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MAXIMIZATION OF A CONVEX QUADRATIC FUNCTION UNDER LINEAR CONSTRAINTS

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Abstract

This paper addresses itself to the maximization of a convex quadratic function subject to linear constraints. We first prove the equivalence of this problem to the associated bilinear program. Next we apply the theory of bilinear programming developed in [9] to compute a local maximum and to generate a cutting plane which eliminates a region containing that local maximum. Then we develop an iterative procedure to improve a given cut by exploiting the symmetric structure of the bilinear program. This procedure either generates a point which is strictly better than the best local maximum found, or generates a cut which is deeper (usually much deeper) than Tui's cut. Finally the results of numerical experiments on small problems are reported.

1. Introduction

Since the appearance of a pioneering paper by H. Tui [14], maximization of a convex function over a polytope has attracted much attention. Two algorithms were proposed in his paper: one cutting-plane and the other enumerative. The idea of his cutting plane is admittedly very attractive. Unfortunately, the numerical experiments reported in [16] on a naive cutting plane approach were discouraging enough to shift the researchers more into the direction of enumerative approaches ([7,8,17]).

In this paper, however, we will propose a cutting plane algorithm for maximizing a convex quadratic function subject to linear constraints by fully exploiting the special structure of the problem. We will first prove the equivalence of the original quadratic program and an associated bilinear program. We will then discuss the ways to generate a valid cut and develop the iterative improvement procedure of a given valid cut by using the theory of bilinear programming (see [9] for details). The algorithm has been tested on CYBER 74 up to the problem of size 11 × 22, the results of which are summarized at the end of the paper. It turned out that the iterative improvement procedure is quite powerful in generating a deep cut. This work is closely related to [9], whose results will be frequently referred to without proof. Also some of our results parallel those established in [2].

2. ε-Locally Maximum Basic Feasible Solution and Equivalent Bilinear Program

We will consider the following quadratic program:

$$\max f(x) = 2c^t x + x^t Qx$$

$$\text{s.t.} \quad Ax = b \quad , \quad x \geq 0 \quad , \tag{2.1}$$

where $c, x \in F^n$, $b \in R^m$, $A \in R^{m\times n}$ and $Q \in R^{n\times n}$ is a symmetric positive semi-definite matrix. We will assume that the feasible region

$$X = \{x \in R^n | Ax = b, x \geq 0\} \tag{2.2}$$

is non-empty and bounded. It is well known that in this case (2.1) has an optimal solution among basic feasible solutions.

Given a feasible basis B of A, we will partition A as (B,N) assuming, without loss of generality, that the first m columns of A are basic. Partition x correspondingly, i.e. $x = (x_B, x_N)$. Premultiplying B^{-1} to the constraint equation $Bx_B + Nx_N = b$ and suppressing basic variables x_B, we get the following system which is totally equivalent to (2.1):

$$\max \bar{f}(x_N) = 2\bar{c}_N^t x_N + x_N^t \bar{Q} x_N + \phi_o$$

$$\text{s.t.} \quad B^{-1}Nx_N \leq B^{-1}b \quad , \quad x_N \geq 0 \quad . \tag{2.3}$$

Here, $\phi_o = f(x^o)$ where $x^o = (x_B^o, x_N^o) = (B^{-1}b, 0)$ and

$$\bar{c}_N = c_N - \bar{N}^t c_B - 2\bar{N}^t Q_{BB} b + 2Q_{BN}^t x_B^o \quad ,$$

$$\bar{Q} = Q_{NN} + \bar{N}^t Q_{BB} \bar{N} - 2\bar{N}^t Q_{BN} \quad ,$$

where $\bar{N} = B^{-1}N$ and $Q = \begin{pmatrix} Q_{BB} & Q_{BN} \\ Q_{BN}^t & Q_{NN} \end{pmatrix}$.

