arbitrary	≤ 5	5/8	3/4
arbitrary	≤ 4	2/3	3/4
arbitrary	3	3/4	3/4
arbitrary	2	3/4	3/4
3	3	5/6	5/6
2	2	1	1

Figure 1: Numbers in the last two columns are the bounds we established in this paper for the largest fraction of maximin share value that is guaranteed for all instances satisfying conditions specified in the first two columns, when goods are vertices of a cycle.

have some non-standard features. The reason is that for most well-known problems that are NP-complete, it is rather easy to give examples of broad classes of both YES and NO instances. A reduction from such an NP-complete problem would in particular give rise to a broad class of NO instances to the maximin share allocation problem on a cycle. But, as we noted, so far we have not found rich classes of such NO instances. This may suggest that the problem has a polynomial-time solution. If so, this would be a significant positive result opening a possibility for further expansions of the class of graphs admitting a polynomial solution to the maximin-share allocation problem. It is then clear that understanding that problem for the case of cycles is a fundamental first step to more general results.

Our focus on a specific class of graphs is in line with other recent work on fair allocations of indivisible goods (sometimes chores) on graphs. Suksompong (2019) proposes specific concepts of *approximate* allocations of "connected" bundles of goods on a path for several fairness criteria (including proportional and envy-free ones). He then shows bounds on the parameter controlling the "closeness" of approximation that guarantee the existence of an approximate allocation. Bilò et al. (2019) propose a concept of an allocation that is envy-free *up to k goods*, that is, is envy-free if agents are willing to disregard up to k goods in bundles assigned to other agents when comparing bundles. In their paper, Bilò et al. study the existence of allocations on a path that are envy-free up to one and two goods, and obtain several highly non-trivial positive results. Igarashi and Peters (2019) study the problem of allocations of connected bundles that are Pareto-optimal. In particular, they study the complexity of deciding the existence of such allocations. Their work is primarily concerned with specific classes of graphs such as trees, paths, and stars. Bouveret et al. (2018) study a related problem of fair allocation of indivisible *chores* (items that generate disutility). They assume chores are placed in the nodes of a graph and investigate the existence of fair allocations of connected bundles of chores under some common fairness criteria. They show that deciding the existence is hard even for some very simple underlying graphs such as paths or stars.

We conclude this section by outlining the organization of the remainder of the paper. In Section 2 we introduce key concepts and the terminology used throughout the paper; in Section 3, we present and prove some basic results on mms-allocations of goods forming arbitrary connected graphs. We then turn to cycles. We show three utility functions on nine goods on a cycle, where mms-allocations (to three agents with these utility functions) do not exist. We also show that such

allocations (also to three agents) can always be found if the number of goods is eight. In Section 4, we study the complexity of the problem to decide the existence of an mms-allocation of goods on a connected graph. One of our main results in that section shows that for arbitrary connected graphs, the problem is in the class Δ_2^P, improving on the previously established upper complexity bound of Σ_2^P. Another result shows a polynomial-time algorithm to compute maximin share values in the case when goods are placed on a unicyclic graph, which immediately implies that the problem of the existence of an mms-allocation of goods on a unicyclic graph (in particular, on a cycle) is in NP. The last key result of Section 4 presents a dynamic programming polynomial-time algorithm for finding an mms-allocation (if one exists) in the case of goods on a cycle and agents of a constant number of types. In Section 5, we study the problem of allocations that assign each agent a bundle valued at a fixed fraction of her maximin share value. In particular, we aim at proving positive results showing possibly tight bounds on the fraction that can be guaranteed. We show that in general, when allocating goods on a cycle, each agent can be guaranteed at least $\frac{\sqrt{5}-1}{2} \approx 0.61803$ of her maximin share value. In Section 5 we impose no limits on the number of agents nor on the number of types of the agents. We discuss several cases that restrict these numbers in Section 6. In particular, we obtain stronger results for the case of three agents, and for the case of agents of three types. Our proof of this last result is highly technical and we present it in the appendix.

This paper is an extension of an earlier conference paper (Lonc & Truszczynski, 2018). The present version provides proofs of all results presented in the conference paper. In addition, it contains new results. They assert the existence of allocations guaranteeing each agent a fraction of at least 0.61803 of her maximin share value, when no restriction on the number of agent types is imposed (Theorem 5.4 and Corollary 5.5), and a fraction of at least $3/4$ of her maximin share value, when agents are of exactly three types (Theorem 6.4). Finally, the paper now also includes a result that allows us to extract polynomial time algorithms from proofs of the results in Section 6 (Corollary 4.12).

2. Definitions

We start by formally introducing the concepts we study in the paper. A *utility* function on a set V of *goods* (items) is a function assigning *non-negative* reals (utilities) to goods in V. We extend utility functions to subsets of V by assuming additivity. That is, for a utility function u defined on V and for every $X \subseteq V$, we define $u(X) = \sum_{x \in X} u(x)$.

Following Bouveret et al. (2017), we consider the case when goods form nodes of a certain *connected* graph $G = (V, E)$. We adopt the assumption of the connectedness of the graph for the entire paper and do not mention it explicitly again. In particular, whenever we use the term *graph*, we mean a *connected graph*. We write $V(G)$ and $E(G)$ for the sets of nodes and edges of G, respectively, and refer to G as the graph of goods. Given a graph G of goods, we often refer to utility functions on (the set) $V(G)$ as utility functions on (the graph) G.

Let G be a graph and $V = V(G)$ a set of goods. A *G-bundle* is a subset of V that induces in G a connected subgraph. A *(G, n)-split* is a sequence $P_1, \ldots, P_n$ of pairwise disjoint G-bundles such that

$$\bigcup_{i=1}^{n} P_i = V.$$

When G or n are clear from the context, we drop them from the notation and speak about bundles and splits (occasionally, G-splits and n-splits). We use the term *split* rather than partition for two

reasons. First, elements of partitions are required to be non-empty, while we allow bundles to be empty. Second and more important, partitions are commonly defined as sets not sequences, that is, the order of elements in a partition does not matter. Here the order matters, as we later use it to define an allocation of bundles in a split to agents. The first bundle in a split goes to the first agent, the second one to the second agent, and so on (for some adopted enumeration of agents).

Let G be a graph of goods, u a utility function on G (in most cases, attached to some agent i), q a non-negative real, and n a positive integer. We call a split *q-strong for a utility function u* (or for an agent with that utility function) if every bundle in the split has value at least q under u. The *maximin share* for G, u and n, written $mms^{(n)}(G,u)$, is defined by setting

$$mms^{(n)}(G,u) = \max\{q : \text{there is an } n\text{-split of } G \text{ that is } q\text{-strong for } u\}.$$

An equivalent definition is given by

$$mms^{(n)}(G,u) = \max_{P_1,\ldots,P_n} \min_{i=1,\ldots,n} u(P_i),$$

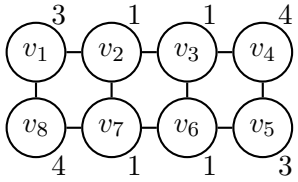
where the maximum is taken over all n-splits $P_1,\ldots,P_n$ of G. A split for which the maximum is attained is a *maximin share split* or an *mms-split* for n and u or for n and an agent with a utility function u. We often leave n and G implicit when they are clearly determined by the context. By the definition, for every bundle P in an mms-split for n and u, $u(P) \geq mms^{(n)}(G,u)$. Similarly as above, when G and n are clear from the context, we leave them out and write $mms(u)$ for $mms^{(n)}(G,u)$. When considering an agent i with a utility function u_i we write $mms(i)$ for $mms(u_i)$.

Let G be a graph of goods. A *(G,n)-allocation* is an assignment of pairwise disjoint G-bundles of goods to n agents $1,2,\ldots,n$ so that all goods are assigned. Clearly, (G,n)-allocations can be represented by (G,n)-splits, where we understand that the ith bundle in the split is the bundle assigned to agent i. Let u_i be the utility function of agent i, $1 \leq i \leq n$. A (G,n)-allocation $P_1,\ldots,P_n$ is a *maximin share* allocation, or an *mms-allocation*, if for every $i \in \{1,\ldots,n\}$ we have $u_i(P_i) \geq mms^{(n)}(G,u_i)$.

To illustrate the concepts introduced above, let us consider the graph in Figure 2. We will call this graph G. The graph G defines the set of goods $\{v_1,\ldots,v_8\}$ and their adjacency relation. The table in the figure shows three utility functions u_1, u_2 and u_3 of three different agents on the set of goods. The values of the utility function u_1 are also shown by the corresponding goods in the graph (it will facilitate our discussion below).

In our example, the set $\{v_1,v_2,v_6,v_7\}$ of goods is a G-bundle (or, simply a bundle, as G is clear). Indeed, the subgraph of G induced by $\{v_1,v_2,v_6,v_7\}$ is connected. On the other hand, the set $\{v_1,v_2,v_5,v_6\}$ of goods is not a bundle as the corresponding induced graph is not connected. Further, the sequence $\{v_1,v_7,v_8\},\{v_2,v_3\},\{v_4,v_5,v_6\}$ is a $(G,3)$-split (or simply a 3-split). Indeed, all sets in the split are bundles. It is also easy to see that the sequence $\{v_1,v_7,v_8\},\{v_2,v_4\},\{v_3,v_5,v_6\}$ is not a split as the set $\{v_2,v_4\}$ is not a bundle.

Let us now consider the utility function u_1. The total utility of all goods under u_1 is 18. Further, it is easy to see that 3-splits into bundles of value 6 (under u_1) do not exist. To this end, we note that v_1 and v_8 cannot be in the same bundle valued at 6 as their total value is 7. Thus, the bundle containing v_8 must be either $\{v_6,v_7,v_8\}$ or $\{v_2,v_7,v_8\}$. In each case the value of the bundle containing v_1 is not 6. Thus, $mms^{(3)}(G,u_1) < 6$. On the other hand, the sequence



	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
u_1	3	1	1	4	3	1	1	4
u_2	2	2	0	3	1	3	1	3
u_3	1	3	2	3	0	3	2	3

Figure 2: A graph of goods and three utility functions.

$\{v_1, v_2, v_3\}, \{v_4, v_5\}, \{v_6, v_7, v_8\}$ is a 3-split with its bundles having values 5, 7 and 6, respectively. Thus, $mms^{(3)}(G, u_1) = 5$.

Reasoning in a similar way one can show that maximin share values for the two other utility functions are also 5, that is, $mms^{(3)}(G, u_2) = mms^{(3)}(G, u_3) = 5$. Let us now observe that the sequence $\{v_1, v_7, v_8\}, \{v_4, v_5, v_6\}, \{v_2, v_3\}$ is a 3-split (indeed, its components are bundles). Moreover, $u_1(\{v_1, v_7, v_8\}) = 8$, $u_2(\{v_4, v_5, v_6\}) = 7$ and $u_3(\{v_2, v_3\}) = 5$. Thus, this 3-split defines an mms-allocation of goods on G to three agents with the utility functions u_1, u_2 and u_3. Later, we will also see examples when mms-allocations of goods on graphs (specifically, cycles) do not exist.

3. Basic General Observations

We are now ready to present a few basic results on the problem of the existence of mms-allocations. Two agents are of the *same type* if they have the same utility functions. The maximin share allocation problem is easy to analyze when all but one agent are of the same type.

Proposition 3.1. *Let $G = (V, E)$ be a graph of goods. If we have n agents and at most one of them is of a different type than the others, then an mms-allocation exists.*

Proof. The result is obvious for $n = 1$. Thus, we will assume $n \geq 2$. Let u be the utility function of agents $1, \ldots, n-1$ and u' a utility function of n. Let Π be an mms-split for u, and P_{max} a bundle of Π most valuable under u'. Then,

$$u'(P_{max}) \geq \frac{\sum_{v \in V} u'(v)}{n} \geq mms(u').$$

The first inequality follows from the fact that

$$n \cdot u'(P_{max}) \geq \sum_{P \in \Pi} u'(P) = \sum_{v \in V} u'(v).$$

The second inequality follows from the fact that in an mms-split for u', say Π',

$$\sum_{v \in V} u'(v) = \sum_{P \in \Pi'} u'(P) \geq n \cdot mms(u').$$

Assign P_{max} to agent n. Next, allocate the remaining parts of Π to agents $1, \ldots, n-1$ in an arbitrary way. Since for each bundle Q in Π, $u(Q) \geq mms(u)$, the resulting allocation is an mms-allocation. □

In particular, this result applies to the case of two agents.

Corollary 3.2. *When there are two agents, mms-allocations of goods on a graph always exist.*

Let $G = (V, E)$ be a graph of goods. An agent with a utility function u on G (a utility function u on G) is n-*proportional* if

$$mms^{(n)}(G, u) = \frac{\sum_{v \in V} u(v)}{n}.$$

We often omit n from the notation if it is clear from the context. A set $\{1, \ldots, n\}$ of agents (a set $\{u_1, \ldots, u_n\}$ of utility functions) is *proportional* if every agent i (utility function u_i) is n-proportional.

An agent (a utility function) is n-*regular* if it is n-proportional and its maximin share (with respect to n) is 1. A collection of n agents (utility functions) is *regular* if every agent in the collection is n-regular. It is clear that for every agent in an n-element regular collection of agents, the total value of all goods for that agent is n.

Let c be a positive real. A bundle $P \subseteq V$ is c-*approximate* for an agent i if $u_i(P) \geq c \cdot mms(i)$. An allocation $P_1, \ldots, P_n$ is c-*approximate* if for every $i \in \{1, \ldots, n\}$, P_i is c-approximate for i. Clearly, a 1-approximate allocation is an mms-allocation. The next result shows that when studying the existence of c-approximate allocations (and so, in particular, mms-allocations) one can restrict considerations to the case when all agents are regular. We use it to prove Theorem 3.4 later on in this section but its full power becomes clear in Section 5.

Proposition 3.3. *Let G be a graph of goods and c a positive real such that for every regular collection of n agents (for every regular collection of n agents of at most t-types) a c-approximate allocation exists. Then a c-approximate allocation exists for every collection of arbitrary n agents (for every collection of arbitrary n agents of at most t types).*

Proof. Let $N = \{1, \ldots, n\}$ be the set of agents and $\{u_1, \ldots, u_n\}$ the set of their utilities. If for every $i \in N$, $mms^{(n)}(u_i) = 0$, then a c-approximate allocation exists. In fact, any split of G into n bundles is a c-approximate allocation.

Thus, let us assume that for at least one agent, say k, $mms^{(n)}(u_k) > 0$. For each agent $i \in N$ such that $mms^{(n)}(u_i) = 0$, we change her utility function to u_k. We denote the new set of utilities by $v_1'', \ldots, v_n''$. It is clear that if $P_1, \ldots, P_n$ is a c-approximate allocation with respect to the utility functions v_i'', then it is also a c-approximate allocation with respect to the original utilities u_i. It is then enough to prove that a c-approximate allocation with respect to the functions v_i'' exists.

For every agent $i \in N$, we denote by Π_i any mms-split into n bundles with respect to the utility function v_i''. For each bundle of Π_i valued more than $mms^{(n)}(v_i'')$ we decrease the value of some elements in that bundle to bring its value down to $mms^{(n)}(v_i'')$. In this way, we produce a new utility function, say v_i'. Since for every item x, $v_i'(x) \leq v_i''(x)$, $mms^{(n)}(v_i') \leq mms^{(n)}(v_i'')$. On the other hand, by the construction, for every bundle $P \in \Pi_i$, $v_i'(P) = mms^{(n)}(v_i'')$. Thus, each utility function v_i' is n-proportional and $mms^{(n)}(v_i') = mms^{(n)}(v_i'') > 0$.

Next, for each agent $i \in N$ we set $k_i = n/(\sum_{y \in V(G)} v_i'(y))$ and, for every $x \in V(G)$, we define

$$v_i(x) = k_i v_i'(x),$$

Clearly, each utility function v_i, $i \in N$, is n-regular. Moreover, it follows directly from the construction that if the original utility functions are of at most t types, then the resulting utility functions are of at most t types, too.

Let $\Pi = P_1, \ldots, P_n$ be a c-approximate allocation for $v_1, \ldots, v_n$ (its existence is guaranteed by the assumption). Therefore, for every agent $i \in N$, $v_i(P_i) \geq c \cdot mms^{(n)}(v_i)$. Since $mms^{(n)}(v_i) = k_i mms^{(n)}(v'_i)$ and $v_i(P_i) = k_i v'_i(P_i)$, it follows that $v'_i(P_i) \geq c \cdot mms^{(n)}(v'_i)$. Next, we recall that for every $x \in V(G)$, $v''_i(x) \geq v'_i(x)$. Consequently, for every bundle P, $v''_i(P) \geq v'_i(P)$. Since for every $i \in N$, $mms^{(n)}(v'_i) = mms^{(n)}(v''_i)$, it follows that

$$v''_i(P_i) \geq v'_i(P_i) \geq c \cdot mms^{(n)}(v'_i) = c \cdot mms^{(n)}(v''_i).$$

Thus, Π is a c-approximate allocation of goods to agents in N with respect to the utility functions v''_i and the result follows. □

Corollary 3.2 states that for two agents and for any connected graph of goods an mms-allocation exists. On the other hand, Bouveret et al. (2017) gave an example of nonexistence of an mms-allocation for a cycle and four agents. In fact, as we show in Figure 3, even for three agents it may be that mms-allocations of goods on a cycle do not exist. In the figure, $v_1, \ldots, v_9$ denote consecutive nodes of a cycle and the numbers in each row represent the utility functions. Observe that the maximin shares for the agents 1 and 2 are 5. For the agent 3, it is 6. Moreover, no two consecutive nodes have the total value satisfying any of the agents. Therefore, if an mms-allocation existed, it would have to be a split into three bundles of three consecutive nodes each. It is simple to check that none of the three possible partitions of this type is an mms-allocation.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
agent 1	0	3	1	3	1	3	0	2	2
agent 2	2	2	0	3	1	3	1	3	0
agent 3	1	3	2	3	0	3	2	3	1

Figure 3: An example of nonexistence of an mms-allocation for a cycle with 9 nodes and 3 agents.

On the other hand, for three agents and at most 8 goods on a cycle mms-allocations always exist.

Theorem 3.4. *For three agents and at most* 8 *goods on a cycle, mms-allocations always exist.*

Proof. Let C be a cycle with at most 8 nodes and let $N = \{1, 2, 3\}$ be the set of agents. By Proposition 3.3, we assume without loss of generality that the agents are 3-regular. In particular, they are proportional and $mms(i) = 1$, for every $i \in N$.

Let us consider mms-splits for agents 1, 2 and 3, each into three bundles. By regularity, all bundles in those splits are non-empty. Thus, each split determines three edges of the cycle that connect adjacent bundles of the split. Since C has 8 edges and there are three splits, there is an edge in C that connects adjacent bundles in two different splits, say for agents 1 and 2. It follows that for some bundles A and B of the mms-splits for agents 1 and 2, respectively, $A \subseteq B$ or $B \subseteq A$.

