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MAXIMIZATION OF DISTANCES OF REGULAR POLYGONS ON A CIRCLE

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The intense development of transport and efforts to achieve maximal efficiency raise various practical everyday problems; their generalization and formulation determines new interesting mathematical problems. For the best utilization of the existing means of transport the application of the available mathematical knowledge in the organization and the management of transport is often decisive.

The main purpose of this paper is to extend the set of already known mathematical methods to the solution of some transport problems.

This paper presents the solution of the basic problem defined in [1] and [2], which solves the concrete problem in railway and road transport (the problem of the optimization of time-tables by some criteria).

THE BASIC PROBLEM

The Basic Problem. *Let us have a circle c with a length T and positive integers* $m_1, m_2, \ldots, m_s$ $(s > 1)$.

It is necessary to locate

a regular m_1*-gon* $\mathbf{A}_1 = \{A_{11}, \ldots, A_{1m_1}\}$,
a regular m_2*-gon* $\mathbf{A}_2 = \{A_{21}, \ldots, A_{2m_2}\}$,
$\vdots$ *and*
a regular m_s*-gon* $\mathbf{A}_s = \{A_{s1}, \ldots, A_{sm_s}\}$

on the circle c in order that the number

$d = \min A_{ij}A_{\bar{i}\bar{j}}$ (*where* $i, \bar{i} = 1, 2, \ldots, s; j = 1, \ldots, m_i;$

$\bar{j} = 1, \ldots, m_{\bar{i}}; i \neq \bar{i}$) *be maximal;*

the symbol $A_{ij}A_{\bar{i}\bar{j}}$ denotes the length of the arc from the vertex A_{ij} to the vertex $A_{\bar{i}\bar{j}}$ in the positive direction.

Let us have a circle c with a length T and positive integers $m_1, m_2, \ldots, m_s$, $(s > 1)$ which are arbitrary but given and fixed through the whole paper. Let us introduce now the following definition to avoid possible misunderstandings later.

Definition 1. *Every system of regular polygons $\mathbf{A}_1, \ldots, \mathbf{A}_s$ located on the circle c will be called an* M-*system.*

Let us introduce a coordinate system on the circle c in the following way. One arbitrary point on the circle is denoted by O and to each point X we determine a coordinate x, which is equal to the length of the arc OX in the positive direction.

Every M-system $\mathbf{A}_1, \ldots, \mathbf{A}_s$ can be uniquely determined by s-tuple coordinates $(a_{11}, a_{21}, \ldots, a_{s1})$ of the vertices $A_{11}, \ldots, A_{s1}$.

For the sake of simplicity we shall study in the following only M-systems with $a_{s1} = 0$ and $0 \leqq a_{i1} < T/m_i$ for every $i = 1, \ldots, s - 1$. We can do it, because every general M-system is identical in geometrical sense with some M-system from the chosen class of M-systems which we shall study.

Now we can characterize every M-system by the $(s - 1)$-tuple $U \equiv (a_{11}, a_{21}, \ldots \ldots, a_{s-1,1})$ and the vectors of coordinates

$$\mathbf{a}_1 = (a_{11}, a_{12}, \ldots, a_{1m_1})$$
$$\vdots$$
$$\mathbf{a}_s = (a_{s1}, \ldots, a_{sm_s})$$

fulfil

$$a_{ij} = a_{i1} + (j - 1)\, T/m_i \quad \text{for} \quad 1 \leqq j \leqq m_i\,.$$

Definition 2. *Let $\mathbf{A}_i, \mathbf{A}_{\bar{i}}$ be polygons on the circle c. The number $\varrho(\mathbf{A}_i, \mathbf{A}_{\bar{i}}) = \min_{j,\bar{j}} (a_{\bar{i}\bar{j}} - a_{ij} + T) \bmod T$ will be called the distance between the polygon $\mathbf{A}_i$ and the polygon $\mathbf{A}_{\bar{i}}$.*

Definition 3. *If no two vertices of the* M-*system have the same position, i.e. if $\varrho(\mathbf{A}_i, \mathbf{A}_j) > 0$ holds for all $i \neq j$, the* M-*system will be called a free* M-*system.*

Definition 4. *The* M-*system U will be called a fixed* M-*system, if there exists a sequence of positive integers $s = i_1, i_2, \ldots, i_n = j$ $(1 \leqq i_k \leqq s$ for all $k)$ for every j $(1 \leqq j \leqq s)$ so that $\varrho(\mathbf{A}_{i_k}, \mathbf{A}_{i_{k+1}}) = 0$ holds for all k $(1 \leqq k < n)$.*

CHARACTERISTIC VECTORS AND M-GRAPHS

Construction 1. Let us have a free M-system $U \equiv (a_{11}, \ldots, a_{s-1,1})$. We shall construct a directed graph $G = (V, H)$ with a vertex set $V = \{X_1, X_2, \ldots, X_s\}$. We shall carry out the construction in steps.

The 0-th step

We define ${}_0a_{ij} = a_{ij}$ for all $i = 1, \ldots, s$ and $j = 1, \ldots, m_i$, then we define the sets $B_0 = \{s\}$ and $C_0 = \{1, 2, \ldots, s-1\}$, the directed graph $G_0 = (V_0, H_0)$, where $V_0 = \{X_s\}$ and $H_0 = \emptyset$.