Introducing the notations:

$$\ell = n - m \quad , \quad d = \bar{c}_N \quad , \quad y = x_N \quad , \quad F = B^{-1}N \quad ,$$

$$f = B^{-1}b \quad , \quad D = \bar{Q} \quad ,$$

we will rewrite (2.3) as:

$$\max\ g(y) = 2d^t y + y^t D y + \phi_o$$

$$\text{s.t.} \quad Fy \leq f \quad , \quad y \geq 0 \quad , \qquad (2.4)$$

and call this a 'canonical' representation of (2.1) relative to a feasible basis B. To express the dependence of vectors in (2.4) on B, we occasionally use the notation d(B), etc.

Definition 2.1. Given a basic feasible solution $x \in X$, let $N_X(x)$ be the set of adjacent basic feasible solutions which can be reached from x in one pivot step.

Definition 2.2. Let ε be a non-negative scalar. A basic feasible solution $x^* \in X$ is called an ε-locally maximum basic feasible solution of (2.1) if

(i) $d \leq 0$

(ii) $f(x^*) > f(x) - \varepsilon \quad , \forall\, x \in N_X(x^*)$.

Let us introduce here a bilinear program associated with (2.1), which is essential for the development of cutting planes:

$$\begin{aligned} \max\ \phi(x_1,x_2) &= c^t x_1 + c^t x_2 + x_1 Q x_2 \\ \text{s.t.} \quad Ax_1 &= b\ ,\quad x_1 \geq 0 \\ Ax_2 &= b\ ,\quad x_2 \geq 0\ . \end{aligned} \tag{2.5}$$

<u>Theorem 2.1</u> [9]. If X is non-empty and bounded, then (2.5) has an optimal solution (x_1^*,x_2^*) where x_1^* and x_2^* are basic feasible solutions of X.

Moreover, two problems (2.1) and (2.5) are equivalent in the following sense:

<u>Theorem 2.2</u>. If x^* is an optimal solution of (2.1), then $(x_1,x_2) = (x^*,x^*)$ is an optimal solution of (2.5). Conversely, if (x_1^*,x_2^*) is optimal for (2.5), then <u>both</u> x_1^*,x_2^* are optimal for (2.1).

<u>Proof</u>. Let x^* be optimal for (2.1) and (x_1^*,x_2^*) be optimal for (2.5). By definition $f(x^*) \geq f(x), \forall x \in X$. In particular,

$$f(x^*) \geq f(x_i^*) = \phi(x_i^*,x_i^*) \quad , \quad i = 1,2 \quad ;$$

also

$$\begin{aligned} \phi(x_1^*,x_2^*) &= \max\{\phi(x_1,x_2)\,|\,x_1 \in X,\ x_2 \in X\} \\ &\geq \max\{\phi(x,x)\,|\,x \in X\} = f(x^*) \quad . \end{aligned}$$

To establish the theorem, it suffices therefore to prove that

$$f(x_i^*) = \phi(x_1^*,x_2^*), \quad i = 1,2 \tag{2.6}$$

because we then have $f(x_i^*) \geq f(x^*)$, $i = 1,2$ and $\phi(x^*,x^*) = f(x^*) = \phi(x_1^*,x_2^*)$. Let us now prove (2.6). Since (x_1^*,x_2^*) is optimal for (2.5), we have

$$0 \leq \phi(x_1^*,x_2^*) - \phi(x_1^*,x_1^*) = c^t(x_2^* - x_1^*) + (x_1^*)^t\, Q(x_2^* - x_1^*)$$

$$0 \leq \phi(x_1^*,x_2^*) - \phi(x_2^*,x_2^*) = c^t(x_1^* - x_2^*) + (x_2^*)^t\, Q(x_1^* - x_2^*) \quad .$$

Adding these two inequalities, we obtain

$$(x_1^* - x_2^*)^t\, Q(x_1^* - x_2^*) \leq 0 \quad .$$

Since Q is positive semi-definite, this implies $Q(x_1^* - X_2^*) = 0$. Putting this into the inequality above, we get $c^t(x_1^* - x_2^*) = 0$. Hence $\phi(x_1^*,x_1^*) = \phi(x_1^*,x_2^*) = \phi(x_2^*,x_2^*)$ as was required.