Let us assume without loss of generality that $A \subseteq B$. Clearly, the elements of the set $B - A$ can be added to one of the two other parts of the mms-split for the agent 2 to form a 2-split of the path $C - A$, say B', B'', with both parts of value at least 1 for the agent 2.

We now construct an mms-allocation as follows. If the set A has value at least 1 for the agent 3, then she receives it. Clearly, the value of $C - A$ for the agent 1 is equal to 2. Thus, one of the parts

B', B'' has value at least 1 to agent 1. We allocate this part to agent 1. Finally, the agent 2 receives the part that remains.

If A does not satisfy the agent 3, then the agent 1 gets this part. The value of $C - A$ for the agent 3, is larger than 2. Thus, we allocate the parts B', B'' to the agents 3 and 2 in the same way as to the agents 1 and 2 in the previous case. □

We conclude this section with a comment on the notation. Formally, whenever we talk about allocations of goods on a subgraph of a graph we consider, we should use restrictions of the utility functions u_i to the set of nodes of the subgraph. Continuing to refer to u_i's is simpler and does not introduce ambiguity. Therefore, we adopt this convention in the paper.

4. Complexity and Algorithms

This section presents three groups of results. First, we study the complexity of the problem of deciding the existence of an mms-allocation both in the case of goods on arbitrary graphs and on cycles (and more generally, on unicyclic graphs). Second, we identify some special cases of the problem when goods are located on a cycle and mms-allocations can be found in polynomial time. Lastly, we present an algorithm for a technical problem of regularizing utility functions that allows us to extract polynomial-time algorithms for finding c-approximate allocations from proofs that show their existence.

4.1 Complexity

We assume familiarity with basic concepts of computational complexity and, in particular, with the classes NP, Δ_2^P and Σ_2^P. We recall that a decision problem is in the class NP if it can be solved by a polynomial-time non-deterministic algorithm. Further, a decision problem is in the class Δ_2^P (Σ_2^P, respectively) if it can be solved by a polynomial-time (polynomial-time non-deterministic, respectively) algorithm making calls to oracles for deciding some problems in NP or coNP, with each call to an oracle assumed to take constant time. A decision problem is *hard* for a complexity class, if every problem in that complexity class can be reduced to that decision problem in polynomial time. A problem is *complete* for a complexity class if it belongs to that class and is hard for that class. We refer to the books by Garey and Johnson (1979) and Papadimitriou (1993) for details.

We start with general comments on how instances of fair division of indivisible goods on graphs are given. First, we assume any standard representation of graphs. Because our primary objective is to understand when problems we consider admit efficient algorithms (and not necessarily to find the most efficient one), the details of how graphs are represented are not critical. Moreover, many of our results concern particular graphs such as cycles, when explicit representations of nodes and edges are not even needed (such graphs can be represented implicitly by specifying how many nodes they have, say n, with the assumption that the nodes are represented by integers $1, 2, \ldots, n$, and that edges connect consecutive nodes, with "wrap around"). Finally, we assume that all utilities and other parameters that may be part of the input are rational, the least restrictive assumption for studies of algorithms and complexity.

We now formally define several problems related to maximin share allocations of indivisible goods on graphs and derive bounds on their complexity.

MMS-VALUES-G: Given a graph G on m goods, a utility function u represented by a sequence $u = (u_1, \ldots, u_m)$ of non-negative rational numbers, an integer $n > 1$ and a rational number $k \geq 0$, decide whether $mms^{(n)}(G, u) \geq k$.

The problem was studied in the original case (with G being a complete graph) by Bouveret and Lemaître (2016). Their complexity result and its proof extend to the general case of an arbitrary graph. The hardness argument uses a reduction from the well-known NP-complete problem PARTITION (Garey & Johnson, 1979):

PARTITION: Given a set U of m non-negative integers, decide whether there is a set $X \subseteq U$ such that $\sum_{u \in X} u = \sum_{u \in U \setminus X} u$.

Proposition 4.1. *The problem* MMS-VALUES-G *is NP-complete.*

Proof. The problem is clearly in NP. Indeed, to solve the problem, we guess an n-split and verify that each of its parts induces in G a connected subgraph that has value at least k (under u). The problem is NP-hard even in the case when G is a complete graph, $n = 2$ and the utility function is integer valued. We can show this by a reduction from PARTITION. Indeed, an instance of PARTITION consisting of a set of non-negative integers $\{u_1, \ldots, u_m\}$ yields an instance of MMS-VALUES-G with the complete graph of goods, $n = 2$, and $k = \lceil (\sum_{i=1}^{m} u_i)/2 \rceil$. It is easy to observe that an instance of PARTITION is a YES instance if and only if the corresponding instance of MMS-VALUES-G is a YES instance. Thus, NP-hardness follows. □

Another related problem concerns the existence of an allocation meeting given bounds on the values of its individual bundles.

ALLOC-G: Given a graph G on m goods, n utility functions $u_1, \ldots, u_n$ (each u_i is a sequence of length m consisting of non-negative rational numbers) and n rational numbers $q_1, \ldots, q_n$, decide whether an allocation $P_1, \ldots, P_n$ for G exists such that for every $i \in \{1, \ldots, n\}$, $u_i(P_i) \geq q_i$.

Proposition 4.2. *The problem* ALLOC-G *is NP-complete.*

Proof. The membership in NP is evident. The problem is NP-hard even if G is a complete graph, $n = 2$, $q_1 = q_2$, and $u_1 = u_2$. Indeed, it becomes then the MMS-VALUES-G problem with G being a complete graph, $n = 2$, $k = q_1$ $(= q_2)$, and $u = u_1$ $(= u_2)$. Thus, the proof of hardness for Proposition 4.1 can be applied here, too. □

We use these observations to show an upper bound on the complexity of the problem of the existence of an mms-allocation in the graph setting. We formally define the problem as follows.

MMS-ALLOC-G: Given a graph G of goods and n utility functions $u_1, \ldots, u_n$ (each u_i is a sequence of length m of non-negative rational numbers), decide whether an mms-allocation for G and $u_1, \ldots, u_n$ exists.

Theorem 4.3. *The problem* MMS-ALLOC-G *is in the class* Δ_2^P.

Proof. The key step in the proof is the observation that we can rescale each utility function so that all its values are integers. To this end, given a utility function u_i, $1 \leq i \leq n$, we compute the product, say k_i, of all denominators of the rational numbers used by u_i as its values. Next, we multiply all values of u_i by k_i to produce a new utility function u_i'. By the construction, all values of u_i' are indeed integers.

Let us denote by I the original input instance of the problem and by I' the instance obtained by rescaling the utility functions in the way described above. We note that the products k_i and the rescaled utility functions u'_i can be computed in time $O(nm^2M^2)$, where M is the number of bits in the binary representation of the largest integer appearing as the numerator or the denominator in rational numbers specifying the original utility functions u_i. Since the size, say S, of the input instance satisfies $S = \Omega(m + n + M)$, I' can be computed in time polynomial in S.

Finally, we note that I is a YES instance of the problem MMS-ALLOC-G if and only if I' is a YES instance of the problem. In fact, an allocation Π is an mms-allocation for I if and only if Π is an mms-allocation for I'.

Let us now consider an auxiliary problem where, given a graph G on m nodes and a sequence $s = s_1, \ldots, s_m$ of m non-negative integers (that is, a utility function), the goal is to compute $mms^{(n)}(G, s)$. We will design for this problem an algorithm with an NP-oracle. We will then show that our algorithm runs in time polynomial in the size of the representation of s, where we consider each call to an oracle to be a single step taking a constant amount of time.

To this end, we observe that $mms^{(n)}(G, s) \leq (\sum_{j=1}^{m} s_j)/n$. Thus, we can compute $mms^{(n)}(G, s)$ by binary search on the range $[0..\lfloor(\sum_{j=1}^{m} s_j)/n\rfloor]$ of possible values for $mms^{(n)}(G, s)$. Each step starts with a range, say $[p..r]$, that contains $mms^{(n)}(G, s)$. We narrow down this range by making a call to an oracle for the problem MMS-VALUES-G, which we proved to be NP-complete above (Proposition 4.1). The call to the oracle is made on the instance consisting of G, s and $k = \lceil(p + r)/2\rceil$. Depending on the oracle output, it results in a new smaller range — either $[p..k - 1]$ or $[k..r]$. The process stops when the range is reduced to just one element and that element is returned as $mms^{(n)}(G, s)$.

The number of range-narrowing steps is given by $\log_2(\lfloor(\sum_{j=1}^{m} s_j)/n\rfloor + 1)$. Clearly,

$$\log_2(\lfloor(\sum_{j=1}^{m} s_j)/n\rfloor + 1) \leq \log_2(\sum_{j=1}^{m} s_j + 1) \leq \sum_{j=1}^{m} \log_2(s_j + 1).$$

Since the number of bits needed to represent a non-negative integer x is given by $\max(1, \lceil\log_2(x + 1)\rceil)$, it follows that the size, say S', of the representation of a problem instance satisfies $S' \geq \sum_{j=1}^{m} \max(1, \lceil\log_2(s_j + 1)\rceil$. Thus, the number of range-narrowing steps is bounded by S'. Consequently, the oracle algorithm we described runs indeed in time polynomial in S'.

It follows that the problem MMS-ALLOC-G can be solved for an instance I by a procedure consisting of these three steps:

1. Compute the instance I'.

2. Compute the values $q_i = mms^{(n)}(G, u'_i)$, where $u'_1, \ldots, u'_n$ are the rescaled utility functions computed in step (1). This is done using the oracle algorithm described above. It can be applied as all utilities in I' are integer-valued.

3. Decide whether there is an allocation $P_1, \ldots, P_m$ for G such that $u'_i(P_i) \geq q_i$, for every $i \in \{1, \ldots, m\}$, by invoking once an NP-oracle for the problem ALLOC-G (cf. Proposition 4.2).

We recall that I' can be computed in time polynomial in S (the size of the original instance I). Based on that and on our discussion of the auxiliary problem, step (2) of our algorithm also runs in

time polynomial in S, counting each oracle call as taking a unit amount of time. Finally, the last step takes a single call to an oracle. Thus, the entire algorithm runs in time polynomial in S, where we count each oracle call as taking the unit amount of time. Since the oracles used by the algorithm are for NP-complete problems, the problem MMS-ALLOC-G is in Δ_2^P. □

The membership in Δ_2^P, established by Theorem 4.3, holds also in the case when G is assumed implicitly, for instance, when it is a path, a cycle or a complete graph represented by its set of nodes (but not edges). It is because the number of oracle calls is bounded by the size of the representation of the utility functions only. In the case of complete graphs, Theorem 4.3 yields an improvement on the complexity bound Σ_2^P obtained by Bouveret and Lemaître (2016). We do not know if the upper complexity bound of Δ_2^P can be improved in general and, in particular, whether it can be improved for complete graphs. On the other hand, we do know that it can be improved for trees. Bouveret et al. (2017) proved the following two results.

Theorem 4.4 (Bouveret et al. (2017))**.** *There is a polynomial time algorithm that computes $mms^{(n)}(T, u)$ and a corresponding mms-split given a tree of goods T, a non-negative rational utility function u on the nodes (goods) of T, and an integer $n \geq 1$.*

Theorem 4.5 (Bouveret et al. (2017))**.** *For every tree T of goods and every set of n agents with non-negative rational utility functions on the goods in T, an mms-allocation exists. Moreover, there is a polynomial-time algorithm to find it.*

The results we presented suggest a question of the relationship between the complexity of the MMS-ALLOC-G problem and the properties of the underlying graph, as it becomes more complex than trees. The first step towards understanding how the complexity grows is to analyze the case of cycles and unicyclic graphs (that is, graphs with exactly one cycle).

First, we show that as in the case of trees (cf. (Bouveret et al., 2017)), the maximin share values and mms-splits for the case when goods form a cycle (or a unicyclic graph) can be computed in polynomial time.

Theorem 4.6. *There is a polynomial time algorithm for computing $mms^{(n)}(U, u)$ and a corresponding mms-split, where U is a unicyclic graph and u is a rational-valued utility function.*

Proof. Using the rescaling technique discussed in the proof of Theorem 4.3, we can reduce in polynomial time the general problem to the problem when all utilities are integers. To this end, we compute the product, say k, of all denominators of the rational numbers that are values of goods under u (if all values are integers, we assume the denominator is 1; this applies, in particular, to the case when all utilities are 0). We then define a new utility function by multiplying the values of u by k. Once the maximin share value for the rescaled utility function is computed, it is then used as the numerator of the maximin share for u and k is used as the denominator. Thus, to complete the proof it suffices to describe a method of computing $mms^{(n)}(U, u)$ and an mms-split under the assumption that all values of u are integers.

Let C be the unique cycle of U. Every U-split has a bundle that contains C or is a split of the graph $U - e$ for some edge $e \in C$. Thus,

$$mms^{(n)}(U, u) = \max(\max_{e \in C} mms^{(n)}(U - e, u), mms_C),$$

where mms_C stands for

$$\max_{P_1,\ldots,P_n} \min_{i=1,\ldots,n} u(P_i),$$

with the maximum taken over all splits $P_1, \ldots, P_n$ that have a bundle containing C. To compute mms_C, we proceed as follows. We construct a tree U_C by contracting C in U to a single "supernode," say c. Thus, the nodes of U_C are all nodes of $U - C$ and the supernode c. We define a utility function u' on the nodes of U_C by setting

$$u'(x) = \begin{cases} u(x) & \text{if } x \text{ is a node of } U - C \text{ (not the supernode)} \\ \sum_{y \in C} u(y) & \text{if } x = c \text{ (is a super node).} \end{cases}$$

Clearly, $mms_C = mms^{(n)}(U_C, u')$. We note that U_C and u' can be computed in polynomial time. Moreover, U_C and all graphs $U - e$, $e \in C$ are trees. We now apply the algorithm by Bouveret et al. (2017) (cf. Theorem 4.4) to compute mms_C and $mms^{(n)}(U - e, u)$, for all edges $e \in C$, as well as the corresponding mms-splits. This takes polynomial time. We then select the largest value among them and return it together with its mms-split (in the case, the largest value is mms_C, in the bundle containing the supernode, we replace it with the nodes of C). □

It is now a matter of routine to show that the MMS-ALLOC-G problem for cycles and, more generally, for unicyclic graphs is in the class NP.

Corollary 4.7. *The problem* MMS-ALLOC-G *for unicyclic graphs is in NP.*

Proof. Let U be a unicyclic graph and $u_1, \ldots, u_n$ rational-valued utility functions defined on the nodes of U. The following non-deterministic polynomial-time algorithm decides the problem MMS-ALLOC-G: (1) guess an allocation $P_1, \ldots, P_n$, (2) compute the values $mms^{(n)}(U, u_i)$, for $i \in \{1, \ldots, n\}$, and (3) verify that $u_i(P_i) \geq mms^{(n)}(U, u_i)$, for $i \in \{1, \ldots, n\}$. By Theorem 4.6, the step (2) can be accomplished in polynomial time. Thus, the entire algorithm runs in polynomial time. □

We do not have an argument for NP-hardness of the problem MMS-ALLOC-G. Thus, we do not know whether the complexity bound obtained in Corollary 4.7 is tight in general. We do know, however, that in some special cases it can be improved. We will now discuss these results.

4.2 Cases when Polynomial-time Algorithms Exist

Our first group of results is concerned with the case when the number m of goods is small with respect to the number n of agents, specifically, when $m \leq 2n$. Our results rely on the following simple observation.

Proposition 4.8. *Let C be a cycle, u a utility function on C, and $n \geq 2$ an integer. For every node x of C, $mms^{(n-1)}(C - x, u) \geq mms^{(n)}(C, u)$.*

Proof. Let Π be an mms-split of C for u and n. Let P be the bundle in Π containing x, and let P' and P'' be the two bundles in Π inducing in C segments neighboring the one induced by P, respectively preceding it and succeeding it when traversing C clockwise ($P' = P''$ if $n = 2$). We move all goods in P that precede x to P' and those that succeed x to P''. In this way each bundle still spans a connected segment in C. Next, we remove the "new" P (at this point, P consists of x only). The result is a split Π' of $C - x$ into $n - 1$ bundles, in which every bundle has value at least $mms^{(n)}(C, u)$. Thus, $mms^{(n-1)}(C - x, u) \geq mms^{(n)}(C, u)$. □

We are now ready to show that, when there are m goods on a cycle and n agents, and $m < 2n$, then mms-allocations are guaranteed to exist. Moreover, we also show that this result is tight — having exactly $m = 2n$ goods allows for situations when mms-allocations do not exist.

Theorem 4.9. *If $m < 2n$, then an mms-allocation of m goods on a cycle C to n agents exists and it can be computed in polynomial time.*

Proof. Let us consider agents $1, \ldots, n$ with utility functions $u_1, \ldots, u_n$. Let Π be an mms-split for the agent n. Clearly, for every bundle $P \in \Pi$, $u_n(P) \geq mms^{(n)}(C, u_n)$. Since $m < 2n$, at least one bundle in Π consists of only one element, say x, or some bundle in Π is empty. In the latter case, obviously, $mms^{(n)}(C, u_n) = 0$ and we define x to be any vertex of C. In both cases it follows that $u_n(x) \geq mms^{(n)}(C, u_n)$. Since $C - x$ is a path (and so, a tree), Theorem 4.5 implies that there is an allocation giving each agent j, $1 \leq j \leq n-1$, a bundle valued at least $mms^{(n-1)}(C-x, u_j)$. By Proposition 4.8, each such bundle is valued at least $mms^{(n)}(C, u_j)$. Thus, this allocation extended by the bundle $\{x\}$, allocated to the agent n, forms an mms-allocation of the goods on C to n agents.

For an algorithm, we (1) compute the maximin share $mms^{(n)}(C, u_n)$; (2) select an item x such that $u_n(x) \geq mms^{(n)}(C, u_n)$ (as argued above, such an x exists); (3) construct an mms-allocation of goods on the path $C - x$ to agents $1, \ldots, n-1$ (possible as $C - x$ is a tree); and (4) extend the allocation constructed in (3) by giving $\{x\}$ to n.

By our argument above, the allocation constructed in step (4) is an mms-allocation. Steps (1) and (3) can be accomplished in polynomial time by Theorems 4.6 and 4.5, respectively. The same obviously holds for steps (2) and (4). Thus, the algorithm we described runs in polynomial time. □

This result is tight. If $m = 2n$ and $n > 3$ then an mms-allocation of m goods on a cycle to n agents may not exist as shown by the example in Figure 4 (similarly to Figure 3, $v_1, v_2, \ldots, v_{2n}$ are consecutive vertices of the cycle).

	v_1	v_2	v_3	v_4	v_5	v_6	$\ldots$	v_{2n-1}	v_{2n}
agents $1, \ldots, n-2$	n	1	$n-1$	2	$n-2$	3	$\ldots$	1	n
agents $n-1, n$	n	n	1	$n-1$	2	$n-2$	$\ldots$	$n-1$	1

Figure 4: An example showing that the result in Theorem 4.9 is tight.