The general $(v+1)$-st step

We have from the preceding v-th step

$$\begin{aligned} {}_v\boldsymbol{a}_1 &= ({}_va_{11}, \ldots, {}_va_{1m_1}) \\ &\vdots \\ {}_v\boldsymbol{a}_s &= ({}_va_{s1}, \ldots, {}_va_{sm_s}) \end{aligned}$$

and the sets $B_v \subset \{i\}_{i=1}^s$, $C_v \subset \{i\}_{i=1}^s$, the directed graph $G_v = (V_v, H_v)$, where $V_v = \{X_i; i \in B_v\}$. Now let us define

$$d_{v+1} = \min_{\substack{i \in B_v \\ \bar{i} \in C_v}} \min_{\substack{j=1,\ldots,m_i \\ \bar{j}=1,\ldots,m_{\bar{i}}}} ({}_va_{\bar{i}\bar{j}} - {}_va_{ij} + T) \bmod T .$$

The M-system U is free and this implies $d_{v+1} > 0$. We define $O_{v+1} = \{(i, \bar{i}); i \in B_v, \bar{i} \in C_v$, so that there exist $j, \bar{j}$ such that $({}_va_{\bar{i}\bar{j}} - {}_va_{ij} + T) \bmod T = d_{v+1}\}$. We define $P_{v+1} = \{\bar{i} \in C_v$, so that there exists i such, that $(i, \bar{i}) \in O_{v+1}\}$. We denote $B_{v+1} = B_v \cup P_{v+1}$, $C_{v+1} = C_v - P_{v+1}$. We define vectors ${}_{v+1}\boldsymbol{a}_i = ({}_{v+1}a_{i1}, \ldots, {}_{v+1}a_{im_i})$ for $i = 1, \ldots, s$ in the following way

$${}_{v+1}a_{ij} = {}_va_{ij} \quad \text{if} \quad i \in B_v$$

and

$${}_{v+1}a_{ij} = {}_va_{ij} - d_{v+1} \quad \text{if} \quad i \in C_v .$$

We denote $H_{v+1} = H_v \cup \{(X_i, X_{\bar{i}}); (i, \bar{i}) \in O_{v+1}\}$, $V_{v+1} = \{X_i \in V; i \in B_{v+1}\}$. We define the directed graph $G_{v+1} = (V_{v+1}, H_{v+1})$.

The last t-th step

There exists $t \leqq s - 1$ such that $C_{t-1} \neq \emptyset$ and $C_t = \emptyset$, because card $(C_0) = s - 1$ and $C_{i+1} \subsetneqq C_i$. Thus we obtain the last set of directed edges H_t and the last vectors ${}_t\boldsymbol{a}_i$. We denote $H = H_t$ and

$$\begin{aligned} {}_*\boldsymbol{a}_1 &= ({}_*a_{11}, \ldots, {}_*a_{1m_1}) , \\ &\vdots \\ {}_*\boldsymbol{a}_s &= ({}_*a_{s1} \ldots, {}_*a_{sm_s}) , \end{aligned}$$

where ${}_*a_{ij} = {}_ta_{ij}$ for all i, j. The directed graph which we obtained in this construction from the free M-system U is the graph $G = (V, H)$. The numbers ${}_*a_{ij}$ are the coordinates of the fixed M-system ${}_*U$ which we also obtained as the result of Construction 1. We define a function $f: H \to N$, $N = \{i\}_{i=1}^{\infty}$ in this way: $f(X_i, X_j) = \min\{\bar{i}; \exists \bar{j}$, such that ${}_*a_{i\bar{i}} = {}_*a_{j\bar{j}}\}$, then $f(X_i, X_j) \leqq m_i / D(m_i, m_j)$ is fulfilled, where $D(m_i, m_j)$ is the greatest common divisor of m_i and m_j.

Definition 5. *Let us have a free* M-*system* $U \equiv (a_{11}, \ldots, a_{s-1,1})$. *Let* $g = \sum_{i=1}^{s} m_i$ *hold and let* $(c_1, c_2, \ldots, c_g)$ *be such a permutation of the vector* $(a_{11}, \ldots, a_{1m_1}, a_{21}, \ldots, a_{2m_2}, \ldots, a_{sm_s})$ *that* $c_i \leqq c_{i+1}$ *holds for all* i $(1 \leqq i < g)$. *Let* $(b_1, b_2, \ldots, b_g)$ *be such a vector that for every* i $(1 \leqq i \leqq g)$ *there exists* k *such that* $c_i = a_{b_i k}$. *This vector* $(b_1, b_2, \ldots, b_g)$ *will be called the characteristic vector of the free* M-*system* U.

Remark 1. It is obvious that there exists a unique characteristic vector for every free M-system, because $a_{i\bar{i}} \neq a_{j\bar{j}}$ holds for every $(i, \bar{i}) \neq (j, \bar{j})$.

Definition 6. *Let us have a free* M-*system* U. *Let* $G_v = (V_v, H_v)$ *be a graph and* ${}_v\mathbf{a}_1, \ldots, {}_v\mathbf{a}_s$ *the vectors we obtained in the* v-*th step of Construction 1. Let* $g = \sum_{i=1}^{s} m_i$ *hold and let* $({}_vc_1, {}_vc_2, \ldots, {}_vc_g)$ *be such a permutation of the vector* $({}_va_{11}, \ldots, {}_va_{1m_1}, {}_va_{21}, \ldots, {}_va_{2m_2}, \ldots, {}_va_{sm_s})$ *that* ${}_vc_i \leqq {}_vc_{i+1}$ *holds for all* i $(1 \leqq i < g)$. *Let* $({}_vb_1, \ldots, {}_vb_g)$ *be such a vector that*

1. *for every* i $(1 \leqq i \leqq g)$ *there exists such a* k *that* ${}_vc_i = {}_va_{{}_vb_i,k}$ *and*
2. *if* ${}_vc_i = {}_vc_{i+1}$ *then there exists a directed edge from* X_{b_i} *to* $X_{b_{i+1}}$ *in* G_v.

This vector $({}_vb_1, \ldots, {}_vb_g)$ *will be called the general characteristic vector of the* v-*th step of Construction 1 from the free* M-*system* U.

Lemma 1. *Let* G_v *be a directed graph of the* v-*th step of Construction 1. Then* G_v *contains no directed cycle.*

Proof. We shall use mathematical induction.

1. It is obvious for $v = 0$, because there exists no directed cycle in G_0.

2. Let us have now G_{v-1} with no directed cycle. For $G_v = (V_v, H_v)$ we have $V_v = V_{v-1} \cup W_v$, where $W_v = \{X_i; i \in P_v\}$ and $H_v = H_{v-1} \cup \{(X_i, X_j); (i, j) \in O_v\}$, which means that $X_i \in V_{v-1}$ and $X_j \in W_v$ for all new edges (X_i, X_j) in the v-th step. Hence there exists no edge in G_v between two vertices X_i, X_j, both from W_v.