As before, we will define a canonical representation of (2.5) relative to a feasible basis B:

$$\begin{aligned} \max\ \psi(y_1,y_2) &= d^t z_1 + d^t z_2 + z_1^t D z_2 + \phi_o \\ \text{s.t.} \qquad Fz_1 &\leq f \quad , \quad z_1 \geq 0 \\ Fz_2 &\leq f \quad , \quad z_2 \geq 0 \quad , \end{aligned} \tag{2.7}$$

which is equivalent to (2.4). Also let

$$Y = \{y \in R^\ell \,|\, Fy \leq f, \quad y \geq 0\} \quad . \tag{2.8}$$

Now that we have established the equivalence of (2.1) and (2.5), we can use all the results developed in [9].

3. Valid Cutting Planes and Iterative Improvement Procedure

We will assume in this section that an ε-locally maximum basic feasible solution x^0 and corresponding basis B_0 have been obtained. Also, let ϕ_{max} be the best feasible solution obtained so far by one method or another.

Given a canonical representation (2.4) relative to B_0, we will proceed to introduce a 'valid' cutting plane in the sense that it

(i) does eliminate the current ε-locally maximum basic feasible solution, i.e., the point $y = 0$;

(ii) does not eliminate any point y in Y for which $g(y) > \phi_{max} + \varepsilon$.

Theorem 3.1 [14]. Let θ_i be the larger root of the equation:

$$2d_i\lambda + d_{ii}\lambda^2 = \phi_{max} - \phi_0 + \varepsilon \quad . \tag{3.1}$$

Then the cut

$$H(\theta): \quad \sum_{i=1}^{\ell} y_i/\theta_i \geq 1$$

is a valid cut.

This theorem is based upon the convexity of $g(y)$ and the simple geometric observation illustrated below for the two dimensional case.

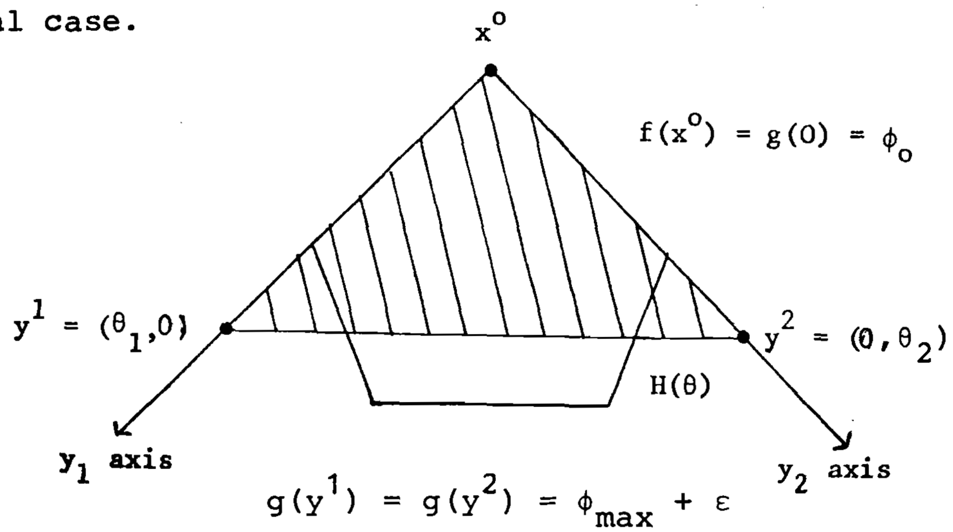

Figure 3.1

Though this cut is very easy to generate and attractive from the geometric point of view, it tends to become shallower as the dimension increases, and the results of numerical experiments reported in [16] were quite disappointing. In this section, we will demonstrate that if we fully exploit the structure, then we can generate a cut which is generally much deeper than Tui's cut.