Indeed, all maximin share values are $n + 1$ Thus, every bundle in any mms-allocation would have to contain at least two elements. Since $m = 2n$, every bundle would have to consist of exactly two elements. There are only two splits of a cycle with $m = 2n$ nodes into n bundles of two consecutive goods. None of them is an mms-allocation. The assumption that $n > 3$ is essential. We have seen earlier that if $n = 2$ then mms-allocations exist for every number m of goods. Thus, they exist for $m = 2n = 4$. If $n = 3$, mms-allocations of m goods on a cycle exist for every $m \leq 8$ (cf. Theorem 3.4). In particular, they exist if $m = 2n = 6$.

As our example shows, if $m = 2n$, there are cases when mms-allocations do not exist. However, whether an mms-allocation exists in the case when $m = 2n$ can be decided in polynomial time and, if so, an mms-allocation can be computed efficiently.

Corollary 4.10. *There is a polynomial time algorithm for deciding the existence of an mms-allocation of $2n$ goods on a cycle to n agents, and computing one, if one exists.*

Proof. Let C be the cycle and $u_1, \ldots, u_n$ the rational-valued utility functions of the agents $1, \ldots, n$. We first compute the values $mms^{(n)}(C, u_i)$, for $i \in \{1, \ldots, n\}$ (cf. Theorem 4.6). If for some item x and agent i, $u_i(x) \geq mms^{(n)}(C, u_i)$, then reasoning as in the proof of Theorem 4.9 one can show that an mms-allocation for C exists, and that it can be found in polynomial time. Otherwise, if there is an mms-allocation of goods on C to n agents, every bundle in this allocation consists of two consecutive goods. There are only two candidates for such allocations and one can check whether any of them is an mms-allocation in polynomial time (as the values $mms^{(n)}(C, u_i)$ are known). □

The next result of this section concerns the case of n agents of t types, where t is constant and is not a part of the input. Before we present our result and prove it, we discuss how inputs and outputs are represented.

As we already noted earlier, instances to an mms-allocation problem on a cycle are specified by a non-negative integer m representing the number of goods, and n m-element sequences of non-negative rational numbers, each sequence representing a utility function. We note that m implicitly defines the goods as $v_1, v_2, \ldots, v_m$, as well as the cycle as having these goods appear on it in this order. Further, we adopt a natural convention that for every $i \in \{1, \ldots, m\}$, the ith element in every utility sequence provides the utility value for the good i. This representation consists of $nm + 1$ rational numbers (one of them, m, an integer).[4]

In the case when agents can be grouped into t types, where t is constant and not part of the input, the case we are now considering, input instances can be represented more concisely by a non-negative integer m, t sequences of m non-negative rational numbers (t utility functions) and t non-negative integers $n_1, \ldots, n_t$, where each represents the number of agents of *type* r (that is, having u_r as their utility function). Thus, an instance consists of $tm = \Theta(m)$ rational numbers represented by pairs of integers, and $t + 1 = \Theta(1)$ integers. We will use M to denote the length of the binary representation of the largest of the integers appearing in this representation, and S to denote the total size of the binary representations of the integers in the instance. In particular, we have that $S = \Omega(m + M)$.

We observe that if $n > m$, then the maximin share for each agent is 0 and, consequently, every allocation of m goods to n agents is an mms-allocation. Since t is constant, computing $n = n_1 + \ldots + n_t$ takes time $O(M)$ and, consequently, also $O(S)$. Thus, this case can be recognized in time $O(S)$ and then handled directly. Namely, we return a sequence of t 0's as the maximin shares for agents of types $1, 2 \ldots, t$ and, if an mms-allocation is required, we allocate all goods to agent 1 and empty bundles of goods to all other agents. We do not generate any explicit representation for this allocation. It is implicitly identified by the all-0's output of the maximin shares. This is to avoid having to output explicitly an n-element sequence of bundles, as n, the length of this sequence, may be exponential in the size of input. It follows that the key case is then the case $n \leq m$.

Theorem 4.11. *For every constant integer $t \geq 1$, there is a polynomial-time algorithm for deciding the existence of an mms-allocation of m goods on a cycle to n agents of t types (and computing one, if one exists).*

Proof. We will use a dynamic programming approach. Let us consider an instance of the problem given by an integer m (the number of goods), t sequences $u_1, \ldots, u_t$, each consisting of m non-negative rational numbers (t utility functions), and t non-negative integers $n_1, \ldots, n_t$ (the numbers

4. We note that we do not include the number of agents, n, in the input; if we need it, we can compute it by counting in the input the sequences representing the utility functions.

of agents of type $1, \ldots, t$, respectively). We start by computing $n = n_1 + n_2 + \ldots + n_t$, which can be accomplished in $O(S)$ time, as t is constant (we recall that we write S for the size of the input instance). The case $n > m$ has been discussed above. Thus, in what follows we assume that $n \leq m$. An important consequence of this assumption is that for every $r = 1, \ldots, t$, $n_r \leq m$.

To decide whether there is an mms-allocation and, if so, to compute it, we compute the values $q_r = mms^{(n)}(C, u_r)$, $r = 1, \ldots, t$. Theorem 4.6 implies that all these values can be computed in time bounded by a polynomial in $|u|$, where $|u|$ represents the largest size of the representation of an input utility function and, consequently, also by a polynomial in S.

We now observe that an allocation Π of goods on C to agents of t types given by $u_1, \ldots, u_t$ is an mms-allocation on C if and only if for some edge e of C, Π is an allocation of goods on the path $C - e$ to these agents such that for every agent of type r, the bundle $P \in \Pi$ allocated to that agent satisfies $u_r(P) \geq q_r$.

Thus, to prove the assertion, it is enough to show that the following problem has a polynomial-time solution: Given a *path* F of m goods $v_1, v_2, \ldots, v_m$, appearing in this order on F, t rational-valued utility functions $u_1, \ldots, u_t$, t non-negative integers $n_1, \ldots, n_t$ such that $n = n_1 + \ldots + n_t \leq m$, and t non-negative integers $q_1, \ldots, q_t$, find an allocation Π of goods on F to $n_1 + \ldots + n_t$ agents such that for every agent of type r, the bundle P assigned to that agent satisfies $u_r(P) \geq q_r$.

To describe our method, we define $\mathcal{H}$ to be the set of all sequences $(h_1, \ldots, h_t)$ such that for every $r = 1, \ldots, t$, $0 \leq h_r \leq n_r$. Clearly, the number of sequences in $\mathcal{H}$ is bounded by $(n_1 + 1) \cdot \ldots \cdot (n_t + 1) \leq (m + 1)^t = O(m^t) = O(S^t)$. If H denotes a sequence $(h_1, \ldots, h_t) \in \mathcal{H}$ and $h_r > 0$, where $1 \leq r \leq t$, then we write H_r^- for the sequence $(h_1, \ldots, h_{r-1}, h_r - 1, h_{r+1}, \ldots, h_t)$ which, we note, also belongs to $\mathcal{H}$.

For a sequence $(h_1, \ldots, h_t) \in \mathcal{H}$, we define $T(h_1, \ldots, h_t)$ to be the smallest j such that there is an allocation of goods $v_1, \ldots, v_j$ to $h_1 + \ldots + h_t$ agents, exactly h_r of them of type r, $1 \leq r \leq t$, so that each agent of type r obtains a bundle worth at least q_r. If such an allocation does not exist, $T(h_1, \ldots, h_t)$ is set to ∞.

Next, for every $j \in \{1, \ldots, m\}$ and every $r \in \{1, \ldots, t\}$, we define $k(j, r)$ to be the minimum $k \geq j$ such that $u_r(\{v_{j+1}, \ldots, v_k\}) \geq q_r$. We set $k(j, r) = \infty$ if such a k does not exist. Clearly, all values $k(j, r)$ can be computed in time $O(tmM) = O(mM) = O(S^2)$.

Let $(h_1, \ldots, h_t) \in \mathcal{H}$. We will show that $T(h_1, \ldots, h_t)$ can be efficiently computed. Clearly, $T(0, \ldots, 0) = 0$ (there are no agents to get any bundles and so, even the empty path suffices). Thus, let us consider $H = (h_1, \ldots, h_t) \in \mathcal{H}$ and let us assume that $h_1 + \ldots + h_t > 0$. Let us define $I = \{r : h_r > 0,\ 1 \leq r \leq t\}$. It is easy to see that

$$T(H) = \min\{k(T(H_r^-), r) \colon r \in I\}. \tag{1}$$

Assuming that all values $k(T(H_r^-), r)$, for $r \in I$, are known, $T(H)$ can be computed in time $O(t) = O(1)$. It follows that with the initial value of $T(0, \ldots, 0) = 0$, considering sequences $H \in \mathcal{H}$ in any order consistent with non-decreasing sums of their elements and using the formula (1) to compute $T(H)$, one can compute $T(n_1, \ldots, n_r)$ in polynomial time, in fact, in time $O(S^{\max(2,t)})$ (we recall that all values $k(j, r)$ can be computed in time $O(S^2)$; also, the number of entries in T is $O(S^t)$ and each entry can be computed in $O(1)$ time).

If $T(n_1, \ldots, n_t) = \infty$, then there is no split Π of F such that for every agent of type r, its bundle $P \in \Pi$ satisfies $u_r(P) \geq q_r$. Otherwise, $T(n_1, \ldots, n_t) \neq \infty$ and a split solving the problem exists. Moreover, it can be computed in the standard way for dynamic programming algorithms. To this

end, each time the formula (1) is applied we have to record an index $r \in I$ that minimizes the expression $k(T(H_r^-), r))$. This information allows us to construct the split in time polynomial in S.

As all algorithmic tasks we presented can be accomplished in time bounded by a polynomial in S, the assertion follows. □

4.3 Regularizing Utility Functions on Unicyclic Graphs in Polynomial Time

We close this section with yet another corollary to Theorem 4.6. It concerns a possibility of regularizing utility functions on unicyclic graphs in polynomial time by converting a method used in the proof of Proposition 3.3 into an algorithm. Let us call a collection $\{u_1, \ldots, u_n\}$ *trivial* if for every i, $mms^{(n)}(u_i) = 0$. We will call all other collections non-trivial.

Corollary 4.12. *There is a polynomial time algorithm that, given a unicyclic graph U of goods and a non-trivial collection of rational-valued utility functions u_i, $1 \leq i \leq n$, on U, produces a rational-valued regular collection of utility functions u_i' on U such that for every c, if a split Π is a c-approximate allocation with respect to u_i''s then it is a c-approximate allocation with respect to u_i's. If the original utility functions are of at most t types, then the resulting utility functions are of at most t types.*

Proof. The algorithm follows the method we used in the proof of Proposition 3.3. It consists of the following key steps.

1. Rescale the utilities u_i as described in the proof of Theorem 4.3 to produce an equivalent collection of integer-valued utilities w_i (by equivalent we mean determining the same splits as mms-allocations).

2. Compute the maximin shares $q_i = mms^{(n)}(U, w_i)$, $1 \leq i \leq n$.

3. Select one utility function w_k such that $q_k > 0$. For each agent i such that $q_i = 0$, replace w_i with w_k; denote the new utility functions v_i, $1 \leq i \leq n$.

4. For a good x and a utility function v_i, use the binary search method, similar to that used in the proof of Theorem 4.3, to find the smallest a such that $mms^{(n)}(U, v_i^{x \to a}) = q_i$; replace v_i with $v_i^{x \to a}$; repeat for all goods as long as it is possible to decrease a value of some utility function on some good.[5]

5. Rescale the computed utility functions so that the values of each of them sum up to n. Call the resulting functions u_i'.

The correctness of the method, both for the main statement and under the restriction to utility functions of at most t types, follows from the argument used to prove Proposition 3.3. For the running time, we note that step (1) runs in polynomial time (we argued this in the proof of Theorem 4.3). Step (2) runs in polynomial time (Theorem 4.6). Step (3) runs in time $O(nm)$. Next, we note that the argument we used in the proof of Theorem 4.3 to estimate the running time of the "range narrowing" binary search and the fact that the maximin shares can be computed in polynomial time (Theorem 4.6) together imply that step (4) runs in polynomial time. Step (5) consists of $O(nm)$ additions and multiplications involving utility values produced in step (4) and the integer n. Thus, it also runs in time polynomial in the size of the original instance. □

5. Here, $v_i^{x \to a}$ stands for the utility function obtained from v_i by setting its value on x to a and keeping all other values as they are in v_i.

5. Approximate Maximin Share Allocations on a Cycle — the General Case

Throughout the remainder of the paper, we study the problem of the existence of c-approximate allocations of goods on cycles. In the present section, we consider the case of arbitrary sets of agents. In particular, we do not limit the number of agents nor the number of their types. We start with a simple observation that $\frac{1}{2}$-approximate allocations always exist and, moreover, can be found in polynomial time.

Proposition 5.1. *There is a polynomial time algorithm which, for any number of agents and any number of goods on a cycle, constructs a $\frac{1}{2}$-approximate allocation.*

Proof. We remove any edge e from the cycle C and get a path, say P. The maximin share for each agent on P is at least half of the original maximin share for C. Indeed, let us consider an arbitrary agent i and her mms-split of C. By removing e, no more than one part of this split may break into two pieces. The value for the agent i of one of these pieces is at least $\frac{1}{2}mms(i)$. We adjoin the other piece, if it is present, to its neighboring part in the mms-split of C. In this way, we obtain a split of P. Clearly, in this split of P every piece is worth to i at least $\frac{1}{2}mms(i)$, so the claim follows.

Thus, a $\frac{1}{2}$-approximate allocation can be found by (1) removing any edge from the cycle, (2) applying to the resulting path the algorithm by Bouveret et al. (2017) that constructs in polynomial time an mms-allocation for trees (cf. Theorem 4.5). □

To find a better guarantee than $\frac{1}{2}$ turns out to be non-trivial. We will show that it can be improved to $(\sqrt{5}-1)/2 \approx 0.61803...$, and that further improvements are possible if we restrict the number of agent types.

Let c be a positive real and let $N = \{1, \ldots, n\}$, where $n \geq 2$, be a set of agents with arbitrary utility functions $u_1, \ldots, u_n$ on a set of goods on a path, say P. Figure 5 shows an algorithm $allocate$ that assigns to some (possibly all) agents in N bundles they value at c or more. The algorithm uses the concept of a c-strong split. We recall that a split is *c-strong* for a utility function u (or for an agent with the utility function u) if every bundle in the split has value at least c under u.

The algorithm *allocate* is the key to our main result in this section (Theorem 5.4). We use it to argue the existence of c-approximate allocations for cycles. Moreover, under the restriction to rational-valued utilities and a rational c, the algorithm runs in polynomial time. Thus, we also use it to argue the existence of polynomial-time algorithms for constructing c-approximate allocations.

Let us first give an intuition on how the algorithm *allocate* will be used in the proof of Theorem 5.4. The objective of the algorithm is to construct (under some assumptions formulated in Theorem 5.3) an allocation that assigns goods on the input path P to n agents so that each agent receives a bundle she values at c or more. A prefix Q of the path P is also a part of the input of the algorithm $allocate$. Its role will be made clear in the proof of Theorem 5.4. Roughly speaking it will correspond to the first bundle that can be assigned to an agent while constructing a c-approximate allocation for a cycle in the proof of that theorem. Obviously, after assigning this bundle, we are left with a path which should be distributed properly among the remaining $n-1$ agents.

A general idea of the algorithm $allocate$ is as follows. To start, the algorithm divides the set of agents into two subsets S and R. The set S consists of agents for which there is an $(n-1)$-split of $P-Q$ that is c-strong and the set R consists of the remaining agents. In each step of the algorithm a shortest prefix of the path worth at least c to some agent is assigned. If there are agents from both sets S and R for whom this prefix is acceptable as a bundle of value at least c, then an agent from the set R gets it.

$allocate(N, P, Q, c)$

% P is a path; we fix its direction so that prefixes (*initial* subpaths) of P are well defined
% $N = \{1, \ldots, n\}$, $n \geq 2$, is a set of agents, each with a utility function on P
% c is a positive real
% Q is a prefix of P valued at least c by at least one agent in N

1 $S := \{i \in N$: there is an $(n-1)$-split of $P - Q$ that is c-strong for $i\}$;
2 $R := N - S$;
3 $j := 1$;
4 **while** $j \leq n$ and P has value at least c for some agent in $S \cup R$ **do**
5 $Q_j :=$ the shortest prefix of P worth at least c for some agent in $S \cup R$
6 **if** Q_j is worth at least c to an agent $i \in R$ **then**
7 assign Q_j to i;
8 $R := R - \{i\}$
9 **else**
10 assign Q_j to an agent $i \in S$ such that Q_j is worth at least c for i;
11 $S := S - \{i\}$;
12 $P := P - Q_j$;
13 $j := j + 1$

Figure 5: Algorithm $allocate$

Let us now discuss the algorithm $allocate$ still informally but in a more detailed way. After defining the sets S and R in the way described above the algorithm sets j to 1 and proceeds to the loop (4–13). Throughout the execution of the loop, P denotes the path consisting of unallocated goods, and $R \cup S$ contains all agents that are as yet unassigned. In each iteration j, the algorithm attempts to allocate a bundle to an "unassigned" agent. At the start of that iteration $j-1$ agents have received bundles selected as prefixes of the paths being considered in earlier iterations. If $j > n$, then the loop terminates and all agents are assigned bundles. If $j \leq n$, some agents are unassigned, $R \cup S$ contains all unassigned agents and P is a path on unallocated goods. If the value of P for each unassigned agent is less than c, no further assignments are possible and the loop terminates. Otherwise P has value at least c for at least one unassigned agent and an assignment can be made. The bundle for the assignment, denoted by Q_j is chosen as a *shortest prefix* of P that has value at least c for some unassigned agent. Selecting Q_j as a prefix ensures that the remaining goods form a path. Selecting for Q_j a shortest prefix that has value c or more for some unassigned agent is essential for a key property of the algorithm, which we will discuss later. Once the bundle Q_j is constructed, it is assigned. By construction, there are unassigned agents that value Q_j at c or more. We select one such agent, say i. We first check if such an agent i can be found in R and if so, assign Q_j to i. Only if no agent in R values Q_j at c or more, we select i from S (this selection is possible as at least one unassigned agent values Q_j at c or more), and assign Q_j to that i.

The following proposition gives the key property of the algorithm $allocate$.