Then the directed cycle must contain some vertices from V_{v-1} and also some vertices from W_v. This is a contradiction, as there exist no $X_i \in V_{v-1}$ and $X_j \in W_v$ such that $(X_j, X_i) \in H_v$.

Lemma 2. *For an arbitrary step* v *of Construction 1 there exists no more than one general characteristic vector* $({}_vb_1, {}_vb_2, \ldots, {}_vb_g)$.

Proof. For the vector ${}_v\mathbf{c} = ({}_vc_1, \ldots, {}_vc_g)$ there exist such numbers $0 = g_0 < < g_1 < g_2 < \ldots < g_e = g$ that ${}_vc_j = {}_vc_{\bar{j}}$ holds for all i $(0 < i \leqq e)$ and for all $j, \bar{j}$; $(g_{i-1} < j \leqq g_i;\ g_{i-1} < \bar{j} \leqq g_i)$. Let us have two different general characteristic vectors $({}_vb_1, \ldots, {}_vb_g) \neq ({}_vb'_1, \ldots, {}_vb'_g)$ of the same free M-system U and of the same step v. Then there exists such i $(1 \leqq i \leqq e)$ that $({}_vb_{g_{i-1}+1}, \ldots, {}_vb_{g_i}) \neq ({}_vb'_{g_{i-1}+1}, \ldots \ldots, {}_vb'_{g_i})$. For every $j < \bar{j}$, $k < \bar{k} \in \langle g_{i-1} + 1; g_i \rangle$ the relations ${}_vb_j \neq {}_vb_{\bar{j}}$ and ${}_vb'_k \neq {}_vb'_{\bar{k}}$ hold because in the opposite case there would exist a cycle in G_v

$$(X_{{}_vb_j} \to X_{{}_vb_{j+1}} \to \ldots \to X_{{}_vb_{\bar{j}}} = X_{{}_vb_j})$$

or

$$(X_{{}_vb'_k} \to X_{{}_vb'_{k+1}} \to \ldots \to X_{{}_vb'_{\bar{k}}} = X_{{}_vb'_k}) .$$

This implies that the vectors $({}_vb_{g_{i-1}+1}, \ldots, {}_vb_{g_i})$ and $({}_vb'_{g_{i-1}+1}, \ldots, {}_vb'_{g_i})$ are two different permutations of the same set with $(g_i - g_{i-1})$ elements and this implies that there exist $a, \bar{a}$ such that $a = {}_vb_j = {}_vb'_{j'}$, and $\bar{a} = {}_vb_k = {}_vb'_{k'}$ so that $g_{i-1} + 1 \leqq j < < k \leqq g_i$ and $g_{i-1} + 1 \leqq k' < j' \leqq g_i$.

This yields that there exists a directed path from X_a to $X_{\bar{a}}$ in G_v and there exists also a directed path from $X_{\bar{a}}$ to X_a in G_v and this means that there exists a directed cycle in G_v, which is a contradiction – the lemma is proved.

Lemma 3. *For every v there exists just one general characteristic vector of the v-th step and it is equal to the characteristic vector of the free* M-*system U.*

Proof. We shall use mathematical induction.

1. The lemma is obvious for $v = 0$, because ${}_0a_{ij} = a_{ij}$ for all i, j and $a_{ij} \neq a_{\bar{i}\bar{j}}$ holds for all $(i, j) \neq (\bar{i}, \bar{j})$.

2. Let ${}_{v-1}\mathbf{b} = ({}_{v-1}b_1, \ldots, {}_{v-1}b_g) = (b_1, \ldots, b_g)$ be the general characteristic vector of the $(v - 1)$-st step and the characteristic vector of the free M-system U. For the v-th step, ${}_va_{ij} = {}_{v-1}a_{ij}$ holds for all $i \in B_{v-1}$ and ${}_va_{ij} = {}_{v-1}a_{ij} - d_v$ holds for all $i \in C_{v-1}$. This implies that ${}_vc_i = {}_{v-1}c_i$ holds for all i such that ${}_{v-1}b_i \in B_{v-1}$ and that ${}_vc_i = {}_{v-1}c_i - d_v$ holds for all i such that ${}_{v-1}b_i \in C_{v-1}$, because for all i such that ${}_{v-1}b_i \in C_{v-1}$ and ${}_{v-1}b_{i-1} \in B_{v-1}$ the inequality ${}_{v-1}c_i \geqq {}_{v-1}c_{i-1} + d_v$ holds. Now we prove that ${}_{v-1}\mathbf{b}$ is also a general characteristic vector of the v-th step.

It follows from the preceding that the condition is fulfilled that for every i there exists $\bar{k}$ such that ${}_vc_i = {}_va_{{}_{v-1}b_i,\bar{k}}$ because for every i there exists $k = \bar{k}$ so that ${}_{v-1}c_i = = {}_{v-1}a_{{}_{v-1}b_i,k}$. If ${}_vc_i = {}_vc_{i+1}$ and

A) if ${}_{v-1}c_i = {}_{v-1}c_{i+1}$, then it follows from the preceding $(v - 1)$-st step that there exists a directed edge from $X_{{}_{v-1}b_i}$ to $X_{{}_{v-1}b_{i+1}}$ in G_v.

B) if ${}_{v-1}c_i \neq {}_{v-1}c_{i+1}$ holds, then ${}_{v-1}c_i + d_v = {}_{v-1}c_{i+1}$ holds. This implies ${}_{v-1}b_i \in B_{v-1}$, ${}_{v-1}b_{i+1} \in C_{v-1}$ and $({}_{v-1}b_i;\ {}_{v-1}b_{i+1}) \in O_v$ and so there exist a new edge $(X_{{}_{v-1}b_i}; X_{{}_{v-1}b_{i+1}})$ in G_v.

From the preceding facts and from lemma 2 it follows that $({}_v b_1, \ldots, {}_v b_g) = ({}_{v-1} b_1, \ldots, {}_{v-1} b_g) = (b_1, \ldots, b_g)$ is a single general characteristic vector of the v-th step.