Let us start by stating the results proved in [9], taking into account the symmetric property of the bilinear programming problem (2.7) associated with (2.4).

<u>Theorem 3.2</u>. Let θ_i be the maximum of λ for which

$$\max_{z_2} \max_{z_1} \{\psi z_1, z_2) \mid 0 \leq z_{1i} \leq \lambda,\ z_{1j} = 0,\ j \neq i,\ z_2 \in Y\} \leq \phi_{max} + \varepsilon \quad .$$

Then the cut

$$H(\theta): \quad \sum_{j=1}^{\ell} y_j/\theta_j \geq 1$$

is a valid cut (relative to (2.4)).

<u>Theorem 3.3</u>. θ_i of Theorem 3.2. is given by solving a linear program:

$$\theta_i = \min \left[-d^t z + (\phi_{max} - \phi_o + \varepsilon)\ z_o \right]$$

$$\text{s.t.} \qquad Fz - fz_o \leq 0 \qquad (3.2)$$

$$d_{i\cdot}^t z + d_i z_o = 1$$

$$z \geq 0 \quad , \quad z_o \geq 0 \quad ,$$

where $d_{i\cdot}$ is the ith column vector of D.

The readers are referred to section 3 of reference [9] (in particular Theorem 3.3 and 3.5) for the proof of these theorems. E. Balas and C.-A. Burdet [2] obtained the same results by applying the theory of generalized outer polars, while our approach is based upon bilinear programming.

Though the bilinear programming cut (BLP cut) of Theorem 3.2. is usually stronger (eliminates more feasible region) than the corresponding Tui's cut, it need not always be so. Therefore, we will proceed further to improve this cut or <u>any</u> given valid cut to generate a cut which is always stronger (and usually much stronger) than Tui's cut by using <u>local</u> information only.

For a given positive vector $\theta = (\theta_1,\ldots,\theta_\ell) > 0$, let

$$\Delta(\theta) = \{y \in R^\ell \mid \sum_{j=1}^{\ell} y_j/\theta_j \leq 1,\ y_j \geq 0,\quad j = 1,\ldots,\ell\} \quad . \tag{3.3}$$

<u>Theorem 3.4</u>. Let $\tau \geq \theta > 0$. If

$$\max\{\psi(z_1,z_2) \mid z_1 \in \Delta(\theta),\ z_2 \in Y\} \leq \phi_{max} + \varepsilon \tag{3.4}$$

and if

$$\max\{\psi(z_1,z_2) \mid z_1 \in \Delta(\tau),\ z_2 \in Y\setminus\Delta(\theta)\} \leq \phi_{max} + \varepsilon \quad , \tag{3.5}$$

then

$$H(\tau):\quad \sum_{j=1}^{\ell} y_j/\tau_j \geq 1$$

is a valid cut (relative to 2.4)).

<u>Proof</u>. Let $Y_1 = \Delta(\theta) \cap Y$, $Y_2 = (\Delta(\tau)\setminus\Delta(\theta)) \cap Y$, $Y_3 = Y\setminus\Delta(\tau)$. Obviously $Y = Y_1 \cup Y_2 \cup Y_3$. By (3.4) and (3.5), we have that:

$$\max\{\psi(z_1,z_2)\,|\,z_1 \in Y_1,\ z_2 \in Y_1 \cup Y_2 \cup Y_3\} \le \phi_{max} + \varepsilon$$

$$\max\{\psi(z_1,z_2)\,|\,z_1 \in Y_1 \cup Y_2,\ z_2 \in Y_2 \cup Y_3\} \le \phi_{max} + \varepsilon \quad .$$

By symmetry of function ψ, we have that

$$\max\{\psi(z_1,z_2)\,|\,z_1 \in Y_2,\ z_2 \in Y_1\} = \max\{\psi(z_1,z_2)\,|\,z_1 \in Y_1,\ z_2 \in Y_2\} \quad ,$$

and hence

$$\max\{\psi(z_1,z_2)\,|\,z_1 \in Y_1 \cup Y_2,\ z_2 \in Y_1 \cup Y_2\} \le \phi_{max} + \varepsilon \quad .$$

Referring to Theorem 2.2, this implies that

$$\max\{g(y)\,|\,y \in Y_1 \cup Y_2\} \le \phi_{max} + \varepsilon \quad .$$

This, in turn, implies that $H(\tau)$ is a valid cut.