Proposition 5.2. *Let P be a path of goods and $N = \{1, \ldots, n\}$, where $n \geq 2$, a set of agents, each with a utility function on P, and c a positive real. Further, let Q be a prefix of P valued at least c by*

at least one agent in N, and S the set computed by the algorithm $allocate(N, P, Q, c)$ in line (1). When the algorithm $allocate(N, P, Q, c)$ terminates, $S = \emptyset$, that is, all agents included in S in line (1) are assigned bundles they value at c or more.

Proof. We will denote by P_0 the original path P and by S_0 the set S as computed in line (1). Let us assume that when the algorithm $allocate(N, P_0, Q, c)$ terminates, $S \neq \emptyset$. Let $i \in S$. Since after line (1) the algorithm never includes elements in S, it follows that $i \in S_0$.

Let us consider the value, say j_t of the variable j when the loop (4–13) terminates. Clearly, the body of the loop was executed $j_t - 1$ times and $j_t - 1$ agents are assigned bundles $Q_1, \ldots, Q_{j_t-1}$. By the conditions on the input parameters, the body of the loop executes for $j = 1$ and defines a bundle Q_1. This bundle is a prefix of Q (because of how we select prefixes Q_j). Since $i \in S_0$, $P_0 - Q$ has an $(n-1)$-split that is c-strong for i. Since Q_1 is a prefix of Q, $P_0 - Q_1$ has an $(n-1)$-split that is c-strong for i. Let us denote this split by $D_2, \ldots, D_n$. Since i has not been assigned a bundle in iteration 1 (in any iteration, in fact) and D_2 has value at least c for i, at the beginning of iteration 2 we have that $i \in S \cup R$ and the value of the path P for i is at least c. Thus, the body of the loop executes for the second time. It follows that $3 \leq j_t$. Further, since i is not assigned a bundle, $j_t - 1 \leq n - 1$. Thus, $3 \leq j_t \leq n$.

Let us assume that $(Q_1 \cup \ldots \cup Q_{j_t-1}) \cap D_{j_t} = \emptyset$. It follows that in the iteration of the loop when $j = j_t$, $D_{j_t} \subseteq P$. Thus, P has value at least c for i. Consequently, the body of the loop would execute for $j = j_t$, a contradiction. Therefore, we have $(Q_1 \cup \ldots \cup Q_{j_t-1}) \cap D_{j_t} \neq \emptyset$.

Let k be the smallest integer such that $3 \leq k \leq j_t$ and $(Q_1 \cup \ldots \cup Q_{k-1}) \cap D_k \neq \emptyset$. From our observation above it follows that k is well defined. Moreover, $(Q_1 \cup \ldots \cup Q_{k-2}) \cap D_{k-1} = \emptyset$. Let P be the path of unallocated goods when $j = k-1$, that is, $P = P_0 - (Q_1 \cup \ldots \cup Q_{k-2})$. It follows that D_{k-1} is a subpath of P. Let D be the shortest prefix of P containing D_{k-1}. It follows that D is a strictly shorter prefix of P than Q_{k-1} and D has value at least c to i. This is a contradiction with the algorithm selecting Q_{k-1} when $j = k - 1$. □

Extending the notation we used for splits, we call an allocation *q-strong* if it assigns to each agent a bundle worth at least q to that agent. We note that for regular sets of agents, q-strong and q-approximate allocations coincide.

Theorem 5.3. *Let P be a path of goods, $N = \{1, \ldots, n\}$, $n \geq 2$, a* regular *set of agents, and c a real such that $c \leq 1$. Further, let Q be a prefix of P valued at least c by at least one agent in N. Let R be the set computed by the algorithm $allocate(N, P, Q, c)$ in line (2). If no single good in P has value at least c for any agent in N and $|R| \leq \frac{1-c}{c}n + 1$, then the algorithm $allocate(N, P, Q, c)$ finds a c-strong allocation of goods in P to agents in N.*

Proof. Let us assume that when the algorithm terminates, some agents are left without a bundle. Let us denote $s = |S|$, $r = |R|$, where S and R are computed in the lines (1) and (2) of the algorithm $allocate(N, P, Q, c)$ and let k be the number of agents that have no bundle when the algorithm terminates. By Proposition 5.2, these agents are members of R. Moreover, by our assumption, $k \geq 1$.

Let ℓ be an unassigned agent. Clearly, in each iteration that assigns a bundle to an agent from S, the value of that bundle is smaller than c for the agent ℓ (it is so because agents in R are prioritized over agents in S when bundles are assigned). Moreover, in each iteration j when an agent from R is assigned a bundle, we recall this bundle is referred to as Q_j, the value of Q_j for the agent ℓ is smaller than $2c$. Indeed, otherwise the prefix of Q_j formed by removing the last node of Q_j would

have value at least c for ℓ (since, by assumption, that node is worth less than c to ℓ). This contradicts the property that Q_j is a shortest prefix of the path P in the iteration j.

It follows that the total value for ℓ of all bundles constructed and allocated by the algorithm is less than $cs + 2c(r-k)$. Consequently, the value for ℓ of all unallocated goods when the algorithm terminates, say v_ℓ, satisfies $v_\ell > n - cs - 2c(r-k)$ (since N is a regular collection of agents, for every agent in N, the total value of goods on P to that agent is n). On the other hand, by the stopping condition, $v_\ell < c$. Thus, $c > n - cs - 2c(r-k)$. Since $s = n - r$, it follows that $r > \frac{1-c}{c}n + 2k - 1 \geq \frac{1-c}{c}n + 1$, a contradiction.

Thus, when the algorithm terminates, all agents are assigned bundles and each bundle the algorithm allocates to an agent has value at least c for that agent. In other words, the allocation defined by the algorithm is c-strong. □

The results we presented above allow us to prove the main result of this section, which establishes a bound on c guaranteeing the existence of c-approximate allocations of goods on a cycle.

Theorem 5.4. *Let C be a cycle of goods and $N = \{1, \ldots, n\}$, where $n \geq 2$, a set of agents, each with a utility function on C. Let*

$$c = \max_{d=n,n+1,\ldots} \min\left(\frac{n}{d}, \frac{n}{\left\lceil \frac{n^2}{d} \right\rceil + n - 2}\right)$$

Then, there is a c-approximate allocation of goods on C to agents in N.

Proof. By Proposition 3.3, it suffices to show the result under the assumption that N is a regular collection of agents. In particular, c-approximate allocations and c-strong allocations coincide.

If some node x of C has value at least c to some agent j then a c-approximate allocation exists. Indeed, we assign x to the agent j. By Proposition 4.8, the values of the maximin shares for the remaining $n-1$ agents and the path $C - x$ do not drop. Applying the algorithm of Bouveret et al. (2017), we construct an mms-allocation of goods on the path $C - x$ to those $n-1$ agents. This allocation together with an assignment of x to the agent j is a c-approximate allocation of goods on C to agents in N (j receives a bundle worth at least at c and all other agents receive bundles worth at least their maximin share). So, we assume that no single-element bundle is c-approximate for any agent in N.

For every agent $i \in N$ we select any of her mms-splits of C. By regularity, all bundles in these splits are non-empty. Thus, each selected split can be obtained by removing n different edges, say $e^i_1, \ldots, e^i_n$, from C. We arrange all edges e^i_j, $i, j \in \{1, \ldots, n\}$, into a sequence $E = e_0, e_1, \ldots, e_{n^2-1}$. To this end, we start in any place in C and inspect the edges of C moving clockwise. Each time we find an edge e used to obtain mms-splits for, say, k agents, we place k occurrences of e in the sequence.

For an integer $d = n, n+1, \ldots$, we define

$$f(d) = \min\left(\frac{n}{d}, \frac{n}{\left\lceil \frac{n^2}{d} \right\rceil + n - 2}\right).$$

We then define p to be that integer $d \geq n$, for which $f(d)$ achieves its maximum (in case of ties we pick for p the smallest of those values d). Since $f(n) > 1/2$ and, for $d \geq n^2$, $f(d) \leq 1/n$, it

follows that $n \leq p < n^2$. It is also clear that $f(p) = c$ (as defined in the statement of the theorem). Moreover, $c \leq \frac{n}{\lceil n^2/p \rceil + n - 2} \leq 1$.

Let us define $\overline{h} = \left\lceil \frac{n^2}{p} \right\rceil$, $\underline{h} = \left\lfloor \frac{n^2}{p} \right\rfloor$ and $r = n^2 - p \cdot \underline{h}$. Clearly, we have $\underline{h} \geq 1$ and $0 \leq r < p$. We define a split of the cycle C into p parts (some of them possibly empty) by removing p edges e_i, where $i = 0, \overline{h}, 2\overline{h}, \ldots, r\overline{h}, r\overline{h} + \underline{h}, r\overline{h} + 2\underline{h}, \ldots, r\overline{h} + (p - r - 1)\underline{h}$. It is easy to check that between two consecutive removed edges there are $\overline{h} - 1$ or $\underline{h} - 1$ edges of the sequence E.

Let Q be a part of this split with the largest value to agent n (any other agent could be chosen for n, too). This value is at least $\frac{n}{p} \geq f(p) = c$. Let P be the path obtained from C by removing an edge so that Q is a prefix of P. We will show that the call $allocate(N, P, Q, c)$ produces a c-strong allocation for N.

First, we note that Q contains at most $\overline{h} - 1$ edges of the sequence E. This means that there are at most $\overline{h} - 1$ agents such that Q intersects more than one part of their mms-split. For each of the remaining agents their mms-split gives rise to an $(n-1)$-split of the path $P - Q$ that is 1-strong for them. Since $c \leq 1$, these splits are c-strong for the corresponding agents and, consequently, all these agents are in S — the set defined in line (1) of the algorithm $allocate(N, P, Q, c)$. It follows that $|S| \geq n - (\overline{h} - 1)$. Let R be the set defined in line (2) of $allocate(N, P, Q, c)$. Clearly, $|R| = n - |S| \leq \overline{h} - 1$. Hence, by the definition of p,

$$|R| \leq \overline{h} - 1 = \left\lceil \frac{n^2}{p} \right\rceil - 1 \leq \frac{n}{f(p)} - n + 1 = \frac{n}{c} - n + 1 = \frac{1 - c}{c} n + 1.$$

By Theorem 5.3, there is a c-strong allocation for P. This allocation is also a c-strong allocation for C. □

This result yields a corollary that gives a specific value c for which the existence of c-approximate allocations of goods on cycles is guaranteed.

Corollary 5.5. *For any number n of agents there is a $\frac{\sqrt{5}-1}{2}$-approximate (that is, $0.61803..$-approximate) allocation for a cycle.*

Proof. Let

$$\varphi = \frac{\sqrt{5} + 1}{2} \quad \text{and} \quad \psi = \frac{1}{\varphi} = \frac{\sqrt{5} - 1}{2} \approx 0.61803.. .$$

Clearly, the corollary holds for $n = 1$, so we assume that $n \geq 2$. Since $\lfloor \varphi n \rfloor \geq n$, by Theorem 5.4, it suffices to show that

$$\min \left(\frac{n}{\lfloor \varphi n \rfloor}, \frac{n}{\left\lceil \frac{n^2}{\lfloor \varphi n \rfloor} \right\rceil + n - 2} \right) \geq \psi.$$

Clearly, $\frac{n}{\lfloor \varphi n \rfloor} \geq \frac{n}{\varphi n} = \psi$. Thus, it remains to prove that

$$\psi \leq \frac{n}{\left\lceil \frac{n^2}{\lfloor n\varphi \rfloor} \right\rceil + n - 2}.$$

To this end, we observe that $n^2 \leq n^2 + n - 1 = (\varphi n - 1)(\psi n + 1)$, so

$$\psi \cdot \left(\left\lceil \frac{n^2}{\lfloor \varphi n \rfloor} \right\rceil + n - 2 \right) \leq \psi \cdot \left(\left\lceil \frac{(\varphi n - 1)(\psi n + 1)}{\varphi n - 1} \right\rceil + n - 2 \right)$$

$$\leq \psi \cdot (\psi n + 2 + n - 2) = \psi(\psi + 1)n = n.$$

□

6. Approximate Maximin Share Allocations on a Cycle with Limits on the Number or the Number of Types of Agents

We now turn our attention to the problem of c-approximate allocations when some agents have the same utility function, that is, are of the same type. First, we consider the general case when the number of types is fixed to some $t \geq 4$. The result we obtain, Theorem 6.2, shows that for any number of types $t \geq 4$, a $\frac{t}{2t-2}$-approximate allocation of goods on a cycle to n agents exists. We then study approximate allocations when $t = 2$ and $t = 3$. In each case, we show the existence of $\frac{3}{4}$-approximate allocations. Finally, we study approximate allocations for three agents and show that in that case $\frac{5}{6}$-approximate allocations exist.

6.1 The Case when Agents Are of $t \geq 4$ Types

We start with an auxiliary result necessary to prove the main result of this subsection, Theorem 6.2.

Lemma 6.1. *Let C be a cycle of goods, N a regular set of agents, $t \geq 2$ an integer, and N_i, $1 \leq i \leq t$, pairwise disjoint subsets of N such that $\bigcup_{i=1}^{t} N_i = N$, $|N_1| \geq |N_2| \geq \ldots \geq |N_t|$ and, for every $i \in \{1, \ldots, t\}$, all agents in N_i are of the same type. If $N_1, N_2 \neq \emptyset$ and there are splits Π_1 and Π_2 such that Π_i is $\frac{t}{2t-2}$-strong for agents in N_i, for $i \in \{1, 2\}$, and there is a bundle in Π_1 and a bundle in Π_2 whose intersection has value at least $\frac{t}{2t-2}$ to some agent in N, then there is a $\frac{t}{2t-2}$-strong allocation of goods on C to agents in N.*

Proof. Let us define $n = |N|$, $c_t = \frac{t}{2t-2}$, and $n_i = |N_i|$, $1 \leq i \leq t$. We will call agents in a set N_i, $1 \leq i \leq t$, to be of type i.

If there is a single good, say x, in C of value at least c_t to some agent $k \in N$, then agent k receives x. It follows from Theorem 4.5 that there is an mms-allocation for the path $C - x$ and the remaining $n - 1$ agents. We distribute the goods of $C - x$ to these $n - 1$ agents according to this allocation. By Proposition 4.8, each agent $i \in N \setminus \{k\}$ receives a bundle worth to her at least $mms^{(n)}(i)$. Since each agent is regular, for every $i \in N$ we have $mms^{(n)}(i) = 1 \geq c_t$. Thus, each agent in N is allocated a bundle worth at least c_t to her. In other words, the allocation we constructed is c_t-strong.

Thus, let us assume that no single good in C has value at least c_t for any agent in N. Let Q be the intersection of two bundles A and B, where A is a bundle from Π_1 and B is a bundle from Π_2, such that Q has value at least c_t for some agent in N. It is clear that Q is a proper subpath of C. Let P be the path obtained from C by removing an edge so that Q is a prefix of P (that is, an initial subpath of P directed according to the, say, clockwise order of nodes in C). We will consider the call $allocate(N, P, Q, c_t)$ and follow the notation introduced in the description of the algorithm.

Since $Q \subseteq A, B$, all agents of types 1 and 2 are in S. It follows that $|R| \leq n - (n_1 + n_2)$. Since $n_1 \geq n_2 \geq \ldots \geq n_t$, $n_1 + n_2 \geq 2n_j$, for $j = 3, \ldots, t$. Thus,

$$(t - 2)(n_1 + n_2) \geq 2(n_3 + \ldots + n_t) = 2(n - (n_1 + n_2)).$$

Consequently, we have $n_1 + n_2 \geq \frac{2}{t}n$. We use this inequality to estimate $|R|$ getting

$$|R| \leq n - \frac{2}{t}n = \frac{t-2}{t}n = \frac{1 - c_t}{c_t}n.$$

By Theorem 5.3, there is a c_t-strong allocation of goods on C to agents in N. □

We will use this lemma to obtain a general result about the existence of c-approximate allocations for any number of agents of $t \geq 4$ types. Afterwards, we will obtain results for the two specific cases of $t = 2$ and $t = 3$.

Theorem 6.2. *Let C be a cycle of goods, N a set of agents of at most t types, where $t \geq 4$. Then, a $\frac{t}{2t-2}$-approximate allocation of goods on C to agents in N exists.*

Proof. By Proposition 3.3 we may assume that all agents in N are regular. Thus, we have that the maximin share for all agents is 1.

Let $n_1, \ldots, n_t$ be the numbers of agents of types $1, \ldots, t$, respectively, and let $n = n_1 + \ldots + n_t$ be the number of all agents in N. As before, we define $c_t = \frac{t}{2t-2}$ and assume that $n_1 \geq n_2 \geq \ldots \geq n_t$. If all agents are of the same type, Proposition 3.1 ensures the existence of an mms-allocation. Since $c_t < 1$, this mms-allocation is also a $\frac{t}{2t-2}$-approximate allocation. Thus, we assume that $n_2 > 0$ (and so, obviously, $n_1 > 0$, as well).

Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be mms-splits of the cycle for agents of type 1 and 2, respectively. In particular, since $c_t < 1$, the two splits are c_t-strong for agents of type 1 and 2, respectively.

If for some i and j, where $1 \leq i, j \leq n$, the path $A_i \cap B_j$ has value at least c_t to some agent in N, then we are done by Lemma 6.1. Indeed, by the regularity of the set of agents, c_t-strong and c_t-approximate allocations coincide.

So, let us assume that no path $A_i \cap B_j$ has value c_t or more to any agent. Then, in particular, $A_i \not\subseteq B_j$ and $B_j \not\subseteq A_i$ for all i and j, $1 \leq i, j \leq n$. Thus, each of the sets $A_1, \ldots, A_n$ intersects exactly two consecutive sets of the mms-split $B_1, \ldots, B_n$. We can assume without loss of generality that for every i, $1 \leq i \leq n$, the set A_i intersects the sets B_i and B_{i+1} (the arithmetic on indices is modulo n, adjusted to the range $[1..n]$).

We claim that the mms-split $A_1, \ldots, A_n$ for agents of type 1 is a c_t-approximate split for agents of type 2. To prove it, denote by u_2 the utility function for agents of type 2. Since for every i, $1 \leq i \leq t$, $A_i \subseteq B_i \cup B_{i+1}$, it follows that

$$2 = u_2(B_i \cup B_{i+1}) = u_2(A_{i-1} \cap B_i) + u_2(A_i) + u_2(A_{i+1} \cap B_{i+1}).$$

By our assumption the sets $A_{i-1} \cap B_i$ and $A_{i+1} \cap B_{i+1}$ have value less than c_t to any agent. Since for $t \geq 4$ we have $c_t \leq \frac{2}{3}$, it follows that $u_2(A_i) > 2 - 2c_t \geq c_t$. We proved that $A_1, \ldots, A_n$ is a c_t-strong split for any agent of type 2. As we noted, it is also a c_t-strong split for agents of type 1. Therefore, we can take $A_1, \ldots, A_n$ for Π_1 and Π_2 in Lemma 6.1. Moreover, A_1 is clearly the intersection of a bundle in Π_1 with a bundle in Π_2 simply because A_1 is a bundle in each of these splits. Moreover, as $A_1, \ldots, A_n$ is an mms-split for agents of type 1, A_1 has value at least 1 and, consequently, at least c_t for agents of type 1. Thus, the assumptions of Lemma 6.1 are satisfied and a c_t-strong allocation exists. This completes the proof as c_t-strong and c_t-approximate allocations coincide. □

Note that $\frac{t}{2t-2} < \frac{\sqrt{5}-1}{2}$ for $t \geq 6$, so only for agents of 4 or 5 types the result given in Theorem 6.2 is stronger than the general result in Corollary 5.5 that holds for any number of agent types.