Definition 7. *Given a directed graph $G = (V, H)$; $V = \{X_i\}_{i=1}^s$, $H \subset V \times V$, then every vertex X_j such that $(X_i, X_j) \notin H$ for all i will be called an initial vertex of the directed graph G. Every vertex X_i such that $(X_i, X_j) \notin H$ for all j will be called a terminal vertex of the directed graph G.*

Definition 8. *Let U be a free* M-*system. Let $G = (V, H)$ be a directed graph and let f be the function f: $H \to N$, which we obtain in Construction 1 from the free* M-*system U. The ordered triplet (V, H, f) (a weighted directed graph) will be called an* M-*graph of the free* M-*system U.*

Remark 2. Construction 1 evidently implies that X_s is a single initial vertex in the M-graph G and that every M-graph is connected. From Lemma 1 it follows that there exists no directed cycle in the M-graph G. It is also true that there exists just one corresponding fixed M-system to every M-graph.

Theorem 1. *There exists a unique* M-*graph $G = (V, H, f)$ for every free* M-*system U.*

Proof. The proof of this theorem follows from the fact that Construction 1 is uniquely determined.

FINDING ALL M-GRAPHS OF FREE M-SYSTEMS THE M-TREE OF THE FREE M-SYSTEM

Construction 2. Let us have an M-graph $G = (V, H, f)$ of the free M-system U. Let us construct a new directed graph (V', H') with the function $f' : H' \to N$ so that $V = V'$, $H' \subset H$ and we delete edges which do not fulfil the following conditions. An arbitrary edge $(X_i, X_j) \in H$ will be contained also in H' if and only if:

1. There exists no directed path from X_s to X_j in G which does not contain (X_i, X_j) and which is longer than all the directed paths from X_s to X_j which contain (X_i, X_j).

2. There exists no directed path from X_s to X_j in G

a) which does not contain $(X_i X_j)$, but contains (X_k, X_j) and

b) which is as long as the longest directed path from X_s to X_j, which contains (X_i, X_j) and

c) $k < i$ holds.

Now we define the function $f' : H' \to N, f'(X_i, X_j) = f(X_i, X_j)$ for all $(X_i, X_j) \in H'$.

Definition 9. *Let $G = (V, H, f)$ be an M-graph of the free M-system U. The ordered triplet $G' = (V', H', f')$ (a directed weighted graph, which we obtain in Construction 2) will be called the M-tree of M-graph G.*

Theorem 2. *There exists a unique M-tree to every M-graph of some free M-system U.*

Proof. It follows from the uniquely determined Construction 2.

Remark 3. If follows from Construction 2 that every M-tree is a directed weighted tree with a single initial vertex X_s. Thus it is in general a simpler graph than the M-graph. But we must note that not every directed weighted tree with a single initial vertex X_s is an M-tree of some M-graph. We shall investigate this fact in Theorem 3.

Theorem 3. *Let us have a directed weighted graph $G' \equiv (V', H', f')$ which is a tree with a single initial vertex X_s. Then we can uniquely determine whether G' is an M-tree of some M-graph or not. If G' is an M-tree we can uniquely determine the corresponding M-graph G and the coordinates ${}_*a_{ij}$ of the corresponding fixed M-system ${}_*U$.*

Proof. If the condition $1 \leqq f'(X_i X_j) \leqq m_i / D(m_i, m_j)$ is fulfilled, then we shall determine the vectors ${}_*\boldsymbol{a}_i$ in several steps (if the condition is not fulfilled, then there exists no M-graph G with an M-tree G').

The 0-th step

For ${}_*\boldsymbol{a}_s = ({}_*a_{s1}, \ldots, {}_*a_{sm_s})$ the condition that ${}_*a_{si} = (i - 1)\, T/m_s$ for all $i = 1, \ldots, m_s$ is obviously fulfilled.

The general $(v + 1)$-st step

We know already all vectors ${}_*\boldsymbol{a}_i$ such that there exists a directed path X_s to X_i which is not longer than v. Let R_v be a set of all vertices to which the directed path from X_s has the length just v. In this $(v + 1)$-st step we determine vectors to which the directed path from X_s has the length just $(v + 1)$. Let ${}_*\boldsymbol{a}_j$ be such a vector. The directed path with the length $(v + 1)$ from X_s to X_j must contain some edge (X_i, X_j), where $X_i \in R_v$. Then one element of the vector ${}_*\boldsymbol{a}_j$ will be ${}_*a_{i,f'(X_i,X_j)}$, the others will be

$$({}_*a_{i,f'(X_i,X_j)} + kT/m_j) \bmod T \quad \text{for} \quad k = 0, 1, \ldots, m_j - 1$$

We order these elements into a sequence from the least element to the greatest and so we obtain the vector ${}_*\boldsymbol{a}_j$.

In this way we successively determine all vectors ${}_*\boldsymbol{a}_i$, $i = 1, \ldots, s$; we shall not make more than s steps.

Then we investigate all such cases of the couples i, j that there exist $k, \bar{k}$ that ${}_*a_{ik} = {}_*a_{j\bar{k}}$ and $(X_i, X_j) \notin G'$, $(X_j, X_i) \notin G'$. Let d_i (resp. d_j) be the length of the directed path from X_s to X_i (resp. X_j) in G' and let $d_j \geqq d_i$ be fulfilled.

If follows from Construction 1 that if ${}_*a_{ik} = {}_*a_{j\bar{k}}$, there must exist either an edge $(X_i X_j)$ or (X_j, X_i) in the M-graph G.

1. If $d_i = d_j$ and we add the edge $(X_i X_j)$, then the length of the directed path from X_s through X_i to X_j will be greater and it is a contradiction to condition 1 of construction 2. We obtain the same contradiction if we add the edge (X_j, X_i). This implies that there exists no M-graph with an M-tree G'.

2. If $d_j = d_i + 1$ and $k > i$, where (X_k, X_j) is contained in the directed path of the length d_j from X_s to X_j in G', we can add neither the edge (X_i, X_j) because of condition 2 of Construction 2 nor the edge $(X_j X_i)$ because of condition 1 of Construction 2. This yields that there exists no M-graph with the M-tree G'.