This theorem gives us a technique to improve a given valid cut (e.g. Tui's cut or the cut defined in Theorem 3.2). Given a cut $H(\theta)$, let τ_i be

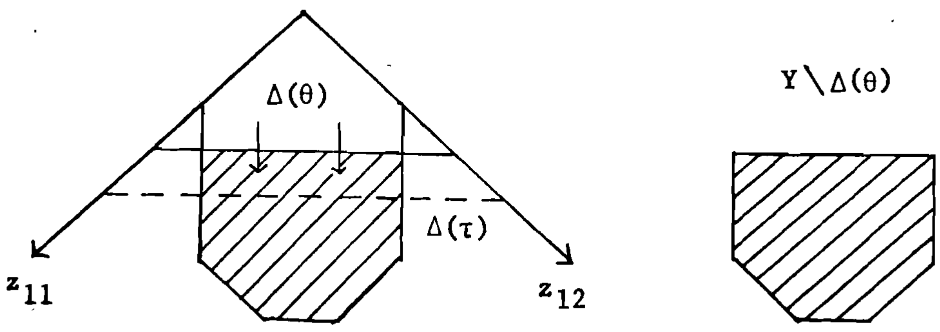

Figure 3.2

the maximum of λ for which :

$$\max\{\psi(z_1,z_2)\,|\,0 \le z_{1i} \le \lambda,\ z_{1j} = 0,\ j \ne i,\ z_2 \in Y \setminus \Delta(\theta)\} \le \phi_{max} + \varepsilon \quad ;$$

then $H(\tau)$ is also a valid cut as is illustrated in Figure 3.2.

It is easy to prove (see [9], Theorems 3.2 and 3.3) that τ_i defined above is equal to the optimal objective value of the following linear program:

$$\tau_i = \min\left[-d^t z + (\phi_{max} - \phi_o + \varepsilon)\ z_o\right]$$

$$\begin{aligned} \text{s.t.} \quad & Fz - fz_o \leq 0 \\ & \sum_{j=1}^{\ell} d_{ij} z_j + d_i z_o = 1 \\ & \sum_{j=1}^{\ell} z_j/\theta_j - z_o \geq 0 \ . \end{aligned} \tag{3.6}$$

Note that since $d \leq 0$ and $\phi_{max} - \phi_o + \varepsilon > 0$, $(z, z_o) = (0,0)$ is a dual feasible solution with only one constraint violated, and that it usually takes only several pivots to solve this linear program starting from this dual feasible solution. Also it should be noted that the objective value is monotonically increasing during the dual simplex procedure and hence we can stop pivoting whenever the objective functional value exceeds some specified level.

Lemma 3.5.

(i) $\phi(x_1, x_2) \leq \max\{\phi(x_1, x_1),\ \phi(x_2, x_2)\},\ \forall x_1 \in X,\ x_2 \in X.$

(ii) If Q is positive definite and $x_1 \neq x_2$, then

$$\phi(x_1, x_2) < \max\{\phi(x_1, x_1),\ \phi(x_2, x_2)\} \ .$$

Proof.

(i) Assume not. Then

$$0 < \phi(x_1,x_2) - \phi(x_1,x_1) = c^t(x_2 - x_1) + x_1^t Q(x_2 - x_1)$$

$$0 < \phi(x_1,x_2) - \phi(x_2,x_2) = c^t(x_1 - x_2) + (x_1 - x_2)^t Qx_2 \quad .$$

Adding these two inequalities, we obtain

$$-(x_1 - x_2)^t \, Q(x_1 - x_2) > 0 \quad ,$$

which is a contradiction since Q is positive semi-definite.