6.2 The Case of Agents of at Most Three Types

The main result of this subsection concerns the case when agents are of three types (Theorem 6.4). This result shows that $\frac{3}{4}$-approximate allocations of goods on a cycle to n agents of three types are always possible. The proof of this result is highly technical, complex and long (enough so that we provide it in an appendix). To offer a glimpse of the difficulties, we present instead a proof of a weaker result, offering the same guarantee of the quality of an allocation but for a more restrictive case when agents are of two types only. This proof is relatively simple yet and of substantial interest in its own right to be presented.

Theorem 6.3. *Let C be a cycle of goods and N a set of agents of no more than two types. Then, a $\frac{3}{4}$-approximate allocation of goods on C to agents in N exists.*

Proof. By Proposition 3.3 we may assume that the set N of agents is regular. In particular, $\frac{3}{4}$-strong and $\frac{3}{4}$-approximate allocations coincide. Let n be the number of agents and let n_1 and n_2 be the numbers of agents of type 1 and type 2, respectively. Clearly, $n = n_1 + n_2$. We will write u_1 and u_2 for the utility functions for the agents of type 1 and type 2, respectively. Without loss of generality, we will assume that $n_1 \geq n_2$. If $n_2 = 0$, then all agents are of the same type and an mms-allocation (or, equivalently, a 1-approximate allocation) exists by Proposition 3.1. This implies the assertion. Therefore, we will assume that $n_2 > 0$. Consequently, $n_1 > 0$, too.

Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be mms-splits for agents of type 1 and type 2, respectively. If some set $A_i \cap B_j$, where $1 \leq i, j \leq n$, is of value at least $\frac{3}{4}$ to some agent, then we allocate this set $A_i \cap B_j$ to this agent. Let $P = C - (A_i \cap B_j)$. Clearly, P is a path. Moreover, since every bundle in the split $A_1, \ldots, A_n$ other than A_i is included in P, $mms^{(n-1)}(P, u_1) \geq mms^{(n)}(C, u_1) = 1$. Similarly, we have $mms^{(n-1)}(P, u_2) \geq 1$. Thus, we can assign the goods in P to the remaining $n - 1$ agents so that each agent receives a bundle that she values at least at 1. To this end, we may use the algorithm by Bouveret et al. (2017) (cf. Theorem 4.5). The resulting allocation of goods on C is $\frac{3}{4}$-strong and so, $\frac{3}{4}$-approximate.

Therefore, we assume from now on that no set $A_i \cap B_j$, $1 \leq i, j \leq n$, is of value $\frac{3}{4}$ or more for any of the agents. In particular, $A_i \not\subseteq B_j$ and $B_j \not\subseteq A_i$ for all i and j, $1 \leq i, j \leq n$. Thus, without loss of generality, we can assume that each set A_i has a nonempty intersection with the sets B_i and B_{i+1} (addition modulo n adjusted for $[1..n]$) and with no other set B_j.

We will show that the sets $A_1, \ldots, A_n$ can be allocated so that each agent receives a set of value at least $\frac{3}{4}$ to this agent. Suppose that fewer than $\frac{n}{2}$ of the sets $A_1, \ldots, A_n$ have value at least $\frac{3}{4}$ to agents of type 2. Then, there are two sets A_i, A_{i+1} such that each of them is valued at less than $\frac{3}{4}$ by agents of type 2. Thus, we have

$$\begin{aligned} 3 &= u_2(B_i \cup B_{i+1} \cup B_{i+2}) \\ &= u_2(A_{i-1} \cap B_i) + u_2(A_i \cup A_{i+1}) + u_2(A_{i+2} \cap B_{i+2}) \\ &< \frac{3}{2} + u_2(A_{i-1} \cap B_i) + u_2(A_{i+2} \cap B_{i+2}). \end{aligned}$$

This inequality implies that at least one of the sets $A_{i-1} \cap B_i$ and $A_{i+2} \cap B_{i+2}$ has value larger that $\frac{3}{4}$ for agents of type 2, a contradiction.

It follows that at least $\frac{n}{2}$ of the sets $A_1, \ldots, A_n$ are of value at least $\frac{3}{4}$ to agents of type 2. Since there are at most $\frac{n}{2}$ agents of this type (we recall that $n_1 \geq n_2$), we have sufficiently many such sets for them. The remaining sets are allocated to agents of type 1. The resulting allocation is $\frac{3}{4}$-strong and so, also $\frac{3}{4}$-approximate. □

Theorem 6.3 is tight. Let us consider a cycle $v_1, v_2, \ldots, v_{12}$, and six agents of two types with the utility functions shown in Figure 6. It is easy to check that in this case $\frac{3}{4}$-approximate allocation exists but c-approximate allocation for $c > \frac{3}{4}$ does not.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
agents 1, 2, 3	3	3	1	2	2	1	3	3	1	2	2	1
agents 4, 5, 6	3	1	2	2	1	3	3	1	2	2	1	3

Figure 6: An example showing that the result in Theorem 6.3 is tight.

As we pointed out above, the $\frac{3}{4}$ fraction of maximin shares can be guaranteed in a more general setting when agents are of three types. Specifically, the following theorem holds. We provide its proof in the appendix.

Theorem 6.4. *Let C be a cycle of goods and N a set of agents of at most three types. Then, a $\frac{3}{4}$-approximate allocation of goods on C to agents in N exists.*

Theorem 6.4 is tight too. In Figure 7 we present an example where a $\frac{3}{4}$-approximate allocation of goods on a cycle exists but there is no c-approximate allocation for any $c > \frac{3}{4}$.

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}
2	0	2	1	2	1	2	0	2	1	2	1	2	0	2	1	2	1
2	1	2	1	2	0	2	1	2	1	2	0	2	1	2	1	2	0
2	1	2	0	2	1	2	1	2	0	2	1	2	1	2	0	2	1

Figure 7: An example showing that the result in Theorem 6.4 is tight. The rows two, three and four in the table specify the utility functions for agents 1 and 2, 3 and 4, and 5 and 6, respectively.

An interesting property of the example in Figure 7 is that the set $\{0, 1, 2\}$ of values of the utility functions is very small. The problem of existence and construction of an mms-allocation with this set of values was studied in the original version of the mms-allocation problem (i.e. for complete graphs in our terminology). Amanatidis et al. (2017) proved that unlike in the case of a cycle (see Figure 7) for a complete graph an mms-allocation with $\{0, 1, 2\}$ as the set of values of the utility functions always exists.

It is also easy to show that if $\{0, 1\}$ is the set of values of the utility functions, then an mms-allocation for a cycle always exists and can be constructed in polynomial time. An analogous statement for complete graphs was observed earlier by Bouveret and Lemaître (2016).

6.3 The case of Three Agents

As we have seen earlier, even for three agents mms-allocations of goods on a cycle may not exist. Theorem 6.4 implies that when allocating goods on a cycle to three agents, we can guarantee that each agent receives a bundle worth at least $\frac{3}{4}$ of her maximin share. We will now show that this guarantee can be strengthened to $\frac{5}{6}$. Re-examining the example in Figure 3 shows that this is the best guarantee we can get. To see this, we recall that in this case the maximin share value for agents 1 and 2 is 5 and for agent 3 is 6. In particular, any split in which an agent 1 or 2 obtains a bundle consisting of two or fewer items is at best a $\frac{4}{5}$-approximate allocation (indeed, any two consecutive items have the total value of no more than 4). Thus, in any c-approximate allocation with $c \geq \frac{5}{6}$,

agents 1 and 2 receive bundles of at least three items. If agent 3 receives at least three items, then all agents obtain bundles of exactly three items. There are only three such allocations. It is easy to see that one of them is $\frac{4}{5}$-approximate and the other two are $\frac{5}{6}$-approximate. Since any two consecutive items have the total value at most 5 for agent 3, any allocation in which agent 3 receives no more than 2 items is at best $\frac{5}{6}$-approximate (some of them are actually $\frac{5}{6}$-approximate; for instance, the allocation $\{v_4, v_5, v_6\}, \{v_7, v_8, v_9, v_1\}, \{v_2, v_3\}$.

To show that a $\frac{5}{6}$-approximate allocation of goods on a cycle always exists for 3 agents we start by introducing some additional terminology and two lemmas. Let C be a cycle, $N = \{1, 2, 3\}$ a set of agents and u_i the utility function of an agent $i \in \{1, 2, 3\}$. To simplify notation, we call an mms-split for $(C, 3, u_i)$ an $mms(i)$-*split* and we recall that we write $mms(i)$ as a shorthand for $mms^{(3)}(C, u_i)$.

Lemma 6.5. *Let C be a cycle and $N = \{1, 2, 3\}$ a set of agents. Let $c \in (0, 1]$ be a real number. If for some agent $i \in N$ two different bundles of an $mms(i)$-split of C are of value at least $c \cdot mms(j)$ to an agent j, where $j \neq i$, then there is a c-approximate allocation of goods on C to agents in N.*

Proof. We assume without loss of generality that $i = 1$ and $j = 2$. Let us observe that for any agent and any 3-split of the cycle there is a bundle of value at least one third of the total value of all goods for this agent. The value of this bundle is larger than or equal to the maximin share for this agent. In particular, the value of one of the bundles of any $mms(1)$-split is greater than or equal to $mms(3)$. The following protocol finds a c-approximate allocation. Agent 3 picks a bundle of the $mms(1)$-split that has value at least $mms(3)$ to her. At least one of the remaining two bundles has value at least $c \cdot mms(2)$ to agent 2. That bundle is allocated to agent 2. The remaining bundle of the split is assigned to agent 1. □

Lemma 6.6. *Let C be a cycle and $N = \{1, 2, 3\}$ a set of agents. Let $c \in (0, 1]$ be a real number. If for some two different agents $i, j \in N$, the intersection of a bundle of an $mms(i)$-split and a bundle of an $mms(j)$-split of C has value at least $c \cdot mms(i)$ to the agent i, then there is a c-approximate allocation of goods in C to agents in N.*

Proof. Without loss of generality, we assume that $i = 1$ and $j = 2$. Let A and B be bundles of mms-splits for agents 1 and 2, respectively, such that $Q = A \cap B$ has value at least $c \cdot mms(1)$ to the agent 1. The set $B - Q$ consists of one interval, or of two intervals (may happen when $A \subset B$), or is empty (when $B \subset A$). Clearly, the elements in $B - Q$ (if $B - Q \neq \emptyset$) can be added to the remaining two bundles (different from B) of the $mms(2)$-split so as to form a 2-split of $C - Q$, say B', B'', with both parts of value at least $mms(2)$ for the agent 2.

We construct a c-approximate allocation as follows. If $u_3(Q) \geq c \cdot mms(3)$, then we allocate Q to agent 3. Since $Q \subseteq A$, $C - Q$ contains the remaining two bundles (different from A) of the $mms(1)$-split. Hence, $u_1(C - Q) \geq 2 \cdot mms(1)$. Since $C - Q = B' \cup B''$, at least one of the bundles B' and B'' is worth $mms(1)$ or more to the agent 1. We allocate that bundle to that agent and we allocate the remaining one to the agent 2.

If $u_3(Q) < c \cdot mms(3)$, then the agent 1 gets Q. Since $u_3(C) \geq 3 \cdot mms(3)$ and $c \leq 1$, $u_3(C - Q) > 2 \cdot mms(3)$. Thus, at least one bundle of B' and B'' is worth $mms(3)$ or more to agent 3. That bundle goes to the agent 3 and the remaining one to agent 2. □

We are now ready to prove that if goods on a cycle are allocated to three agents, then each agent can be guaranteed a bundle worth at least $\frac{5}{6}$ of her maximin share.

Theorem 6.7. *Let C be a cycle and $N = \{1, 2, 3\}$ a set of agents. Then, a $\frac{5}{6}$-approximate allocation of goods in C to agents in N exists.*

Proof. Because of Proposition 3.3, we restrict attention to the case when the set N of agents is regular. It follows that each agent values all goods on C at 3 and that all agents are proportional. In particular, $mms(i) = 1$, for every agent $i \in N$.

Let $\Pi_1 = A_1, A_2, A_3$, $\Pi_2 = B_1, B_2, B_3$ and $\Pi_3 = C_1, C_2, C_3$ be mms-splits for agents 1, 2 and 3, respectively. If a bundle of one of these splits is contained in a bundle of another split, then an mms-allocation (and consequently a $\frac{5}{6}$-approximate allocation) exists, by Lemma 6.6 (applied to the two bundles and $c = 1$).

From now on we assume that there are no such containments. Then, we can relabel the parts of the splits Π_1, Π_2 and Π_3 so that the sets $A_{\lceil (k+2)/3 \rceil} \cap B_{\lceil (k+1)/3 \rceil} \cap C_{\lceil k/3 \rceil}$, $k = 1, 2, \ldots, 9$ (indices are computed modulo 3 with respect to the set $\{1, 2, 3\}$) form a split of the cycle. Let us call these 9 sets *chunks*.

Since each agent $i \in N$ values C at 3, any split of C into three bundles contains at least one bundle of value at least 1 to i, that is, of value at least $mms(i)$ to i. In particular, such a bundle can be found in the mms-splits of the two agents other than i. Moreover, if for some two agents $i, j \in N$, $i \neq j$, more than one part in Π_i has value at least 1 ($= mms(j)$) for j, then we are done by Lemma 6.5.

Let $i, j, k \in N$ be three different agents. We observe that if a bundle of Π_k of value at least 1 ($= mms(i)$) for the agent i is different from a part of Π_k of value at least 1 ($= mms(j)$) for the agent j, then we can easily distribute the parts of Π_k among the agents to get an mms-allocation.

So, we assume that for every agent i there is a unique bundle in Π_i whose value for each of the remaining two agents is at least 1. Let us call this bundle *i-significant*.

One can readily observe that, for $i, j \in N$, $i \neq j$, the intersection of the i-significant bundle and the j-significant bundle consists of either two chunks, one chunk or is empty. Moreover, it is not possible that the three pairs of different significant bundles intersect on single chunks. Thus, there is a pair $i, j \in N$ of agents such that the i-significant bundle and the j-significant bundle are disjoint or intersect on two chunks. We will consider these two cases separately. Clearly, we can assume without loss of generality that $i = 1$, $j = 2$ and that A_s and B_t, $s, t \in \{1, 2, 3\}$, are the 1-significant and 2-significant bundles, respectively,

Case 1 A_s and B_t are disjoint.

We can assume without loss of generality that $A_s = A_1$ and $B_t = B_2$. Let us consider the mms-split Π_3. By the definition of significant bundles, the value for the agent 3 of each of the sets A_1 and B_2 is at least 1.

If the value of A_2 or A_3 for the agent 3 is at least $\frac{5}{6}$, then a $\frac{5}{6}$-approximate allocation exists by Lemma 6.5. Moreover, if the value for the agent 3 of the set $A_1 \cap C_3$ is at least $\frac{5}{6}$, then a $\frac{5}{6}$-approximate allocation exists by Lemma 6.6.

So, let us assume that the values for the agent 3 of the sets A_2, A_3 and $A_1 \cap C_3$ are all smaller than $\frac{5}{6}$. Then the value of the chunk $A_1 \cap B_1 \cap C_1$ is larger than $3 - 3 \cdot \frac{5}{6} = \frac{1}{2}$.

In a very similar way we show that if any of the sets B_1, B_3, $B_2 \cap C_2$ has value at least $\frac{5}{6}$ for the agent 3, then a $\frac{5}{6}$-approximate allocation exists. Otherwise, the value for the agent 3 of the chunk $A_2 \cap B_2 \cap C_1$ is larger than $\frac{1}{2}$.

Then, however, the value for the agent 3 of the part C_1, which contains the chunks $A_1 \cap B_1 \cap C_1$ and $A_2 \cap B_2 \cap C_1$ is larger than 1, a contradiction because the agents are proportional.

Case 2 A_s and B_t intersect on two chunks.

We assume without loss of generality that $A_s = A_1$ and $B_t = B_1$. By the definition of significant bundles, the value for the agent 3 of each of the bundles A_1 and B_1 is at least 1.

Reasoning in exactly the same way as in the first case, we conclude that either a $\frac{5}{6}$-approximate allocation exists or the value for the agent 3 of the chunk $A_1 \cap B_1 \cap C_1$ is larger than $\frac{1}{2}$.

Let us then assume the latter. We observe that as in Case 1, if any of the bundles $B_2, B_3, B_1 \cap C_1$ has value at least $\frac{5}{6}$ for the agent 3, then a $\frac{5}{6}$-approximate allocation exists (by Lemma 6.5 or 6.6). Otherwise, the value for the agent 3 of the chunk $A_1 \cap B_1 \cap C_3$ is larger than $\frac{1}{2}$.

Then, however, the value for the agent 3 of the set $A_1 \cap B_1 = (A_1 \cap B_1 \cap C_1) \cup (A_1 \cap B_1 \cap C_3)$ is larger than 1. In this case an mms-allocation exists because we can assign A_2 to the agent 1, $B_3 \cup (A_3 \cap B_2 \cap C_2)$ to the agent 2 and $A_1 \cap B_1$ to the agent 3. □

For more than three agents a $\frac{5}{6}$-approximate allocation of goods on a cycle may not exist. The example in Figure 6 shows that this may be the case even for agents of two types only. For this example, we have already observed that there is a $\frac{3}{4}$-approximate allocation but no c-approximate allocation for $c > \frac{3}{4}$.

6.4 Comments on Algorithms

We close this section by commenting on the problem of computing c-approximate allocations. First, we note that all proofs we presented in this section, as well as the proof of Theorem 6.4, which we give in the appendix, yield algorithms for constructing c-approximate allocations. These constructions apply to the case of regular sets of agents and require that both maximin shares and mms-splits be computed for every agent. Given those, the actual construction of a c-approximate allocation provided in each proof can clearly be implemented to run in polynomial time. We recall that in the case of goods on a cycle, maximin shares and mms-splits can be computed in polynomial time. It follows then by Corollary 4.12 that in all cases we discussed in this section when c-approximate allocations exist, they can be computed by polynomial-time algorithms.