If conditions 1 and 2 are not fulfilled we add the edge (X_i, X_j). We obtain similar conditions if $d_i \geqq d_j$; if they are not fulfilled, we add the edge (X_j, X_i). If conditions 1 and 2 are not fulfilled for all cases $({}_*a_{ik} = {}_*a_{j\bar{k}})$, G' is an M-tree of the M-graph G which we obtain by the mentioned adding of edges and by defining $f(X_i X_j) = \min\{\bar{i};\ \exists \bar{j}$ such that ${}_*a_{i\bar{i}} = {}_*a_{j\bar{j}}\}$ for all new edges (X_i, X_j).

Theorem 4. *Let us have two free* M-*systems* U, U' *which have the same* M-*graph* $G = G'$. *Then* U *and* U' *have also the same characteristic vector* $(b_1, b_2, \ldots, b_g) = (b'_1, \ldots, b'_g)$. *(Also the reverse implication is true, but we do not need it to solve our problem.)*

Proof. Let Construction 1 be applied to U (resp. U') in just t (resp. t') steps and let $({}_tb_1, \ldots, {}_tb_g)$ (resp. $({}_{t'}b'_1, \ldots, {}_{t'}b'_g)$) be the general characteristic vector of the last t-th (resp. t'-th) step of Construction 1 from the free M-system U (resp. U'). The values of the vectors ${}_*\boldsymbol{a}_i$ and ${}_*\boldsymbol{a}'_i$ (for $i = 1, \ldots, s$), the coordinates of the corresponding fixed M-systems can be obtained from the M-graphs G, G' in the same way as in the proof of Theorem 3.

U and U' have the same M-graph $G = G'$, which implies ${}_*a_{ij} = {}_*a'_{ij}$ for all i, j. This gives

$$({}_tc_1, \ldots, {}_tc_g) = ({}_{t'}c'_1, \ldots, {}_{t'}c'_g)\,.$$

The condition $G_t = G = G' = G'_t$ is fulfilled, which implies $({}_tb_1, \ldots, {}_tb_g) = ({}_{t'}b'_1, \ldots, {}_{t'}b'_g)$. According to Lemma 3 $(b_1, \ldots, b_g) = ({}_tb_1, \ldots, {}_tb_g)$, $(b'_1, \ldots, b'_g) = ({}_{t'}b'_1, \ldots, {}_{t'}b'_g)$ and this implies $(b_1, \ldots, b_g) = (b'_1, \ldots, b'_g)$, which we wanted to prove.

If we want to find all M-graphs, it is sufficient according to Theorem 3 to find the set of all directed trees (with s vertices) with a single initial vertex X_s, the edges of which are weighted with the function f' so that $1 \leqq f'(X_i, X_j) \leqq m_i/D(m_i, m_j)$. Every directed tree with a single initial vertex X_s is uniquely determined by the set of terminal vertices and by the set of directed paths from X_s to the terminal vertices. Starting from this we shall successively construct all directed trees with a single initial vertex X_s in the following way:

Construction 3

I. We determine successively the number of terminal vertices $p = 1, 2, \ldots, s - 1$.

II. To every number p we successively choose the set M of terminal vertices, i.e. we construct all the combinations of the p-th class from the elements $X_1, X_2, \ldots, X_{s-1}$.

III. We arrange the elements of the set M into an increasing sequence $X_{b_1}, X_{b_2}, \ldots, X_{b_p}$ according to the numbers of the subscripts and we divide the set $V - (M \cup \{X_s\})$ into p disjoint sets $M_{b_1}, \ldots, M_{b_p}$ (which can be also empty), which determine for every vertex X_{b_i} the part of the directed path from X_s to X_{b_i} in the following way:

for X_{b_1}, M_{b_1} means just the set of vertices contained in the directed path from X_s to X_{b_1} (except X_s, X_{b_1}) and for X_{b_i} $(i > 1)$ M_{b_i} means just the set of vertices contained in the directed path from X_{g_i} to X_{b_i} (except X_{g_i}, X_{b_i}), where X_{g_i} is such a vertex of the directed path from X_s to X_{b_i} that the vertices between X_s and X_{g_i} (also X_{g_i}) are contained in $\{X_s\} \cup \bigcup_{j=1}^{i-1} M_{b_j}$ and the directed path from X_{g_i} to X_{b_i} contains no vertex from the set $\bigcup_{j=1}^{i-1} M_{b_j} - \{X_{g_i}\}$.

In this way we assign an integer $1, 2, \ldots, p$ to each element of the set $V - (M \cup \{X_s\})$, i.e. we perform all permutations with repetitions of the $(s - p - 1)$-st class from p elements.

IV. We choose successively all $(p - 1)$-tuples of the vertices $X_{g_2}, X_{g_3}, \ldots, X_{g_p}$ where $X_{g_i} \in \{X_s\} \cup \bigcup_{j=1}^{i-1} M_{b_j}$.

V. We determine successively all the possible directed paths from X_{g_i} to X_{b_i} (for given sets M_{b_i}), i.e. we make all permutations of the sets M_{b_i}.

VI. We determine successively all possible values of the function f', i.e. we assign to every edge an integer which fulfils $1 \leqq f'(X_i, X_j) \leqq m_i/D(m_i, m_j)$.

If we change successively all parameters in cycles I, II, III, IV, V and VI, we obtain all suitable weighted directed trees with the vertex set V and a single initial vertex X_s.

ESTIMATE OF THE NUMBER OF DIRECTED TREES WHICH HAVE A SINGLE INITIAL VERTEX X_s AND WHICH ARE WEIGHTED WITH THE FUNCTION f'

For this estimate we shall use the following one to one mapping φ of the set of undirected trees with the vertex set V onto the set of directed trees with the same vertex V and a single initial vertex X_s.

Let $G = (V, H)$ be an undirected tree, then $\varphi(G) = (V', H')$ will be the directed tree, where $V = V'$ and $(X_i, X_j) \in H'$ if and only if there exists a path from X_s to X_j containing (X_i, X_j) in G.