(ii) Assume not. As in (i) above, we get

$$-(x_1 - x_2)^t \, Q(x_1 - x_2) \geq 0 \quad ,$$

which is a contradiction to the assumption that $x_1 - x_2 \neq 0$ and that Q is positive definite.

Theorem 3.6. If Q is positive definite, then the iterative improvement procedure either generates a point $y \in Y$ for which $g(y) \geq \phi_{max} + \varepsilon$ or else generates a cut which is strictly deeper than corresponding Tui's cut.

Proof. Let $H(\theta)$ be Tui's cut and let $H(\tau)$ be the cut resulting from iterative improvement starting from a valid cut $H(\omega)$ where $\omega \geq 0$. Let

$$z_1^i = (0,\ldots,0,\ \tau_i,\ 0,\ldots,0), \quad i = 1,\ldots,\ell \quad .$$

Let $z_2^i \in Y \setminus \Delta(\omega)$ satisfy

$$\psi(z_1^i,z_2^i) \equiv \max\{\psi(z_1^i,z_2) \,|\, z_2 \in Y \setminus \Delta(\omega)\} = \phi_{max} + \varepsilon \quad . \qquad (3.7)$$

Case 1. $\psi(z_2^i, z_2^i) \geq \psi(z_1^i, z_1^i)$. It follows from Lemma 3.5 and (3.7) that

$$g(z_2^i) = \psi(z_2^i, z_2^i) \geq \psi(z_1^i, z_2^i) = \phi_{max} + \varepsilon \quad .$$

Note that $z_2^i \in Y$.

Case 2. $\psi(z_1^i, z_1^i) > \psi(z_2^i, z_2^i)$. Again by Lemma 3.5 and (3.7), we have

$$\psi(z_1^i, z_1^i) \geq \psi(z_1^i, z_2^i) = \phi_{max} + \varepsilon \quad .$$

We will prove that this inequality is indeed a strong one. Suppose that $\psi(z_1^i, z_1^i) = \psi(z_1^i, z_2^i)$; then

$$c^t(z_1^i - z_2^i) + z_1^t D(z_1^i - z_2^i) = 0 \quad .$$

From $\psi(z_1^i, z_2^i) > \psi(z_2^i, z_2^i)$ we obtain

$$c^t(z_2^i - z_1^i) + z_2^t D(z_2^i - z_1^i) > 0 \quad .$$

Adding these two, we have that $(z_1^i - z_2^i)^t D(z_1^i - z_2^i) < 0$, which is a contradiction. Thus we have established

$$g(z_1^i) > \phi_{max} + \varepsilon \quad ,$$

which, in turn, implies that $\tau_i > \theta_i$, since θ_i is defined (see (3.1)) as a point at which $g(\cdot)$ attains the value $\phi_{max} + \varepsilon$. If, on the other hand, z_2^i satisfying (3.7) does not exist, then $\tau_i = \infty$ and therefore $\tau_i > \theta_i$ as before.

It turns out that this iterative improvement procedure quite often leads to a substantially deep cut. Figure 3.3 shows a typical example.

The deeper the cut $H(\theta)$ gets, the better is the chance that some of the non-negativity constraints $y_i \geq 0$, $i = 1,\ldots,\ell$

$$
\begin{cases}
\max -2z_1 - 3z_2 + 2z_1^2 - 2z_1 z_2 + 2z_2^2 \\
\text{s.t.} \quad -z_1 + z_2 \le 1 \\
\quad z_1 - z_2 \le 1 \\
\quad -z_1 + 2z_2 \le 3 \\
\quad 2z_1 - z_2 \le 3 \\
\quad z_1 \ge 0 \;,\; z_2 \ge 0
\end{cases}
$$