7. Conclusions

We investigated maximin share allocations in the graph setting for the fair division problem of indivisible goods proposed by Bouveret et al. (2017). That paper settled the case of trees by showing that maximin share allocations of goods on trees always exist and can be computed in polynomial time. It also gave an example of goods on a cycle to be distributed to four agents where no maximin share allocation exists.

Our work focused on cycles. We found several cases when maximin share allocations on cycles exist and can be found in polynomial time. For some other cases, when maximin share allocations are not guaranteed to exist, we found polynomial-time algorithms for deciding the existence of maximin share allocations and, if they do exist, computing them, too. Interestingly, we do not know the complexity of deciding the existence of maximin share allocations on cycles. We proved that the problem is in NP but whether it is NP-hard is open.

In general, understanding the complexity of deciding the existence of maximin share allocations of goods on graphs is a major challenge. We improved an earlier upper bound from Σ_2^P down to Δ_2^P but, as in the case of cycles, we do not have any hardness results. Establishing such results and characterizing classes of graphs for which deciding the existence of maximin share allocations

is in P, is NP-complete or goes beyond NP (under the assumption, of course, that the polynomial hierarchy does not collapse) are important open problems.

Perhaps our most interesting results concern the existence of allocations guaranteeing each agent a given fraction of her maximin share. For instance, we show that for three agents one can always find an allocation giving each agent at least $5/6$ of her maximin share. For an arbitrary number of agents of two or three types, we show allocations giving each agent $3/4$ of the maximin share, and we also obtain some results for the case when the number of types is larger than three. Moreover, in each case, these allocations can be found by polynomial-time algorithms. We conjecture that in the general case of any number of agents there exist allocations guaranteeing every agent at least $3/4$ of her maximin share. Theorems 6.3 and 6.4 show the conjecture holds if agents are of three or fewer types. However, the methods we developed so far seem too weak to prove it in full generality.

It is perhaps interesting to extend the problem of existence and approximation of mms-allocations to disconnected graphs of goods G. In this extension we still require that G-bundles are sets of vertices of connected subgraphs of G. However, we should slightly change the definition of the maximin share. In this definition we should replace the notion of n-split, by the notion of *n-packing* by which we mean a sequence $P_1, \ldots, P_n$ of pairwise disjoint G-bundles such that $\bigcup_{i=1}^{n} P_i \subseteq V$. The notions of q-strong split and mms-split should be replaced, in a standard way, by the notions of q-strong packing and mms-packing, respectively. In this extension in the definition of (G, n)-allocation we no longer assume that *all* goods are distributed among the agents.

The main reason of this change is an observation that if the number of connected components of the graph G is larger than the number n of agents, then no (G, n)-split exists. Moreover, even if G has only two components A and B, and for all agents the total value of B is by far smaller than the total value of A, then the agents will certainly be more satisfied by receiving a small but valuable piece of A, than a large by worthless piece of B. In this case the goods in B will not be assigned to any agent.

We checked that some of the results included in this paper (e. g. Proposition 3.1 and Corollary 3.2) can be easily extended to disconnected graphs of goods. We think studying of mms-allocations for disconnected graphs of goods may be an interesting research direction, however, it is beyond the scope of this paper.

There are more general open questions concerning maximin share allocations of goods on graphs that are worth of study. For instance it is not known if there is an absolute constant $c > 0$ such that for every connected graph and any number of agents there exists a c-approximate allocation. A reasoning similar to the one used in the proof of Proposition 5.1 shows that such a constant exists if we restrict our attention to unicyclic graphs and other authors proved existence of such a constant for complete graphs and trees.

Another general open problem concerns existence of a maximin share allocation for a "small" number of goods. Let $\mathcal{G}$ be some family of connected graphs. We define $f_{\mathcal{G}}(n)$ to be the largest k, if it exists, such that for any instance with n agents and any graph of goods $G \in \mathcal{G}$ with no more than k vertices there is a maximin share allocation. This parameter was defined by Kurokawa et al. (2018) for the original problem (that is, when $\mathcal{G}$ is the set of complete graphs). They proved that $n + 4 \leq f_{\mathcal{G}}(n) \leq 3n + 3$ in this case. It follows from our Theorem 4.9 and the fact that it is tight that $f_{\mathcal{G}}(n) = 2n - 1$ when $\mathcal{G}$ is the set of cycles. It would be interesting to establish the parameter $f_{\mathcal{G}}(n)$, where $\mathcal{G}$ is the set of all connected graphs.

In this work, we were focused on fairness not on efficiency. It is a straightforward observation that whenever an mms-allocation or a c-approximate allocation exists, such an allocation that is also

Pareto optimal exists as well. Our results on existence of mms-allocations and their approximate versions also imply polynomial-time algorithms for finding them. A natural question arises whether there are polynomial-time algorithms for finding Pareto optimal mms-allocations. We leave it for the future work.

Acknowledgments

A preliminary version of the paper appeared in the proceedings of IJCAI 2018. The work of the second author was partially supported by the NSF grant IIS-1618783.

Appendix A — Proof of Theorem 6.4

In this section we prove Theorem 6.4. We adhere to the notation we used throughout the paper. In particular, we consider a set $N = \{1, 2 \ldots, n\}$ of agents of three types and a cycle C of m goods.

The proof consists of many cases. In each case we provide a different construction of a $\frac{3}{4}$-approximate allocation. In the constructions we usually combine mms-splits (defined as sequences of consecutive bundles) for agents of two different types to produce a sequence of n disjoint bundles that we distribute among agents. In some cases one of the distributed bundles is an intersection of two parts of mms-splits for agents of two different types. In Lemmas 7.1-7.5 we formulate properties of these combinations of two splits that are used many times in the proof. Lemma 7.6 covers a very special case of Theorem 6.4. This is the most complicated case so we decided to formulate and prove it separately to improve readability of the proof. The main part of the proof of Theorem 6.4 consists of 9 cases and in each case we assume that none of the cases considered earlier applies.

In the proof, as in several other places in the paper, we work under the assumption that agents are regular. Thus, their mms-splits consist of non-empty bundles. Therefore, in our auxiliary results, concerning properties of splits, we restrict attention to splits into non-empty bundles, that is, to splits that are partitions.

We select and fix an element a in C and call it an *anchor*. Whenever we say that $\mathcal{X} = X_1, \ldots, X_n$ is a split, we mean that it is a split of C and, assume that $a \in X_1$ and that the parts $X_1, \ldots, X_n$ are enumerated according to their location on the cycle as we traverse it clockwise. Further, for every $i \in \{1, \ldots, n\}$, we write $\mathcal{X}_i$ for the set of all elements in the segment of C extending from the anchor a clockwise to the last element of X_i (inclusive). We also assume that arithmetic expressions appearing in indices labeling bundles in splits are evaluated modulo n; in particular, for every $i \in \{1, 2, \ldots, n\}$, $i + n = i$. Finally, we call any set of a split $\mathcal{X}$ an *X-set*.

Assuming these conventions, let $\mathcal{X} = X_1, \ldots, X_n$ and $\mathcal{Y} = Y_1, \ldots, Y_n$ be two splits. A bundle X_i is a *jump to the split* $\mathcal{Y}$ if $\mathcal{X}_i \subseteq \mathcal{Y}_i$. In such case, we will also say that X_i is a jump to Y_{i+1}. We usually omit the reference to the target split of a jump, if it is clear from the context. The following property follows directly from the definition.

Lemma 7.1. *Let $\mathcal{X} = X_1, \ldots, X_n$ and $\mathcal{Y} = Y_1, \ldots, Y_n$ be two splits. Then for every i, $1 \leq i \leq n$, at least one of the sets X_i, Y_i is a jump (to the other split).*

Lemma 7.2. *Let $\mathcal{X} = X_1, \ldots, X_n$ and $\mathcal{Y} = Y_1, \ldots, Y_n$ be two splits. If for some i, $1 \leq i < n$, X_i is a jump and X_{i+1} is not a jump, then $Y_{i+1} \subseteq X_{i+1}$ and Y_{i+1} is a jump.*

Proof. Since X_i is a jump, $\mathcal{X}_i \subseteq \mathcal{Y}_i$. Further, since X_{i+1} is not a jump, Y_{i+1} is a jump (Lemma 7.1). Thus, $\mathcal{Y}_{i+1} \subseteq \mathcal{X}_{i+1}$. Since $X_{i+1} = \mathcal{X}_{i+1} \setminus \mathcal{X}_i$ and $Y_{i+1} = \mathcal{Y}_{i+1} \setminus \mathcal{Y}_i$, $Y_{i+1} \subseteq X_{i+1}$ follows. □

Let $\mathcal{X} = X_1, \ldots, X_n$ be a sequence of pairwise disjoint n subsets of C. An *$\mathcal{X}$-interval* is any sequence $X_i, X_{i+1} \ldots, X_j$ of up to n consecutive elements of $\mathcal{X}$ (with X_1 being a successor of X_n).

Let $\mathcal{X} = X_1, \ldots, X_n$ and $\mathcal{Y} = Y_1, \ldots, Y_n$ be two splits. A sequence $\mathcal{Z} = Z_1, \ldots, Z_n$ of subsets of C is *$\mathcal{XY}$-useful* if for every $i \in \{1, 2, \ldots, n\}$, at least one of the following conditions holds:

1. $Z_i = X_i$ and $Z_{i+1} = X_{i+1}$
2. $Z_i = Y_i$ and $Z_{i+1} = Y_{i+1}$
3. $Z_i = X_i$ and X_i is a jump to Y_{i+1}
4. $Z_i = Y_i$ and Y_i is a jump to X_{i+1}.

Lemma 7.3. *Let $\mathcal{X} = X_1, \ldots, X_n$ and $\mathcal{Y} = Y_1, \ldots, Y_n$ be splits. If a sequence $\mathcal{Z} = Z_1, \ldots, Z_n$ is $\mathcal{XY}$-useful then the sets in $\mathcal{Z}$ are pairwise disjoint.*

Proof. Let us consider the sets Z_i and Z_j, where $1 \leq i < j \leq n$. From the definition of a $\mathcal{XY}$-useful sequence, it follows that for every k, $1 \leq k \leq n$, $Z_k = X_k$ or $Z_k = Y_k$. Thus, $Z_i = X_i$ and $Z_j = X_j$, $Z_i = Y_i$ and $Z_j = Y_j$, $Z_i = X_i$ and $Z_j = Y_j$, or $Z_i = Y_i$ and $Z_j = X_j$. In the first two cases, the sets Z_i and Z_j are disjoint, because they are two different members of a split. The other two cases are symmetric. Without loss of generality, we will restrict attention to the last one: $Z_i = Y_i$ and $Z_j = X_j$. Clearly, we may assume that $Z_i \neq X_i$ and $Z_j \neq Y_j$ (otherwise, Z_i and Z_j would be members of the same split, the case we already considered).

Let k be the largest integer such that $i \leq k < j$, $Z_k = Y_k$, and $Z_{k+1} \neq Y_{k+1}$. Since $Z_i = Y_i$, k is well defined. It follows that Y_k is a jump. Thus, $\mathcal{Y}_k \subseteq \mathcal{X}_k$. Since $k + 1 \leq j \leq n$, $X_j \cap \mathcal{X}_k = \emptyset$. Consequently, $X_j \cap \mathcal{Y}_k = \emptyset$. Since $i \leq k$, if $i \geq 2$ then $Z_i \subseteq \mathcal{Y}_k$ (indeed, it follows from the equality $Z_i = Y_i$). Thus, Z_i and Z_j are disjoint.

Let us assume then that $i = 1$. Since the anchor a belongs to Y_1, Y_1 is the union of two segments, $A = C \setminus \mathcal{Y}_n$ and $B = \mathcal{Y}_1$. Since $X_j \cap \mathcal{Y}_k = \emptyset$, $X_j \cap B = \emptyset$. Let p be the smallest integer such that $j \leq p \leq n$, $Z_p = X_p$ and $Z_{p+1} \neq X_{p+1}$. Since $Z_j = X_j$ and $Z_1 \neq X_1$ (we proved $Z_i \neq X_i$ above and here $i = 1$), p is well defined. It follows that X_p is a jump. Therefore, $\mathcal{X}_p \subseteq \mathcal{Y}_p$. Since $\mathcal{Y}_p \subseteq \mathcal{Y}_n$, $A \cap \mathcal{Y}_p = \emptyset$. Since $X_j \subseteq \mathcal{X}_p \subseteq \mathcal{Y}_p$, $X_j \cap A = \emptyset$. Thus, $Z_j (= X_j)$ and $Z_1 (= Y_1)$ are disjoint. □

Splits $\mathcal{X}$ and $\mathcal{Y}$ are *proper relative to each other* if for every $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, $X \not\subseteq Y$ and $Y \not\subseteq X$. The following lemma provides some basic properties of splits that are proper relative to each other (they are easy to see and we omit the proof).

Lemma 7.4. *Let splits $\mathcal{X} = X_1, \ldots, X_n$ and $\mathcal{Y} = Y_1, \ldots, Y_n$, with both sequences enumerated according to our convention clockwise starting with sets containing the anchor a, be proper relative to each other. Then for every $i \in \{1, \ldots, n\}$,*

1. *$X_i \subseteq Y_i \cup Y_{i+1}$ and $Y_i \subseteq X_{i-1} \cup X_i$, and*
2. *$X_{i-1} \cap Y_i \neq \emptyset$, $X_i \cap Y_i \neq \emptyset$, and $X_i \cap Y_{i+1} \neq \emptyset$.*

or for every $i \in \{1, \ldots, n\}$,

1. $X_i \subseteq Y_{i-1} \cup Y_i$ *and* $Y_i \subseteq X_i \cup X_{i+1}$, *and*
2. $X_i \cap Y_{i-1} \neq \emptyset$, $X_i \cap Y_i \neq \emptyset$, *and* $X_{i+1} \cap Y_i \neq \emptyset$.

From now on we consider splits that are mms-splits for a set N of regular agents. In particular, all agents are proportional and for every agent the total utility of all goods in C is n. We also use the following terminology. A subset X of C that induces in C a connected subgraph is *acceptable* to an agent x if the total value of the goods in X to x is at least $\frac{3}{4}$.

Lemma 7.5. *Let x and y be two regular agents from N. Let $\mathcal{X}$ and $\mathcal{Y}$ be mms-splits of a cycle for agents x and y, respectively, and let $\mathcal{X}$ and $\mathcal{Y}$ be proper relative to each other. If no set $X \cap Y$, where $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, is acceptable to y, then at least one of each pair of consecutive sets of $\mathcal{X}$ is acceptable to y.*

Proof. To prove the lemma let us consider any two consecutive sets in $\mathcal{X}$, say X and X', with X' directly following X in $\mathcal{X}$. Since $\mathcal{X}$ and $\mathcal{Y}$ are proper relative to each other, Lemma 7.4 implies that there are three consecutive sets $Y, Y', Y'' \in \mathcal{Y}$ such that $X \cup X' \subseteq Y \cup Y' \cup Y''$. Let X_{-1} and X'_{+1} be the predecessor of X in $\mathcal{X}$ and the successor of X' in $\mathcal{X}$, respectively. By our assumption, $u_y(X_{-1} \cap Y) < \frac{3}{4}$ and $u_y(X'_{+1} \cap Y'') < \frac{3}{4}$, where u_y is the utility function for agent y. It follows that

$$\begin{aligned} 3 &= u_y(Y \cup Y' \cup Y'') \\ &= u_y(X_{-1} \cap Y) + u_y(X) + u_y(X') + u_y(X'_{+1} \cap Y'') \\ &< 2 \cdot \frac{3}{4} + u_y(X) + u_y(X'). \end{aligned}$$

Thus, $u_y(X) \geq \frac{3}{4}$ or $u_y(X') \geq \frac{3}{4}$, that is, at least one of X and X' is acceptable to y. □

In the next lemma and in the proof of Theorem 6.4, we will use the following notation. We write n_1, n_2, n_3 for the number of agents of types i_1, i_2, i_3, respectively. In particular, $n = n_1 + n_2 + n_3$. Further, we assume without loss of generality that $n_1 \geq n_2 \geq n_3 \geq 1$.

Lemma 7.6. *Let $\mathcal{A} = A_1, \ldots, A_n$ (resp. $\mathcal{B} = B_1, \ldots, B_n$ and $\mathcal{C} = C_1, \ldots, C_n$) be mms-splits of the cycle C for agents of type i_1 (resp. i_2 and i_3). If $n_1 = n_2$, $n_3 \geq 1$, no set $A_i \cap B_j$ is acceptable to any agent, and some C-set is contained in some A-set or B-set, then there is a $\frac{3}{4}$-approximate allocation for a cycle and any number of agents of three types.*

Proof. Since no set $A_i \cap B_j$ is acceptable to any agent, we have that for every i and j, where $1 \leq i, j \leq n$, $A_i \not\subseteq B_j$ and $B_i \not\subseteq A_j$. Thus, the splits $\mathcal{A}$ and $\mathcal{B}$ are proper relative to each other.

Our assumptions do not distinguish between the splits $\mathcal{A}$ and $\mathcal{B}$. Therefore, without loss of generality, we may assume that a C-set, say C_k, is contained in an A-set, say A_i. Let B_j and B_{j+1} be two B-sets that intersect A_i. It follows that $C_k \subseteq B_j \cup B_{j+1}$. If $C_k \subseteq B_{j+1}$, then $C_k \subseteq A_i \cap B_{j+1}$. But then $A_i \cap B_{j+1}$ would be acceptable to agents of type i_3 (and there is at least one such agent). This is a contradiction. Thus, $C_k \cap B_j \neq \emptyset$. Consequently, $C_k \cap B_j \cap A_i \neq \emptyset$. Without loss of generality, we may assume that the anchor a belongs to $C_k \cap B_j \cap A_i$. Assuming the sets in the splits $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are enumerated with respect to that anchor, we have $k = j = i = 1$, $C_1 \subseteq A_1$ and $B_1 \subseteq A_n \cup A_1$. The latter implies (we recall that $\mathcal{A}$ and $\mathcal{B}$ are proper relative to each other and so, Lemma 7.4 applies) that for every $i \in \{1, \ldots, n\}$,

$$A_i \subseteq B_i \cup B_{i+1} \text{ and } B_i \subseteq A_{i-1} \cup A_i.$$

In what follows we will be applying Lemmas 7.1 - 7.5. We can do so, as we follow here the same convention for enumerating sets in splits with respect to the same fixed anchor.

Case 1 Some two consecutive A-sets are jumps to $\mathcal{C}$.

If all A-sets are jumps to $\mathcal{C}$, then the sequence

$$\mathcal{Z} = C_1, A_2, \ldots, A_{2n_1+1}, C_{2n_1+2}, \ldots, C_n$$

is $\mathcal{AC}$-useful. Indeed, C_1 is a jump as it is a subset of A_1 (thus, $\mathcal{C}_1 \subseteq \mathcal{A}_1$, in the notation we introduced to define jumps), and A_{2n_1+1} is a jump by assumption. By Lemma 7.3, all sets in $\mathcal{Z}$ are pairwise disjoint.