As G is a tree, the condition is fulfilled that for every $(X_i X_j) \in H$ either $(X_i, X_j) \in H'$ or $(X_j, X_i) \in H'$. This implies that $\varphi(G)$ is also a tree. As there exists a directed path from X_s to every $X_k \neq X_s$, there exists some $\bar{k}$ such that $(X_{\bar{k}}, X_k) \in H'$ and so no $X_k \neq X_s$ is an initial vertex. Let $\bar{\varphi}$ be a mapping which assigns to every directed graph its undirected graph by means of deleting the direction of edges. Then the condition is fulfilled that the mappings φ, $\bar{\varphi}$ are inverse and $\bar{\varphi}(\varphi(G)) = G$ and so φ and $\bar{\varphi}$ are one to one mappings. According to [3] the number of all undirected trees with s vertices is s^{s-2}, so the number of all directed trees with a single initial vertex X_s will be the same. To every edge we can assign maximum $(\max_i m_i)$ various values, so the number of all directed trees with the vertex set V, with a single initial vertex X_s and with the edge-weight of the function f' will not be greater than $s^{s-2} (\max_i m_i)^{s-1}$.

DETERMINING THE OPTIMAL FREE M-SYSTEM

Theorem 5. *Let us have a free* M-*system* $U \equiv \{\mathbf{A}_1, \ldots, \mathbf{A}_s\}$ *and let* $d = \min_{i \neq i} \varrho(\mathbf{A}_i, \mathbf{A}_i)$. *Let* $\mathbf{A}_{b_1}, \mathbf{A}_{b_2}, \ldots, \mathbf{A}_{b_t}$ $(2 \leqq t \leqq s)$ *be the sequence of polygons which satisfies* $d = \varrho(\mathbf{A}_{b_i}, \mathbf{A}_{b_{i+1}})$ *for all* $i = 1, 2, \ldots, t-1$ *as well as* $d = \varrho(\mathbf{A}_{b_t}, \mathbf{A}_{b_1})$. *Let* $U' \equiv \{\mathbf{A}'_1, \ldots, \mathbf{A}'_s\}$ *be another free* M-*system which has the same* M-*graph* G. *Let there be* $d' = \min_{i \neq i} \varrho(\mathbf{A}'_i, \mathbf{A}'_i)$. *Then* $d' \leqq d$.

Proof. It follows from the assumption of the theorem that there exist $i_1, i_2, \ldots, i_t$ and $j_1, j_2, \ldots, j_t$ so that $d = A_{b_1 i_1} A_{b_2 j_2} = A_{b_2 i_2} A_{b_3 j_3} = \ldots = A_{b_{t-1} i_{t-1}} A_{b_t j_t} = A_{b_t i_t} \cdot A_{b_1 j_1}$. We shall carry out the proof indirectly. Let $d' > d$. Let us have U as assumed in Theorem 5, i.e. U and U' have the same M-graph and this implies (Theorem 4) that they have the same characteristic vector, too.

I. Let $\alpha = a'_{b_1 1} - a_{b_1 1} = A_{b_1 1} A'_{b_1 1} \geqq 0$. Hence $A_{b_1 i_1} A_{b_2 j_2} = d$, $A'_{b_1 i_1} A'_{b_2 j_2} \geqq d'$. It follows that $A_{b_2 j_2} A'_{b_2 j_2} + A_{b_1 i_1} A_{b_2 j_2} = A_{b_1 i_1} A'_{b_1 i_1} + A'_{b_1 i_1} A'_{b_2 j_2}$ and then $A_{b_2 j_2} \cdot A'_{b_2 j_2} \geqq \alpha + (d' - d)$. This evidently implies that $A_{b_2 k} A'_{b_2 k} \geqq \alpha + (d' - d)$ for all $k = 1, 2, \ldots, m_{b_2}$.

In the same way we obtain $A_{b_3 k} A'_{b_3 k} \geqq \alpha + 2(d' - d)$ for all $k = 1, 2, \ldots, m_{b_3}$, and finally we obtain $A_{b_t k} A'_{b_t k} \geqq \alpha + (t-1)(d' - d)$ for all $k = 1, \ldots, m_{b_t}$, and also $A_{b_t i_t} A'_{b_t i_t} \geqq \alpha + (t-1)(d' - d)$. Since U and U' have the same characteristic vector and $a_{s1} = a'_{s1} = 0$, we have $A'_{b_t i_t} A'_{b_1 j_1} + A_{b_t i_t} A'_{b_t i_t} = A_{b_t i_t} A_{b_1 j_1} + A_{b_1 j_1} A'_{b_1 j_1}$ and then $A'_{b_t i_t} A'_{b_1 j_1} \leqq d + \alpha - [\alpha + (t-1)(d' - d)] < d < d'$, which is a contradiction, as we must have $A'_{b_t i_t} A'_{b_1 j_1} \geqq d'$.

II. The proof is very similar in the case of $a'_{b_1 1} - a_{b_1 1} < 0$.

The theorem is proved.

In the next part we shall construct to every M-graph a free M-system, which contains the sequence $\boldsymbol{A}_{b_1}, \ldots, \boldsymbol{A}_{b_t}$, which fulfils the conditions from Theorem 5 and so this free M-system is the best of all those which have the same M-graph.

Construction 4. Let us have an M-graph $G = (V, H, f)$. We determine the coordinates ${}_{*}a_{ij}$ of the corresponding fixed M-system ${}_{*}U$ in the same way as in the proof of Theorem 3. Now we shall carry out Construction 4 in steps.

The 0-th step

We define ${}_0\boldsymbol{a}_i$ for all i so that ${}_0a_{ij} = {}_{*}a_{ij}$ for all i, j. We define the directed graph $G_0 = (V, H_0)$, where $H_0 \subset H$ so that $(X_i X_j) \in H_0$ if and only if $(X_i, X_j) \in H$ and in G there does not exist a longer directed path from X_s to X_j not containing $(X_i X_j)$ than the longest one, which contains $(X_i X_j)$. We define also $e_0 = 0$ and then we have $(X_i, X_j) \in H_0$ implies $\varrho(\boldsymbol{A}_i, \boldsymbol{A}_j) = 0 = e_0$.