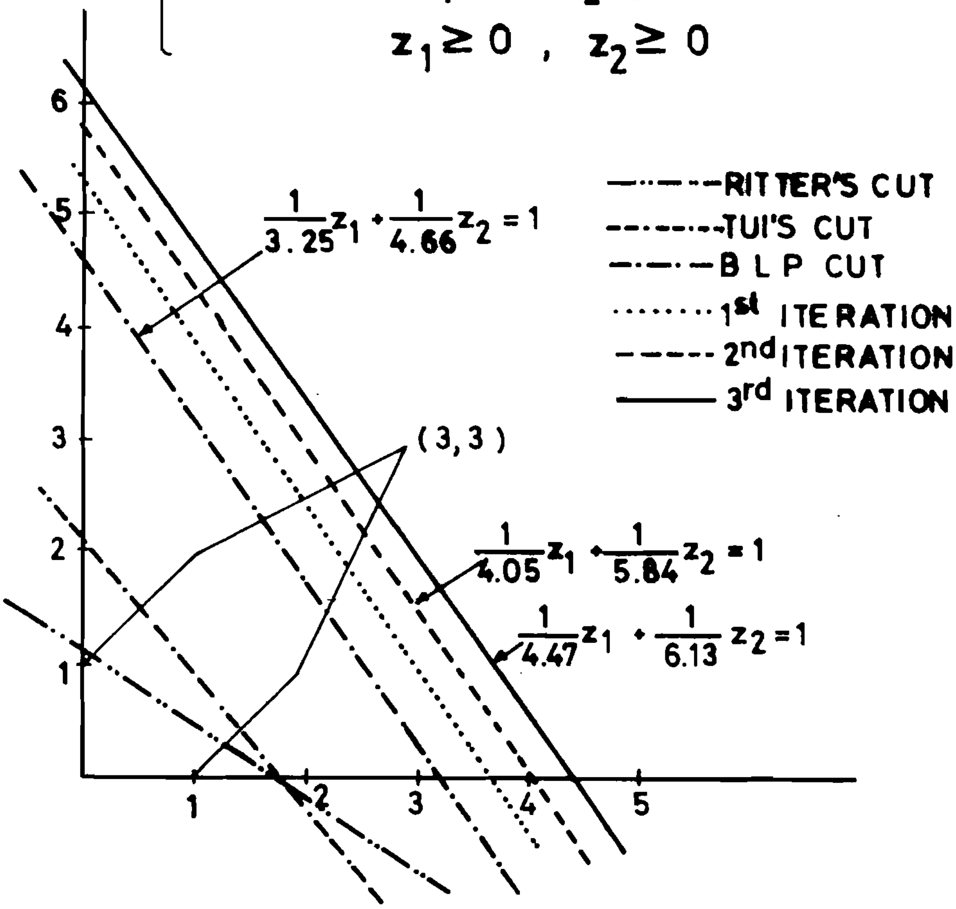

Figure 3.3. Illustrative example of iterative improvement.

become redundant for specifying the reduced feasible region $Y\setminus\Delta(\tau)$. Such redundant constraints can be identified by solving the following linear program:

$$\min\{y_i | Fy \leq f,\ y \geq 0,\ \Sigma y_j/T_j \geq 1\} \quad .$$

If the minimal value of y_i is positive, then the constraint $y_i \geq 0$ is redundant and we can reduce the size of the problem. This procedure is certainly costly and its use is recommended only when there is a very good chance of success, i.e., when τ is sufficiently large.

4. Cutting Plane Algorithm and the Results of Experiments

We will describe below one version of the cutting plane algorithm which has been coded in FORTRAN IV for CYBER 74.

Cutting Plane Algorithm

Step 1. Let $\ell = 0$ and $X_0 = X$, $Y_0 = Y$.

Step 2. If $\ell > \ell_{max}$ then stop. Otherwise go to Step 3.

Step 3. Let $k = 0$ and let $x^0 \in X_\ell$ be a basic feasible solution and let $\phi_{max} = f(x^0)$.