By assumption, no set $A_i \cap B_j$ is acceptable to any agent of type i_2. Thus, by Lemma 7.5, some $n_1 (= n_2)$ of different sets in the interval $A_2, \ldots, A_{2n_1+1}$ in $\mathcal{Z}$ are acceptable to agents of type i_2. We assign them to those agents. We assign the remaining n_1 sets in $A_2, \ldots, A_{2n_1+1}$ to agents of type i_1. Finally, we assign the sets C_j, $j \in \{1, 2n_1 + 2, \ldots, n\}$, to agents of type i_3. In this way all agents are allocated sets that are acceptable to them. If there are any unallocated goods ($\mathcal{Z}$ does not have to cover all goods in C), they can be distributed to agents so that all agents receive bundles inducing in C a path. This, yields an assignment that is a $\frac{3}{4}$-approximate allocation.

So, assume that some of A-sets are not jumps. Since some two consecutive A-sets are jumps, there is an integer i such that A_{i-1} and A_i are jumps and A_{i+1} is not. By Lemma 7.2, C_{i+1} is a jump and $C_{i+1} \subseteq A_{i+1}$. The latter property implies that we may relabel sets A_i and C_i so that A_{n-1} and A_n are jumps and A_1 is not. Indeed, under this relabeling we have $C_1 \subseteq A_1$, which we assumed to hold when we started the proof (of course, we also need to relabel sets B_i correspondingly).

We now consider the case when C_{n_3} or C_{n_3-1} is a jump. In the first case, we set

$$\mathcal{Z} = C_1, \ldots, C_{n_3}, A_{n_3+1}, \ldots, A_n$$

and, in the second case,

$$\mathcal{Z} = C_n, C_1, \ldots, C_{n_3-1}, A_{n_3}, \ldots, A_{n-1}.$$

Since A_n and A_{n-1} are jumps, in either case $\mathcal{Z}$ is $\mathcal{AC}$-useful. Thus, by Lemma 7.3, its elements are pairwise disjoint. Reasoning as above, we distribute the sets $A_{n_3+1}, \ldots, A_n$ (respectively, $A_{n_3}, \ldots, A_{n-1}$) among $2n_1$ agents of types i_1 and i_2 (since $n = 2n_1 + n_3$, in each case there are $2n_1$ sets in that sequence), and then allocate the remaining n_3 sets, all of them C-sets, to agents of type i_3. Clearly, the resulting allocation (after possibly attaching goods not "covered" by $\mathcal{Z}$ to appropriate sets in $\mathcal{Z}$) is $\frac{3}{4}$-approximate.

Next, let us assume that neither C_{n_3} nor C_{n_3-1} is a jump. Then let k be the largest integer such that $1 \leq k < n_3 - 1$ and C_k is a jump (such k exists because C_1 is a jump). Moreover, let ℓ be the smallest integer j such that $n_3 < j \leq n$ and C_j is a jump, if such j exists, or $\ell = n$, otherwise. Observe that the sequence

$$\mathcal{Z} = C_1, \ldots, C_k, A_{k+1}, \ldots, A_{\ell-(n_3-k)}, C_{\ell-(n_3-k)+1}, \ldots, C_\ell, A_{\ell+1} \ldots, A_n$$

is $\mathcal{AC}$-useful. Indeed, for $k < j < \ell$ the sets C_j are not jumps. By Lemma 7.1, the sets A_j, where $k < j < \ell$ are jumps. Since $k < \ell - (n_3 - k) < \ell$, $A_{\ell-(n_3-k)}$ is a jump. Moreover, C_k is a jump by our choice of k, C_ℓ is a jump by our choice of ℓ, if ℓ is not assigned to n by default, and A_n is a jump by assumption. By Lemma 7.3, all sets in $\mathcal{Z}$ are pairwise disjoint.

Clearly, if ℓ and $n(= 2n_1 + n_3)$ have the same parity, then the numbers of sets in the intervals $A_{k+1}, \ldots, A_{\ell-(n_3-k)}$ and $A_{\ell+1}, \ldots, A_n$ are even (the latter is present and needs to be considered only if $\ell < n$). Moreover, the total number of sets in these two intervals is $n - n_3 = 2n_1 = 2n_2$. By Lemma 7.5, we can select among these A-sets in $\mathcal{Z}$ $n_1(= n_2)$ sets that are acceptable to agents of type i_2. We allocate these sets to those agents. We then allocate the remaining n_1 of these A-sets to the agents of type i_1. Finally, agents of type i_3 receive the n_3 C-sets of $\mathcal{Z}$. Extending $\mathcal{Z}$ to an allocation of C, yields an allocation that is $\frac{3}{4}$-approximate.

So, assume now that ℓ and n have different parity. In this case, we define ℓ' to be the smallest integer j such that $n_3 < j \leq n-1$ and C_j is a jump, if such j exists, or $\ell' = n-1$ otherwise. Obviously, $\ell' = \ell$ or $\ell' = n-1$, so in both cases ℓ' and n have different parity. Since A_{n-1} is a jump, the same reasoning as above yields that the sequence

$$\mathcal{Z} = \\ C_n, C_1, \ldots, C_k, A_{k+1}, \ldots, A_{\ell'-(n_3-k)+1}, C_{\ell'-(n_3-k)+2}, \ldots, C_{\ell'}, A_{\ell'+1} \ldots, A_{n-1}$$

is $\mathcal{AC}$-useful. Thus, its elements are pairwise disjoint. Moreover, the A-intervals $A_{k+1}, \ldots, A_{\ell'-(n_3-k)+1}$ and $A_{\ell'+1}, \ldots, A_{n-1}$ (if $\ell' < n-1$) consist of even numbers of sets and the total number of sets in these two intervals is $2n_1 = 2n_2$. Therefore, we can define $\frac{3}{4}$-approximate allocation in a similar way as when ℓ and n have the same parity.

Case 2 Some two consecutive C-sets are jumps to $\mathcal{A}$.

This case is symmetric to the previous one if we enumerate sets counterclockwise rather than clockwise. Indeed, if a C-set X is a jump to an A-set Y under the clockwise enumeration, then Y is a jump to X under the counterclockwise enumeration.

Case 3 No two consecutive A-sets are jumps to $\mathcal{C}$, and no two consecutive C-sets are jumps to $\mathcal{A}$.

Since $C_1 \subseteq A_1$, C_1 is a jump. It follows that $C_n \cap A_1 \neq \emptyset$ as otherwise, C_n would be a jump. In particular, $C_n \not\subseteq A_n$. Applying Lemma 7.2 repeatedly, yields that for all odd i, $1 \leq i \leq n$, $C_i \subseteq A_i$ and for every even i, $1 \leq i \leq n$, $A_i \subseteq C_i$. Further, since $C_n \not\subseteq A_n$, it follows that n is even.

We will now show that the splits $\mathcal{B}$ and $\mathcal{C}$ are proper relative to each other. To this end, we need to show that for every i and j, $1 \leq i, j \leq n$, $C_i \not\subseteq B_j$ and $B_j \not\subseteq C_i$. First, let us assume that $C_i \subseteq B_j$, for some i and j, $1 \leq i, j \leq n$. If $A_i \subseteq C_i$, then $A_i \subseteq B_j$, a contradiction as the splits $\mathcal{A}$ and $\mathcal{B}$ are proper relative to each other. If $C_i \subseteq A_i$, then $C_i \subseteq A_i \cap B_j$. Therefore, $A_i \cap B_j$ is acceptable to agents of type i_3, a contradiction with the assumptions of the lemma. Thus, for every i, j such that $1 \leq i, j \leq n$, $C_i \not\subseteq B_j$.

Next, let us assume that $B_j \subseteq C_i$, for some i and j, $1 \leq i, j \leq n$. If i is odd, then $C_i \subseteq A_i$. It follows that $B_j \subseteq A_i$, a contradiction with the fact that $\mathcal{A}$ and $\mathcal{B}$ are proper relative to each other. Thus, i is even, so $B_j \subseteq C_i \subseteq C_{i-1} \cup C_i \cup C_{i+1} \subseteq A_{i-1} \cup A_i \cup A_{i+1}$. We recall that we have $B_j \subseteq A_{j-1} \cup A_j$. Hence, $j = i$ or $j = i+1$.

Let us assume that $j = i$. Since i is even, $C_{i-1} \subseteq A_{i-1} \subseteq B_{i-1} \cup B_i$. Moreover, $B_i \subseteq C_i$, so $B_i \cap C_{i-1} = \emptyset$. Thus, $C_{i-1} \subseteq B_{i-1}$, a contradiction with what we already proved above. It follows that we must have $j = i+1$. In this case we similarly get, $C_{i+1} \subseteq A_{i+1} \subseteq B_{i+1} \cup B_{i+2}$ and $B_{i+1} \subseteq C_i$ (so, $B_{i+1} \cap C_{i+1} = \emptyset$). Thus, $C_{i+1} \subseteq B_{i+2}$, a contradiction as in the case $j = i$. This completes the proof that for every i, j such that $1 \leq i, j \leq n$, $B_j \not\subseteq C_i$, and shows that the splits $\mathcal{B}$ and $\mathcal{C}$ are proper relative to each other.

Let us observe that $B_1 \subseteq A_n \cup A_1$, $A_n \subseteq C_n$, $B_1 \cap C_1 \neq \emptyset$ and $C_1 \not\subseteq B_1$. It follows that $B_1 \subseteq C_n \cup C_1$. Since the splits $\mathcal{B}$ and $\mathcal{C}$ are proper relative to each other, Lemma 7.4 implies that for every i, $1 \leq i \leq n$, $B_i \subseteq C_{i-1} \cup C_i$.

Let us suppose that some set $B_i \cap C_j$, where $1 \leq i, j \leq n$, is acceptable for agents of type i_2. Then, there is i, $1 \leq i \leq n$, such that $B_i \cap C_{i-1}$ or $B_i \cap C_i$ has this property.

Let us assume the former. We define a $\frac{3}{4}$-approximate allocation as follows. We consider the sequence $C_i, C_{i+1}, \ldots, C_{i+n_3}$ of $n_3 + 1$ C-sets. One of the sets C_i, C_{i+1} contains an A-set and that set is assigned to an agent of type i_1. The remaining n_3 sets of that sequence go to agents of type i_3. The sequence $B_{i-1}, B_{i-2}, \ldots, B_{i-(2n_1-2)}$ contains at least $n_1 - 1$ sets acceptable for agents of type i_1 (Lemma 7.5). We select some $n_1 - 1$ of such sets and assign them to agents of type i_1. We allocate the remaining $n_1 - 1$ sets from this sequence and the set $B_i \cap C_{i-1}$ to agents of type i_2. In this way, all agents are allocated sets that are acceptable to them. Moreover, these sets are pairwise disjoint. Indeed, the sets C_{i+n_3} and $B_{i-(2n_1-2)}$ do not overlap because $i - (2n_1 - 2) = i + n - 2n_1 + 2 = i + n_3 + 2$ (we recall that the arithmetic of indices is modulo n), and $B_{i+n_3+2} \subseteq C_{i+n_3+1} \cup C_{i+n_3+2}$. Thus, the assignment is a $\frac{3}{4}$-approximate allocation.

The latter case, when $B_i \cap C_i$ is acceptable to an agent of type i_2 can be handled similarly by considering sequences $C_{i-1}, C_{i-2}, \ldots, C_{i-(n_3+1)}$ and $B_{i+1}, B_{i+2}, \ldots, B_{i+2n_1-2}$.

Assume now that no set $B_i \cap C_j$ is acceptable for agents of type i_2. By Lemma 7.5 one in every two consecutive sets of the mms-split $C_1, \ldots, C_n$ is acceptable for agents of type i_2. Clearly, A_n is a jump to the split $\mathcal{C}$ because $C_1 \subseteq A_1$. Moreover, $C_{n_3} \subseteq A_{n_3}$ or $C_{n_3+1} \subseteq A_{n_3+1}$. Therefore, C_{n_3+j} is a jump for some $j \in \{0, 1\}$. It follows that the sequence

$$\mathcal{Z} = C_1, \ldots, C_{n_3+j}, A_{n_3+j+1}, \ldots, A_n$$

is $\mathcal{AC}$-useful. By the comment above, at least j sets in $C_1, \ldots, C_{n_3+j}$ are acceptable to agents of type i_2 (since $n_3 \geq 1$). We assign j of these sets to agents of type i_2. The remaining n_3 sets of $C_1, \ldots, C_{n_3+j}$ are assigned to agents of type i_3. Next, the sets $A_{n_3+j+1}, \ldots, A_n$ are distributed among agents of types i_1 and i_2. Agents of type i_2 take $n_2 - j$ sets of this interval (there are this many sets acceptable to agents of type i_2 because $\left\lfloor \frac{n-n_3-j}{2} \right\rfloor = \left\lfloor \frac{2n_2-j}{2} \right\rfloor \geq n_2 - j$). The agents of type i_1 receive the remaining n_1 sets of the A-interval. By construction, this allocation is $\frac{3}{4}$-approximate. □

Theorem 4.9. *Let C be a cycle of goods and N a set of agents of at most three types. Then, a $\frac{3}{4}$-approximate allocation of goods on C to agents in N exists.*

Proof. We adhere to the notation from Lemma 7.6. Further, as in that lemma, we assume that $n_3 \geq 1$ (otherwise, we are in the case of agents of two types settled by Theorem 6.3). We now consider several cases. In each case we assume that none of the cases considered earlier applies.

Case 1. Some set $A_i \cap B_j$ is acceptable to some agent.
In this case, the existence of a $\frac{3}{4}$-approximate allocation follows directly from Lemma 6.1.

From now on, we assume Case 1 does not apply, that is, no set $A_i \cap B_j$ is acceptable to any agent. It follows that for every $1 \leq i, j \leq n$, $A_i \not\subseteq B_j$ and $B_j \not\subseteq A_i$. Thus, the splits $\mathcal{A}$ and $\mathcal{B}$ are proper relative to each other. This means, in particular, that from now on we may apply Lemma 7.5 to splits $\mathcal{A}$ and $\mathcal{B}$, as well as to $\mathcal{B}$ and $\mathcal{A}$.

Case 2. Some C-set is contained in some A-set.

Given our comments above, if $n_1 = n_2$, Lemma 7.6 applies and we are done. Thus, we will assume that $n_1 > n_2 \geq n_3$. We can also assume without loss of generality that $C_1 \subseteq A_1$. Then, the sets A_n and C_1 are jumps.

If the set C_{n_3} is a jump too, then the sequence

$$\mathcal{Z} = C_1, \ldots, C_{n_3}, A_{n_3+1}, \ldots, A_n$$

is $\mathcal{AC}$-useful. By Lemma 7.5 (which also applies now), at least $\lfloor \frac{n_1+n_2}{2} \rfloor \geq n_2$ of the sets that form the sequence $A_{n_3+1}, \ldots, A_n$ are acceptable for agents of type i_2. We assign n_2 of them to these agents. We assign the remaining n_1 sets of this interval to agents of type i_1. Agents of type i_3 receive the sets $C_1, \ldots, C_{n_3}$. In this way, we construct a $\frac{3}{4}$-approximate allocation.

If C_{n_3} is not a jump, then let k be the largest integer such that $1 \leq k < n_3$ and C_k is a jump (such k exists because C_1 is a jump). Moreover, let ℓ be the smallest integer j such that $n_3 < j \leq n$ and C_j is a jump, if such j exists, or $\ell = n$ otherwise. We define

$$\mathcal{Z} = C_1, \ldots, C_k, A_{k+1}, \ldots, A_{\ell-(n_3-k)}, C_{\ell-(n_3-k)+1}, \ldots, C_\ell, A_{\ell+1} \ldots, A_n$$

and observe that it is $\mathcal{AC}$-useful. Indeed, C_k is a jump by the construction. Further, $k < \ell - (n_3 - k) < \ell$ and for $k < j < \ell$ the sets C_j are not jumps. Thus, $A_{\ell-(n_3-k)}$ is a jump by Lemma 7.1. Finally, if $\ell < n$, C_ℓ is a jump by the choice of ℓ and A_n is a jump by assumption.

By Lemma 7.5, there are at least $\left\lfloor \frac{\ell-n_3}{2} \right\rfloor + \lfloor \frac{n-\ell}{2} \rfloor \geq \frac{\ell-n_3-1}{2} + \frac{n-\ell-1}{2} = \frac{n_1+n_2}{2} - 1 > n_2 - 1$ sets in the intervals $A_{k+1}, \ldots, A_{\ell-(n_3-k)}$ and $A_{\ell+1} \ldots, A_n$ (the latter present only if $\ell < n$) that are acceptable to agents of type i_2. We assign n_2 of them to these agents. The remaining sets of these intervals (there are n_1 of them) are assigned to agents of type i_1. Finally, agents of type i_3 receive the sets in the intervals $C_1, \ldots, C_k$ and $C_{\ell-(n_3-k)+1}, \ldots, C_\ell$. This yields a $\frac{3}{4}$-approximate allocation.

Case 3. Some A-set is contained in some C-set.

Without loss of generality we may assume that $A_1 \subseteq C_1$. Since no C-set is contained in an A-set (that possibility is excluded by Case 2), a simple inductive argument shows that for every $i = 2, \ldots, n$, A_i ends *strictly* before C_i does. In particular, A_n ends strictly before C_n does. This implies that $A_1 \cap C_n \neq \emptyset$, a contradiction with $A_1 \subseteq C_1$ (and the fact that different sets in a split are disjoint).

Given Cases 2 and 3, from now on we may assume that the splits $\mathcal{A}$ and $\mathcal{C}$ are proper relative to each other.

Case 4. Some set $A_i \cap C_j$ is acceptable to agents of type i_3.

As we noted, we may assume that the splits $\mathcal{A}$ and $\mathcal{C}$ proper are relative to each other. Thus, we can relabel the sets in the split $\mathcal{C}$ if necessary so that for all i, $A_i \subseteq C_i \cup C_{i+1}$ and, consequently, also $C_i \subseteq A_{i-1} \cup A_i$.

We can assume without loss of generality that $A_1 \cap C_1$ or $A_n \cap C_1$ is acceptable to agents of type i_3. Let us assume the former. We allocate $A_1 \cap C_1$ and the sets $C_2, C_3, \ldots, C_{n_3}$ to agents of type i_3. Clearly, all these sets are included in $A_1 \cup \ldots \cup A_{n_3}$. The sequence $A_{n_3+1}, \ldots, A_n$ of the remaining A-sets contains $n_1 + n_2$ sets. By Lemma 7.5, at least $\lfloor \frac{n_1+n_2}{2} \rfloor \geq n_2$ of them are acceptable for agents of type i_2. We allocate any n_2 of those sets to agents of type i_2 and the remaining n_1 of them to the agents of type i_1. The allocation we defined in this way extends to a $\frac{3}{4}$-approximate allocation for C.