The general $(v + 1)$-st step

We have ${}_v\boldsymbol{a}_i = ({}_va_{i1}, \ldots, {}_va_{im_i})$ for $i = 1, \ldots, s$ and the directed graph $G_v = (V, H_v)$. We define ${}_vk_i$ to be equal to the length of the directed path from X_s to X_i in G_v. The condition is fulfilled that if $(X_i, X_j) \in H_v$, then $\varrho({}_v\boldsymbol{A}_i, {}_v\boldsymbol{A}_j) = \sum_{k=0}^{v} e_k$ and if $(X_i, X_j) \notin H_v$ then $\varrho({}_v\boldsymbol{A}_i, {}_v\boldsymbol{A}_j) \geqq \sum_{k=0}^{v} e_k$. We define $h_v(X_i, X_j) = \min_{\substack{(\bar{i},\bar{j}) \\ {}_va_{i\bar{i}} \neq {}_va_{j\bar{j}}}} ({}_va_{j\bar{j}} - {}_va_{i\bar{i}} + T)$. $\operatorname{mod} T - \sum_{k=0}^{v} e_k$ for all $1 \leqq i, j \leqq s$. We define

$$e_{v+1} = \min_{i=1,\ldots,s-1} \min_{\substack{j \\ {}_vk_i \geqq {}_vk_j}} \frac{h_v(X_i, X_j)}{{}_vk_i - {}_vk_j + 1} \quad \text{and} \quad R_{v+1} = \{(X_i, X_j) \in H_v$$

such that ${}_vk_i \geqq {}_vk_j$ and $e_{v+1} = h_v(X_i, X_j)/({}_vk_i - {}_vk_j + 1)\}$. We define ${}_{v+1}\boldsymbol{a}_i$ so that ${}_{v+1}a_{ij} = {}_va_{ij} + {}_vk_i \cdot e_{v+1}$. We denote $H'_v = H_v \cup R_{v+1}$ and $G'_v = (V, H'_v)$. If the graph G'_v does not contain a directed cycle, we determine the graph $G_{v+1} = (V, H_{v+1})$; $H_{v+1} \subset H'_v$ so that $(X_i, X_j) \in H_{v+1}$ if and only if $(X_i, X_j) \in H'_v$ and in G'_v there does not exist a longer directed path from X_s to X_j, not containing (X_i, X_j) than the longest one, which contains (X_i, X_j).

From the definition of e_{v+1} it follows that if $(X_i, X_j) \in H_{v+1}$, then $\varrho({}_{v+1}\boldsymbol{A}_i, {}_{v+1}\boldsymbol{A}_j) = \sum_{k=0}^{v+1} e_k$ and if $(X_i, X_j) \notin H_{v+1}$, then $\varrho({}_{v+1}\boldsymbol{A}_i, {}_{v+1}\boldsymbol{A}_j) \geqq \sum_{k=0}^{v+1} e_k$.

We can begin the $(v + 2)$-nd step. If the graph G'_v contains a directed cycle $X_{b_1}, X_{b_2}, \ldots, X_{b_t}$ we finish Construction 4 and we can determine the resulting vectors

$\boldsymbol{a}_i = (a_{i1}, \dots, a_{im_i})$, where $a_{ij} = {}_{v+1}a_{ij}$ for all i, j. These vectors determine the free M-system which satisfies $d = \min_{i \neq i} \varrho(\boldsymbol{A}_i, \boldsymbol{A}_i) = \sum_{k=0}^{v+1} e_k$ and $d = \varrho(\boldsymbol{A}_{b_t}, \boldsymbol{A}_{b_1})$ and for all $i = 1, \dots, t-1$, $d = \varrho(\boldsymbol{A}_{b_i}, \boldsymbol{A}_{b_{i+1}})$.

End of Construction 4.

In this way we have solved the basic problem, because we can successively construct all suitable M-trees (according to Construction 3); to every M-tree we determine its M-graph (Theorem 3) and then we find the best M-system (one of the best M-systems with the same number d) of the whole group of M-systems, which have the same M-graph (Construction 4). Then we only choose the optimal M-system (or M-systems) from a finite number of M-systems which are the best ones of the whole class of M-systems with the same M-graph.

A SHORT EXAMPLE

For better understanding of the algorithm solving the problem we shall show one part of it.

Let us have $s = 4$, $m_4 = 2$, $m_1 = 3$, $m_2 = 1$, $m_3 = 5$, $T = 30$. We choose some M-tree by Construction 3, e.g. let $p = 2$, let $X_{b_1} = X_1$, $X_{b_2} = X_3$, let $M_1 = \emptyset$, $M_3 = \{X_2\}$, let $X_{g_2} = X_4$, let $f'(X_4, X_1) = 2$, $f'(X_4, X_2) = 1$, $f'(X_2, X_3) = 1$.

In this way we have chosen an M-tree G' (see fig. 1). The corresponding M-graph G (see fig. 2) determines the fixed M-system ${}_*U$ (see fig. 7), whose coordinates are ${}_*\boldsymbol{a}_4 = (0, 15)$; ${}_*\boldsymbol{a}_1 = (5, 15, 25)$; ${}_*\boldsymbol{a}_2 = (0)$; ${}_*\boldsymbol{a}_3 = (0, 6, 12, 18, 24)$.

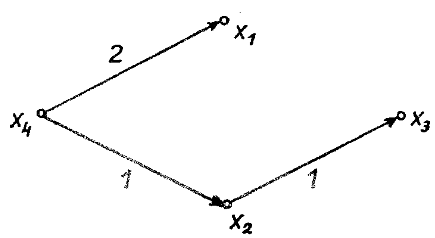

Figure 1 — G′.

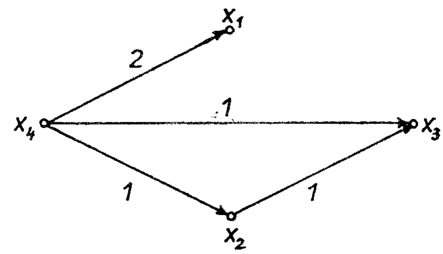

Figure 2 — G.

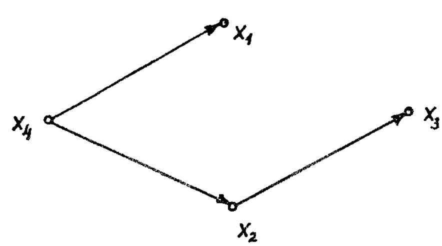

Figure 3 — G_0.