Step 4. Solve a subproblem: $\max\{\phi z, x^k) | z \in X_\ell\}$, and let x^{k+1} and B^{k+1} be its optimal basic feasible solution and corresponding basis.

Step 5. Compute $d(B_{k+1})$, the coefficients of the linear term of (2.7) relative to B_{k+1}. If $d(B_{k+1}) \not\leq 0$, then add 1 to k and go to Step 4. Otherwise let $B^* = B_{k+1}$, $x^* = x^{k+1}$ and go to Step 6.

Step 6. Compute matrix D in (2.7) relative to B^*. If x^* is an ε-locally maximum basic feasible solution (relative to X), then let $\phi_{max} := \max\{\phi_{max}, f(x^*)\}$, $\phi_0 = f(x^*)$ and go to Step 7. Otherwise move to a new basic feasible solution $\hat{x}$ where $f(\hat{x}) = \max\{f(x) | x \in N_{X_\ell}(x^*)\}$. Let $k = 0$, $x^0 = x$ and go to Step 4.

Step 7. Let $j = 0$ and let $Y^{0}_{\ell+1} = Y_{\ell}$.

Step 8. Compute $\theta(Y^{j}_{\ell+1})$ and let $Y^{j+1}_{\ell+1} = Y^{j}_{\ell+1} \setminus \Delta(\theta(Y^{j}_{\ell+1}))$. If $Y^{j+1}_{\ell+1} = \phi$ then stop. Otherwise go to Step 9.

Step 9. Let $\alpha = \| \theta(Y^{j+1}_{\ell+1}) - \theta(Y^{j}_{\ell+1}) \|$. If $\alpha > \alpha_0$ (where a_0 is a given constant), then add 1 to j and go to Step 8. Otherwise let $X_{\ell+1}$ be the feasible region in X corresponding to $Y^{j+1}_{\ell+1}$; add 1 to ℓ and go to Step 2.

When this algorithm stops at Step 8 with $Y^{j+1}_{\ell+1}$ becoming empty, then $x_{max} \varepsilon X$ corresponding to ϕ_{max} is actually an ε-optimal solution of (2.1). For the finite convergence of Steps 4 and 5, readers are referred to [9]. Though this algorithm may stop at Step 2 rather than at Step 8 and thus may fail to identify an ε-optimal solution, all the problems tested were solved successfully. Table 4.1 summarizes some of the results for smaller problems.

Table 4.1

Problem No.	Size of the Problem		ε/ϕ_{max}	No. of Local Maxima Identified	Approximate CPU time (sec)
	m	n			
1	3	6	0.0	1	0.2
2	5	8	0.0	2	0.6
3	6	11	0.0	1	0.3
4	7	11	0.0	1	0.5
5	9	19	0.0	2	3.0
6-1	6	12	0.05	5	2.5
6-2	6	12	0.01	6	3.0
6-3	6	12	0.0	6	3.0
7	11	22	0.1	8	28.0

Problems 1 $\sim$ 5 have no particular structure, while problems 6-1, 6-2, 6-3 and 7 have the following data structure:

$$\max\{c_m^t x + \tfrac{1}{2}x^t Q_m x \mid A_m x \leq b_m,\ x \geq 0\} \quad ,$$

where

$$A_m = \begin{bmatrix} 1 & 2 & \cdots & m-1 & m \\ 2 & 3 & \cdots & m & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ m & 1 & \cdots & m-2 & m-1 \end{bmatrix}, \quad Q_m = \begin{bmatrix} 2 & -1 & & & \bigcirc \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ \bigcirc & & & -1 & 2 \end{bmatrix}$$

$$c_m = (0,\ldots,0)^t \quad , \quad b_m = (m(m+1)/2,\ldots,m(m+1)/2)^t \quad .$$

They have m local maxima with the same objective functional values. All of them are, in fact, global maxima.

The experiments for larger problems are now under way using a more sophisticated version of the primal simplex (to be used in Step 4) and dual simplex algorithm (to be used in Step 8). These results will be reported subsequently.

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