The other case, when $A_n \cap C_1$ is acceptable to agents of type i_3, can be handled similarly by considering sequences $C_n, C_{n-1}, \ldots, C_{n-n_3+2}$ and $A_1, A_2, \ldots, A_{n-n_3}$.

Excluding Cases 1-4 means in particular that $\mathcal{A}$ and $\mathcal{C}$ are proper relative to each other and that no set $A_i \cap C_j$ is acceptable to agents of type i_3. Therefore, from now on we may apply Lemma 7.5 also to the splits $\mathcal{A}$ and $\mathcal{C}$.

Case 5. $n_1 \geq \lfloor \frac{n}{2} \rfloor$.

Let A_i be any A-set acceptable to an agent of type i_3. Such sets exist by Lemma 7.5. Without loss of generality we may assume that A_n is acceptable to an agent of type i_3. By Lemma 7.5, the sequence $A_1, A_2, \ldots, A_{2n_2}$ contains n_2 sets that are acceptable to agents of type i_2. We allocate these sets to them. Next, we consider the sequence $A_{2n_2+1}, A_{2n_2+2}, \ldots, A_{n-1}$ of $n-1-2n_2$ sets that follow A_{2n_2}. Since $n_1 \geq \lfloor \frac{n}{2} \rfloor = \lceil \frac{n-1}{2} \rceil$, we have $\lfloor \frac{n-1}{2} \rfloor \geq n_2 + n_3 - 1$. Thus, $\lfloor \frac{n-1-2n_2}{2} \rfloor \geq n_3 - 1$. By Lemma 7.5, it follows that some $n_3 - 1$ sets in the sequence $A_{2n_2+1}, A_{2n_2+2}, \ldots, A_{n-1}$ are acceptable to agents of type i_3. We allocate these sets and the set A_n to agents of type i_3. We allocate the remaining n_1 sets in $A_1, \ldots, A_n$ to agents of type i_1. This defines a $\frac{3}{4}$-approximate allocation for C.

Case 6. Some set $A_i \cap C_j$ is acceptable to agents of type i_1 and $n_1 > n_2$.

By the same argument as in Case 4, we may assume that $C_i \subseteq A_{i-1} \cup A_i$ for all i and that $A_1 \cap C_1$ or $A_n \cap C_1$ is acceptable to agents of type i_1. If $A_1 \cap C_1$ is acceptable to agents of type i_1, we proceed as follows. We allocate the sets $C_2, C_3, \ldots, C_{n_3+1}$ to agents of type i_3. Next, we note that, by Lemma 7.5, at least one of each consecutive pair of sets in the sequence $A_{n_3+2}, A_{n_3+3}, \ldots, A_n$ is acceptable to agents of type i_2. Since the sequence contains $n - n_3 - 1 = n_1 + n_2 - 1 \geq 2n_2$ sets (recall that we assume here that $n_1 > n_2$), some n_2 of the sets in the sequence are acceptable to agents of type i_2. We allocate these n_2 sets to agents of type i_2. We allocate the remaining $n_1 - 1$ sets in that sequence and the set $A_1 \cap C_1$ to agents of type i_1. In this way, all agents get sets that are acceptable to them. Further, we note that $C_2 \cup \ldots \cup C_{n_3+1} \subseteq A_1 \cup \ldots \cup A_{n_3+1}$. It follows that all sets used in the allocation are pairwise disjoint and so give rise to a $\frac{3}{4}$-approximate allocation for C.

The case when $A_1 \cap C_n$ is acceptable to an agent of type i_1 follows from the one we just considered by the same argument as that used in Case 4.

Case 7. Some C-set is contained in some B-set.

By Lemma 7.6, we can assume that $n_1 > n_2$ (because Case 1 is excluded, $\mathcal{A}$ and $\mathcal{B}$ are proper relative to each other and no set $A_i \cap B_j$ is acceptable to any agent). We now recall that splits $\mathcal{C}$ and $\mathcal{A}$ are proper relative to each other. Moreover, no set $A_i \cap C_j$ is acceptable for agents of type i_1 (Case 6 is excluded). Thus, Lemma 7.5 applies to splits $\mathcal{C}$ and $\mathcal{A}$ and implies that every pair of consecutive sets of the mms-split $C_1, \ldots, C_n$ contains at least one set acceptable to agents of type i_1.

We can assume without loss of generality that $C_1 \subseteq B_1$. It follows that the sets B_n and C_1 are jumps. If the set C_{2n_3} is a jump, then the sequence

$$\mathcal{Z}_1 = C_1, \ldots, C_{2n_3}, B_{2n_3+1}, \ldots, B_n$$

is $\mathcal{BC}$-useful. Similarly, if the set C_{2n_3-1} is a jump, then the sequence

$$\mathcal{Z}_2 = C_1, \ldots, C_{2n_3-1}, B_{2n_3}, \ldots, B_n$$

is $\mathcal{BC}$-useful.

Assume now that the sets C_{2n_3-1} and C_{2n_3} are not jumps. Let k, $1 \leq k < 2n_3 - 1$, be the largest index such that C_k is a jump (since C_1 is a jump, k is well defined), and let ℓ be the smallest integer j such that $2n_3 < j \leq n$ and C_j is a jump, if such j exists, or $\ell = n$ otherwise.

We observe that the sequence

$$\mathcal{Z}_3 = C_1, \ldots, C_k, B_{k+1}, \ldots, B_{\ell-(2n_3-k)}, C_{\ell-(2n_3-k)+1}, \ldots, C_\ell, B_{\ell+1} \ldots, B_n$$

is $\mathcal{BC}$-useful. In particular the set $B_{\ell-(2n_3-k)}$ is a jump. Indeed, $k < \ell - (2n_3 - k) < \ell$ and for $k < j < \ell$ the sets C_j are not jumps. By Lemma 7.1, the sets B_j are jumps.

It can be shown similarly that the sequence

$$\mathcal{Z}_4 = C_1, \ldots, C_k, B_{k+1}, \ldots, B_{\ell-(2n_3-k)+1}, C_{\ell-(2n_3-k)+2}, \ldots, C_\ell, B_{\ell+1} \ldots, B_n$$

is $\mathcal{BC}$-useful.

We now observe that the sequence $\mathcal{Z}_1$ is a special case of the sequence $\mathcal{Z}_3$ for $k = 2n_3$, and the sequence $\mathcal{Z}_2$ is a special case of the sequence $\mathcal{Z}_4$ for $k = 2n_3 - 1$. Thus, to complete this case, it is enough to describe allocations based on sequences $\mathcal{Z}_3$ and $\mathcal{Z}_4$.

For k even, we construct an allocation based on the sequence $\mathcal{Z}_3$ which, we recall is $\mathcal{BC}$-useful. Since no set $A_i \cap C_j$ is acceptable for agents of type i_1, by Lemma 7.5 half of the sets $C_1, \ldots, C_k$ and half of the sets $C_{\ell-(2n_3-k)+1}, \ldots, C_\ell$ (the number of sets in each of these intervals is even), n_3 sets in total, are acceptable for agents of type i_1. We allocate these sets to agents of type i_1. Agents of type i_3 receive the remaining n_3 sets from these intervals. Since at least every other set of the intervals $B_{k+1}, \ldots, B_{\ell-(2n_3-k)}$ and $B_{\ell+1} \ldots, B_n$ in $\mathcal{Z}_5$ is acceptable for agents of type i_1 (by Lemma 7.5), these two intervals contain at least $\left\lfloor \frac{\ell-2n_3}{2} \right\rfloor + \left\lfloor \frac{n-\ell}{2} \right\rfloor = \left\lfloor \frac{\ell}{2} \right\rfloor + \left\lfloor \frac{n-\ell}{2} \right\rfloor - n_3 \geq \left\lfloor \frac{n}{2} \right\rfloor - 1 - n_3 \geq n_1 - n_3$ sets (because Case 5 is excluded, $\left\lfloor \frac{n}{2} \right\rfloor \geq n_1 + 1$) that are acceptable to agents of type i_1. We allocate $n_1 - n_3$ of these sets to the agents of type i_1 that have not been allocated any set in the previous stage. The remaining sets, there are n_2 of them, are allocated to agents of type i_2. Clearly, this allocation gives rise to a $\frac{3}{4}$-approximate allocation for C.

For k odd, we construct an allocation based on the $\mathcal{BC}$-useful sequence $\mathcal{Z}_4$. By the same argument as above, at least $\lfloor \frac{k}{2} \rfloor$ of the sets $C_1, \ldots, C_k$ and at least $\lfloor \frac{2n_3-k-1}{2} \rfloor$ of the sets $C_{\ell-(2n_3-k)+2}, \ldots, C_\ell$ are acceptable for agents of type i_1. Since k is odd, the total number of sets in the two sequences that are acceptable to agents of type i_1 is at least $\frac{k-1}{2} + \frac{2n_3-k-1}{2} = n_3 - 1$. We select some $n_3 - 1$ of these sets and allocate them to $n_3 - 1$ agents of type i_1. The number of sets remaining in the two sequences is n_3. We allocate them to agents of type i_3. Moreover, the intervals $B_{k+1}, \ldots, B_{\ell-(2n_3-k)+1}$ and $B_{\ell+1} \ldots, B_n$ contain at least $\left\lfloor \frac{\ell-2n_3+1}{2} \right\rfloor + \left\lfloor \frac{n-\ell}{2} \right\rfloor = \left\lfloor \frac{\ell+1}{2} \right\rfloor + \left\lfloor \frac{n-\ell}{2} \right\rfloor - n_3 \geq \left\lfloor \frac{n}{2} \right\rfloor - n_3 \geq n_1 - n_3 + 1$ sets that are acceptable to agents of type i_1 (as before, we use the fact that $\left\lfloor \frac{n}{2} \right\rfloor \geq n_1 + 1$). We allocate some $n_1 - n_3 + 1$ of these sets to agents of type i_1 that have not been allocated a set before. Finally, we allocate the remaining sets in these two sequences, there are n_2 of them, to agents of type i_2. This allocation gives rise to a $\frac{3}{4}$-approximate allocation for C.

From now on, we will assume that no C-set is included in any B-set. Reasoning as in Case 3, we can prove that also no B-set is included in any C-set. Thus, in the remaining part of the proof we may assume that the splits $\mathcal{B}$ and $\mathcal{C}$ are proper relative to each other.

Case 8. Some set $B_i \cap C_j$ is acceptable to agents of type i_3.

Since the splits $\mathcal{B}$ and $\mathcal{C}$ are proper relative to each other, we may relabel the sets if necessary so that for all i, $B_i \subseteq C_i \cup C_{i+1}$, and that $B_1 \cap C_1$ or $B_n \cap C_1$ is acceptable for agents of type i_3. We will

consider the first case, that is, that $B_1 \cap C_1$ is acceptable to agents of type i_3. As in several places before, the other case can be handled in a similar way.

Suppose first that $n_1 > n_2$. Since the conditions formulated in Cases 1 and 6 are not satisfied, by Lemma 7.5 every other set in mms-splits $\mathcal{B}$ and $\mathcal{C}$ is acceptable to agents of type i_1 (recall that $\mathcal{B}$ and $\mathcal{A}$ are proper relative to each other and so are $\mathcal{C}$ and $\mathcal{A}$). Thus, there are at least $n_3 - 1$ sets in the interval $C_2, C_3, \ldots, C_{2n_3-1}$ that are acceptable to agents of type i_1. We allocate these sets to those agents. We allocate the remaining $n_3 - 1$ the sets in this sequence and the set $B_1 \cap C_1$ to agents of type i_3. Similarly, we note that the sequence $B_{2n_3}, B_{2n_3+1}, \ldots, B_n$ contains at least $\left\lfloor \frac{n-2n_3+1}{2} \right\rfloor \geq \left\lfloor \frac{n}{2} \right\rfloor - n_3 \geq n_1 - n_3 + 1$ sets that are acceptable to agents of type i_1. We allocate some $n_1 - n_3 + 1$ of such sets to those agents of type i_1 that have not yet been allocated a set. The remaining sets in the sequence, there are n_2 of them, are allocated to agents of type i_2. All sets in the assignment are pairwise disjoint as $B_{2n_3} \subseteq C_{2n_3} \cup C_{2n_3+1}$. Thus, the assignment we defined gives rise to a $\frac{3}{4}$-approximate allocation for C.

Thus, assume that $n_1 = n_2$. In this case agents of type i_3 receive $B_1 \cap C_1$ and the sets $C_2, C_3, \ldots, C_{n_3}$. Agents of types i_1 and i_2 distribute among themselves the sets $B_{n_3+1}, B_{n_3+2}, \ldots, B_n$. This is possible as the number of those sets is even, in fact, equal to $2n_1$ ($= 2n_2$), and every other set of this sequence is acceptable for agents of type i_1. Since $B_{n_3+1} \subseteq C_{n_3+1} \cup C_{n_3+2}$, these sets are disjoint with the sets assigned to agents of type i_3 and, clearly, they are pairwise disjoint themselves. It follows that this allocation gives rise to a $\frac{3}{4}$-approximate allocation for C.

Case 9. None of the conditions formulated in Cases 1-8 holds.

We may assume that any pair of splits $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are proper relative to each other. Moreover, relabeling the sets if necessary, we may also assume that $A_i \subseteq C_i \cup C_{i+1}$ and $B_i \subseteq C_i \cup C_{i+1}$. Thus, A_i and B_i overlap and, consequently, $A_i \subseteq B_i \cup B_{i+1}$ or $A_i \subseteq B_{i-1} \cup B_i$. Both cases can be dealt with similarly, so we assume that $A_i \subseteq B_i \cup B_{i+1}$.

We note that for every i

$$C_i \cup C_{i+1} \cup C_{i+2} = (A_{i-1} \cap C_i) \cup A_i \cup (A_{i+1} \cap B_{i+1}) \cup (B_{i+2} \cap C_{i+2}).$$

The value of $C_i \cup C_{i+1} \cup C_{i+2}$ for agents of type i_3 is 3. The conditions formulated in Cases 1, 4 and 8 do not hold, so the sets $A_{i-1} \cap C_i$, $A_{i+1} \cap B_{i+1}$ and $B_{i+2} \cap C_{i+2}$ are not acceptable for agents of type 3. Thus, the value of A_i for agents of type i_3 is larger than $3 - 3 \cdot \frac{3}{4} = \frac{3}{4}$. As i was arbitrary, it follows that every A-set is acceptable for agents of type i_3.

We have $n_2 \leq n_1 < \left\lfloor \frac{n}{2} \right\rfloor$, where the latter inequality holds because of Case 5 being excluded. Since at least one in every two consecutive sets A_i, A_{i+1} is acceptable to agents of type i_2, we assign n_2 of such sets to agents of type i_2. All remaining A-sets are acceptable for agents of types i_1 and i_3. Thus, they can be allocated among them arbitrarily yielding a $\frac{3}{4}$-satisfying allocation. □

References

Amanatidis, G., Markakis, E., Nikzad, A., & Saberi, A. (2017). Approximation algorithms for computing maximin share allocations. *ACM Transactions on Algoritms (TALG)*, *13*(4), 52:1–52:28.

Barman, S., & Murthy, S. K. K. (2017). Approximation algorithms for maximin fair division. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, pp. 647–664. ACM.

Bilò, V., Caragiannis, I., Flammini, M., Igarashi, A., Monaco, G., Peters, D., Vinci, C., & Zwicker, W. S. (2019). Almost envy-free allocations with connected bundles. In Blum, A. (Ed.), *Proceedings of the 10th Innovations in Theoretical Computer Science Conference, ITCS 2019*, Vol. 124 of *LIPIcs*, pp. 14:1–14:21. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik.

Bouveret, S., Cechlárová, K., Elkind, E., Igarashi, A., & Peters, D. (2017). Fair division of a graph. In Sierra, C. (Ed.), *Proceedings of the 26th International Joint Conference on Artificial Intelligence, IJCAI 2017*, pp. 135–141. IJCAI.org.

Bouveret, S., Cechlárová, K., & Lesca, J. (2018). Chore division on a graph. *CoRR*, *abs/1812.01856*.

Bouveret, S., & Lemaître, M. (2016). Characterizing conflicts in fair division of indivisible goods using a scale of criteria. *Autonomous Agents and Multi-Agent Systems*, *30*(2), 259–290.

Brams, S. J., & Taylor, A. D. (1996). *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press.

Budish, E. (2011). The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, *119(6)*, 1061–1103.

Garey, M. R., & Johnson, D. S. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Co.

Garg, J., McGlaughlin, P., & Taki, S. (2019). Approximating maximin share allocations. In *Proceedings of the 2nd Symposium on Simplicity in Algorithms (SOSA), volume 69*, pp. 20:1–20:11. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik.

Garg, J., & Taki, S. (2019). An improved approximation algorithm for maximin shares. *CoRR*, *abs/1903.00029*.

Ghodsi, M., Hajiaghayi, M., Seddighin, M., Seddighin, S., & Yami, H. (2018). Fair allocation of indivisible goods: Improvements and generalizations. In *Proceedings of the 2018 ACM Conference on Economics and Computation, EC-2018*, pp. 539–556, New York, NY, USA. ACM.

Igarashi, A., & Peters, D. (2019). Pareto-optimal allocation of indivisible goods with connectivity constraints. In *The 33rd AAAI Conference on Artificial Intelligence, AAAI 2019*, pp. 2045–2052. AAAI Press.

King, R., & Burton, S. (1982). Land fragmentation: Notes on a fundamental rural spatial problem. *Progress in Human Geography*, *6(4)*, 475–494.

Kurokawa, D., Procaccia, A. D., & Wang, J. (2018). Fair enough: Guaranteeing approximate maximin shares. *J. ACM*, *65*(2), 8:1–8:27.

Lonc, Z., & Truszczynski, M. (2018). Maximin share allocations on cycles. In Lang, J. (Ed.), *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI 2018*, pp. 410–416. IJCAI.org.

Papadimitriou, C. H. (1993). *Computational Complexity*. Pearson.

Procaccia, A. (2016). Cake-cutting algorithms. In Brandt, F., Conitzer, V., Endriss, U., Lang, J., & Procaccia, A. D. (Eds.), *Handbook of Computational Social Choice*, pp. 311–329. Cambridge University Press.

Procaccia, A. D., & Wang, J. (2014). Fair enough: guaranteeing approximate maximin shares. In Babaioff, M., Conitzer, V., & Easley, D. (Eds.), *Proceedings of the ACM Conference on Economics and Computation, EC-2014*, pp. 675–692. ACM.

Suksompong, W. (2019). Fairly allocating contiguous blocks of indivisible items. *Discrete Applied Mathematics*, *260*, 227–236.