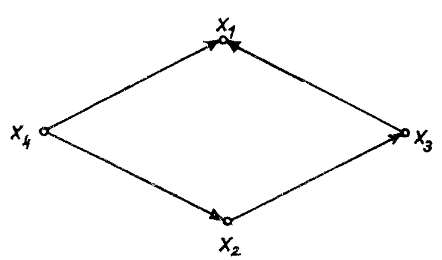

Figure 4 — G_0'.

In the 0-th step of Construction 4 we define ${}_0\boldsymbol{a}_1 = {}_*\boldsymbol{a}_1$; ${}_0\boldsymbol{a}_2 = {}_*\boldsymbol{a}_2$; ${}_0\boldsymbol{a}_3 = {}_*\boldsymbol{a}_3$; ${}_0\boldsymbol{a}_4 = {}_*\boldsymbol{a}_4$ and we determine the graph G_0 (see fig. 3).

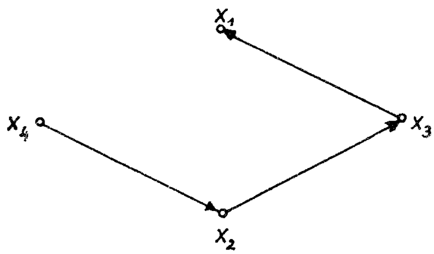

Figure 5 — G_1.

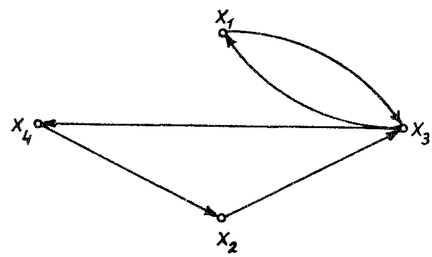

Figure 6 — G_1'.

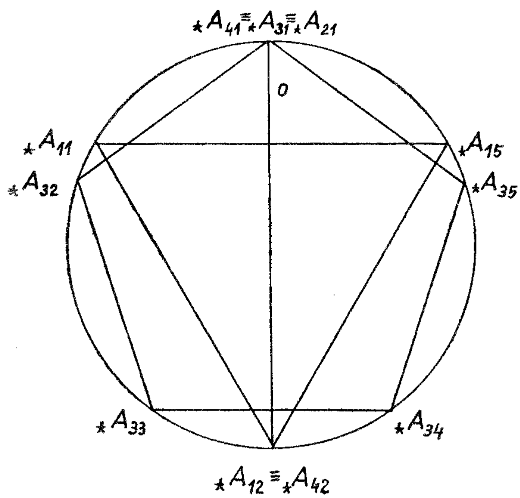

Figure 7 — ${}_*$U.

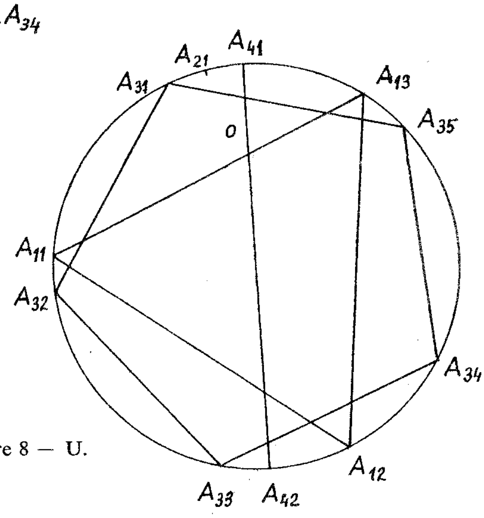

Figure 8 — U.

In the first step we find that $e_1 = \frac{1}{2}$; ${}_1\boldsymbol{a}_4 = (0, 15)$; ${}_1\boldsymbol{a}_1 = (5\frac{1}{2}; 15\frac{1}{2}; 25\frac{1}{2})$; ${}_1\boldsymbol{a}_2 = (\frac{1}{2})$; ${}_1\boldsymbol{a}_3 = (1, 7, 13, 19, 25)$. We determine the graph G_0' (see fig. 4) and the graph G_1 (see fig. 5).

In the second step we find that $e_2 = \frac{1}{2}$; ${}_2\boldsymbol{a}_4 = (0, 15)$; ${}_2\boldsymbol{a}_1 = (7, 17, 27)$; ${}_2\boldsymbol{a}_2 = (1)$; ${}_2\boldsymbol{a}_3 = (2, 8, 14, 20, 26)$. The graph G_1' (see fig. 6) contains two directed cycles, thus Construction 4 is finished. The result is that the best free M-system U (see fig. 8) of those which have the same M-graph G has the vectors with coordinates $\boldsymbol{a}_4 = (0, 15)$; $\boldsymbol{a}_1 = (7, 17, 27)$; $\boldsymbol{a}_2 = (1)$; $\boldsymbol{a}_3 = (2, 8, 14, 20, 26)$ and that $1 = d = \min_{i \neq \bar{i}} \varrho(A_i A_{\bar{i}})$ $(1 \leqq i, \bar{i} \leqq 4)$ holds.

The above problem corresponds e.g. to a network of 4 bus lines which have regular time-tables (the first line has the interval 10 minutes between two vehicles, the second has 30 minutes, the third has 6 minutes, the fourth has 15 minutes). The bus lines have one common segment on which we optimize the transport by the solution of the basic problem.

References

[1] *J. Černý:* Problems for systems of regular polygons on a circumference and their application in transport (Czech). Matematické obzory (1972), 51—59.

[2] *J. Černý:* Applied mathematics and transport (Czech). Pokroky matematiky, fyziky a astronomie 19 (1974), 316—323.

[3] *F. Harrary:* Graph Theory. Addison-Wesley, 1969.

Súhrn

MAXIMALIZÁCIA VZDIALENOSTÍ PRAVIDELNÝCH MNOHOUHOLNÍKOV NA KRUŽNICI

FILIP GULDAN

Tento článok uvádza riešenie základného problému, definovaného v [1] a [2] a riešiaceho konkrétny problém v železničnej a cestnej doprave (problém optimalizácie cestovných grafikonov na základe určitých kriterií).

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